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Mappings of domains connected with the Dirichlet problem for the equation of vibrating string

By

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1. Introduction

In the present paper we continue investigations we have begun in [L-S]. In paper [L-S] we studied solvability of the Dirichlet problem for the vibrating string equation

$$(1)\qquad \begin{aligned} &u_{xx}-u_{yy}+f(x, y, u)=0, \quad (x, y)\in\Omega \\ &u|_{\partial\Omega}=0 \end{aligned}$$

for a class of bounded domains $\Omega\subset\mathbf{R}^2$ with piecewise smooth boundaries. Under some symmetry and smoothness assumptions on the boundary $\partial\Omega$, which will be described in section 2, it was shown that there exist piecewise smooth increasing functions h, g: $\mathbf{R}\overset{\text{on}}{\to}\mathbf{R}$ such that

$$(2)\qquad \Omega=\left\{(h(z+t)+g(z-t), h(z+t)-g(z-t))\mid z\in(0, \pi), t\in(0, T)\right\}$$

for some $T>0$ with $\frac{T}{\pi}$ rational. Therefore problem (1) can be rewritten in the following equivalent form

$$(3)\qquad \begin{aligned} &w_{tt}-w_{zz}+\tilde{f}(z, t, w)=0, \quad (z, t)\in\Pi_T=(0, \pi)\times(0, T) \\ &w|_{\partial\Pi_T}=0 \end{aligned}$$

where

$$(4)\qquad w(z, t)=u(h(z+t)+g(z-t), h(z+t)-g(z-t))$$

We notice that F. John [Jo] was among the first who suggested to use a change of variables of the form

$$x_1=h(x+y)+g(x-y), \quad y_1=h(x+y)-g(x-y)$$

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to reduce problem (1) to a problem of the same form in a simpler domain.

Applying the results of [Ra], [B-N], [Sm] we obtained existence, uniqueness and regularity of weak solutions of (3) under some assumptions on $\tilde{f}$ [L-S]. Because of (4), the regularity of a solution $u(x, y)$ of problem (1) is determined by regularity of the solution $w(z, t)$ of problem (3) and by regularity properties of the functions h, g. Our main goal in the present work is to derive necessary and sufficient conditions for existence of functions $h, g \in C^k(\mathbf{R})$ satisfying (2).

The outline of the paper is as follows. In section 2 the class of domains we consider is introduced. In section 3 we show existence and describe general structure of functions h, g satisfying (2). A necessary condition for the functions h, g to belong $C^k(\mathbf{R})$ are derived in section 4. Finally, in section 5 we obtain a necessary and sufficient condition for existence h, g$\in C^k(\mathbf{R})$ satisfying (2).

2. A class of domains

For the sake of convenience we rewrite problem (1) in the characteristic form

$$u_{xy}+f(x, y, u)=0, \quad (x, y)\in\Omega$$

$$u|_{\partial\Omega}=0$$

We will be interested in existence and regularity of increasing functions h, g such that

(5) $$\Omega=\{(h(x), g(y)) \mid 0<x+y<T_1,\ 0<x-y<T_2\}$$

for some $T_1, T_2>0$.

The domain $\Omega\subset\mathbf{R}^2$ is assumed to be bounded, with a boundary $\Gamma=\partial\Omega$ satisfying:

(A1) $\Gamma=\partial\Omega=\bigcup_{j=1}^{4}\Gamma_j$, $\Gamma_j=\{(x, y_j(x))\,|\,x_j^0\leq x\leq x_j^1\}$, $y_j(x)\in C^k[x_j^0, x_j^1]$ for any $j=1, 2, 3, 4$ and for some $k\geq 2$;

(A2) $|y_j'(x)|>0$, $x\in[x_j^0, x_j^1]$, $j=1, 2, 3, 4$;

(A3) The endpoints $P_j=(x_j^0, y_j(x_j^0))$ of the curves $\Gamma_1, \ldots, \Gamma_4$ are the vertices of Γ with respect to the lines $x=const$, $y=const$. By this we mean that for any $j=1, \ldots, 4$ one of the two lines $x=x_j^0$, $y=y_j(x_j^0)$ has empty intersection with Ω and there are no other points on Γ with this property.

Conditions (A1)-(A3) imply that the domain Ω is strictly convex relative to the lines $x=const$, $y=const$. Therefore, following [Jo], we can define homeomorphisms T^+, T^- on the boundary Γ as follows: T^+ assigns to a point

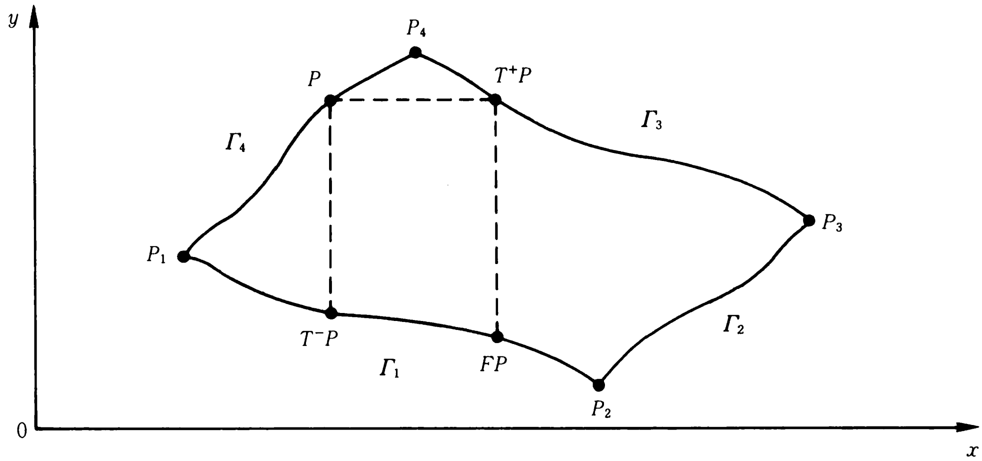

FIGURE 1

on the boundary the other boundary point with the same y coordinate; T^- assigns to a point on the boundary the other boundary point with the same x coordinate. Notice that each vertex P_j is a fixed point of either T^+ or T^-. We set $F = T^- \circ T^+$. It is easy to see that F preserves the orientation of the boundary (see Figure 1).

Let $\Gamma = \{(x(s), y(s)) \mid 0 \leq s < l\}$ be the parametrization of Γ by the arc length parameter, so that l is the total length of Γ. For each point $P \in \Gamma$ we denote its coordinate by $S(P) \in [0, l)$. Then the homeomorphism F can be lifted [Ni] to a continuous map $f_1 : \mathbf{R} \to \mathbf{R}$, which is an increasing function onto $\mathbf{R}$ such that $0 \leq f_1(0) < l$,

$$f_1(s+l) = f_1(s) + l,\ s \in \mathbf{R}, \quad \text{and} \quad S(FP) = f_1(S(P)) \pmod{l},\ P \in \Gamma.$$

The function $f_1(s)$ is called the lift of F [Ni]. If we inductively set $f_k(s) = f_1(f_{k-1}(s))$ for integer $k \geq 2$, then it is known [Ni] that the limit

$$\lim_{k \to \infty} \frac{f_k(s)}{kl} \overset{def}{=} \alpha(F) \in [0, 1]$$

exists and is independent of $s \in \mathbf{R}$. The number $\alpha(F)$ is called the rotation number of F [Ni]. The following cases are possible:

(A) $\alpha(F) = \dfrac{m}{n}$ is a rational number, and $F^n = I$ where I is the identity mapping of Γ onto itself;

(B) $\alpha(F) = \dfrac{m}{n}$ is a rational number, F^n has a fixed point on Γ, but $F^n \neq I$;

(C) $\alpha(F)$ is an irrational number, and F^k has no fixed points on Γ for any $k \in \mathbf{N}$.

Here we will consider only case (A), and so we make a fourth assumption on Ω:

(A4) $\alpha(F)=\frac{m}{n}$ is a rational number and $F^n=I$.

Henceforth we consider the class of bounded domains $\Omega\subset\mathbf{R}^2$ such that the boundary $\Gamma=\partial\Omega$ satisfies conditions (A1)-(A4). This class of domains will be denoted by Σ.

We point out that condition (A4) can be regarded as a symmetry condition on the boundary. If (A4) holds then the boundary Γ can be divided into two parts Γ^1, Γ^2 in such a way that Γ^2 is completely determined by Γ^1 and the number $\alpha(F)=\frac{m}{n}$. This follows, for example, from the results of [Jo] (see also section 3).

3. Existence and general structure

Notice that the collection of domains Σ satisfying (A1)-(A4) is composed of classes $E(m, n, k)$, where for a given triple of natural numbers m, n, k we denote by $E(m, n, k)$ the set of domains Ω satisfying (A1)-(A4) with smoothness k, rotation number $\alpha(F)=\frac{m}{n}$, and $F^n=I$. Correctness of the definition of classes $E(m, n, k)$ follows from the following lemma.

Lemma 1 ([S-L]). *1) $E(mj, nj, k)=E(m, n, k)$ for any $m, n, k, j\in\mathbf{N}$; 2) $E(m, n, k)=\emptyset$ for any $m, n, k\in\mathbf{N}$, $m\geq n$.*

Henceforth we will always assume that $m<n$ and $(m, n)=1$. The simplest representative of the class $E(m,n,k)$ is the rectangle

$$\Pi_n^m=\{(x, y)\mid 0<x+y<\frac{m}{\sqrt{2}},\ 0<x-y<\frac{n-m}{\sqrt{2}}\}$$

Indeed, it is easy to see that the length l of $\Gamma_n^m=\partial\Pi_n^m$ is equal to n and $f_1(s)=s+m$ is the lift of F. Thus $f_k(s)=km+s$ and $\alpha(F)=\frac{m}{n}$. Therefore $E(m, n, k)\neq\emptyset$ if $m<n$.

The following theorem implies that for any $\Omega\in E(m, n, k)$ there exist increasing functions h, g satisfying

$$\Omega=\{(h(x), g(y))\mid (x, y)\in\Pi_n^m\} \tag{6}$$

Theorem 1 ([L-S]). *Let $\Omega_1, \Omega_2\in E(m, n, k)$ for some $m, n, k\in\mathbf{N}$, $m<n$, $k\geq 2$. Then there exist functions $h(x)$, $g(y)$ such that*

$$\Omega_2=\{(h(x), g(y))\mid (x, y)\in\Omega_1\} \tag{7}$$

and

$$(8)\qquad \begin{cases} h(x) \in C(\mathbf{R}) \cap C^k(-\infty, x_1] \cap C^k[x_{n-1}, +\infty) \bigcap_{j=1}^{n-2} C^k[x_j, x_{j+1}] \\ g(y) \in C(\mathbf{R}) \cap C^k(-\infty, y_1] \cap C^k[y_{n-1}, +\infty) \bigcap_{j=1}^{n-2} C^k[y_j, y_{j+1}] \\ 0 < \delta \leq h'(x), g'(y) \leq C, x \notin \{x_1, \dots, x_{n-1}\}, y \notin \{y_1, \dots, y_{n-1}\} \end{cases}$$

for some points $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$ *satisfying* $x_j < x_{j+1}, y_j < y_{j+1}, j = 1, \dots, n-2$ *and for some positive constants* δ, C.

Although Theorem 1 was proved in [L-S] we will give the complete proof here because it will be used further, and because it contains description of general structure of the functions h, g.

Proof of Theorem 1. We shall use the notations $T^{\pm}$, F introduced in section 2. Observe that from the definitions it follows that $T^+T^+ = T^-T^- = I$, so that $F = T^-T^+$ has inverse $F^- = T^+T^-$. Let $\Omega \in E(m, n, k)$; thus $F^n = I$, $\alpha(F) = \frac{m}{n}$. Without loss of generality we assume that the vertices $P_1, \dots, P_4$ (see (A3)) are numbered such that $P_1(P_3)$ has minimal (maximal) x coordinate on $\Gamma = \partial\Omega$, and $P_2(P_4)$ has minimal (maximal) y coordinate on Γ (see Figure 1). Then

$$T^+P_2 = P_2,\ T^+P_4 = P_4,\ T^-P_1 = P_1,\ T^-P_3 = P_3$$

and there are no other fixed points of T^+ and T^-. For any $P \in \Gamma$ we set

$$O(P) = \{P, T^+P, FP, T^+FP, F^2P, \dots, F^{n-1}P, T^+F^{n-1}P\}$$

Because of (A4) the set $O(P)$ is invariant with respect to the homeomorphisms $T^{\pm}$, F, i. e. $T^+(O(P)) = T^-(O(P)) = F(O(P)) = O(P)$. Following [Fo] we call the set $O(P)$ the cycle generated by $P \in \Gamma$. It is easy to see that for any $P, Q \in \Gamma$ either $O(P) \cap O(Q) = \emptyset$ or $O(P) = O(Q)$, and if $Q \in O(P)$ then $O(Q) = O(P)$.

Consider the vertex P_2. Since $T^+P_2 = P_2$ and $T^+F^l = F^{-l}T^+$ for any integer l, we have

$$T^+F^lP_2 = F^{-1}P_2 = F^{n-1}P_2$$

If n is even the choice $l = \frac{n}{2}$ shows that $F^{\frac{n}{2}}P_2$ is a fixed point of T^+, and consequently $F^{\frac{n}{2}}P_2 \in \{P_2, P_4\}$. Since $(m,n) = 1$ the minimal period of any point $P \in \Gamma$ is n (otherwise $\alpha(F) = \frac{m'}{n'}$ for some $m' < m$, $n' < n$), and so we must have $F^{\frac{n}{2}}P_2 = P_4$ and $F^{\frac{n}{2}}P_4 = P_2$. Using the same arguments we obtain $F^{\frac{n}{2}}P_1 = P_3$, $F^{\frac{n}{2}}P_3 = P_1$.

If n is odd then the choice $l=\frac{n-1}{2}$ shows $T^-F^{\frac{n+1}{2}}P_2=T^-T^+F^{\frac{n-1}{2}}P_2=F^{\frac{n+1}{2}}$ P_2. Thus $F^{\frac{n+1}{2}}P_2\in\{P_1, P_3\}$. Applying the same arguments we obtain $F^{\frac{n+1}{2}}P_4\in\{P_1, P_3\}$ and $F^{\frac{n-1}{2}}P_1$, $F^{\frac{n-1}{2}}P_3\in\{P_2, P_4\}$. We have shown that

$$\begin{array}{ll} F^{\frac{n}{2}}P_2=P_4, \quad F^{\frac{n}{2}}P_1=P_3, & n \text{ even} \\ F^{\frac{n+1}{2}}P_2=P_1, \quad F^{\frac{n+1}{2}}P_4=P_3 \quad \text{or} \quad F^{\frac{n+1}{2}}P_2=P_3, F^{\frac{n+1}{2}}P_4=P_1, & n \text{ odd} \end{array} \tag{9}$$

From (9) it follows that $O(P_2)=O(P_4)$, $O(P_1)=O(P_3)$ if n is even; $O(P_2)=O(P_1)$, $O(P_4)=O(P_3)$ or $O(P_2)=O(P_3)$, $O(P_4)=O(P_1)$ if n is odd.

If n is even then $O(P_2)\cap O(P_1)=\emptyset$. Indeed, if $O(P_2)\cap O(P_1)\neq\emptyset$ then $O(P_2)=O(P_1)$ and there exists $l<n$ such that $F^lP_2=P_1$ or $T^+F^lP_2=P_1$. If $T^+F^lP_2=P_1$ then $F^{l+1}P_2=T^-P_1=P_1$. Hence we only need to consider the case $F^lP_2=P_1$ for some $l<n$, and in this case we have

$$F^{2l}P_2=F^lP_1=F^lT^-P_1=T^-F^{-l}P_1=T^-P_2=T^-T^+P_2=FP_2$$

Hence $F^{2l-1}P_2=P_2$. But then $2l-1=jn$ for some $j\in\mathbf{N}$ which is impossible since n is even. Therefore $O(P_2)\cap O(P_1)=\emptyset$ when n is even. Using the same arguments we obtain $O(P_1)\cap O(P_3)=\emptyset$ if n is odd. Thus

$$\left\{\begin{array}{ll} O(P_1)=O(P_3),\ O(P_2)=O(P_4),\ O(P_1)\cap O(P_2)=\emptyset, & n \text{ even} \\ O(P_1)=O(P_2),\ O(P_3)=O(P_4),\ O(P_1)\cap O(P_3)=\emptyset & \\ \text{or} & ,\ n \text{ odd} \\ O(P_1)=O(P_4),\ O(P_2)=O(P_3),\ O(P_1)\cap O(P_3)=\emptyset & \end{array}\right. \tag{10}$$

Following [Ze] we next define the so-called generating set for the homeomorphisms $T^{\pm}$, F. For any points $P, Q\in\Gamma$ we denote by $(P, Q)_\Gamma$ the open arc of Γ from P to Q according to the positive orientation on Γ; we also denote $(P, Q]_\Gamma=(P, Q)_\Gamma\cup\{Q\}$. If n is even then we denote by P_* the point from the finite nonempty set $O(P_2)\cap(P_1, P_2]_\Gamma$ with the property $(P_1, P_*)_\Gamma\cap O(P_2)=\emptyset$. If n is odd then we denote by P_* the point from the finite nonempty set $O(P_3)\cap(P_1, P_2]_\Gamma$ such that $(P_1, P_*)_\Gamma\cap O(P_3)=\emptyset$. By the generating set for homeomorphisms $T^{\pm}$, F we shall mean the arc $M_0=[P_1, P_*]_\Gamma$.

Lemma 2 ([L-S]). I) *For any $P, Q\in M_0$, with $P\neq Q$, we have $O(P)\cap O(Q)=\emptyset$;*
II) $\bigcup_{P\in M_0} O(P)=\Gamma$, *i. e. $O(P)\cap M_0\neq\emptyset$ for any $P\in\Gamma$.*

We define, for $l=0,\ldots,n-1$, the sets

$$M_{2l}=F^l(M_0)=[F^lP_1, F^lP_*]_\Gamma, \quad \overset{o}{M}_{2l}=(F^lP_1, F^lP_*)_\Gamma$$

$$M_{2l+1}=T^+F^l(M_0)=[T^+F^lP_*, T^+F^lP_1]_\Gamma, \quad \overset{o}{M}_{2l+1}=(T^+F^lP_*, T^+F^lP_1)_\Gamma$$

Then Lemma 2 can be written as follows.

Lemma 2.′ I) $\overset{o}{M}_{l_1} \cap \overset{o}{M}_{l_2} = \emptyset$, $l_1, l_2 \in \{0, \ldots, 2n-1\}$, $l_1 \neq l_2$;
II) $\bigcup_{j=1}^{2n-1} M_j = \Gamma$.

Let us now introduce the constants

$$a = \min\{x \mid (x, y) \in \Gamma\}, \quad b = \max\{x \mid (x, y) \in \Gamma\}$$
$$c = \min\{y \mid (x, y) \in \Gamma\}, \quad d = \max\{y \mid (x, y) \in \Gamma\}$$

and the functions $X(P): \Gamma \xrightarrow{on} [a, b]$, $Y(P): \Gamma \xrightarrow{on} [c, d]$, such that

$$(X(P), Y(P)) = P, \quad \forall P \in \Gamma$$

Next we define intervals

$$X_j = X(M_j) = \{X(P) \mid P \in M_j\}, \quad \overset{o}{X}_j = X(\overset{o}{M}_j), \quad j = 0, \ldots, 2n-1$$
$$Y_j = Y(M_j) = \{Y(P) \mid P \in M_j\}, \quad \overset{o}{Y}_j = Y(\overset{o}{M}_j), \quad j = 0, \ldots, 2n-1$$

Clearly X_j, Y_j ($\overset{o}{X}_j$, $\overset{o}{Y}_j$) are closed (open) intervals satisfying the properties

$$X_{2j} = X_{2j-1},\ \overset{o}{X}_{2j} = \overset{o}{X}_{2j-1},\ \overset{o}{X} = X_{2n-1},\ \overset{o}{X}_0 = \overset{o}{X}_{2n-1},\ j = 1, \ldots, n-1$$
$$Y_{2j} = Y_{2j+1},\ \overset{o}{Y}_{2j} = \overset{o}{Y}_{2j+1},\ j = 0, \ldots, n-1$$
$$\bigcup_{j=0}^{n-1} X_{2j} = [a, b], \quad \bigcup_{j=0}^{n-1} Y_{2j} = [c, d]$$
$$\overset{o}{X}_{2j} \cap \overset{o}{X}_{2l} = \emptyset,\ \overset{o}{Y}_{2j} \cap \overset{o}{Y}_{2l} = \emptyset,\ j \neq l$$
$$[a, b] \setminus \left(\bigcup_{j=0}^{n-1} \overset{o}{X}_{2j} \right) = \bigcup_{l=1}^{4} X(O(P_l)) = \bigcup_{l=1}^{4} \{X(P) \mid P \in O(P_l)\}$$
$$[c, d] \setminus \left(\bigcup_{j=0}^{n-1} \overset{o}{Y}_{2j} \right) = \bigcup_{l=1}^{4} Y(O(P_l)) = \bigcup_{l=1}^{4} \{Y(P) \mid P \in O(P_l)\}$$

We define functions $\gamma_j(x): X_0 \xrightarrow{on} X_{2j}$, $\beta_j(x): X_0 \xrightarrow{on} Y_{2j}$ by the following formulae

$$\gamma_j(X(P)) = X(F^j P), \quad \beta_j(X(P)) = Y(F^j P),\ P \in M_0, \quad j = 0, \ldots, n-1$$

where $F^0 = I$. Since $\overset{o}{M}_j \cap O(P_l) = \emptyset$, $j = 0, \ldots, 2n-1$, $l = 1, 2, 3, 4$ then because of (A1), (A2) we have

$$\gamma_j, \beta_j \in C^k(X_0), \quad 0 < C_1 \leq |\gamma_j'|, |\beta_j'| \leq C_2, \quad j = 0, \ldots, n-1$$

for some positive constants C_1, C_2. In addition, for any $P \in M_0$

$$F^l P = (\gamma_l(X(P)), \beta_l(X(P))),\ T^+ F^l P = (\gamma_{l+1}(X(P)), \beta_l(X(P))),\ l = 0, \ldots, n-1$$

Proposition 1 ([L-S]). *Let $\Omega \in E(m, n, k)$, $(m, n)=1$, $m<n$, and the arcs M_j be defined as above. Let us define a permutation $j_i=\sigma(i)$ of the numbers* 0, 1, . . . , $2n-1$ *according to the order of arcs $M_0, M_1, \ldots, M_{2n-1}$ on Γ, so that*

$$M_i \prec M_d \text{ if and only if } j_i < j_d$$

where $M_i \prec M_d$ if $P_1 \notin (P, Q)_\Gamma$ for any $P \in \overset{o}{M}_i$, $Q \in \overset{o}{M}_d$.
Then

$$j_{2i}=\sigma(2i)=2mi \pmod{2n}, \quad i=0, \ldots, n-1 \tag{11}$$

Corollary 1. *The order of the intervals X_{2j} (resp. Y_{2j}) on $[a,b]$ (resp. $[c, d]$) depends only on the numbers n, m.*

Using the fact that F preserves orientation on Γ we have $\gamma_j'(x)>0$ (resp. $\gamma_j'(x)<0$) if and only if $M_{2j} \subset [P_1, P_3]_\Gamma$ (resp. $M_{2j} \subset [P_3, P_1]_\Gamma$); $\beta_j'(x)>0$ (resp. $\beta_j'(x)<0$) if and only if $M_{2j} \subset [P_2, P_4]_\Gamma$ (resp. $M_{2j} \subset [P_4, P_2]_\Gamma$). Hence from Proposition 1 and Corollary 1 we obtain a second result.

Corollary 2. *The sign of the functions $\gamma_j'(x)$, $\beta_j'(x)$ depends only on the numbers m, n, j and does not depend on the shape of the domain $\Omega \in E(m, n, k)$.*

Let $\Omega_1, \Omega_2 \in E(m, n, k)$, $\Gamma_j=\partial\Omega_j$, $j=1, 2$. According to the formulae above we define for each boudary Γ_1, Γ_2 intervals

$$X_j^{\Gamma_1}, \quad X_j^{\Gamma_2}, \quad Y_j^{\Gamma_1}, \quad Y_j^{\Gamma_2}, \quad j=0, \ldots, 2n-1$$

and functions

$$X^{\Gamma_1}(P) : \Gamma_1 \overset{on}{\longrightarrow} [a_1, b_1], \quad Y^{\Gamma_1}(P) : \Gamma_1 \overset{on}{\longrightarrow} [c_1, d_1]$$
$$X^{\Gamma_2}(P) : \Gamma_2 \overset{on}{\longrightarrow} [a_2, b_2], \quad Y^{\Gamma_2}(P) : \Gamma_2 \overset{on}{\longrightarrow} [c_2, d_2]$$
$$\gamma_j^{\Gamma_1} : X_0^{\Gamma_1} \overset{on}{\longrightarrow} X_{2j}^{\Gamma_1}, \quad \beta_j^{\Gamma_1} : X_0^{\Gamma_1} \overset{on}{\longrightarrow} Y_{2j}^{\Gamma_1}, \quad j=0, \ldots, n-1$$
$$\gamma_j^{\Gamma_2} : X_0^{\Gamma_2} \overset{on}{\longrightarrow} X_{2j}^{\Gamma_2}, \quad \beta_j^{\Gamma_2} : X_0^{\Gamma_2} \overset{on}{\longrightarrow} Y_{2j}^{\Gamma_2}, \quad j=0, \ldots, n-1$$

Let $h_0(x)$ be an arbitrary function $h_0(x) : X_0^{\Gamma_1} \overset{on}{\longrightarrow} X_0^{\Gamma_2}$ satisfying the conditions

$$h_0 \in C^k(X_0^{\Gamma_1}); \quad h_0'(x) \geq \delta > 0, \quad x \in X_0^{\Gamma_1}$$

We define

$$\begin{cases} h(x)=\gamma_j^{\Gamma_2}\Big(h_0\Big((\gamma_j^{\Gamma_1})^{-1}(x)\Big)\Big), & x \in X_{2j}^{\Gamma_2} \\ g(y)=\beta_j^{\Gamma_2}\Big(h_0\Big((\beta_j^{\Gamma_1})^{-1}(y)\Big)\Big), & y \in Y_{2j}^{\Gamma_1} \end{cases} \tag{12}$$

Then using the properties of the functions $\gamma_j^{\Gamma_i}$, $\beta_j^{\Gamma_i}$ we obtain

$$h(x) : [a_1, b_1] \xrightarrow{on} [a_2, b_2], \quad h \in \bigcap_{j=0}^{n-1} C^k(X_{2j}^{\Gamma_1})$$

$$g(y) : [c_1, d_1] \xrightarrow{on} [c_2, d_2], \quad g \in \bigcap_{j=0}^{n-1} C^k(Y_{2j}^{\Gamma_1})$$

$$|h'(x)|, |g'(y)| \geq \delta > 0, \quad x \in \bigcup_{j=0}^{n-1} \overset{O}{X}{}_{2j}^{\Gamma_1}, \quad y \in \bigcup_{j=0}^{n-1} \overset{O}{Y}{}_{2j}^{\Gamma_1}$$

From Corollary 2 it follows that $\gamma_j^{\Gamma_1}$, $\gamma_j^{\Gamma_2}$ are either both increasing or both decreasing. The same is true for $\beta_j^{\Gamma_1}$, $\beta_j^{\Gamma_2}$. Therefore

$$h'(x), g'(y) \geq \delta > 0, \quad x \in \bigcup_{j=0}^{n-1} \overset{O}{X}{}_{2j}^{\Gamma_1}, \quad y \in \bigcup_{j=0}^{n-1} \overset{O}{Y}{}_{2j}^{\Gamma_1}$$

From Corollary 1 it follows that $h(x)$ is continuous at the points $[a_1, b_1] \setminus \bigcup_{j=0}^{n-1} \overset{O}{X}{}_{2j}^{\Gamma_1}$, and $g(y)$ is continuous at the points $[c_1, d_1] \setminus \bigcup_{j=0}^{n-1} \overset{O}{Y}{}_{2j}^{\Gamma_1}$.
Thus

$$h(x) \in C[a_1, b_1] \cup C^k\left(X_{2j}^{\Gamma_1}\right), \quad j=0, \dots, n-1$$

$$g(y) \in C[c_1, d_1] \cup C^k\left(Y_{2j}^{\Gamma_1}\right), \quad j=0, \dots, n-1$$

$$h'(x), g'(y) \geq \delta > 0, \quad x \in \bigcup_{j=0}^{n-1} \overset{O}{X}{}_{2j}^{\Gamma_1}, \quad y \in \bigcup_{j=0}^{n-1} \overset{O}{Y}{}_{2j}^{\Gamma_1}$$

Extending $h(x)$, $g(y)$ to $x \in \mathbf{R} \setminus [a_1, b_1]$, $y \in \mathbf{R} \setminus [c_1, d_1]$ as functions from C^k we obtain that h, g satisfy (8) where

$$\{x_1, \dots, x_{n-1}\} = (a_1, b_1) \cup \{X^{\Gamma_1}(P) | P \in \bigcup_{j=1}^{4} O^{\Gamma_1}(P_j^{\Gamma_1})\}$$

$$\{y_1, \dots, y_{n-1}\} = (c_1, d_1) \cup \{Y^{\Gamma_1}(P) | P \in \bigcup_{j=1}^{4} O^{\Gamma_1}(P_j^{\Gamma_1})\}$$

It remains to prove (7). Since for any $j=1, 2$, $l=0, \dots, n-1$, $P \in \overset{O}{M}{}^{\Gamma_j}$

$$F_{\Gamma_j}^l P = \left(\gamma_l^{\Gamma_j}(X^{\Gamma_j}(P)), \beta_l^{\Gamma_j}(X^{\Gamma_j}(P))\right),$$

$$T_{\Gamma_j}^+ F_{\Gamma_j}^l P = \left(\gamma_{l+1}^{\Gamma_j}(X^{\Gamma_j}(P)), \beta_l^{\Gamma_j}(X^{\Gamma_j}(P))\right)$$

it follows from (12) that if $P \in M_0^{\Gamma_1}$, $Q \in M_0^{\Gamma_2}$ and

(13) $\qquad X^{\Gamma_2}(Q) = h_0(X^{\Gamma_1}(P))$

then

$$F^l_{\Gamma_2}Q = (h(X^{\Gamma_1}(F^l_{\Gamma_1}P)), g(Y^{\Gamma_1}(F^l_{\Gamma_1}P)))$$

$$T^+_{\Gamma_2} F^l_{\Gamma_2} Q = (h(X^{\Gamma_1}(T^+_{\Gamma_1} F^l_{\Gamma_1}P)), g(Y^{\Gamma_1}(T^+_{\Gamma_1} F^l_{\Gamma_1}P)))$$

for any $l=0,\dots,n-1$. Thus mapping (12) transforms any cycle $O^{\Gamma_1}(P)$, $P\in M_0^{\Gamma_1}$, into a cycle $O^{\Gamma_2}(Q)$, where $Q\in M_0^{\Gamma_2}$ is uniquely determined by (13). Since

$$(X^{\Gamma_2})^{-1}(h_0(X^{\Gamma_1}(P))) : M_0^{\Gamma_1} \xrightarrow{on} M_0^{\Gamma_2}$$

it follows from Lemma 2 that mapping (12) transforms Γ_1 onto Γ_2. Since Ω_1, Ω_2 are convex relative to the lines $x=const$, $y=const$ and $h(x)$, $g(y)$ are increasing continuous functions (7) holds. This completes the proof of Theorem 1.

It easy to see that any functions h, g satisfying (7) can be determined by (12) where $h_0(x)=h(x)$, $x\in X_0^{\Gamma_1}$. Therefore the proof of Theorem 1 gives us the general structure of functions h, g satisfying (7).

4. A necessary condition

From Theorem 1 it follows that for any $\Omega\in E(m, n, k)$ there exist increasing functions h, g satisfying (8) and such that

$$\Pi^m_n = \{(h(x), g(y)) \mid (x, y)\in\Omega\} \tag{14}$$

In the present section we derive a necessary condition for the functions h, g to belong to $C^k(\mathbf{R})$.

Let h, g satisfy (8), (14). From the definition of rectangle Π^m_n it follows

$$h(X(P)) + h(X(T^+P)) = \begin{cases} \dfrac{n-m}{\sqrt{2}}, & P\in\Gamma,\ |P-P_2|<\varepsilon \\ \dfrac{m}{\sqrt{2}}, & P\in\Gamma,\ |P-P_4|<\varepsilon \end{cases} \tag{15}$$

$$g(Y(P)) + g(Y(T^-P)) = \begin{cases} 0, & P\in\Gamma,\ |P-P_1|<\varepsilon \\ \dfrac{2m-n}{\sqrt{2}}, & P\in\Gamma,\ |P-P_3|<\varepsilon \end{cases} \tag{16}$$

for $\varepsilon>0$ small enough. Using (A1) we rewrite (15), (16) as follows

$$\begin{cases} h(y_1^{-1}(y)) + h(y_2^{-1}(y)) = \dfrac{n-m}{\sqrt{2}}, & y\in[c, c+\varepsilon] \\ h(y_3^{-1}(y)) + h(y_4^{-1}(y)) = \dfrac{m}{\sqrt{2}}, & y\in[d-\varepsilon, d] \end{cases} \tag{17}$$

$$
(18) \qquad \begin{cases} g(y_1(x)) + g(y_4(x)) = 0, & x \in [a, a+\varepsilon] \\ g(y_2(x)) + g(y_3(x)) = \dfrac{2m-n}{\sqrt{2}}, & x \in [b-\varepsilon, b] \end{cases}
$$

where

$$
\begin{aligned} a &= \min\{x \mid (x, y) \in \Gamma\}, & b &= \max\{x \mid (x, y) \in \Gamma\} \\ c &= \min\{y \mid (x, y) \in \Gamma\}, & d &= \max\{y \mid (x, y) \in \Gamma\} \end{aligned}
$$

Equalities (17), (18) imply some necessary conditions for the functions h, g to belong to C^k. To derive these conditions we consider the following auxiliary problem.

Assume functions $w_1(x)$, $w_2(x)$ satisfy

i) $w_1, w_2 \in C^k[0, 1]$,
ii) $0 < \delta \le w_1'(x), (-w_2'(x)) \le C, \quad x \in [0, 1]$,
iii) $w_1(0) = w_2(0) = 0$.

Assume there exists a real-valued function $g \in C^k(\mathbf{R})$, $g'(0) > 0$ satisfying

$$
(19) \qquad g(w_1(x)) + g(w_2(x)) = C_1, \quad x \in [0, 1]
$$

for some constant $C_1 \in \mathbf{R}$. Equality (19) implies

$$
(20) \qquad \frac{d^j}{dx^j}\Big(g(w_1(x)) + g(w_2(x))\Big)\Big|_{x=+0} = 0, \quad j = 1, \dots, k
$$

We set for $j = 1, \dots, k$

$$
(21) \qquad g_j = \frac{1}{j!}\frac{d^j g}{dy^j}(0), \quad \alpha_j = \frac{1}{j!}\frac{d^j w_1}{dx^j}(+0), \quad \beta_j = \frac{1}{j!}\frac{d^j w_2}{dx^j}(+0)
$$

and define functions $a_{ij}(\bar{\gamma})$, $\bar{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbf{R}^k$ by the following formula

$$
(22) \qquad \Big(\sum_{l=1}^{k} \gamma_l x^l\Big)^j = \sum_{j \le i \le jk} a_{ij}(\bar{\gamma}) x^i, \quad j = 1, \dots, k, \quad x \in \mathbf{R}
$$

Then (20) can be written in the form

$$
(23) \qquad \begin{cases} g_1 c_{11} = 0 \\ g_1 c_{21} + g_2 c_{22} = 0 \\ g_1 c_{31} + g_2 c_{32} + g_3 c_{33} = 0 \\ \\ \cdots\cdots\cdots\cdots \\ g_1 c_{k1} + g_2 c_{k2} + g_3 c_{k3} + \cdots + g_k c_{kk} = 0 \end{cases}
$$

where

(24) $$c_{ij}=c_{ij}(\bar{\alpha},\bar{\beta})=a_{ij}(\bar{\alpha})+a_{ij}(\bar{\beta}),\quad 1\leq j\leq i\leq k$$

It is easy to check that

(25) $$\begin{cases} c_{j1}=(\alpha_j+\beta_j),\quad c_{jj}=(\alpha_1^j+\beta_1^j),\quad j=1,\dots,k \\ c_{j(j-1)}=(j-1)(\alpha_1^{j-2}\alpha_2+\beta_1^{j-2}\beta_2),\quad j=2,\dots,k \\ c_{j(j-2)}=(j-2)(\alpha_1^{j-3}\alpha_3+\beta_1^{j-3}\beta_3) \\ \qquad +\dfrac{(j-2)(j-3)}{2}(\alpha_1^{j-4}\alpha_2^2+\beta_1^{j-4}\beta_2^2),\quad j=3,\dots,k \end{cases}$$

Since $g'(0)>0$ then $g_1>0$. Using (23),(25) we obtain $c_{11}=(\alpha_1+\beta_1)=0$. Hence

(26) $$\beta_1=-\alpha_1$$

From ii) it follows $\alpha_1\neq 0$. Therefore

$$c_{jj}=\begin{cases} 0, & j \text{ is odd} \\ 2\alpha_1^j, & j \text{ is even} \end{cases}$$

Let $k\geq 3$. Consider the second and the third equation of (23). Since $g_1\neq 0$ then using (25), (26) we obtain

$$0=c_{21}c_{32}-c_{22}c_{31}=2\alpha_1((\alpha_2^2-\beta_2^2)-\alpha_1(\alpha_3+\beta_3))$$

Hence

(27) $$\alpha_1(\alpha_3+\beta_3)=(\alpha_2^2-\beta_2^2)$$

Let $k\geq 5$. Consider the fourth and the fifth equation of (23)

$$g_1c_{41}+g_2c_{42}+g_3c_{43}+g_4c_{44}=0$$
$$g_1c_{51}+g_2c_{52}+g_3c_{53}+g_4c_{54}=0$$

Using (25),(26) we obtain

$$c_{43}c_{54}-c_{53}c_{44}=3\alpha_1^2(\alpha_2+\beta_2)\cdot 4\alpha_1^3(\alpha_2-\beta_2)-2\alpha_1^4[3\alpha^2{}_1(\alpha_3+\beta_3)$$
$$+3\alpha_1(\alpha_2^2-\beta_2^2)]=6\alpha_1^5(\alpha_2^2-\beta_2^2)-6\alpha_1^6(\alpha_3+\beta_3)=0$$

by force of (27). Hence

$$g_1(c_{41}2(\alpha_2-\beta_2)-\alpha_1c_{51})+g_2(c_{42}2(\alpha_2-\beta_2)-\alpha_1c_{52})=0$$

Since $g_1\neq 0$ then using the second equation of (23) we obtain

$$0=(c_{41}2(\alpha_2-\beta_2)-\alpha_1c_{51})\cdot c_{22}-(c_{42}2(\alpha_2-\beta_2)-\alpha_1c_{52})\cdot c_{21}$$

One can check that the last equation can be written as follows

(28) $$2\alpha_1^2(\alpha_2-\beta_2)(\alpha_4+\beta_4)-\alpha_1^3(\alpha_5+\beta_5)+\alpha_1^2(\alpha_4-\beta_4)(\alpha_2+\beta_2)$$
$$+\alpha_1(\alpha_2\alpha_3+\beta_2\beta_3)(\alpha_2+\beta_2)-(\alpha_2^4-\beta_2^4)-2\alpha_1^2(\alpha_3^2-\beta_3^2)=0$$

Let $k \geq 5$. Then for any $l \geq 2$, $2l+1 \leq k$ we have by force of (25) - (27)

$$\det\begin{pmatrix} c_{2l(2l-1)} & c_{2l2l} \\ c_{(2l+1)(2l-1)} & c_{(2l+1)2l} \end{pmatrix} = (2l-1)\alpha_1^{2l-2}(\alpha_2+\beta_2) \cdot 2l\alpha_1^{2l-1}(\alpha_2-\beta_2)$$
$$-2\alpha_1^{2l}[(2l-1)\alpha_1^{2l-2}(\alpha_3+\beta_3)+(2l-1)(l-1)\alpha_1^{2l-3}(\alpha_2^2-\beta_2^2)]$$
$$=2(2l-1)\alpha_1^{4l-3}(\alpha_2^2-\beta_2^2)-2(2l-1)\alpha_1^{4l-2}(\alpha_3+\beta_3)=0$$

Since $\alpha_1 \neq 0$ then the system of two equations

$$g_1c_{2l1}+\cdots+g_{2l}c_{2l2l}=0$$
$$g_1c_{(2l+1)1}+\cdots+g_{2l}c_{(2l+1)2l}=0$$

is equivalent to

$$g_1\tilde{c}_{(2l-2)1}+\cdots+g_{2l-2}\tilde{c}_{(2l-2)(2l-2)}=0$$
$$g_1c_{2l1}+\cdots\cdots+g_{2l}c_{2l2l}=0$$

where

(29) $$\tilde{c}_{(2l-2)j}=c_{2lj}l(\alpha_2-\beta_2)-c_{(2l+1)j}\alpha_1, \quad j=1,\ldots,2l-2$$

Assume (26) - (28) hold and consider the system

(30) $$\begin{cases} g_1c_{21}+g_2c_{22}=0 \\ g_1c_{41}+\cdots+g_4c_{44}=0 \\ g_1\tilde{c}_{41}+\cdots+g_4\tilde{c}_{44}=0 \\ \cdots\cdots\cdots\cdots \\ g_1c_{2l1}+\cdots\cdots+g_{2l}c_{2l2l}=0 \\ g_1\tilde{c}_{2l1}+\cdots\cdots+g_{2l}\tilde{c}_{2l2l}=0 \\ \cdots\cdots\cdots\cdots \\ g_1c_{(k-2)1}+\cdots+g_{k-2}c_{(k-2)(k-2)}=0 \\ g_1c_{k1}+\cdots\cdots+g_kc_{kk}=0 \end{cases}$$

if k is even, or the system

(31) $$\begin{cases} g_1c_{21}+g_2c_{22}=0 \\ g_1c_{41}+\cdots+g_4c_{44}=0 \\ g_1\tilde{c}_{41}+\cdots+g_4\tilde{c}_{44}=0 \\ \cdots\cdots\cdots\cdots \\ g_1c_{2l1}+\cdots\cdots+g_{2l}c_{2l2l}=0 \\ g_1\tilde{c}_{2l1}+\cdots\cdots+g_{2l}\tilde{c}_{2l2l}=0 \\ \cdots\cdots\cdots\cdots \\ g_1c_{(k-1)1}+\cdots+g_{k-1}c_{(k-1)(k-1)}=0 \end{cases}$$

if k is odd.

Since $c_{2l2l}>0$ and solvality of (30), (31) does not depend on the magnitude of $g_1>0$ then we can assume $g_1=1$ and rewrite (30), (31) in the following equivalent form

$$(30') \quad \begin{cases} g_2 d_{22} = -d_{21} \\ g_2 d_{32} + g_3 d_{33} = -d_{31} \\ g_2 d_{42} + g_3 d_{43} + g_4 d_{44} = -d_{41} \\ \cdots\cdots\cdots\cdots \\ g_2 d_{(k-4)2} + \cdots\cdots + g_{k-4} d_{(k-4)(K-4)} = -d_{(k-4)1} \\ g_1 c_{(k-2)1} + \cdots\cdots + g_{k-2} c_{(k-2)(k-2)} = -c_{(k-2)1} \\ g_1 c_{k1} + \cdots\cdots + g_k c_{kk} = -c_{k1} \end{cases}$$

$$(31') \quad \begin{cases} g_2 d_{22} = -d_{21} \\ g_2 d_{32} + g_3 d_{33} = -d_{31} \\ g_2 d_{42} + g_3 d_{43} + g_4 d_{44} = -d_{41} \\ \cdots\cdots\cdots \\ g_2 d_{(k-3)2} + \cdots\cdots + g_{k-3} d_{(k-3)(k-3)} = -d_{(k-3)1} \\ g_1 c_{(k-1)1} + \cdots\cdots + g_{k-1} c_{(k-1)(k-1)} = -c_{(k-1)1} \end{cases}$$

where

$$(32) \quad d_{2lj} = c_{2lj}, \quad d_{(2l-1)j} = c_{2lj} \cdot \tilde{c}_{2l2l} - \tilde{c}_{2lj} \cdot c_{2l2l}$$

Consider the system

$$(33) \quad \begin{cases} g_2 d_{22} = -d_{21} \\ g_2 d_{32} + g_3 d_{33} = -d_{31} \\ g_2 d_{42} + g_3 d_{43} + g_4 d_{44} = -d_{41} \\ \cdots\cdots\cdots\cdots \\ g_2 d_{l2} + \cdots\cdots + g_l d_{ll} = -d_{ll} \end{cases}$$

where

$$(34) \quad l = l(k) = \begin{cases} (k-4), & k \text{ is even} \\ (k-3), & k \text{ is odd} \end{cases}$$

Then system (30′) (resp. (31′)) is solvable if and only if (33) is solvable. Indeed, if k is even and $(g_2, \ldots, g_{k-4})$ satisfy (33) then for arbitrary $g_{k-3}, g_{k-1} \in \mathbf{R}$ the values of g_{k-2}, g_k are uniquely determined by the last two equations of (30′). If k is odd and $(g_2, \ldots, g_{k-3})$ satisfy (33) then for any arbitrary $g_{k-2}, g_k \in \mathbf{R}$ the value of g_{k-1} is uniquely determined by the last equation of (31′).

For any $k \geq 5$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $\bar{\beta} = (\beta_1, \ldots, \beta_k)$ we denote

$$(35) \quad D(\bar{\alpha}, \bar{\beta}, k) = \begin{pmatrix} d_{22} & 0 & \cdots & 0 \\ d_{32} & d_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ d_{l2} & d_{l3} & \cdots & d_{ll} \end{pmatrix}$$

where $d_{ji} = d_{ji}(\bar{\alpha}, \bar{\beta})$ are determined by (24), (29), (32), $l = l(k)$ is determined by (34).

We have obtained that if there exists $g \in C^k(\mathbf{R})$, $g'(0) > 0$ satisfying (19) then (26)-(28) hold and there exists $\overline{X} \in \mathbf{R}^{l(k)-1}$ such that

$$D(\overline{\alpha}, \overline{\beta}, k) \cdot \overline{X} = -d(\overline{\alpha}, \overline{\beta}, k)$$

where

$$d(\overline{\alpha}, \overline{\beta}, k) = \begin{pmatrix} d_{21}(\overline{\alpha}, \overline{\beta}) \\ \vdots \\ d_{l(k)1}(\overline{\alpha}, \overline{\beta}) \end{pmatrix} \tag{36}$$

Let us determine $\overline{\alpha}(P_j) = (\alpha_1(P_j), \ldots, \alpha_k(P_j))$, $\overline{\beta}(P_j) = (\beta_1(P_j), \ldots, \beta_k(P_j))$, $j=1, 2, 3, 4$ by the following formulae

$$\begin{cases} \alpha_l(P_1) = \dfrac{1}{l\,!} \dfrac{d^l y_4}{dx^l}(a+0), & \beta_l(P_1) = \dfrac{1}{l\,!} \dfrac{d^l y_1}{dx^l}(a+0) \\ \alpha_l(P_2) = \dfrac{1}{l\,!} \dfrac{d^l (y_1^{-1})}{dy^l}(c+0), & \beta_l(P_2) = \dfrac{1}{l\,!} \dfrac{d^l (y_2^{-1})}{dy^l}(c+0) \\ \alpha_l(P_3) = \dfrac{1}{l\,!} \dfrac{d^l y_2}{dx^l}(b-0), & \beta_l(P_3) = \dfrac{1}{l\,!} \dfrac{d^l y_3}{dx^l}(b-0) \\ \alpha_l(P_4) = \dfrac{1}{l\,!} \dfrac{d^l (y_3^{-1})}{dy^l}(d-0), & \beta_l(P_4) = \dfrac{1}{l\,!} \dfrac{d^l (y_4^{-1})}{dy^l}(d-0) \end{cases} \tag{37}$$

The next theorem follows from the above arguments.

Theorem 2. *Let $\Omega \in E(m, n, k)$ for some $m, n, k \in \mathbf{N}$, $m < n$, $k \geq 5$. If there exist functions $h(x) \in C^k[a, b]$, $g(y) \in C^k[c, d]$ satisfying* (8), (14) *then for any $j=1, 2, 3, 4$ vectors $\overline{\alpha}(P_j)$, $\overline{\beta}(P_j)$ satisfy* (26)-(28) *and there exists $\overline{X}_j \in \mathbf{R}^{l(k)-1}$ satisfying*

$$D(\overline{\alpha}(P_j), \overline{\beta}(P_j), k) \cdot \overline{X}_j = -d(\overline{\alpha}(P_j), \overline{\beta}(P_j), k)$$

where $D(\overline{\alpha}, \overline{\beta}, k)$ is determined by (35), *$d_{ji}(\overline{\alpha}, \overline{\beta})$ are determined by* (24), (29), (32), *$l(k)$ is determined by* (34), *$d(\overline{\alpha}, \overline{\beta}, k)$ is determined by* (36).

5. A necessary and sufficient condition

In the present section we derive a necessary and sufficient condition for existence $h \in C^k[a, b]$, $g \in C^k[c, d]$ satisfying (8), (14).

Let $\Omega \in E(m, n, k)$ for some $m, n, k \in \mathbf{N}$, $k \geq 2$. Denote $\Gamma = \partial\Omega, \Gamma_n^m = \partial\prod_n^m$.

It is easy to see that from the proof of Theorem 1 it follows that if $h_0 : X_0^\Gamma \xrightarrow{on} X_0^{\Gamma_n^m}$, $h_0 \in C^k(X_0^\Gamma)$ is chosen such that function $h(x)$, $g(y)$ defined by (12) satisfy for some $\varepsilon > 0$

$$(38)\qquad \begin{aligned} &h \in C^k(X(P_2)-\varepsilon, X(P_2)+\varepsilon) \cap C^k(X(P_4)-\varepsilon, X(P_4)+\varepsilon)\\ &g \in C^k(Y(P_1)-\varepsilon, Y(P_1)+\varepsilon) \cap C^k(Y(P_3)-\varepsilon, Y(P_3)+\varepsilon)\end{aligned}$$

then $h \in C^k[a, b]$, $g \in C^k[c, d]$. Because of (8) conditions (38) are equivalent to

$$(39)\qquad \begin{aligned} &\frac{d^l h}{dx^l}(X(P_j)-0) = \frac{d^l h}{dx^l}(X(P_j)+0), \quad j=2, 4, l=1, \dots, k\\ &\frac{d^l g}{dy^l}(Y(P_j)-0) = \frac{d^l g}{dy^l}(Y(P_j)+0), \quad j=1, 3, l=1, \dots, k\end{aligned}$$

Consider the rectangle Π_n^m. It is easy to see that the vertices of Π_n^m

$$(40)\qquad \begin{aligned} &P_1^{\Gamma_{nm}} = (0, 0), \qquad && P_2^{\Gamma_{nm}} = \left(\frac{n-m}{2\sqrt{2}}, \frac{m-n}{2\sqrt{2}}\right)\\ &P_3^{\Gamma_{nm}} = \left(\frac{n}{2\sqrt{2}}, \frac{2m-n}{2\sqrt{2}}\right), && P_4^{\Gamma_{nm}} = \left(\frac{m}{2\sqrt{2}}, \frac{m}{2\sqrt{2}}\right)\end{aligned}$$

and for any $l=0, \dots, n-1, j=0, \dots, 2n-1, P, Q \in M_j^{\Gamma_{nm}}$

$$(41)\qquad \begin{aligned} |X^{\Gamma_{nm}}(P) - X^{\Gamma_{nm}}(Q)| &= |X^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}P) - X^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}Q)|\\ &= |Y^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}P) - Y^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}Q)| = |Y^{\Gamma_{nm}}(P) - Y^{\Gamma_{nm}}(Q)|\end{aligned}$$

Let $h(x)$, $g(y)$ satisfy (8), (14). Since the mapping $(x_1, y_1) = (h(x), g(y))$ transforms any cycle $O(P)$ into the cycle $O^{\Gamma_{nm}}(Q)$ where $Q = (h(X(P)), g(Y(P)))$ then for any $l \in \mathbf{Z}$

$$(42)\qquad \begin{aligned} &X^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}(h(X(P)), g(Y(P)))) = h(X(F^l P)), \quad P \in \Gamma\\ &Y^{\Gamma_{nm}}(F^l_{\Gamma_{nm}}(h(X(P)), g(Y(P)))) = g(Y(F^l P)), \quad P \in \Gamma\end{aligned}$$

Consider two cases: I) n is even, II) n is odd.

I). Let n be an even number. By force of (9) we have $F^{\frac{n}{2}} P_1 = P_3$, $F^{\frac{n}{2}}_{\Gamma_{nm}} P_1^{\Gamma_{nm}} = P_3^{\Gamma_{nm}}$.

Using (40)-(42) we obtain

$$(43)\qquad 0 - g(Y(P)) = g(Y(F^{\frac{n}{2}}(P))) + \frac{n-2m}{2\sqrt{2}}$$

for any $P \in \Gamma$ such that $|P - P_1|$ is sufficiently small.

Since $F^{\frac{n}{2}} P_2 = P_4$, $F^{\frac{n}{2}}_{\Gamma_{nm}} P_2^{\Gamma_{nm}} = P_4^{\Gamma_{nm}}$ then employing (40)-(42) we obtain

$$(44)\qquad h(X(P)) - \frac{n-m}{2\sqrt{2}} = \frac{m}{2\sqrt{2}} - h(X(\tfrac{n}{2}(P)))$$

for any $P \in \Gamma$ such that $|P - P_2|$ is sufficiently small.

Besides, in a neighborhood of each vertex $P_1, \ldots, P_4$ we have one extra functional equation: (15) or (16). We note that if (16) holds then it is sufficient to consider (43) only for $P\in\Gamma$ such that $Y(P)\leq Y(P_1)$. Indeed, let $|P-P_1|\leq\varepsilon$, $Y(P)\leq Y(P_1)$ for a small $\varepsilon>0$. Consider points P, T^-P, $F^{\frac{n}{2}}P$, $T^-F^{\frac{n}{2}}P=F^{\frac{n}{2}}T^-P$. Assume that $g(Y(P))+g(Y(F^{\frac{n}{2}}(P)))=-\frac{n-2m}{2\sqrt{2}}$. Then using (16) we obtain

$$\begin{aligned} g(Y(T^-P))+g(Y(F^{\frac{n}{2}}(T^-P))) &= -g(Y(P))+\left(-\frac{n-2m}{\sqrt{2}}-g(Y(F^{\frac{n}{2}}P))\right) \\ &= -\left(\frac{n-2m}{\sqrt{2}}+g(Y(P))+g(Y(F^{\frac{n}{2}}P)\right)=-\frac{n-2m}{2\sqrt{2}} \end{aligned}$$

Since $Y(P)\leq Y(P_1)$ if and only if $Y(T^-P)\geq Y(P_1)$ then we obtain that it suffices to consider (43) only for $P\in\Gamma$ satisfying $|P-P_1|\leq\varepsilon$, $Y(P)\leq Y(P_1)$ provided (16) holds and ε is sufficiently small. Using the same arguments we obtain that it suffices to consider (44) only for $P\in\Gamma$ satisfying $|P-P_2|\leq\varepsilon$, $X(P)\leq X(P_2)$ provided (15) holds and ε is sufficiently small

For $x\in[X(P_2)-\varepsilon, X(P_2)]$, $y\in[Y(P_1)-\varepsilon, Y(P_1)]$ we define functions $\phi(x)$, $\psi(y)$ using the following formulae

$$\phi(X(P))=X(F^{\frac{n}{2}}P),\quad X(P)\in[X(P_2)-\varepsilon, X(P_2)],\quad P\in\Gamma_1 \tag{45}$$

$$\psi(Y(P))=Y(F^{\frac{n}{2}}P),\quad Y(P)\in[Y(P_1)-\varepsilon, Y(P_1)],\quad P\in\Gamma_1 \tag{46}$$

From (A1), (A2) and Lemma 3′ it follows that for ε sufficiently small

$$\begin{aligned} &\phi(x):[X(P_2)-\varepsilon, X(P_2)]\xrightarrow{on}[X(P_4), X(P_4)+\varepsilon_1] \\ &\psi(y):[Y(P_1)-\varepsilon, Y(P_1)]\xrightarrow{on}[Y(P_3), Y(P_3)+\varepsilon_2] \\ &\phi(x)\in C^k[X(P_2)-\varepsilon, X(P_2)],\quad \psi(y)\in C^k[Y(P_1)-\varepsilon, Y(P_1)] \\ &\phi'(x), \psi'(y)\leq-\delta<0,\quad x\in[X(P_2)-\varepsilon, X(P_2)],\quad y\in[Y(P_1)-\varepsilon, Y(P_1)] \end{aligned}$$

for some constants ε_1, ε_2, $\delta>0$. Equations (43), (44) can be written as follows

$$g(y)+g(\psi(y))=-\frac{n-2m}{2\sqrt{2}},\quad y\in[Y(P_1)-\varepsilon, Y(P_1)] \tag{47}$$

$$h(x)+h(\phi(x))=\frac{n}{2\sqrt{2}},\qquad x\in[X(P_2)-\varepsilon, X(P_2)] \tag{48}$$

Define $\overline{g^{1\pm}}=(g_1^{1\pm},\ldots, g_k^{1\pm})^T$, $\overline{g^{3\pm}}=(g_1^{3\pm},\ldots, g_k^{3\pm})^T$, $\overline{h^{2\pm}}=(h_1^{2\pm},\ldots, h_k^{2\pm})^T$, $\overline{h^{4\pm}}=(h_1^{4\pm},\ldots, h_k^{4\pm})^T$, $\overline{\phi}=(\phi_1,\ldots,\phi_k)$, $\overline{\psi}=(\psi_1,\ldots,\psi_k)$ as follows

$$g_l^{j\pm}=\frac{1}{l\,!}\frac{d^l g}{dy^l}(Y(P_j)\pm 0),\quad j=1,3,\quad l=1,\dots,k \tag{49}$$

$$h_l^{i\pm}=\frac{1}{l\,!}\frac{d^l h}{dx^l}(X(P_i)\pm 0),\quad i=2,4,\quad l=1,\dots,k$$

$$\phi_l=\frac{1}{l\,!}\frac{d^l\phi}{dx^l}(X(P_2)-0),\quad \psi_l=\frac{1}{l\,!}\frac{d^l\psi}{dy^l}(Y(P_1)-0),\quad l=1,\dots,k \tag{50}$$

Equations (47), (48) imply

$$\overline{g^{1-}}=A(\overline{\psi})\cdot\overline{g^{3+}} \tag{51}$$

$$\overline{h^{2-}}=A(\overline{\phi})\cdot\overline{h^{4+}} \tag{52}$$

where

$$-A(\overline{\gamma})=\begin{pmatrix} a_{11}(\overline{\gamma}) & 0 & \cdots & 0 \\ a_{21}(\overline{\gamma}) & a_{22}(\overline{\gamma}) & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ a_{k1}(\overline{\gamma}) & a_{k2}(\overline{\gamma}) & \cdots & a_{kk}(\overline{\gamma}) \end{pmatrix},\qquad \overline{\gamma}\in\mathbf{R}^k \tag{53}$$

and $a_{ji}(\overline{\gamma})$ are determined by (22). It is easy to see that

$$a_{jj}(\overline{\gamma})=(\gamma_1)^j,\quad j=1,\dots,k$$

Since $\phi', \psi'\leq-\delta<0$ then

$$a_{jj}(\overline{\phi})\neq 0,\quad a_{jj}(\overline{\psi})\neq 0,\quad j=1,\dots,k \tag{54}$$

Hence

$$\det A(\overline{\phi})\neq 0,\quad \det A(\overline{\psi})\neq 0$$

As it was shown above, there exists $g\in C^k(Y(P_1)-\varepsilon, Y(P_1)+\varepsilon)\cap C^k(Y(P_3)-\varepsilon, Y(P_3)+\varepsilon)$ satisfying (16) if and only if $\overline{g^{1-}}=\overline{g^{1+}}=\overline{g^1}$, $\overline{g^{3-}}=\overline{g^{3+}}=\overline{g^3}$ satisfy

$$C(\overline{\alpha}(P_j),\overline{\beta}(P_j))\cdot\overline{g^j}=0,\quad j=1,3 \tag{55}$$

where for any $\overline{\alpha}=(\alpha_1,\dots,\alpha_k)$, $\overline{\beta}=(\beta_1,\dots,\beta_k)$

$$C(\overline{\alpha},\overline{\beta})=\begin{pmatrix} c_{11}(\overline{\alpha},\overline{\beta}) & 0 & \cdots & 0 \\ c_{21}(\overline{\alpha},\overline{\beta}) & c_{22}(\overline{\alpha},\overline{\beta}) & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ c_{k1}(\overline{\alpha},\overline{\beta}) & c_{k2}(\overline{\alpha},\overline{\beta}) & \cdots & c_{kk}(\overline{\alpha},\overline{\beta}) \end{pmatrix} \tag{56}$$

and $c_{ji}(\overline{\alpha},\overline{\beta})$ are defined by (24), $\overline{\alpha}(P_j)$, $\overline{\beta}(P_j)$ are defined by (37).

Analogously, there exist $h(x)\in C^k(X(P_2)-\varepsilon, X(P_2)+\varepsilon)\cap C^k(X(P_4)-\varepsilon, X(P_4)+\varepsilon)$ satisfying (15) if and only if $\overline{h^{2-}}=\overline{h^{2+}}=\overline{h^2}$, $\overline{h^{4-}}=\overline{h^{4+}}=\overline{h^4}$ satisfy

(57) $\quad C(\overline{\alpha}(P_j), \overline{\beta}(P_j)) \cdot \overline{h}^j = 0, \quad j=2, 4$

Because of (51), (52) we can rewrite (55), (57) as follows

(58) $$\begin{cases} C(\overline{\alpha}(P_1), \overline{\beta}(P_1)) \cdot \overline{g}^1 = 0 \\ C(\overline{\alpha}(P_3), \overline{\beta}(P_3)) \cdot A^{-1}(\overline{\phi}) \cdot \overline{g}^1 = 0 \end{cases}$$

(59) $$\begin{cases} C(\overline{\alpha}(P_2), \overline{\beta}(P_2)) \cdot \overline{h}^2 = 0 \\ C(\overline{\alpha}(P_4), \overline{\beta}(P_4)) \cdot A^{-1}(\overline{\phi}) \cdot \overline{h}^2 = 0 \end{cases}$$

Thus we obtained that if there exist $h(x) \in C^k[a, b]$, $g(y) \in C^k[c, d]$ satisfying (8), (14) then (58), (59) hold for some $\overline{h}^2, \overline{g}^1 \in \mathbf{R}^k$, $h_1^2, g_1^1 > 0$.

Let us assume that there exist $\overline{G}, \overline{H} \in \mathbf{R}^k$, $G_1, H_1 > 0$ such that $\overline{G}$ satisfies (58) and $\overline{H}$ satisfies (59). Since n is even then $P_* \in O(P_2)$. Hence there exists $l \in \{0, \dots, n-1\}$ such that $P_2 = F^l P_*$. We define $\lambda(x)$, $x \in X_0 = [a, X(P_*)]$ as follows

(60) $\quad \lambda(X(P)) = X(F^l P), \quad X(P) \in X_0, \quad P \in \Gamma_1$

From (A1), (A2) it follows that $\lambda(x) : X_0 \xrightarrow{on} X_{2l} = [X(F^l P_1), X(P_2)]$, $\lambda(x) \in C^k(X_0)$, $\lambda' \geq \delta > 0$. Consider

(61) $\quad \overline{h}^* = -A(\overline{\lambda}^*) \cdot \overline{H}, \quad \overline{h}^a = A(\overline{\lambda}^a) \cdot \overline{G}$

where

$$\overline{\lambda}^* = (\lambda_1^*, \dots, \lambda_k^*), \quad \overline{\lambda}^a = (\lambda_1^a, \dots, \lambda_k^a)$$
$$\lambda_l^* = \frac{1}{l!} \frac{d^l \lambda}{dx^l}(X(P_*) - 0), \quad \lambda_l^a = \frac{1}{l!} \frac{d^l y_1}{dx^l}(a+0), \quad l = 1, \dots, k$$

Since

$$\lambda_1^* = \lambda'(X(P_*) - 0) > 0, \quad \lambda_1^a = y_1'(a+0) < 0$$
$$a_{11}(\overline{\gamma}) = \gamma_1, \quad H_1, G_1 > 0$$

then

(62) $\quad h_1^*, h_1^a > 0$

Let $h_0(x)$, $x \in X_0 = [a, X(P_*)]$ be an arbitrary function satisfying

$$h_0(x) : X_0 \xrightarrow{on} X_0^{\Gamma m_n} = \left[0, \frac{1}{2\sqrt{2}}\right]; \quad h_0 \in C^k(X_0);$$
$$h'_0(x) \geq \delta > 0, \quad x \in X_0 \text{ for some } \delta > 0;$$

(63) $$\frac{1}{l!} \frac{d^l h_0}{dx^l}(a+0) = h_l^a, \quad \frac{1}{l!} \frac{d^l h_0}{dx^l}(X(P_*) - 0) = h_l^*, \quad l = 1, \cdots, k$$

Such function exists because of (62). Let $h(x), g(y)$ be defined by (12) where $\Gamma_1=\Gamma=\partial\Omega$, $\Gamma_2=\Gamma_n^m=\partial\Pi_n^m$. Then h, g satisfy (8), (14). Consider $\overline{g^{1\pm}}$, $\overline{g^{3\pm}}$, $\overline{h^{2\pm}}$, $\overline{h^{4\pm}}$ defined by (49). Because of (12), (41), (42), (60) we have

(64) $$h_0(X(P_*))-h_0(x)=h(X(P_2))-h(\lambda(x)),\quad x\in X_0$$

Hence, by force of (22), (53), (63), (64)

$$\bar{h}^*=\begin{pmatrix}\frac{1}{1!}\frac{dh_0}{dx}(X(P_*)-0\\ \vdots\\ \frac{1}{k!}\frac{d^kh_0}{dx^k}(X(P_*)-0)\end{pmatrix}=-A(\overline{\lambda^*})\cdot\overline{h^{2-}}$$

From (61) it follows

(65) $$\overline{h^{2-}}=\bar{H}$$

Consider $g(y)$, $y\in Y_0$. According to (12) and definition of the functions $\beta_j: X_0 \xrightarrow{on} Y_{2j}$ we have

$$\beta_0(X(P))=Y(P),\quad P\in M_0;\quad \beta_0^{\Gamma_{nm}}(X^{\Gamma_{nm}}(P))=Y^{\Gamma_{nm}}(P),\quad P\in M_0^{\Gamma_{nm}}$$

Hence, according to (A1)

$$\beta_0(x)=y_1(x),\quad \beta_0^{\Gamma_{nm}}(x)=-x$$

Therefore from (12) it follows

$$g(y)=\beta_0^{\Gamma_{nm}}(h_0(\beta_0^{-1}(y)))=-h_0(y_1^{-1}(y)),\quad y\in Y_0=[Y(P_*),Y(P_1)]$$

Last equation can be written in the form

(66) $$h_0(x)=-g(y_1(x)),\quad x\in X_0$$

Using (22), (49), (53), (61), (63), (66) we obtain

$$A(\overline{\lambda^a})\cdot\bar{G}=\bar{h}^a=\begin{pmatrix}\frac{1}{1!}\frac{dh_0}{dx}(a+0)\\ \vdots\\ \frac{1}{k!}\frac{d^kh_0}{dx^k}(a+0)\end{pmatrix}=A(\overline{\lambda^a})\cdot\overline{g^{1-}}$$

Hence

(67) $$\overline{g^{1-}}=\bar{G}$$

According to (51), (52) we have

(68) $$\overline{g^{3+}}=A^{-1}(\bar{\phi})\cdot\overline{g^{1-}},\quad \overline{h^{4+}}=A^{-1}(\bar{\phi})\cdot\overline{h^{2-}}$$

As it was mentioned above, to prove $h\in C^k[a, b]$, $g\in C^k[c, d]$ it is sufficient to show that (39) holds. Equations (39) can be rewritten as follows

$$\overline{h^{2-}}=\overline{h^{2+}},\quad \overline{h^{4-}}=\overline{h^{4+}},\quad \overline{g^{1-}}=\overline{g^{1+}},\quad \overline{g^{3-}}=\overline{g^{3+}} \tag{69}$$

Consider equations (17), (18). Using (17), (18), (22), (37), (49), (53) we obtain

$$\begin{cases} A(\overline{\alpha}(P_2))\cdot\overline{h^{2-}}+A(\overline{\beta}(P_2))\cdot\overline{h^{2+}}=0 \\ A(\overline{\alpha}(P_4))\cdot\overline{h^{4+}}+A(\overline{\beta}(P_4))\cdot\overline{h^{4-}}=0 \\ A(\overline{\beta}(P_1))\cdot\overline{g^{1-}}+A(\overline{\alpha}(P_1))\cdot\overline{g^{1+}}=0 \\ A(\overline{\beta}(P_2))\cdot\overline{g^{3+}}+A(\overline{\alpha}(P_3))\cdot\overline{g^{3-}}=0 \end{cases} \tag{70}$$

From (24) it follows

$$C(\overline{\alpha}, \overline{\beta})=A(\overline{\alpha})+A(\overline{\beta}),\quad \overline{\alpha}, \overline{\beta}\in\mathbf{R}^k \tag{71}$$

Since $\overline{G}$ satisfies (58) and $\overline{H}$ satisfies (59) then using (65), (67), (68), (70), (71) we obtain

$$\begin{cases} A(\overline{\beta}(P_2))\cdot(\overline{h^{2-}}-\overline{h^{2+}})=0 \\ A(\overline{\beta}(P_4))\cdot(\overline{h^{4+}}-\overline{h^{4-}})=0 \\ A(\overline{\alpha}(P_1))\cdot(\overline{g^{1-}}-\overline{g^{1+}})=0 \\ A(\overline{\alpha}(P_3))\cdot(\overline{g^{3+}}-\overline{g^{3-}})=0 \end{cases} \tag{72}$$

By force of (A2) we have $\beta_1(P_j)\neq 0, j=2, 4$; $\alpha_1(P_i)\neq 0, i=1, 3$. Then det $A(\overline{\beta}(P_j))\neq 0, j=2, 4$; det $A(\overline{\alpha}(P_i))\neq 0, i=1, 3$. Therefore (69) holds and $h\in C^k[a, b]$, $g\in C^k[c, d]$.

Thus we verified the following result.

Theorem 3. *Let $\Omega\in E(m, n, k)$ for some $m, n, k\in\mathbf{N}$, $m<n$, $k\geq 2$, and n be even. Then there exist $h(x)\in C^k[a, b]$, $g(y)\in C^k[c, d]$ satisfying* (8), (14) *if and only if there exist vectors $\overline{H}, \overline{G}\in\mathbf{R}^k$, $H_1, G_1>0$ such that $\overline{G}$ satisfies* (58), *$\overline{H}$ satisfies* (59).

II). Let n be an odd number. Then by force of (9) we have $F^{\frac{n-1}{2}}P_1=P_2$, $F^{\frac{n-1}{2}}P_3=P_4$ or $F^{\frac{n-1}{2}}P_1=P_4$, $F^{\frac{n-1}{2}}P_3=P_2$. Using (40) - (42) we obtain for sufficiently small $\varepsilon>0$

$$\begin{cases} 0-g(Y(P))=h(X(F^{\frac{n-1}{2}}P))-\dfrac{n-m}{2\sqrt{2}}, & |P-P_1|\leq\varepsilon \\ g(Y(P))-\dfrac{2m-n}{2\sqrt{2}}=\dfrac{m}{2\sqrt{2}}-h(X(F^{\frac{n-1}{2}}P)), & |P-P_3|\leq\varepsilon \end{cases} \tag{73}$$

if $F^{\frac{n-1}{2}}P_1=P_2$, or

$$(74)\quad \begin{cases} 0-g\,(Y(P))=\dfrac{m}{2\sqrt{2}}-h\,(X(F^{\frac{n-1}{2}}P))\,, & |P-P_1|\leq\varepsilon \\ g\,(Y(P))-\dfrac{2m-n}{2\sqrt{2}}=h\,(X(F^{\frac{n-1}{2}}P))-\dfrac{n-m}{2\sqrt{2}}, & |P-P_3|\leq\varepsilon \end{cases}$$

if $F^{\frac{n-1}{2}}P_1=P_4$.
We define functions $\mu(y)$, $\omega(y)$

$$\mu(Y(P))=X(F^{\frac{n-1}{2}}P),\quad Y(P)\in[Y(P_1)-\varepsilon,\ Y(P_1)],\quad P\in\Gamma_1$$
$$\omega(Y(P))=X(F^{\frac{n-1}{2}}P),\quad Y(P)\in[Y(P_3),\ Y(P_3)+\varepsilon],\quad P\in\Gamma_3$$

From (A1), (A2) and Lemma 3′ it follows that for $\varepsilon>0$ sufficiently small

$$\mu(y)\ :\ [Y(P_1)-\varepsilon,\ Y(P_1)]\xrightarrow{on}\begin{cases}[X(P_2),\ X(P_2)+\varepsilon_1], & F^{\frac{n-1}{2}}P_1=P_2\\ [X(P_4)-\varepsilon_1,\ X(P_4)], & F^{\frac{n-1}{2}}P_1=P_4\end{cases}$$

$$\omega(y)\ :\ [Y(P_3),\ Y(P_3)+\varepsilon]\xrightarrow{on}\begin{cases}[X(P_4)-\varepsilon_1,\ X(P_4)], & F^{\frac{n-1}{2}}P_1=P_2\\ [X(P_2),\ X(P_2)+\varepsilon_1], & F^{\frac{n-1}{2}}P_1=P_4\end{cases}$$

$$\mu\in C^k[Y(P_1)-\varepsilon,\ Y(P_1)],\quad \omega\in C^k[Y(P_3),\ Y(P_3)+\varepsilon]$$
$$|\mu'(y)|\geq\delta\geq0,\quad y\in[Y(P_1)-\varepsilon,\ Y(P_1)];$$
$$|\omega'(y)|\geq\delta\geq0,\quad y\in[Y(P_3),\ Y(P_3)+\varepsilon]$$

for some constants ε_1, $\delta>0$. Equations (73), (74) can be written as follows

$$(75)\quad \begin{cases} g\,(y)+h\,(\mu(y))=\dfrac{n-m}{2\sqrt{2}}, & y\in[Y(P_1)-\varepsilon,\ Y(P_1)],\quad F^{\frac{n-1}{2}}P_1=P_2 \\ g\,(y)+h\,(\omega(y))=\dfrac{3m-n}{2\sqrt{2}}, & y\in[Y(P_3),\ Y(P_3)+\varepsilon],\quad F^{\frac{n-1}{2}}P_1=P_2 \end{cases}$$

$$(76)\quad \begin{cases} g\,(y)-h\,(\mu(y))=-\dfrac{m}{2\sqrt{2}}, & y\in[Y(P_1)-\varepsilon,\ Y(P_1)],\quad F^{\frac{n-1}{2}}P_1=P_4 \\ g\,(y)-h\,(\omega(y))=\dfrac{3m-n}{2\sqrt{2}}, & y\in[Y(P_3),\ Y(P_3)+\varepsilon],\quad F^{\frac{n-1}{2}}P_1=P_4 \end{cases}$$

We define $\bar{\mu}=(\mu_1,\dots,\mu_k)$, $\bar{\omega}=(\omega_1,\dots,\omega_k)$ as follows

$$\mu_l=\frac{1}{l\,!}\frac{d^l\mu}{dy^l}(Y(P_1)-0),\quad \omega_l=\frac{1}{l\,!}\frac{d^l\omega}{dy^l}(Y(P_3)+0),\quad l=1,\dots,k$$

Equations (75), (76) imply

$$(77)\quad \overline{g^{1-}}=\begin{cases} A\,(\bar{\mu})\cdot\overline{h^{2+}}\,, & F^{\frac{n-1}{2}}P_1=P_2\\ -A\,(\bar{\mu})\cdot\overline{h^{4-}}\,, & F^{\frac{n-1}{2}}P_1=P_4\end{cases}$$

$$(78) \qquad \overline{g^{3+}} = \begin{cases} A(\overline{\omega}) \cdot \overline{h^{4-}}, & F^{\frac{n-1}{2}} P_1 = P_2 \\ -A(\overline{\omega}) \cdot \overline{h^{2+}}, & F^{\frac{n-1}{2}} P_1 = P_4 \end{cases} .$$

If $\overline{h^{2-}} = \overline{h^{2+}} = \overline{h}^2$, $\overline{h^{4-}} = \overline{h^{4+}} = \overline{h}^4$, $\overline{g^{1-}} = \overline{g^{1+}} = \overline{g}^1$, $\overline{g^{3-}} = \overline{g^{3+}} = \overline{g}^3$ then using (55), (56), (77), (78) we obtain

$$(79) \qquad \begin{cases} C(\overline{\alpha}(P_1), \overline{\beta}(P_1)) \cdot \overline{g}^1 = 0 \\ C(\overline{\alpha}(P_2), \overline{\beta}(P_2)) \cdot A^{-1}(\overline{\mu}) \cdot \overline{g}^1 = 0 \end{cases}$$

$$(80) \qquad \begin{cases} C(\overline{\alpha}(P_3), \overline{\beta}(P_3)) \cdot \overline{g}^3 = 0 \\ C(\overline{\alpha}(P_4), \overline{\beta}(P_4)) \cdot A^{-1}(\overline{\omega}) \cdot \overline{g}^3 = 0 \end{cases}$$

if $F^{\frac{n-1}{2}} P_1 = P_2$, or

$$(81) \qquad \begin{cases} C(\overline{\alpha}(P_1), \overline{\beta}(P_1)) \cdot \overline{g}^1 = 0 \\ C(\overline{\alpha}(P_4), \overline{\beta}(P_4)) \cdot A^{-1}(\overline{\mu}) \cdot \overline{g}^1 = 0 \end{cases}$$

$$(82) \qquad \begin{cases} C(\overline{\alpha}(P_3), \overline{\beta}(P_3)) \cdot \overline{g}^3 = 0 \\ C(\overline{\alpha}(P_2), \overline{\beta}(P_2)) \cdot A^{-1}(\overline{\omega}) \cdot \overline{g}^3 = 0 \end{cases}$$

if $F^{\frac{n-1}{2}} P_1 = P_4$.

Thus we verified that if there exist $h(x) \in C^k[a, b]$, $g(y) \in C^k[c, d]$ satisfying (8), (14) then there exist $\overline{g}^1, \overline{g}^3 \in \mathbf{R}^k$, $g_1^1, g_1^3 > 0$ satisfying (79), (80) or (81), (82) according to whether $F^{\frac{n-1}{2}} P_1 = P_2$ or $F^{\frac{n-1}{2}} P_1 = P_4$.

Using the same arguments as in the case I) we obtain that if there exist $\overline{G}^1, \overline{G}^3 \in \mathbf{R}^k$, $G_1^1, G_1^3 > 0$ such that $\overline{G}^1$ satisfies (79) or (81) according to whether $F^{\frac{n-1}{2}} P_1 = P_2$ or $F^{\frac{n-1}{2}} P_1 = P_4$, $\overline{G}^3$ satisfies (80) or (82) according to whether $F^{\frac{n-1}{2}} P_1 = P_2$ or $F^{\frac{n-1}{2}} P_1 = P_4$ then there exist $h(x) \in C^k[a, b]$, $g(y) \in C^k[c, d]$ satisfying (8), (14). Thus we verified the following result.

Theorem 4. *Let $\Omega \in E(m, n, k)$ for some $m, n, k \in \mathbf{N}$, $m < n$, $k \geq 2$, and n be odd. Then there exist $h(x) \in C^k[a, b]$, $g(y) \in C^k[c, d]$ satisfying* (8), (14) *if and only if there exist vectors $\overline{G}^1, \overline{G}^3 \in \mathbf{R}^k$, $G_1^1, G_1^3 > 0$ such that $\overline{G}^1$ satisfies* (79) *or* (81) *according to whether $F^{\frac{n-1}{2}} P_1 = P_2$ or $F^{\frac{n-1}{2}} P_1 = P_4$, $\overline{G}^3$ satisfies* (80) *or* (82) *according to whether $F^{\frac{n-1}{2}} P_1 = P_2$ or $F^{\frac{n-1}{2}} P_1 = P_4$.*

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