J. Math. Kyoto Univ. (JMKYAZ)
34-2 (1994) 273-283

Chain conditions on prime ideals in ideal-adically complete Nagata rings

By

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Introduction

Throughout this paper, all rings are commutative with identity. We use the notation of EGA [2], Matsumura [7] and Nagata [9]. Terminology and definitions of [10] and [11] are used freely (see also Appendix).
In this note, we mainly study Lifting Problem on chain conditions of prime ideals in ideal-adically complete Nagata rings. That is:

Lifting Problem. Let A be a noetherian ring and I an ideal of A. Suppose that A is I-complete and that A/I is universally catenary. Is then A also universally catenary?

It would be interesting and useful if one got a positive answer to the problem above. However, S. Greco has found the following surprising and meaningful counter-example:

Greco's Example ([3], cf. [12]). There exist a semi-local domain $(A, \mathfrak{m}_1, \mathfrak{m}_2)$ with two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ and an ideal $I=P_1 \cap P_2$ (=the intersection of two prime ideals P_i) of A such that A is I-complete and A/I is excellent, hence universally catenary, but A itself is *not* universally catenary.

Further, in the same article, since neither A is local nor I is prime, Greco asked (cf. [3, Remark]):

Question 1. Is there a non-universally catenary *local* ring A with an ideal I such that A is I-complete and A/I is universally catenary?

Question 2. Is there a non-universally catenary noetherian domain A with a *prime* ideal P such that A is P-complete and A/P is universally catenary?

Incidentally, we know a lot of good facts concerning Lifting Problem on ideal-adically complete noetherian rings. First of all, J. Marot found a positive answer on Nagata rings:

Theorem 1 ([5]). *Let A be a noetherian ring and I an ideal of A. Sup-*

Received February 25, 1993, Revised September 12, 1993

pose that A is I-complete and that A/I is a Nagata ring. Then A is also a Nagata ring. In particular, any ideal-adic completion of a Nagata ring is again a Nagata ring.

Next, C. Rotthaus solved local Lifting Problem for quasi-excellent rings:

Theorem 2 ([15]). *Let A be a semi-local ring and I an ideal of A. Suppose that A is I-complete and that A/I is quasi-excellent. Then A is also quasi-excellent. In particular, any ideal-adic completion of a (quasi-) excellent semi-local ring is again a (quasi-) excellent ring.*

Further, she obtained Lifting Theorem for (quasi-) excellent rings which contain a field of characteristic zero by proving the so-called Rotthaus Hilfssatz ([16], [1], cf. [11]):

Theorem 3. *Let A be a noetherian ring, containing a field of characteristic zero, and let I be an ideal of A. Suppose that A is I-complete and that A/I is quasi-excellent. Then A also quasi-excellent. In particular, any ideal-adic completion of a (quasi-) excellent ring which contains a field of characteristic zero is again a (quasi-) excellent ring.*

On the other hand, constructing his remarkable counter-example to Ratliff's Chain Conjecture, T. Ogoma observed a good condition on formal fibres which assures a normal Nagata ring to be universally catenary:

Theorem 4 ([13]). *Let A be a Nagata local domain and $\bar{A}$ its derived normal ring. Then the following are equivalent:*

(4.1) *the generic formal fibre of A is locally equidimensional.*

(4.2) *$\bar{A}$ is universally catenary.*

Here, we collect the definitions on equidimensionality of noetherian rings:

Definition 1. A noetherian rign A is *equidimensional* if, for any $\mathfrak{p} \in \mathrm{Min}(A)$, $\dim A/\mathfrak{p} = \dim A$. And a noetherian ring A is called *locally equidimensional* if A_P is equidimensional for any $P \in \mathrm{Spec}(A)$. A local ring A is *formally equidimensional* if its completion $\hat{A}$ is equidimensional. And a noetherian ring A is said to be *locally formally equidimensional* if A_P is formally equidimensional for any $P \in \mathrm{Spec}(A)$.

Then, thanks to Greco, Marot, Ogoma and Rotthaus, we shall show the following:

Proposition 1. *Let A be a noetherian ring with an ideal I. If A is I-complete and A/I is a Nagata ring whose formal fibres are locally equidimensional, then the formal fibres of A are also locally equidimensional.*

Proposition 2. *Let A be a local ring with an ideal I. If A is I-complete and A/I* is *a universally catenary Nagata ring, then A is also universally catenary.*

Proposition 3. *Let A be a P-complete noetherian domain where P is a* prime *ideal. Then, if A/P is a universally catenary Nagata ring, A is also universally catenary.*

Proposition 4. *Let A be a noetherian domain with a non-zero ideal I. If A is I-complete and catenary, then A* is *universally catenary.*

Now we summarize the content of this note. Basic theorem and lemmas are gathered as preliminaries in Section 1. We omit their proofs, which are not straightforward, because these results seem to be well-known. Next, in Section 2, we check that the property of being locally formally equidimensional enjoys some axiomatic conditions, which play key roles in our proof of Proposition 1 for semi-local rings at the beginning of next section. In the last part of Section 3, making use of Nagata's criterion due to Greco-Marinari [4], we shall show the finiteness of the set $\Delta(x)$, which completes via Rotthaus Hilfssatz, our proof of Proposition 1. Then, Propositions 2 and 3 are derived from Proposition 1 in Section 4. Section 5 contains a few related topics, including a proof of Proposition 4, which may be useful. In Appendix, for the reader's convenience, we recall notation and definitions necessary to state Rotthaus Hilfssatz whose proof we do not repeat here, because a complete proof can be found in [11].

We end this introduction by remarking that the original Rotthaus Hilfssatz was proved under a stronger condition, namely, the *universal catenary* hypothesis. Therefore, to get Proposition 1, we should have removed this particular hypothesis in our Rotthaus Hilfssatz, and this is what we have done in [11].

1. Preliminaries

Theorem 1.1 (Nagata-Ratliff, cf. [14, (A11)]). *A local ring A is universally catenary and equidimensional if and only if A is formally equidimensional.*

Lemma 1.2 ([10, (2.4)]). *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local rings and $\psi: A \to B$ a local homomorphism. Suppose that ψ is flat, $\bar{\psi} = \psi \otimes k(\mathfrak{m})$ is reduced and that A is a Nagata ring. Then ψ is also reduced.*

Lemma 1.3 (cf. [13, Theorem 1]). *Let A be a local ring. If A is catenary and not equidimensional, there exists $P \in$* Spec (A) *such that A_P is not equidimensional and* prof $A_P \leq 1$.

Lemma 1.4 ([4], cf. [7, (22.C)]). *If a Nagata ring A satisfies (S_1), then there is a non-zero-divisor y such that A_y satisfies (S_2).*

2. Formal equidimensionally

Lemma 2.1. *Let A be a Nagata local domain with locally equidimensional formal fibres. If A is not formally equidimensional, there is $P \in \operatorname{Spec}(A)$ such that A_P is not formally equidimensional and* $\operatorname{prof} A_P = 1$.

Proof. Since complete local rings are catenary, take $\widehat{P} \in \operatorname{Spec}(\widehat{A})$ such that $\widehat{A}_{\widehat{P}}$ is not equidimensional and $\operatorname{prof} \widehat{A}_{\widehat{P}} = 1$ (cf. Lemma 1.3). Putting $P = \widehat{P} \cap A$, we claim: A_P is not formally equidimensional.

Indeed, assume the contrary, namely, $\overline{A}$ is universally catenary. Then, if we denote by $\overline{A}$ the derived normal ring of A, all heights of the maximal ideals of $\overline{A}_P$ are the same. Thus, since $\widetilde{A} = \widehat{A}_{\widehat{P}} \otimes_A \overline{A}$ is $\overline{A}_P$-flat, all heights of the maximal ideals of $\widetilde{A}$ are also the same, because the formal fibres of A are assumed to be locally equidimensional. Hence, there exists $\widetilde{P} \in \operatorname{Max}(\widetilde{A})$ such that $\widetilde{A}_{\widetilde{P}}$ is not equidimensional. Therefore, we find $\widetilde{Q} \in \operatorname{Spec}(\widetilde{A})$ such that $\widetilde{A}_{\widetilde{Q}}$ is not equidimensional and $\operatorname{prof} \widetilde{A}_{\widetilde{Q}} = 1$. Putting $\overline{Q} = \widetilde{Q} \cap \overline{A}$, we see that $\overline{A}_{\overline{Q}}$ is a DVR. Then, since ψ: $\overline{A}_{\overline{Q}} \rightarrow \widetilde{A}_{\widetilde{Q}}$ is reduced, $\widetilde{A}_{\widetilde{Q}}$ is to be a DVR, too. Contradiction.

Now we set $\operatorname{Eq}(A) = \{P \in \operatorname{Spec}(A) \mid A_P$ is formally equidimensional$\}$ and $\operatorname{Neq}(A) = \operatorname{Spec}(A) \setminus \operatorname{Eq}(A)$. Then

Corollary 2.2. *Let A be a reduced Nagata local ring whose formal fibres are locally equidimensional. If P is minimal in* $\operatorname{Neq}(A)$, *then* $\operatorname{prof} A_P = 1$. *In particular, supposing further that* $\operatorname{Eq}(A)$ *is open, if $\mathfrak{b} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ is the defining ideal of the closed set* $\operatorname{Neq}(A)$, *then* $\operatorname{prof} A_{\mathfrak{p}_i} = 1$ *for any $i = 1, \ldots, r$.*

Now we shall show that the property of being locally formally equidimensional satisfies the following conditons (cf. A.3):

2.3. $\mathbf{P}_{I^*}$: *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be Nagata local rings and ψ: $A \rightarrow B$ a local homomorphism. Suppose that*

(2.3.1) *A is formally equidimensional,*

(2.3.2) *ψ is reduced, and the fibres of ψ are locally equidimensional,*

(2.3.3) *the formal fibres of B are locally equidimensional.*

Then B is also formally equidimensional.

Proof. Assume the contrary. That is, B is not formally equidimensional. We may assume A is a domain. Let $\overline{A}$ be the derived normal ring of A and $\widetilde{B} = B \otimes_A \overline{A}$. Then, all heights of the maximal ideals of $\overline{A}$ are the same, because A is universally catenary. Thus, since $\widetilde{B}$ is $\overline{A}$-flat, all heights of the maximal ideals of $\widetilde{A}$ are also the same, because the closed fibre of ψ is assumed to be locally equidimensional. Hence, there is $\tilde{\mathfrak{n}} \in \mathrm{Max}(\widetilde{B})$ such that $\widetilde{B}_{\tilde{\mathfrak{n}}}$ is not formally equidimensional. Therefore, by Corollary 2.2, we find $\widetilde{Q} \in \mathrm{Spec}(\widetilde{B})$ such that $\widetilde{B}_{\widetilde{Q}}$ is not formally equidensional and prof $\widetilde{B}_{\widetilde{Q}}=1$. Putting $\overline{Q}=\widetilde{Q}\cap\overline{A}$, we see that $\overline{A}_{\overline{Q}}$ is a DVR. Then, since ψ: $\overline{A}_{\overline{Q}} \rightarrow \widetilde{A}_{\widetilde{Q}}$ is reduced, $\widetilde{A}_{\widetilde{Q}}$ is to be a DVR, too. Contradiction.

2.4. $\mathbf{P}_{II}$: *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local rings and ψ: $A \rightarrow B$ a local homomorphism. Suppose that ψ is flat, and B is (formally) equidimensional. Then A is also (formally) equidimensional.*

The following condition is an immediate consequence of Theorem 1.1:

2.5. $\mathbf{P}_{III}$: *If $(A, \mathfrak{m})$ is a formally equidimensional local ring, then A_P is formally equidimensional for any $P \in \mathrm{Spec}(A)$.*

Now, we remark that once conditions $\mathbf{P}_{II}$ and $\mathbf{P}_{III}$ are verified, a formal argument gives:

Proposition 2.6. *Let $(A, \mathfrak{m})$ be a local ring whose formal fibres are locally equidimensional. Then the formal fibres of A_P are locally equidimensional for any $P \in \mathrm{Spec}(A)$.*

We state the following condition without proof, because its proof is standard (cf. [2, (5.12.2)]).

2.7. $\mathbf{P}_{IV*}$: *Let $(A, \mathfrak{m})$ be a catenary local ring and $b \in \mathfrak{m}$ a non-zero-divisor. Suppose that A/bA is equidimensional and satisfies (S_k) $(k \geq 1)$. Then A is equidimensional and satisfies (S_k).*

2.8. $\mathbf{P}_{V}$: *For a complete local ring $(A, \mathfrak{m})$, $\mathrm{Eq}(A)$ is an open set.*

Proof (Seydi [17]). For each $P \in \mathrm{Eq}(A)$, let $X_1, \ldots, X_s$ be the irreducible components of $\mathrm{Spec}(A)$ which do not contain P. Put $V = X_1 \cup \ldots \cup X_s$ and $U = \mathrm{Spec}(A) \setminus V$. Then, to get the assertion, it suffices to show that $U \subset \mathrm{Eq}(A)$. Indeed, take any $Q \in U$ and any $\mathfrak{p} \in \mathrm{Min}(A)$ such that $\mathfrak{p} \subset Q$. Then $\dim A/\mathfrak{p} = \dim A_P + \dim A/P$, because $\mathfrak{p} \subset P$ by definition. And $\dim A_Q/\mathfrak{p}A_Q = \dim A/\mathfrak{p} - \dim A/Q$. Thus, A_Q is equidimensional. Therefore, $Q \in \mathrm{Eq}(A)$.

We include, for the reader's convenience, the conditon $\mathbf{P}_{\mathrm{geom}}$ without proof, because its proof is clear and because it will not be essentially used in the rest of this paper.

2.9. $\mathbf{P}_{\mathrm{geom}}$: *Let k be a field and A a noetherian k-algebra. If A is locally (formally) equidimensional, then $A \otimes_k k'$ is locally (formally) equidimensional for any finite extension field k' of k.*

Proposition 2.10. *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local Nagata rings and ψ: $A \rightarrow B$ a reduced local homomorphism. Suppose that*

(2.10.1) *the closed fibre of ψ is formally equidimensional,* (2.10.2) *the formal fibres of A are locally equidimensional,*

(2.10.3) *the formal fibres of B are locally equidimensional.*

Then the fibres of ϕ are locally formally equidimensional.

Proof. We may assume A is a domain with its quotient field $Q(A)=K$ and the fibres of $\phi\otimes A/\mathfrak{a}$ are locally formally equidimensional for any non-zero ideal $\mathfrak{a}$. Let $\overline{A}$ be the derived normal ring of A and $\widetilde{B}=B\otimes_A \overline{A}$. For $\tilde{\mathfrak{n}} \in \mathrm{Max}(\widetilde{B})$ and for $\overline{\mathfrak{m}}=\tilde{\mathfrak{n}}\cap\overline{A}$, let ϕ': $\overline{A}_{\overline{\mathfrak{m}}} \to \widetilde{B}_{\tilde{\mathfrak{n}}}$ be the induced local homomorphism. Then, since $\overline{A}_{\overline{\mathfrak{m}}}$, $\widetilde{B}_{\tilde{\mathfrak{n}}}$ and ϕ' satisfy the same assumptions as above, we may further assume that A is normal and universally catenary by Theorem 4. Hence, we are only to show that $B\otimes_A K$ is locally formally equidimensional. Indeed, for $\mathfrak{q}\in\mathrm{Spec}(B\otimes_A K)$ $(\subset\mathrm{Spec}(B))$ and a non-zero element $b\in\mathfrak{m}$, take $Q \in\mathrm{Min}(B/\mathfrak{q}+bB))$.

Then, letting $P=Q\cap A$, A_P/bA_P, B_Q/bB_Q and $\phi\otimes A_P/bA_P$: $A_P/bA_P \to B_Q/bB_Q$ satisfy the conditions (2.3.1) - (2.3.3). Moreover, A_P/bA_P is formally equidimensional and satisfies (S_1), because A_P is assumed to be universally catenary and normal. Then, $B_Q\hat{}/bB_Q\hat{}$ is equidimensional and satisfies (S_1). Thus, $B_Q\hat{}$ is also equidimensional. Therefore, $B_{\mathfrak{q}}$ is formally equidimensional.

3. Lifting Theorem (Proof of Proposition 1)

3.1. Proof of Propositon 1 (semi-local case). We checked, in the previous section, that the property of being locally equidimensional satisfies the conditions $\mathbf{P}_{I*}$, $\mathbf{P}_{II}$, $\mathbf{P}_{III}$, and $\mathbf{P}_V$. Further, we have shown, in Proposition 2.10, the localization theorem for local equidimensionality. Thus, the proof of Proposition 1 for semi-local rings, namely, local Lifting Theorem for rings with locally equidimensional formal fibres, can be proved in exactly the same way as that for quasi-excellent rings. So, we refer the reader to [15], [10, Theorem], or [6, Theorem 5.2].

Now, as direct applications of local Lifting Theorem for rings with locally equidimensional formal fibres, by taking the $\mathfrak{m}$-adic completion $\widehat{A}$ of A and the $\mathfrak{m}B$-adic completion B^* of B in place of A and B, respectively, we get the following generalization of Proposition 2.10 and the property $\mathbf{P}_{I*}$.

Corollary 3.2. *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be Nagata local rings and ϕ: $A \to B$ a reduced local homomorphism. Suppose that*

(3.2.1) *the closed fibre of ϕ is formally equidimensional,*

(3.2.2) *the formal fibres of A are locally equidimensional.*

Then the fibres of ϕ are locally formally equidimensional.

3.3. $\mathbf{P}_I$: *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be Nagata local rings and ϕ: $A \to B$ a reduced local homomorphism. Suppose that the fibres of ϕ are locally formally* *equidimensional. Then, if A is formally equidimensional, B is also formally equidimensional.*

Finally, we shall show that Lifting Theorem for rings with locally equidimensional formal fibres holds in general.

3.4. End of the proof of Proposition 1. We use the same notation as in [11], and for the details see also Appendix.

By noetherian induction, we may assume that A is a normal Nagata domain and the formal fibres of $A_{\mathfrak{m}}/\mathfrak{a}$ are locally equidimensional for any $\mathfrak{m}\in\mathrm{Max}(A)$ and for any non-zero ideal $\mathfrak{a}$.

Choose $\mathfrak{m}\in\mathrm{Max}(A)$ and a non-zero element $x\in I$ and fix them. Now take a minimal element $\mathfrak{p}_{\mathfrak{m}}^*$ of $\mathrm{Neq}(A_{\mathfrak{m}}^*)$ and let $\mathfrak{p}_{\gamma}^*=\mathfrak{p}_{\mathfrak{m}}^*\cap A_{\gamma}^*$ for any $\gamma\in\Gamma(\mathfrak{m})$, where, for any A-algebra B, B^* denotes the xB-adic completion of B. Then $(A_{\gamma}^*)_{\mathfrak{p}_{\gamma}^*}$ is not formally eqidimensional (cf. 3.3).

We shall prove $\mathfrak{p}_{\gamma}^*\cap A\neq(0)$. But, to get the assertion, because we checked that the property of being locally formally equidimensional satisfies $\mathbf{P}_I$, $\mathbf{P}_{II}$, and $\mathbf{P}_{III}$ and because we have just proved that local Lifting Theorem for rings with locally equidimensional formal fibres, as remarked in proposition A.10, it is enough to show:

(3.4.1) $\Delta(x)$ is a finite set (for the definition, see (A.7.2)).

Indeed, let $P\in\Delta(x)$. Then, by difinition, there exist $\gamma\in\Gamma(\mathfrak{m})$ and $\bar{Q}\in\mathrm{Ass}(\bar{B}_{\gamma}/x\bar{B}_{\gamma})$ such that $P=\bar{Q}\cap A$. Hence, prof $A_P\leq 2$, because $\mathrm{prof}(A_{\gamma}^*)_{\mathfrak{p}_{\gamma}^*}=1$ for any $\gamma\in\Gamma(\mathfrak{m})$ (cf. Corollary 2.2, [11, (2.8)], [10, (1.3)]). Further, since A is a Nagata normal domain, there exists $y\in A$ such that $(A/xA)_y$ satisfies (S_2) and x, y is an A-sequence (cf. Lemma 1.4). We claim:

(3.4.2) $y\in P$ (, then $\Delta(x)\subset\mathrm{Ass}(A/(x,y))$ and the proof is completed).

Assume the contrary. Then, by the choice of y, $\dim A_P/xA_P\leq 1$. Therefore, $\dim(A_{\gamma}^*)_{\mathfrak{p}_{\gamma}^*}\leq 1$. Contradiction.

Once Lifting Theorem for rings with locally equidimensional formal fibres is proved, the following corollary is an easy exercise.

Corollary 3.5. *Let A be a noetherian ring and I an ideal of A. Suppose that A is I-complete, A satisfies (S_2), and that A/I is a universally catenary Nagata ring. Then A is also universally catenary.*

4. Applications of Lifting Theorem

4.1. Proof of Proposition 2. By noetherian induction, we may assume that A is a domain and $A/\mathfrak{a}$ is universally catenary for any non-zero ideal $\mathfrak{a}$. Let $(\bar{A}, \bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_r)$ be the derived normal ring of A. Then, by Proposition 1 and Corollary 3.5, to get the assertion, it suffices to show:

(4.1.1) $\operatorname{ht} \bar{\mathfrak{m}}_i = \operatorname{ht} \mathfrak{m}$ for $i=1, \dots, r$.

Take a non-zero element $b \in I$. Note that Spec $(\bar{A}/b\bar{A})$ is *connected*, because $\bar{A}$ is a $b\bar{A}$-complete domain. Hence, if $\sqrt{b\bar{A}} = \bar{P}_1 \cap \dots \cap \bar{P}_s$, for any $\bar{\mathfrak{m}}, \bar{\mathfrak{m}}' \in \operatorname{Max}(\bar{A})$, there is a sequence of maximal ideas $\bar{\mathfrak{m}} = \bar{\mathfrak{m}}_{f_0}, \bar{\mathfrak{m}}_{f_1}, \dots, \bar{\mathfrak{m}}_{f_t} = \bar{\mathfrak{m}}'$ such that $\bar{\mathfrak{m}}_{f_i} \cap \bar{\mathfrak{m}}_{f_{i+1}} \supset \bar{P}_{g_i}$, where $f_i \in \{1, \dots, r\}$, $g_i \in \{1, \dots, s\}$ $(i = 0, 1, \dots, t-1)$. Thus we are only to show:

(4.1.2) if $\bar{\mathfrak{m}}, \bar{\mathfrak{m}}' \in \operatorname{Max}(\bar{A})$ and $\bar{\mathfrak{m}} \cap \bar{\mathfrak{m}}' \supset \bar{P}_1$, then $\operatorname{ht} \bar{\mathfrak{m}} = \operatorname{ht} \bar{\mathfrak{m}}'$.

Indeed, $\operatorname{ht} \bar{\mathfrak{m}} = \operatorname{ht} \bar{P}_1 + \operatorname{ht} \bar{\mathfrak{m}}/\bar{P}_1$ and $\operatorname{ht} \bar{\mathfrak{m}}' = \operatorname{ht} \bar{P}_1 + \operatorname{ht} \bar{\mathfrak{m}}'/\bar{P}_1$. Futher, since $A/(\bar{P}_1 \cap A)$ is supposed to be a universally catenary *local* domain and $\bar{A}/\bar{P}_1$ is a finite extension of it, $\operatorname{ht} \bar{\mathfrak{m}}/\bar{P}_1 = \operatorname{ht} \bar{\mathfrak{m}}'/\bar{P}_1$. This gives the assertion.

4.2. Proof of Propositon 3. Let $\bar{A}$ be the derived normal ring. As above, to get the assertion, it is enough to show:

(4.2.1) $\operatorname{ht} \bar{\mathfrak{m}} = \operatorname{ht} \mathfrak{m}$ for any $\bar{\mathfrak{m}} \in \operatorname{Max}(\bar{A})$ and $\mathfrak{m} = \bar{\mathfrak{m}} \cap A$.

Take $\bar{\mathfrak{m}}' \in \operatorname{Max}(\bar{A})$, lying over $\mathfrak{m}$, such that $\operatorname{ht} \bar{\mathfrak{m}}' = \operatorname{ht} \mathfrak{m}$. Note that Spec $(\bar{A}/P\bar{A})$ is *connected*, because $\bar{A}$ is a $P\bar{A}$-complete domain. Hence, if $\sqrt{P\bar{A}} = \bar{P}_1 \cap \dots \cap \bar{P}_r$, then $\bar{P}_i \cap A = P$ and we find a sequence of maximal ideals $\bar{\mathfrak{m}} = \bar{\mathfrak{m}}_0, \bar{\mathfrak{m}}_1, \dots, \bar{\mathfrak{m}}_s = \bar{\mathfrak{m}}'$ such that $\bar{\mathfrak{m}}_0 \cap \bar{\mathfrak{m}}_1 \supset \bar{P}_{f_1}, \dots, \bar{\mathfrak{m}}_{s-1} \cap \bar{\mathfrak{m}}_s \supset \bar{P}_{fs}$, where $f_i \in \{1, \dots, r\}$ $(i=1, \dots, s)$. Consequently, since $\bar{A}$ and A/P are universally catenary, putting $\bar{\mathfrak{m}}_i \cap A = \mathfrak{m}_i$, we have:

$$
\begin{aligned}
\operatorname{ht} \bar{\mathfrak{m}} - \operatorname{ht} \bar{P}_{f_1} &= \operatorname{ht} \bar{\mathfrak{m}}/\bar{P}_{f_1} &&= \operatorname{ht} \mathfrak{m}/P \ , \\
\operatorname{ht} \bar{\mathfrak{m}}_1 - \operatorname{ht} \bar{P}_{f_1} &= \operatorname{ht} \bar{\mathfrak{m}}_1/\bar{P}_{f_1} &&= \operatorname{ht} \mathfrak{m}_1/P \ , \\
\operatorname{ht} \bar{\mathfrak{m}}_1 - \operatorname{ht} \bar{P}_{f_2} &= \operatorname{ht} \bar{\mathfrak{m}}_1/\bar{P}_{f_2} &&= \operatorname{ht} \mathfrak{m}_1/P \ , \\
&\dots\dots \\
&\dots\dots \\
\operatorname{ht} \bar{\mathfrak{m}}_{s-1} - \operatorname{ht} \bar{P}_{fs} &= \operatorname{ht} \bar{\mathfrak{m}}_{s-1}/\bar{P}_{fs} &&= \operatorname{ht} \mathfrak{m}_{s-1}/P \ , \\
\operatorname{ht} \bar{\mathfrak{m}}' - \operatorname{ht} \bar{P}_{fs} &= \operatorname{ht} \bar{\mathfrak{m}}/\bar{P}_{fs} &&= \operatorname{ht} \mathfrak{m}/P \ .
\end{aligned}
$$

Then, $\operatorname{ht} \bar{P}_{f_1} = \cdots = \operatorname{ht} \bar{P}_{f_{s-1}} = \operatorname{ht} \bar{P}_{f_s}$, because $\operatorname{ht} \bar{\mathfrak{m}}_i - \operatorname{ht} \bar{P}_{f_i} = \operatorname{ht} \bar{\mathfrak{m}}_i - \operatorname{ht} \bar{P}_{f_{i+1}} = \operatorname{ht} \mathfrak{m}_i/P$ for $i=1, \dots, s-1$. Therefore, $\operatorname{ht} \bar{\mathfrak{m}} = \operatorname{ht} \bar{P}_{f_1} + \operatorname{ht} \mathfrak{m}/P = \cdots = \operatorname{ht} \bar{P}_{f_s} + \operatorname{ht} \mathfrak{m}/P = \operatorname{ht} \bar{\mathfrak{m}}'$.

5. A few related results

Lemma 5.1 ([8, Corollary 3). *Let A be a noetherian domain and B a finite extension of A with a non-zero ideal $\mathfrak{b}$. Suppose that $(0) \subset Q_1' \subset \cdots \subset Q_{n-1}' \subset Q$ is* *a saturated chain of prime ideals in B. Then there exists a saturated chain of prime ideals in B: $(0)\subset Q_1\subset\cdots\subset Q_{n-1}Q$ such that* $\mathrm{ht}(Q_i\cap A)=i$ *and* $\mathfrak{b}\not\subset Q_i$ $(i=1,\dots,n-1)$.

Propositon 5.2. *Let A be a noetherian domain, I a non-zero ideal and B a finite extension of A. Assume that A is I-complete. Then, if there is a maximal chain of prime ideals of lengh n $(n\geq 1)$ in B: $(0)\subset Q_1\subset\cdots\subset Q_{n-1}\subset\mathfrak{n}$, there exists a maximal chain of prime ideals of length n in A: $(0)\subset P_1\subset\cdots\subset P_1\subset\mathfrak{m}=\mathfrak{n}\cap A$.*

Proof. By Lemma 5.1, we may assume $\mathrm{ht}(Q_i\cap A)=i$ and $IB\not\subset Q_i$ $(i=1,\dots,n-1)$. Then, B/Q_{n-1} is a one-dimensional local domain, because B/Q_{n-1} is $I(B/Q_{n-1})$-complete and has a height-one maximal ideal. Thus, letting $P_i=Q_i\cap A$, $\dim A/P_{n-1}=1$. Therefore, $(0)\subset P_1\subset\cdots\subset P_{n-1}\subset\mathfrak{m}$ is a maximal chain of prime ideals of length n in A.

5.3. Proof of Proposition 4. Let B be a finite extension of A. To get the assertion, it is enough to show that B is catenary and $\mathrm{ht}\,\mathfrak{n}=\mathrm{ht}(\mathfrak{n}\cap A)$ for any $\mathfrak{n}\in\mathrm{Max}(B)$. Indeed, let $(0)\subset Q_1\subset\cdots\subset Q_{n-1}\subset\mathfrak{n}$ be a maximal chain of prime ideals of length n $(n\geq 1)$ in B. Then, by Proposition 5.2, there is a maximal chain of prime ideals of length n in A: $(0)\subset P_1\subset\cdots\subset P_{n-1}\subset\mathfrak{m}=\mathfrak{n}\cap A$. Therefore, $\mathrm{ht}\mathfrak{m}=n$, because A is catenary, and this completes the proof.

A. Appendix

Let $\mathbf{P}$ represent a property of local rings, for example, being regular, Gorenstein, CM, normal, reduced, or formally equidimensional, etc. For a local ring $(A,\mathfrak{m})$, we say $\mathbf{P}(A)$ is true (, $\mathbf{P}$(A) holds, or simply $\mathbf{P}$(A)), when A has the property $\mathbf{P}$.

Definition A.1. Let A, B be noetherian rings. A ring-homomorphism $\psi\colon A\to B$ is called a $\mathbf{P}$-homomorphism if ψ is flat, and $\mathbf{P}((B\otimes_A k')_{P'})$ is true for any $\mathfrak{p}\in\mathrm{Spec}(A)$ and any $P'\in\mathrm{Spec}(B\otimes_A k')$ of any finite extension field k' over $\kappa(\mathfrak{p})$.

Definition A.2. A noetherian ring A is called a $\mathbf{P}$-ring if the canonical map $\rho_{\mathfrak{p}}\colon A_{\mathfrak{p}}\to A_{\mathfrak{p}}\hat{}$ is a $\mathbf{P}$-homomorphism for any $\mathfrak{p}\in\mathrm{Spec}(A)$.

A.3. In many cases, $\mathbf{P}$ satisfies (some of) he following conditions (for the details, see [2, (7.3)], [6]): Let $(A,\mathfrak{m})$, $(B,\mathfrak{n})$ be local rings with local homomorphism $\psi\colon A\to B$. Then

- $\mathbf{P}_I$: $\mathbf{P}(A)$ implies $\mathbf{P}(B)$, if ψ is a $\mathbf{P}$-homomorphism,
- $\mathbf{P}_{II}$: $\mathbf{P}(B)$ implies $\mathbf{P}(A)$, if ψ is flat,
- $\mathbf{P}_{III}$: $\mathbf{P}(A)$ implies $\mathbf{P}(A_{\mathfrak{p}})$ for any $\mathfrak{p}\in\mathrm{Spec}(A)$,
- $\mathbf{P}_{IV}$: $\mathbf{P}(A/aA)$ implies $\mathbf{P}(A)$ for any non-zero divisor $a\in\mathfrak{m}$,
- $\mathbf{P}_V$: $\mathfrak{R}_{\mathbf{P}}(A)=\{\mathfrak{p}\in\mathrm{Spec}(A)\,|\,A_{\mathfrak{p}}\text{ satisfies }\mathbf{P}\}$ is open when $(A,\mathfrak{m})$ is a complete local ring.

A.4. Notation. When A is a ring, let

(A.4.1) $\quad \mathrm{Max}(A) =$ the set of all maximal ideals of A ,

(A.4.2) $\quad \Gamma = \{\gamma \subset \mathrm{Max}(A) \mid \gamma \text{ is a finite set}\}$,

(A.4.3) $\quad \Gamma(\gamma_0) = \{\gamma \in \Gamma \mid \gamma \supset \gamma_0\}$ for a fixed $\gamma_0 \in \Gamma$,

(A.4.4) $\quad A_\gamma = S_\gamma^{-1} A$ with $S_\gamma = A \setminus \bigcup_{\mathfrak{m} \in \gamma} \mathfrak{m}$.

Further, fix an element $x \in A$. And, for any A-algebra B, we denote by B^* the xB-adic completion of B. Then, for $\gamma' \supset \gamma$, $A_\gamma^* = (S_\gamma^{-1} A_{\gamma'}^*)^*$.

Definition A.5. With notation as above, let $\mathfrak{p}_\gamma^*$ be a prime ideal of A_γ^* for each $\gamma \in \Gamma(\gamma_0)$. The set $\{\mathfrak{p}_\gamma^*\}_{\gamma \in \Gamma(\gamma_0)}$ is called *a prime ideal sequence* if

(A.5.1) $\quad \mathfrak{p}_{\gamma'}^* = \mathfrak{p}_\gamma^* \cap A_{\gamma'}^*$ for any $\gamma' \supset \gamma$ $(\supset \gamma_0)$.

Definition A.6. A prime ideal sequence $\{\mathfrak{p}_\gamma^*\}_{\gamma \in \Gamma(\gamma_0)}$ is said to be *good*, if there exists $\gamma_1 \in \Gamma(\gamma_0)$ such that

(A.6.1) $\quad \mathfrak{p}_\gamma^* = \mathfrak{p}_{\gamma'}^* A_\gamma^*$, whenever $\gamma' \supset \gamma$ $(\supset \gamma_1)$.

When a prime ideal squence $\{\mathfrak{p}_\gamma^*\}_{\gamma \in \Gamma(\gamma_0)}$ is given, let $B_\gamma = A_\gamma^* / \mathfrak{p}_\gamma^*$, $\overline{B}_\gamma =$ the derived normal ring of B_γ, and let $C = \bigcap_{\gamma \in \Gamma(\gamma_0)} \overline{B}_\gamma$.

Definition A.7. With notation as above, let

(A.7.1) $\quad \Delta_\gamma(x) = \{Q = \overline{Q} \cap A \mid \overline{Q} \in \mathrm{Ass}(\overline{B}_\gamma / x\overline{B}_\gamma)\}$,

(A.7.2) $\quad \Delta(x) = \bigcup_{\gamma \in \Gamma(\gamma_0)} \Delta_\gamma(x)$.

Definition A.8. A prime ideal sequence $\{\mathfrak{p}_\gamma^*\}_{\gamma \in \Gamma(\gamma_0)}$ is said to be *bounded* if $\Delta(x)$ is a finite set.

Definition A.9. A prime ideal sequence $\{\mathfrak{p}_\gamma^*\}_{\gamma \in \Gamma(\gamma_0)}$ is called *simple* if it is good and bounded.

With notation and definitions above, the next proposition shows that in practice a bounded prime ideal sequence automatically becomes a good one:

Proposition A.10. *Let A be a noetherian ring and $\mathfrak{m} \in \mathrm{Max}(A)$. Fixing an element $x \in A$ and a minimal prime ideal $\mathfrak{p}_\mathfrak{m}^*$* of $\mathfrak{G}_\mathbf{P}(A_\mathfrak{m}^*)$ $(= \mathrm{Spec}(A_\mathfrak{m}^*) \setminus \mathfrak{N}_\mathbf{P}(A_\mathfrak{m}^*))$, *we set* $\mathfrak{p}_\gamma^* = \mathfrak{p}_\mathfrak{m}^* \cap A_\gamma^*$ for $\gamma \in \Gamma(\mathfrak{m})$ $(= \Gamma(\{\mathfrak{m}\}))$. *Suppose that*

(A.10.1) *$\mathbf{P}$ satisfies the conditions $\mathbf{P}_I$, $\mathbf{P}_{II}$, and $\mathbf{P}_V$* ,

(A.10.2) *Local Lifting Theorem holds for $\mathbf{P}$* ,

(A.10.3) *A/xA is a Nagata $\mathbf{P}$-ring* .

Then, if the prime ideal sequence $\{\mathfrak{p}_\gamma^*\}_{\gamma\in\Gamma(m)}$ *is bounded, it is also good.*

Finally, we state:

Theorem A.11 (Rotthaus' Hilfssatz). *Let A be a noetherian ring with $x\in A$ and let $\gamma_0\in\Gamma$. Suppose that*

(A.11.1) *A is an xA-adically complete Nagata ring,*

(A.11.2) $\{\mathfrak{p}_\gamma^*\}_{\gamma\in\Gamma(\gamma_0)}$ *is a simple prime ideal sequence.*

Put $\mathfrak{p}_\gamma=\mathfrak{p}_\gamma^*\cap A$ for each $\gamma\in\Gamma(\gamma_0)$. Then, there exists $\gamma_2\in\Gamma(\gamma_0)$ such that

(A.11.3) $\mathfrak{p}_\gamma^*\in \operatorname{Ass}(A_\gamma^*/\mathfrak{p}_\gamma A_\gamma^*)$ *for any* $\gamma\in\Gamma(\gamma_2)$.

In particular, if $\operatorname{ht}\mathfrak{p}_\gamma^*>0$ *for any* $\gamma\in\Gamma(\gamma_0)$, *then* $\mathfrak{p}_\gamma^*\cap A\neq(0)$.

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