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On the initial-boundary value problems for barotropic motions of a viscous gas in a region with permeable boundaries

By

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1. Introduction

The one-dimensional motion of a viscous polytropic gas is described by the following system of equations [1], [13]:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial p}{\partial y}\ , \tag{1.1}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial y}=0\ , \tag{1.2}$$

$$c_v\rho\left(\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial y}\right)=\kappa\frac{\partial^2 \theta}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2-p\frac{\partial u}{\partial y}\ . \tag{1.3}$$

The system is a simplified form of the Navier-Stokes equations. Here u, ρ, θ and p are the velocity, density, absolute temperature and pressure, respectively — the required characteristics of the medium; y is the Cartesian coordinate; t is the time; μ, c_v, κ are the viscosity, specific heat capacity and thermal conductivity — positive constants.

The system is supplemented with the equation of state

$$p=p(\rho,\ \theta) \tag{1.4}$$

We have a closed set of the equations of an ideal (perfect) gas if the equation of state takes the form

$$p=R\rho\theta\ ,$$

where R is the universal gas constant.

The model called the generalized Burgers' equations of viscous gas is defined by the simplest equation of state:

$$p=\text{const}>0\ .$$

In our paper the main attention will be paid to the equations of a barotro-

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pic motion:

$$p=\rho^{\gamma}\ ,\quad \gamma\geq 1\ .$$

Obviously, in this case the energy equation (1.3) is separated from the system.

The transformation to the Lagrange mass variables plays a great role in our investigation. The importance of the Lagrange formulation of problems for viscous gas equations is based on the fact that the continuity equation (1.2) is an equation for ρ with partial derivatives of first order. Characteristic curves of this equation are integral curves of the ordinary differential equation

$$\frac{dy}{dt}=u(y,t)\ .$$

The method of characteristic curves is the basic one for equations with partial derivatives of first order, and the Lagrange transformation has the same idea. However, it is necessary to note that V. G. Vaigant has recently obtained the interesting results [17], [18], using formulation of problems only in the Cartesian coordinates.

If we formulate the initial-boundary value problem for the system (1.1)-(1.4), then, according to the boundary conditions for the function u, either the side boundaries of a domain of unknowns are characteristic curves of the continuity equation or they simulate permeable walls that is characteristic curves go into or out the domain of definition on these boundaries. In the second case the boundary data have to be prescribed also for ρ if the characteristic curves are going into.

The side boundaries are characteristic curves when the zero (homogeneous) boundary conditions simulate fixed rigid walls or a contact of a viscous gas with vacuum. The main formulation of such boundary value problems for the one-dimensional differential equations of a compressible viscous fluid were investigated by A. Tani [16], A. V. Kazhikhov [4], [5], A. V. Kazhikhov and V. V. Shelukhin [8]. However, there is a great number of physical processes which are described with nonhomogeneous boundary problems: a flow of a gas between moving rigid walls (the double piston problem), flow of a gas through a fixed domain (the flow problem), the filling of a limited volume, the pumping out of a compressible fluid, etc. It is easy to notice that the nonlinearity of equations does not allow to obtain the global in time existence theorems for nonhomogeneous boundary conditions as the consequence of the solvability of homogeneous problems.

Besides, the double piston problem, which was studied first of all, required some additional restriction especially unexpected for the Lagrange formulation.

In the absence of dissipative effect ($\mu=0$), the double piston problem was studied by T. Nishida and J. Smoller [12]. They established that the problem

$$\frac{\partial u}{\partial t}+p(v)=0\ ,\quad v=\rho^{-1}\ ,$$
$$\frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0\qquad \text{in}\quad Q=(0,1)\times(0,T)$$
$$(u(x,0),v(x,0))=(u_0(x),v_0(x))\qquad 0\leq x\leq 1\ ,$$
$$u(0,t)=u_1(t)\ ,\quad u(1,t)=u_2(t)\qquad 0\leq t\leq T;$$

fails to have a solution for any positive time without the following condition:

$$(1.5)\qquad 0<m_0\leq\int_0^1 v_0(x)\,dx+\int_0^T[u_2(t)-u_1(t)]\,dt\leq M_0<\infty\ .$$

Using properties of the solution found by N. Itaya [2], A. V. Kazhikhov [7] constructed an example which showed that the same restriction was necessary in the viscous case. Indeed, it is easy to verify that a set of functions

$$u(x,t)=ax\ ,\quad \rho(x,t)=(1+at)^{-1}\ ,\quad a=\text{const}\ ,$$

is a solution of the system (the Lagrangian record)

$$\frac{\partial u}{\partial t}=\mu\frac{\partial}{\partial x}\left(\rho\frac{\partial u}{\partial x}\right)-\frac{\partial\rho^{\gamma}}{\partial x}\ ,$$
$$\frac{\partial\rho}{\partial t}+\rho^2\frac{\partial u}{\partial x}=0\ ,$$

which satisfies the boundary conditions

$$u(x,0)=u_0(x)=ax\ ,\quad \rho(x,0)=\rho_0(x)=(1+at)^{-1}\qquad 0\leq x\leq 1\ ,$$
$$u(0,t)=u_1(t)=0,\quad u(1,t)=u_2(t)=a\qquad t\geq 0\ .$$

For $a<0$ the condition (1.5) is not valid and the density becomes unbounded at finite time $t_*=-a^{-1}$. That is the solution is destroyed at finite time in spite of the arbitrary smoothness of the data and the arbitrary order of the compatibility conditions.

The example has the obvious explanation if it is reformulated in the Cartesian coordinates: the side boundaries are characteristic curves of the continuity equation and their intersection reduces to destruction at finite time.

The first existence theorems for nonhomogeneous boundary problems were obtained by A. V. Kazhikhov [6], [7], N. Itaya [3], A. Matsumura and T. Nishida [10], T. Nagasawa [11], R. E. Zarnowski [19]. All the studied problems may be called the "characteristic boundary problems", because in every case the boundaries of a domain of definition are characteristic curves of the continuity equation.

This paper deals with the noncharacteristic problems which are more delicate due to the additional boundary conditions for the density ρ and the specific Lagrangian formulations. In the next section 2 we formulate the initial-boundary value problems simulating flows of a viscous gas in regions with permeable walls and the existence theorem for the "flow problem". The theorem is proved by a well-known way: a local solution is continued globally in time by using a priori estimates. The desired a priori estimates are deduced in the Lagrange mass variables. The main feature of the problems is that the side boundaries of a domain of unknowns are not characteristic curves of the continuity equation. Then, although the Lagrange transformation gives the convenient form for the equations but, unlike the homogeneous problems and the characteristic nonhomogeneous ones, the domain of definition is reduced to essentially inconvenient forms. Namely, we obtain curvilinear and unknown boundaries. The Lagrange transformation is described in section 3. Sections 4 and 5 include the presentation of the base of our proof: the estimate of sizes of unknown domain of definition and the step method for the estimates of the density. The final a priori estimates are presented in the last section 6.

2. Formulation of the problems and the existence theorem for the flow problem

We will consider the one-dimensional barotropic motion of a viscous gas inside a certain region with fixed permeable walls. In the first case the gas is constantly pumped in through the left-hand wall and pumped out through the right-hand one. The initial-boundary value problem simulating this process (Problem 1) is called the "flow problem" and has the following formulation.

We have to find a solution of the equations

$$\begin{aligned} &\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial}{\partial y}(\rho^\gamma)\ , \\ &\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial y}=0\ , \end{aligned} \tag{2.1}$$

in some domain $Q_T^Y=\{(y,t): 0<y<Y,\ 0<t<T\}$,
which takes the initial conditions

$$u(y,0)=\tilde{u}_0(y)\ ,\quad \rho(y,0)=\tilde{\rho}_0(y)\qquad \text{for}\quad 0\leq y\leq Y \tag{2.2}$$

and the boundary conditions

$$u(0,t)=u_1(t)>0\ ,\quad \rho(0,t)=\rho_1(t)\ , \tag{2.3}$$

$$u(Y,t)=u_2(t)\geqq 0\qquad \text{for}\quad 0\leq t\leq T\ . \tag{2.4}$$

The additional boundary condition for the function ρ is consistent with the theory of differential equations with partial derivatives of first order: it is necessary to set a boundary condition for a desired solution on the parts with entering characteristic curves.

If the gas is constantly pumped into the region through both of the walls then the process is described by a solution of equations (2.1), which satisfies the conditions (2.2), (2.3) and the following condition

(2.5) $\quad u(Y, t)=u_2(t)<0 \;, \quad \rho(Y, t)=\rho_2(t) \qquad \text{for} \quad 0\leq t\leq T \;.$

This problem will be called Problem 2.

The initial-boundary value problem modelling the process in which the gas is pumped out of the region through both of the walls (Problem 3) has the simplest formulation. We are to find a solution of equations (2.1) with the condition (2.2), (2.4) and the condition

(2.6) $\quad u(0, t)=u_1(t)\leq 0 \qquad \text{for} \quad 0\leq t\leq T \;.$

We will use the notations of well-known functional spaces, which are introduced in [1].

Definition 1. A generalized solution of Problem 1 (2, 3) is a set of functions u, ρ,

$$u(t)\in L_\infty(0, T;\, W_2^1(0, Y))\cap L_2(0, T;\, W_2^2(0, Y)) \;, \quad \frac{\partial u}{\partial t}\in L_2(Q_T^Y) \;,$$
$$\rho(t)\in L_\infty(0, T;\, W_2^1(0, Y)) \;, \quad \frac{\partial \rho}{\partial t}\in L_2(Q_T^Y) \;,$$

obeying equations (2.1) almost everywhere in Q_T^Y and taking the given initial and boundary values in the sense of traces of the functions from the mentioned classes.

Theorem 1. Suppose that

$$\tilde{u}_0\in C^{2+\alpha}(0, Y) \;, \quad \tilde{\rho}_0\in C^{1+\alpha}(0, Y) \;,$$
$$(u_1, u_2)\in C^{1+\frac{\alpha}{2}}(0, T) \;, \quad \rho_1\in C^2(0, T) \;, \quad 0<\alpha<1 \;,$$
$$\tilde{u}_0(0)=u_1(0) \;, \quad \tilde{u}_0(Y)=u_2(0) \;, \quad \tilde{\rho}_0(0)=\rho_1(0) \;,$$
$$0<m_0\leq (u_1, \tilde{\rho}_0, \rho_1)\leq M_0<\infty \;,$$

where m_0 and M_0 are some constants,
and the first order compatibility conditions are satisfied in the points (0, 0) *and* (Y, 0). *Then there exists a unique classical solution of Problem* 1 *such that*

$$u(y, t)\in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T^Y) \;, \quad \rho(y, t)\in C^{1+\alpha, 1+\frac{1+\alpha}{2}}(Q_T^Y) \;, \quad \rho(y, t)>0 \;.$$

If the data belong to a wider class:

$$(\tilde{u}_0, \tilde{\rho}_0)\in W_2^1(0, Y) \;, \quad (u_1, u_2, \rho_1)\in W_2^1(0, T) \;,$$

and

$$\tilde{u}_0(0)=u_1(0) \;, \quad \tilde{u}_0(Y)=u_2(0) \;, \quad \tilde{\rho}_0(0)=\rho_1(0) \;,$$
$$0<m_0\leq (u_1, \rho_0, \rho_1)\leq M_0<\infty \;,$$

then there exists a unique generalized solution of Problem 1 *such that* $\rho>0$.

The existence of a unique classical solution over the whole time interval $[0, T]$ can be obtained by a known procedure: a local solution is continued globally in time by using a priori estimates. The local existence theorem is proved in a way combining arguments presented in [1], [14] [15]. The global in time generalized solution is constructed as a limit of a sequence of classical solutions with smooth and compatible initial and boundary data. The proof of the uniqueness of the generalized solution does not differ from that one which are given in [1] for a homogeneous initial-boundary value problem.

Therefore we will pay the main attention to the a priori estimates. Using Kazhikhov's scheme we will devise the estimates for a solution of the problem formulated in the Lagrange mass variables. The necessary estimates can be classified into three groups:

i. initial integral (energy) relations;

ii. the strict positiveness and the boundedness of the density ρ;

iii. integral estimates for derivatives and bounds for Hölder continuity constants.

The estimates of the first group are not complicated by nonhomogeneous boundary conditions too much. However, the proof of the strict positiveness and the boundedness for ρ takes our attention.

The existence theorems for Problem 2 and Problem 3 have the same formulation as Theorem 1. For these problems we will prove only the estimates of the second group.

3. The Lagrange formulation of the problems

Suppose the conditions of the first part of Theorem 1 are satisfied and the problem has a classical solution with $\rho > 0$. Over a small time interval this is guaranteed by the local existence theorem.

Let us consider the Cauchy problem

$$\frac{dz}{d\tau} = u(z, \tau) \ , \quad z|_{\tau=t} = y \ , \tag{3.1}$$

where $y \in [0, Y]$, $t \in [0, T]$.
The solution $z = z(\tau; y, t)$ defines a characteristic curve passing through a point (y, t). The domain of unknowns is divided into two parts by the characteristic curve $z = z_0(t)$ passing through the "initial" point $(0, 0)$: $z_0(t) = z(t; 0, 0)$. The right-hand part is transformed by a routine procedure [1]. But the mass Lagrange variables for the points of the left-hand part are defined by an original method [6].

If we take a point $(y, t) \in Q_T^Y$ and $y < z_0(t)$ then there exists a number $\xi > 0$ such that

$$z(\xi; y, t) = 0 \ . \tag{3.2}$$

At first we will use new variables ξ and t. The Jacobian of the transformation $J=\dfrac{\partial \xi}{\partial y}$ is obtained from (3.1) and (3.2) by the formula

$$J=-u_1^{-1}(\xi)\exp\left\{-\int_\xi^t \frac{\partial u}{\partial z}(z(\tau;\, y,\, t),\, \tau)\,d\tau\right\} .$$

On the other hand, the continuity equation

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial y}=0$$

can be written along the curve z $(\tau;\, y,\, t)$ as

$$\frac{d\ln\rho}{d\tau}=-\frac{\partial u}{\partial z}(z(\tau;\, y,\, t),\, \tau) .$$

Hence,

$$\rho(\xi,\, t)=-\rho_1(\xi)u_1(\xi)J(\xi,\, t) .$$

Thus, in the new variables system (2.1) assumes the form

$$\frac{\partial u}{\partial t}=\frac{\mu}{(\rho_1 u_1)^2}\frac{\partial}{\partial \xi}\left(\rho\frac{\partial u}{\partial \xi}\right)+\frac{1}{\rho_1 u_1}\frac{\partial}{\partial \xi}(\rho^\gamma) .$$

$$\frac{\partial \rho}{\partial t}-\frac{\rho^2}{\rho_1 u_1}\frac{\partial u}{\partial \xi}=0 .$$

Let us set

$$-\rho_1(\xi)u_1(\xi)d\xi=dx , \quad x(\xi)=-\int_0^\xi \rho_1(\tau)u_1(\tau)d\tau .$$

In the variables x and t the equations take the usual Lagrangian record:

(3.3) $$\frac{\partial u}{\partial t}=\mu\frac{\partial}{\partial x}\left(\rho\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}(\rho^\gamma) ,$$

(3.4) $$\frac{\partial \rho}{\partial t}+\rho^2\frac{\partial u}{\partial x}=0 .$$

Finally, we have to find an image of the boundaries. The left-hand boundary is transformed to a known curve:

$$x=-\int_0^t \rho_1(\tau)u_1(\tau)d\tau .$$

Let us consider a point $(Y,\, t)$. The characteristic curve z $(\tau\colon Y,\, t)$ may intersect the boundary of domain Q_T^Y by two ways. In the first case there exists a number $y_0\in[0,\, Y]$ such that z $(0;\, Y,\, t)=y_0$, and the mass Lagrange variable is equal to

$$x(Y,\, t)=\int_0^{y_0}\tilde{\rho}_0(s)ds .$$

However, integrating the continuity equation over domain
$\{(y,\tau): 0<\tau<t,\ z\ (\tau;\ Y,t)<y<Y\}$,
we see

$$\int_0^{y_0}\tilde{\rho}(s)ds=\int_0^Y\tilde{\rho}_0(s)ds-\int_0^t\rho(Y,\tau)u_2(\tau)d\tau\ .$$

In the second case, there exists a number $\xi>0$ such that $z(\xi;\ Y,t)=0$, and the mass Lagrange variable is defined by the formula:

$$x(Y,t)=-\int_0^{\xi}\rho_1(\tau)u_1(\tau)d\tau\ .$$

Integrating the continuity equation over domain
$\{(y,\tau): 0<y<Y,\ 0<\tau\leq\xi\}\cup\{(y,\tau): \xi<\tau<t,\ z\ (\tau;\ Y,t)<y<Y\}$,
we have

$$-\int_0^{\xi}\rho_1(\tau)u_1(\tau)d\tau=\int_0^Y\tilde{\rho}_0(s)ds-\int_0^t\rho(Y,\tau)u_2(\tau)d\tau\ .$$

Thus, the right-hand boundary is transformed to an unknown curve:

$$x=\int_0^Y\rho_0(s)ds-\int_0^t\rho(Y,\tau)u_2(\tau)d\tau\ .$$

We have obtained the following Lagrange formulation of the problems with "permeable boundaries".

Problem 1. We are to find a solution of equations (3.3), (3.4) in the domain

$$Q_T^1=\{(x,t): 0<t<T,\ a_1(t)<x<b_1(t)\}\ ,$$

where

$$a_1(t)=-\int_0^t\rho_1(\tau)u_1(\tau)d\tau\ ,$$

$$b_1(t)=X-\int_0^t\rho(b_1(\tau),\tau)u_2(\tau)d\tau\ ,$$

$$X=\int_0^Y\tilde{\rho}_0(s)ds\ ,$$

taking the initial conditions

$$(3.5)\qquad u(x,0)=u_0(x)\ ,\quad \rho(x,0)=\rho_0(x)\qquad \text{for}\quad 0\leq x\leq X$$
$$(u_0(x)=\tilde{u}_0(y)\ ,\quad \rho_0(x)=\tilde{\rho}_0(y)\qquad \text{if}\quad x=\int_0^y\tilde{\rho}_0(s)ds)$$

and the boundary conditions

$$(3.6)\qquad u(a_1(t),t)=u_1(t)>0\ ,\quad \rho(a_1(t),t)=\rho_1(t)\qquad \text{for}\quad 0\leq t\leq T\ ,$$

(3.7) $$u(b_1(t), t) = u_2(t) \geq 0 \qquad \text{for} \quad 0 \leq t \leq T .$$

Problem 2. We are to find a solution of equations (3.3), (3.4) in the domain

$$Q_T^2 = \{(x, t): 0 < t < T, a_1(t) < x < b_2(t)\} ,$$

where

$$b_2(t) = X - \int_0^t \rho_2(\tau) u_2(\tau) d\tau ,$$

taking the initial conditions (3.5) and the boundary conditions (3.6) and

(3.8) $$u(b_2(t), t) = u_2(t) < 0 , \quad \rho(b_2(t), t) = \rho_2(t) \qquad \text{for} \quad 0 \leq t \leq T .$$

Problem 3. We are to find a solution of equations (3.3), (3.4) in the domain

$$Q_T^3 = \{(x, t): 0 < t < T, a_2(t) < x < b_1(t)\} ,$$

where

$$a_2(t) = -\int_0^t \rho(a_2(\tau), \tau) \, u_1(\tau) d\tau ,$$

taking the initial conditions (3.5) and the boundary conditions (3.7) and

(3.9) $$u(a_2(t), t) = u_1(t) \leq 0 \qquad \text{for} \quad 0 \leq t \leq T .$$

4. Auxiliary constructions

We will use the Lagrange formulation of the flow problem to prove initial a priori estimates. The constants which depend only on the data of the problems and T will be denoted by C, m, M (with a subscript).

Integrating equation (3.4), written in the form $(\rho^{-1})_t = u_x$, over the region

$$Q_t^1 = \{(x, \tau): 0 < \tau < t, a_1(\tau) < x < b_1(\tau)\}$$

we obtain the following relation:

(4.1) $$\int_{a_1(t)}^{b_1(t)} \rho^{-1}(x, t) dx = \int_0^X \rho_0^{-1}(x) dx = Y \qquad \text{for} \quad 0 \leq t \leq T .$$

To derive the first integral estimate, we substitute $u = w + \tilde{u}$ into equations (3.1), (3.2), where

(4.2) $$\tilde{u}(x, t) = (u_2(t) - u_1(t)) Y^{-1} \int_{a_1(t)}^x \rho^{-1}(s, t) ds + u_1(t) ,$$

multiply the first equation by w and the second equation by $(1 - \rho^\gamma)$, and then integrate their sum over Q_t^1. After simple reduction, estimating the right side with the help of the Cauchy inequality and using Gronwall's inequality, we can deduce

$$(4.3)\qquad \max_{0\le t\le T}\int_{a_1(t)}^{b_1(t)}[u^2(x,t)+\phi(x,t)]\,dx+\iint_{Q_T^1}\rho\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt\le C_1\ ,$$

where

$$\phi(x,t)=\int_1^{\rho^{-1}(x,t)}[1-s^{-\gamma}]\,ds\ge 0\ .$$

If we followed to Kazhiknov's scheme [1] we should obtain a known representation for $\rho(x,t)$ as the next step. However, in our case the domain of definition of the problem is a curvilinear one and we cannot deduce the necessary equality at once.

In what follows, we will use the Lagrange and the Cartesian formulations of the problem simultaneously.

The Lagrange image of any characteristic curve of the continuity equation is a segment of a vertical line $x=const$, which intersects the left-hand boundary at the point $(b^1(t'), t')$ if and only if the characteristic curve reaches to the point (Y, t'). Thus, to estimate the unknown boundary we have to understand "how long" a characteristic curve must be inside the domain.

Using the formula of inverse transformation to the Cartesian variables, for $\xi\ge 0$ and $t>0$ such that $\xi+t\le T$ we have

$$(4.4)\qquad y(\xi+t;0,\xi)=\int_\xi^{\xi+t}u(a_1(\xi),\tau)\,d\tau\le \int_\xi^{\xi+t}\left[u_1(\tau)+\frac{1}{4\varepsilon}\int_{a_1(\tau)}^{b_1(\tau)}\rho^{-1}(x,\tau)\,dx+\varepsilon\int_{a_1(\tau)}^{b_1(\tau)}\rho\left(\frac{\partial u}{\partial x}\right)^2 dx\right]d\tau\ .$$

Let $\varepsilon=\dfrac{Y}{4C_1}$, then

$$y(\xi+t;0,\xi)\le t(M_0+C_1)+\frac{1}{4}Y\ ,$$

where M_0 is the constant listed in Theorem 1. Hence, if $t\le\frac{1}{4}Y(M_0+C_1)^{-1}$, we obtain

$$(4.5)\qquad y(\xi+t;0,\xi)\le\frac{Y}{2}\ ,$$

i.e. the characteristic curve "entering" into the domain of definition at the moment $\xi\ge 0$ cannot "pass" through the domain by the moment $t^0<\min\left\{\xi+\frac{1}{4}Y(M_0+C_1)^{-1},\ T\right\}$.

Let us introduce the notations

$$\tilde{t}=\max\left\{t: t<\min\{T, \frac{1}{4}(M_0+C_1)^{-1}\}, \frac{T}{t}=\left[\frac{T}{t}\right]\right\},$$

$$t_k=k\tilde{t}, \quad k=0, 1, \dots, \frac{T}{\tilde{t}}.$$

Then the result obtained above can be formulated by the following way.

Lemma 1. *If* $0\le t^*\le T-2\tilde{t}$, $0\le t\le\tilde{t}$,
then

$$a_1(t^*)\le b_1(t^*+\tilde{t}+t),$$
$$b_1(t^*+\tilde{t}+t)-a_1(t^*+\tilde{t})\ge m_0^2\tilde{t},$$
$$b_1(t)\ge\frac{1}{2}m_0Y.$$

where m_0 *is the constant listed in Theorem* 1.

Note. To prove the third inequality of the lemma we have to trace for the characterestic curve $y=y(\tau; \frac{1}{2}Y, 0)$

5. The strict positiveness and boundedness of the density

At first we consider the strip

$$Q^1_{(0,t_1]}=\{(x, t): 0<t\le t_1, a_1(t)\le x\le b_1(t)\}.$$

Integrating equation (3. 4), written in the form $(\rho^{-1})_t=u_x$, over the region

$$Q^1_{(0,t)}=\{(x, \tau): 0<\tau<t, 0<x<b_1(\tau)\},$$

we obtain

$$\int_0^{b_1(t)}\rho^{-1}(x, t)dx=Y-\int_0^t u(0, \tau)d\tau.$$

Keeping in mind relations (4.4), (4.5), for $0<t\le t_1$ we have

$$\frac{1}{2}Y\le\int_0^{b^1(t)}\rho^{-1}(x, t)dx\le\frac{3}{2}Y. \tag{5.1}$$

Using the third inequality of Lemma 1 we see that for each $t\in[0, t_1]$ there exists at least one number $x_1(t)\in[0, b_1(t)]$ with the property

$$\frac{1}{3}m_0\le\rho(x_1(t), t)\le 2XY^{-1}. \tag{5.2}$$

Now we may use Kazhikhov's arguments [1] for the region
$Q^{11}_{(0,t_1]}=\{(x, t): 0<t\le t_1, 0\le x\le b^1(t)\}$.

Let us rewrite equation (3.4) as

$$\rho\frac{\partial u}{\partial x}=-\frac{\partial \ln\rho}{\partial t} ,$$

and substitute $\rho\frac{\partial u}{\partial x}$ into equation (3.3):

$$\frac{\partial u}{\partial t}=-\mu\frac{\partial^2}{\partial x\partial t}(\ln\rho)-\frac{\partial}{\partial x}(\rho^{\gamma}) . \tag{5.3}$$

We integrate (5.3) over $(x_1(t), x)\times(0, t)\subset Q^{11}_{(0,t_1]}$ and, taking the exponential, obtain

$$\rho(x, t)\exp\left\{\frac{1}{\mu}\int_0^t\rho^{\gamma}(x, \tau)d\tau\right\}=Y_{11}(t)B_{11}(x, t) , \tag{5.4}$$

where

$$Y_{11}(t)=\frac{\rho(x_1(t), t)}{\rho_0(x_1(t))}\exp\left\{\frac{1}{\mu}\int_0^t\rho^{\gamma}(x_1(t), \tau)d\tau\right\} ,$$
$$B_{11}(x, t)=\rho_0(x)\exp\left\{\frac{1}{\mu}\int_{x_1(t)}^x[u_0(s)-u(s, t)]ds\right\} .$$

Using (4.3), (5.2) and the properties of the initial data, we have

$$C_2\leq B_{11}(x, t)\leq C_3 , \tag{5.5}$$

$$C_4\leq Y_{11}(t) . \tag{5.6}$$

We will show that $Y_{11}(t)$ is bounded from above as well. The following equality holds

$$\frac{d}{dt}\exp\left\{\frac{\gamma}{\mu}\int_0^t\rho^{\gamma}(x, \tau)d\tau\right\}=\frac{\gamma}{\mu}Y_{11}^{\gamma}(t)B_{11}^{\gamma}(x, t) ,$$

which implies

$$\exp\left\{\frac{1}{\mu}\int_0^t\rho^{\gamma}(x, \tau)d\tau\right\}=\left(1+\frac{\gamma}{\mu}\int_0^t Y_{11}^{\gamma}(\tau)B_{11}^{\gamma}(x, \tau)d\tau\right)^{\frac{1}{\gamma}} ,$$

Hence, (5.4) can be rewritten as

$$\rho(x, t)=Y_{11}(t)B_{11}(x, t)\left(1+\frac{\gamma}{\mu}\int_0^t Y_{11}^{\gamma}(\tau)B_{11}^{\gamma}(x, \tau)d\tau\right)^{-\frac{1}{\gamma}} \tag{5.7}$$

and

$$\rho^{-1}(x, t)Y_{11}(t)=B_{11}^{-1}(x, t)\left(1+\frac{\gamma}{\mu}\int_0^t Y_{11}^{\gamma}(\tau)B_{11}^{\gamma}(x, \tau)d\tau\right)^{\frac{1}{\gamma}} . \tag{5.8}$$

Integrating (5.8) with respect to x from 0 to $b_1(t)$ and using (5.1), (5.5), we obtain inequality

$$Y_{11}(t)\leq C_5\left(1+\frac{1}{\mu}\int_0^t Y_{11}^{\gamma}(\tau), d\tau\right)^{\frac{1}{\gamma}} ,$$

from which the estimates from above for $Y_{11}^{\gamma}(t)$ and for $Y_{11}(t)$ follow by Gronwall's inequality. Therefore the bounds for $\rho(x, t)$ are directly derived from (5.7):

$$m \leq \rho(x, t) \leq M \quad \text{for} \quad 0 < t \leq t_1 \ , \quad 0 \leq x \leq b_1(t) \ . \tag{5.9}$$

Now we are able to estimate the density in a region

$$Q^{12}_{(0,t_1]} = \{(x, t) \colon 0 < t \leq t_1, \ a_1(t) < x < 0\} \ .$$

We integrate (5.3) over $(x, 0) \times (t^*(x), t)$, Where $t = t^*(x)$ is the inverse function for $x = a_1(t)$, and, taking the exponential, obtain

$$\rho(x, t) \exp\left\{\frac{1}{\mu}\int_{t^*(x)}^{t} \rho^{\gamma}(x, \tau)\, d\tau\right\} = Y_{12}(x, t) B_{12}(x, t) \ , \tag{5.10}$$

where

$$Y_{12}(x, t) = \frac{\rho(0, t)}{\rho(0, t^*(x))} \exp\left\{\frac{1}{\mu}\int_{t^*(x)}^{t} \rho^{\gamma}(0, \tau)\, d\tau\right\} \ ,$$

$$B_{12}(x, t) = \rho_1(t^*(x)) \exp\left\{\frac{1}{\mu}\int_0^x [u(s, t^*(x)) - u(s, t)]\, ds\right\} \ .$$

We have

$$\exp\left\{\frac{1}{\mu}\int_{t^*(x)}^{t} \rho^{\gamma}(x, \tau)\, d\tau\right\} = \left(1 + \frac{\gamma}{\mu}\int_{t^*(x)}^{t} Y_{12}^{\gamma}(\tau) B_{12}^{\gamma}(x, \tau)\, d\tau\right)^{\frac{1}{\gamma}} \ , \tag{5.11}$$

and

$$\rho(x, t) = Y_{12}(x, t) B_{12}(x, t) \left(1 + \frac{\gamma}{\mu}\int_{t^*(x)}^{t} Y_{12}^{\gamma}(\tau) B_{12}^{\gamma}(x, \tau)\, d\tau\right)^{-\frac{1}{\gamma}} \ , \tag{5.12}$$

Using (4.3), (5.9), we see

$$\begin{aligned} &C_6 \leq B_{12}(x, t) \leq C_7 \ , \\ &C_8 \leq Y_{12}(x, t) \leq C_9 \ , \end{aligned} \tag{5.13}$$

Then equality (5.12) guarantees the strict positiveness and the boundedness of the density in $Q^{12}_{(0,t_1]}$.

Thus, we obtain the necessary estimates in the first strip:

$$m_1 \leq \rho(x, t) \leq M_1 \quad \text{for} \quad (x, t) \in Q^{1}_{(0,t_1]}$$

On the second step we will consider the next strip

$$Q^{1}_{(t_1,t_2]} = \{(x, t) \colon t_1 < t \leq t_2, \ a_1(t) \leq x \leq b_1(t)\} \ .$$

Similarly to the first step we obtain

$$\int_{a_1(t_1)}^{b_1(t)} \rho^{-1}(x, t)\, dx = Y - \int_{t_1}^{t} u(a_1(t_1), \tau)\, d\tau \quad \text{for} \quad t_1 < t \leq t_2 \ .$$

Hence,

$$\frac{1}{2}Y\leq\int_{a_1(t_1)}^{b_1(t)}\rho^{-1}(x,\ t)\,dx\leq\frac{3}{2}Y \qquad \text{for} \quad t_1<t\leq t_2\ .$$

Using Lemma 1, we see that for each $t\in(t_1,\ t_2)$ there exists at least one number $x_2(t)\in[a_1(t_1),\ b_1(t)]$ with the property

$$\frac{2}{3}m_0^2 t_1 Y^{-1}\leq\rho(x_2(t),\ t)\leq 2(X+t_1M_0^2)\,Y^{-1}\ .$$

Repeating the arguments of the first step for the regions

$$Q^{11}_{(t_1,t_2]}=\{(x,\ t):\ t_1<t\leq t_2,\ a_1(t_1)\leq x\leq b_1(t)\}$$

and

$$Q^{12}_{(t_1,t_2]}=\{(x,\ t):\ t_1<t\leq t_2,\ a_1(t)<x<a_1(t_1)\}\ ,$$

we obtain the desired estimates for the density in the second strip:

$$m_2\leq\rho(x,\ t)\leq M_2 \qquad \text{for} \quad (x,\ t)\in Q^1_{(t_1,t_2]}\ .$$

On every following step the arguments are similar to the second one. One n^{th} step we obtain

$$m_n\leq\rho(x,\ t)\leq M_n$$

for $(x,\ t)\in Q^1_{(t_{n-1},t_n]}=\{(x,\ t):\ t_{n-1}<t\leq t_n,\ a_1(t)\leq x\leq b_1(t)\}$.

However, $n\leq\frac{T}{\tilde{t}}$.

Therefore we find the bounds for the density in the whole domain:

$$m\leq\rho(x,\ t)\leq M \qquad \text{for} \quad (x,\ t)\in Q^1_T\ . \tag{5.14}$$

In conclusion of the section, let us consider Problem 2 and Problem 3. Our attention will be paid only to the bounds for the density.

If we use the Lagrange variables then Problem 2 is defined in the domain

$$Q^2_T=\{(x,\ t):\ 0<t\leq T,\ a_1(t)<x<b_2(t)\}\ .$$

The first auxiliary relations have the known form:

$$\int_{a_1(t)}^{b_2(t)}\rho^{-1}(x,\ t)\,dx=\int_0^Y\rho_0^{-1}(x)\,dx=Y \qquad \text{for} \quad 0\leq t\leq T\ , \tag{5.15}$$

$$\max_{0\leq t\leq T}\int_{a_1(t)}^{b_2(t)}u^2(x,\ t)\,dx+\iint_{Q^2_T}\rho\Big(\frac{\partial u}{\partial x}\Big)^2 dx\,dt\leq C_{10}\ . \tag{5.16}$$

We can introduce the quantities similar to the case of Problem 1. But we have to make more narrow decomposition of the domain Q^2_T:

$$\widetilde{t}=\max\left\{t:\ t<\min\{T,\ \frac{1}{8}Y(2C_{10}+M_0)^{-1}\},\ \left[\frac{T}{t}\right]=\frac{T}{t}\right\}\ ,$$

$$t_k=k\widetilde{t}\ ,\quad k=0,\ 1,\ \dots,\ \frac{T}{\widetilde{t}}\ .$$

When we consider the strip

$$Q^2_{(t_{i-1},t_i]}=\{(x,\ t):\ t_{i-1}<t\le t_i,\ a_1(t)\le x\le b_2(t)\}$$

we have already known the bounds for $\rho(x,\ t_{i-1})$, $(a_1(t_{i-1})\le x\le b_2(t_{i-1}))$. We see

$$\int_{a_1(t_{i-1})}^{b_2(t_{i-1})}\rho^{-1}(x,\ t)\,dx=Y-\int_{t_{i-1}}^{t}u(a_1(t_{i-1}),\ \tau)\,d\tau+\int_{t_{i-1}}^{t}u(b_2(t_{i-1}),\ \tau)\,d\tau \quad \text{for}\quad t_{i-1}<t\le t_i\ ,$$

and

$$\frac{1}{2}Y\le\int_{a_1(t_{i-1})}^{b_2(t_{i-1})}\rho^{-1}(x,\ t)\,dx\le\frac{3}{2}Y\qquad\text{for}\quad t_{i-1}<t\le t_i\ .$$

Therefore for each $t\in(t_{i-1},\ t_i]$ there exists at least one number $x_i(t)\in[a_1(t_{i-1}),\ b_2(t_{i-1})]$ with the property

$$\frac{2}{3}Y^{-1}X\le\rho(x_i(t),\ t)\le 2Y^{-1}\left(X+\int_0^T[\rho_1(\tau)u_1(\tau)-\rho_2(\tau)u_2(\tau)]\,d\tau\right)\ .$$

After this we can obtain the known representation for $\rho(x,\ t)$ in the region

$$Q^{21}_{(t_{i-1},t_i]}=\{(x,\ t):\ t_{i-1}<t\le t_i,\ a_1(t_{i-1})\le x\le b_2(t_{i-1})\}$$

and can find the bounds for the density. Considering the subdomains

$$Q^{22}_{(t_{i-1},t_i]}=\{(x,\ t):\ t_{i-1}<t\le t_i,\ a_1(t)<x<a_1(t_{i-1})\}$$

and

$$Q^{23}_{(t_{i-1},t_i]}=\{(x,\ t):\ t_{i-1}<t\le t_i,\ b_2(t_{i-1})<x<b_2(t)\}$$

and using the arguments presented for Problem 1, we have the desired estimates in the whole strip:

$$m_i\le\rho(x,\ t)\le M_i\qquad\text{in}\quad Q^2_{(t_{i-1},t_i]}\ .$$

The number of the strips is bounded. Thus we obtain the estimates for the density from above and from below in the whole domain Q^2_T.

After obtaining the first auxiliary relations for a solution of Problem 3 it is also possible to decompose the domain Q^3_T. But we have to use the most narrow strips:

$$Q_{(t_{i-1},t_i]}=\{(x,\ t):\ t_{i-1}<t\le t_i,\ a_2(t)\le x\le b_1(t)\}\ ,$$

$$\widetilde{t}=\max\{t:\ t<\min\{T,\ \frac{1}{16}(M+4C_{11})^{-1}\},\ \left[\frac{T}{t}\right]=\frac{T}{t}\}\ ,$$

$$t_k=k\tilde{t}\ ,\quad k=0,\,1,\,\dots,\,\frac{T}{\tilde{t}}$$

Here C_{11} is the constant from the first energy estimate for a solution of Problem 3.

When we start to consider the strip $Q^3_{(t_{i-1},t_i]}$ we have already known bounds for $\rho(x,\,t_{i-1})$:

$$m_{i-1}\le\rho(x,\,t_{i-1})\le M_{i-1}\qquad\text{for}\quad a_2(t_{i-1})\le x\le b_1(t_{i-1})\ .$$

Moreover, there exist the numbers $x_i^1\in[a_2(t_{i-1}),\,b_1(t_{i-1})]$ and $x_i^2\in[a_2(t_{i-1}),\,b_1(t_{i-1})]$ such that

$$\int_{a_2(t_{i-1})}^{x_i^1}\rho^{-1}(s,\,t_{i-1})\,ds=\frac{1}{3}Y\qquad\int_{x_i^2}^{b_1(t_{i-1})}\rho^{-1}(s,\,t_{i-1})\,ds=\frac{2}{3}Y$$

and

$$a_2(t)>x_i^1\ ,\quad x_i^2<b_1(t)\qquad\text{for}\quad t\in(t_{i-1},\,t_i]\ .$$

Obviously,

$$\int_{x_i^1}^{x_i^2}\rho^{-1}(s,\,t_{i-1})\,ds=\frac{1}{3}Y\ ,$$

and

$$\frac{1}{3}Ym_{i-1}\le x_i^2-x_i^1\le\frac{1}{3}YM_{i-1}\ .$$

Thus,

$$\int_{x_i^1}^{x_i^2}\rho^{-1}(s,\,t)\,ds=\frac{1}{3}Y+\int_{t_{i-1}}^{t}u(x_i^2,\,\tau)\,d\tau-\int_{t_{i-1}}^{t}u(x_i^2,\,\tau)\,d\tau$$

and

$$\frac{1}{12}Y\le\int_{x_i^1}^{x_i^2}\rho^{-1}(s,\,t)\,ds\le\frac{7}{12}Y\ .$$

Hence, we can successively consider the following subregions

$$Q_i^1=\{(x,\,t):\,t_{i-1}<t\le t_i,\,x_i^1\le x\le x_i^2\}\ ,$$
$$Q_i^2=\{(x,\,t):\,t_{i-1}<t\le t_i,\,a_2(t)\le x<x_i^1\}\ ,$$
$$Q_i^3=\{(x,\,t):\,t_{i-1}<t\le t_i,\,x_i^2<x\le b_1(t)\}\ ,$$

and deduce the desired estimates for the density in the i-th strip. After finite number of the steps we obtain the bounds for ρ in the whole domain Q^3_T.

6. Final a priori estimates

The next integral estimate also has some peculiarities connected with nonhomogeneous boundary conditions and curvilinearity of our domain.

At first, we substitute $u=w+\tilde{u}$ into equations (3.3), (3.4), where

$$\tilde{u}(x,t)=Y^{-1}(u_2(t)-u_1(t))\int_{a^1(t)}^{x}\rho^{-1}(s,t)\,ds+u_1(t)\ ,$$

and differentiate the second equation, written in the form

$$\frac{\partial \ln\rho}{\partial t}=-\rho\frac{\partial w}{\partial x}-(u_2-u_1)Y^{-1}\ ,$$

with respect to x. We obtain the system

$$\frac{\partial w}{\partial t}=\mu\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)-\gamma\rho^{\gamma}\frac{\partial \ln\rho}{\partial x}-K(x,t)\ ,\tag{6.1}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln\rho}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\ ,\tag{6.2}$$

where

$$K(x,t)=Y^{-1}(u'_2(t)-u'_1(t))\int_{a^1(t)}^{x}\rho^{-1}(s,t)\,ds+u'_1(t)+Y^{-1}(u_2-u_1)\,(w+\tilde{u})\ .$$

We multiply equation (6.1) by $\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)$ and equation (6.2) by $\frac{\partial \ln\rho}{\partial x}$ and then we integrate their sum over Q^1_t. After simple reductions we see

$$\begin{aligned}\frac{1}{2}\int_{a_1(\tau)}^{b_1(\tau)}\left[\rho\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]dx\Big|_0^t+\mu\int_0^t\int_{a_1(t)}^{b_1(t)}\left[\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right]^2dx\,d\tau=\\ =\frac{1}{2}\int_0^t\left[2\frac{\partial w}{\partial t}\rho\frac{\partial w}{\partial x}-u\rho^2\left(\frac{\partial w}{\partial x}\right)^2-u\rho\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]\Big|_{x=a_1(\tau)}^{x=b_1(\tau)}d\tau+\\ +\int_0^t\int_{a_1(t)}^{b_1(t)}K\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)dx\,d\tau+\int_0^t\int_{a_1(\tau)}^{b_1(\tau)}(\gamma\rho^{\gamma}-1)\frac{\partial \ln\rho}{\partial x}\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)dx\,d\tau-\\ -\frac{1}{2}\int_0^t\int_{a_1(t)}^{b_1(t)}\rho^2\left(\frac{\partial w}{\partial x}\right)^3dx\,d\tau+\frac{1}{2}Y^{-1}\int_0^t\int_{a_1(t)}^{b_1(t)}(u_2-u_1)\rho\left(\frac{\partial w}{\partial x}\right)^2dx\,d\tau\ .\end{aligned}\tag{6.3}$$

Let us note that the integral

$$\int_0^t\left[u\rho\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]\Big|_{x=b_1(\tau)}d\tau$$

is non-negative and

$$\frac{\partial w}{\partial t}\Big|_{x=a_1(t)}=-a'_1\frac{\partial w}{\partial x}\Big|_{x=a_1(t)}=\rho_1u_1\frac{\partial w}{\partial x}\Big|_{x=a_1(t)}\ ,$$

$$\frac{\partial w}{\partial t}\Big|_{x=b_1(t)}=-b'_1\frac{\partial w}{\partial x}\Big|_{x=b_1(t)}=\left(\rho u\frac{\partial w}{\partial x}\right)\Big|_{x=b_1(t)}\ ,$$

$$\left.\frac{\partial \ln\rho}{\partial x}\right|_{x=a_1(t)}=\frac{1}{a_1{}'\rho_1}\left(\rho'_1-\left.\frac{\partial\rho}{\partial t}\right|_{x=a_1(t)}\right)=$$
$$=\frac{1}{u_1\rho_1^2}\left(\rho_1^2\left.\frac{\partial w}{\partial x}\right|_{x=a_1(t)}+Y^{-1}(u_2-u_1)\rho_1-\rho'_1\right) .$$

Therefore, using the Cauchy inequality, we can estimate the right-hand side of (6.3):

(6.4) $$\int_{a_1(t)}^{b_1(t)}\left[\rho\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial\ln\rho}{\partial x}\right)^2\right]dx+\mu\int_0^t\int_{a_1(\tau)}^{b_1(\tau)}\left[\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right]^2dx\,d\tau\leq$$
$$\leq C_{12}\left[1+\int_0^t\int_{a_1(\tau)}^{b_1(\tau)}\left[\frac{\partial\ln\rho}{\partial x}\right]^2dx\,d\tau+\int_0^t\max_{a_1(\tau)\leq x\leq b_1(\tau)}\left|\rho\frac{\partial w}{\partial x}\right|^2d\tau+\right.$$
$$\left.+\int_0^t\max_{a_1(\tau)\leq x\leq b_1(\tau)}\left|\rho\frac{\partial w}{\partial x}\right|\int_{a_1(t)}^{b_1(t)}\rho\left(\frac{\partial w}{\partial x}\right)^2dx\,d\tau\right] .$$

By the embedding inequality we see

(6.5) $$\max_{a_1(t)\leq x\leq b_1(t)}\left|\rho\frac{\partial w}{\partial x}\right|\leq C\left(\left\|\rho\frac{\partial w}{\partial x}\right\|_{L_2(a_1,b_1)}^{\frac{1}{2}}\left\|\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right\|_{L_2(a_1,b_1)}^{\frac{1}{2}}+\left\|\rho\frac{\partial w}{\partial x}\right\|_{L_2(a_1,b_1)}\right) .$$

Combining (6.4), (6.5) and using the Cauchy inequality, Gronwall's inequality, the representation for $w(x, t)$ and equations (3.3), (3.4), we obtain the following relation

(6.6) $$\max_{0\leq t\leq T}\int_{a^1(t)}^{b^1(t)}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial t}\right)^2\right]dx+$$
$$+\iint_{Q_T^x}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2\rho}{\partial t\partial x}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]dx\,dt\leq C_{13} .$$

In the Cartesian variables for any classical solution we have the same global estimates as (4.3), (5.14), (6.6), which are sufficient to construct a generalized solution by the closure method.

Finally, we can estimate Hölder continuity constants for a classical solution.

At first, using (6.6), we deduce

(6.7) $$\|\rho\|_{C^{\frac{1}{2}}(Q_T^1)}\leq C_{14} .$$

The next desired bounds will be proved only for the first strip $Q^1_{(0,t_1]}$. Other strips are considered by the same way.

For our further considerations it is important to have the exact representation for the partial derivative $\frac{\partial\rho}{\partial x}$.

Using relations (5.7), (5.12), we see

(6.8) $$\frac{\partial\rho}{\partial x}=\rho B_{11}^{-1}\frac{\partial B_{11}}{\partial x}-\frac{\gamma}{\mu}\rho\left(1+\frac{\gamma}{\mu}\int_0^t Y_{11}^{\gamma}(\tau)B_{11}^{\gamma}(x,\tau)d\tau\right)^{-1}\times$$

$$\times \int_0^t Y_{11}^{\gamma}(\tau) B_{11}^{\gamma-1}(x, \tau) \frac{\partial B_{11}}{\partial x}(x, \tau) d\tau$$
$$\text{for} \quad 0<t\leq t_1\ , \quad 0\leq x\leq b_1(t)\ ,$$

where

$$\frac{\partial B_{11}}{\partial x}(x, t)=\rho'_0(x)\rho_0^{-1}(x)B_{11}(x, t)+\frac{1}{\mu}B_{11}(x, t)\,[u_0(x)-u(x, t)];$$

and

$$\frac{\partial \rho}{\partial x}=\rho Y_{12}^{-1}\frac{\partial Y_{12}}{\partial x}+\rho B_{12}^{-1}\frac{\partial B_{12}}{\partial x}-\frac{1}{\mu}\rho\Big(1+\frac{\gamma}{\mu}\int_{t^*(x)}^t Y_{12}^{\gamma}(\tau)B_{12}^{\gamma}(x, \tau)d\tau\Big)^{-1}\times \tag{6.9}$$
$$\times\frac{\partial}{\partial x}\Big(\int_{t^*(x)}^t Y_{12}^{\gamma}(x, \tau)B_{12}^{\gamma}(x, \tau)d\tau\Big)$$
$$\text{for} \quad 0<t\leq t_1\ , \quad a_1(t)<x<0\ ,$$

where

$$\begin{aligned}
\frac{\partial Y_{12}}{\partial x}(x, t)&=Y_{12}(x, t)\,[\rho^{-1}(0, t^*(x))\frac{\partial \rho}{\partial t}(0, t^*(x))+\\
&\quad+\frac{1}{\mu}\rho^{\gamma}(0, t^*(x))]\frac{1}{\rho_1(t^*(x))u_1(t^*(x))}\ ,\\
\frac{\partial B_{12}}{\partial x}(x, t)&=B_{12}\Big[\frac{1}{\mu}(u(x, t^*(x))-u(x, t))-\\
&\quad-\frac{1}{\rho_1(t^*(x))u_1(t^*(x))}\int_0^x\frac{\partial u}{\partial t}(s, t^*(x))ds\Big]-\\
&\quad-B_{12}\rho_1^{-1}(t^*(x))\rho'_1(t^*(x))\frac{1}{\rho_1(t^*(x))u_1(t^*(x))}=\\
&=B_{12}\Big[\frac{1}{\mu}(u(x, t^*(x))-u(x, t))-\frac{1}{\rho_1(t^*(x))u_1(t^*)}\times\\
&\quad\times\Big(\rho\frac{\partial u}{\partial x}(s, t^*(x))-\rho^{\gamma}(s, t^*(x))\Big)\Big|_{s=0}^{s=x}\Big]-\\
&\quad-B_{12}\rho_1^{-1}(t^*(x))\rho'_1(t^*(x))\frac{1}{\rho_1(t^*(x))u_1(t^*(x))}\ .
\end{aligned}$$

Now we will use the Shauder estimates for a solution of a linear parabolic equation. We would not like to be concerned about the smoothness of the right-hand boundary of the domain Q_T^1. Therefore we will use the Cartesian variables. Then we can rewrite the first equation (2.1) in the following form

$$\frac{\partial u}{\partial t}=a(y, t)\frac{\partial^2 u}{\partial y^2}+b(y, t)\frac{\partial u}{\partial y}+f(y, t)\ , \tag{6.10}$$

where

$$a(y, t)=\mu\rho^{-1}(y, t)\ , \quad b(y, t)=-u(y, t)\ , \quad f(y, t)=-\rho^{-1}\frac{\partial \rho^{\gamma}}{\partial y}\ .$$

Obviously, $a(y, t)$ is a continuous and bounded function, and

$$b(y, t) \in L_4(Q_T^Y) \quad , \quad f(y, t) \in L_4(Q_T^Y) \quad .$$

Using the Shauder estimates, we have

$$\|u(y, t)\|_{W_4^{2,1}(Q_T^Y)} \leq C_{15} \quad .$$

By embedding theorem ([9], p.80) $W_4^{2,1}(Q_T^Y) \subset C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^Y)$:

$$\|u(y, t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^Y)} \leq C_{16} \quad .$$

and

$$\|u(x, t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^1)} \leq C_{17} \quad .$$

Then representations (6.8), (6.9) help to obtain the following relation

$$\left\|\frac{\partial \rho}{\partial x}\right\|_{C^{\beta,\frac{\beta}{2}}(Q_T^1)} \leq C_{18} \quad ,$$

where $\beta = \min\{\frac{1}{8}, \alpha\}$.

Hence, equation (6.10) has the coefficients with the properties:

$$\|a(y, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^Y)} + \|b(y, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^Y)} + \|f(y, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^Y)} \leq C_{19} \quad ,$$

and the Shauder estimates for the Hölder continuity constants of a solution of a linear parabolic equation give

$$\|u(y, t)\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_T^Y)} \leq C_{20} \quad , \tag{6.11}$$

If $\beta = \alpha$, than we have the desired estimates. If $\beta < \alpha$, than we use representations (6.8), (6.9) again.

We see

$$\|\rho(x, t)\|_{C^{1+\nu,1+\frac{1+\beta}{2}}(Q_T^1)} \leq C_{21}$$

and

$$\|\rho(y, t)\|_{C^{1+\nu,1+\frac{1+\beta}{2}}(Q_T^Y)} \leq C_{22} \quad ,$$

where $\nu = \min\left\{\frac{1+\beta}{2}, \alpha\right\}$.

Therefore we can repeat the relation (6.11) with ν for β. If $\nu < \alpha$ we possibly repeat the last arguments several times and complete the proof of the a priori estimates.

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