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Stability of Hausdorff foliations of 5-manifolds by Klein bottels

By

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1. Let $Fol_q(M)$ denote the set of codimension q C^∞-foliations of a closed manifold M. $Fol_q(M)$ carries a natural weak C^r-topology $(0 \leq r \leq \infty)$, which is described in [H] and [E2]. We denote this space by $Fol_q^r(M)$. We say a foliation F is C^r-stable if there exists a neighborhood V of $Fol_q^r(M)$ such that every foliation in V has a compact leaf. We say a foliation F is C^r-unstable if not. It seems to be of interest to determine if F is C^r-stable or not. In the previous papers ([F2], [F3]), we studied the stability of foliations of closed 4-manifolds by Klein bottles. In the present paper we study the stability of Hausdoroff foliations of closed 5-manifolds by Klein bottles, where a foliation F of M is said to be *Hausdorff* if the leaf space M/F is Hausdorff. All manifolds and foliations considered here are smooth (i.e., differentiable of class C^∞).

2. Hausdorff foliations of 5-manifolds by Klein bottles

Let M be a closed 5-manifold and F a compact Hausdorff foliation of codimension three. Then we have a nice picture of the local behavior of F as follows.

Proposition 1 (Epstein [E1]). *There is a generic leaf L_0 with property that there is an open dense saturated subset of M, where all leaves have trivial holonomy and are diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite subgroup $G(L)$ of $O(3)$ such that $G(L)$ acts freely on L_0 on the right and $L_0/G(L) \cong L$. Let D^3 be the unit disk. We foliate $L_0 \times D^3$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x, y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and $y \in D^3$, where $G(L)$ acts linearly on D^3. So we have a foliation induced on $U = L_0 \times D^3/G(L)$. The leaf corresponding to $y=0$ is $L_0/G(L)$. Then there is a C^∞-imbedding $\varphi: U \to M$ with $\varphi(U) = U(L)$, which preserves leaves and $\varphi(L_0/G(L)).=L$.*

Remark 2. $U(L)$ can be considered as the total space of a normal

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disk bundle of L in M with structure group $G(L)$ and the restriction map $p: L_0 \to L$ is a finite regular covering with the group $G(L)$ of covering transformations.

Definition 3. A leaf L is *singular* if $G(L)$ is not trivial.

The following fact is well-known (see [S], [I] for instance).

Proposition 4. *The finite subgroups of* $O(3)$ *are listed in the following table*:

G	*order of* G	*structure of* G	*generator*
$G_{\mathrm{I}}(\boldsymbol{Z}_n)$	n	*cyclic group*, $\boldsymbol{Z}_n \subset SO(3)$	u
$G_{\mathrm{I}}(\boldsymbol{D}_{2n})$	$2n$	*dihedral group*, $\boldsymbol{D}_{2n} \subset SO(3)$	u, v
$G_{\mathrm{I}}(A_4)$	12	*alternating group of degree* 4, A_4	
$G_{\mathrm{I}}(S_4)$	24	*symmetric group of degree* 4, S_4	
$G_{\mathrm{I}}(A_5)$	60	*alternating group of degree* 5, A_5	
$G_{\mathrm{II}}(\boldsymbol{Z}_n)$	$2n$	$\boldsymbol{Z}_n \times \boldsymbol{Z}_2$, $G_{\mathrm{I}}(\boldsymbol{Z}_n) \cup J \cdot G_{\mathrm{I}}(\boldsymbol{Z}_n)$	u, J
$G_{\mathrm{II}}(\boldsymbol{D}_{2n})$	$4n$	$\boldsymbol{D}_{2n} \times \boldsymbol{Z}_2$, $G_{\mathrm{I}}(\boldsymbol{D}_{2n}) \cup J \cdot G_{\mathrm{I}}(\boldsymbol{D}_{2n})$	u, v, J
$G_{\mathrm{II}}(A_4)$	24	$A_4 \times \boldsymbol{Z}_2$, $G_{\mathrm{I}}(A_4) \cup J \cdot G_{\mathrm{I}}(A_4)$	
$G_{\mathrm{II}}(S_4)$	48	$S_4 \times \boldsymbol{Z}_2$, $G_{\mathrm{I}}(S_4) \cup J \cdot G_{\mathrm{I}}(S_4)$	
$G_{\mathrm{II}}(A_5)$	120	$A_5 \times \boldsymbol{Z}_2$, $G_{\mathrm{I}}(A_5) \cup J \cdot G_{\mathrm{I}}(A_5)$	
$G_{\mathrm{III}}(\boldsymbol{Z}_n)$	n	$\boldsymbol{Z}_n$ (n : *even*), $G \cap SO(3) = \boldsymbol{Z}_{n/2}$	Ju
$G_{\mathrm{III}}(S_4)$	24	S_4, $G \cap SO(3) = A_4$	
$G^{\mathrm{Z}}_{\mathrm{III}}(\boldsymbol{D}_{2n})$	$2n$	$\boldsymbol{D}_{2n}$, $G \cap SO(3) = \boldsymbol{Z}_n$	u, Jv
$G^{\mathrm{D}}_{\mathrm{III}}(\boldsymbol{D}_{2n})$	$2n$	$\boldsymbol{D}_{2n}$ (n : *even*), $G \cap SO(3) = \boldsymbol{D}_{n/2}$	

$$,\text{where } u=\begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad v=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We consider Hausdorff foliations of 5-manifolds by closed surfaces. A free action of a finite group G on a manifold L_0 is completely determined by a covering map Φ: $(L_0, \tilde{*}) \to (L, *)$ corresponding to a normal subgroup N of $\pi_1(L,*)$ and an epimorphism $\varphi: \pi_1(L,*) \to G$ with $\mathrm{Ker}\varphi = N$. Given N and φ, let Φ_i: $(L_0, \tilde{*}) \to (L, *)$, $i=1, 2$, be covering maps corresponding to N. Then there is an equivariant homeomorphism $((L_0, \tilde{*}), \Phi_1) \to ((L_0, \tilde{*}), \Phi_2)$, where $((L_0, \tilde{*}), \Phi_i)$ is the G-space with the action defined by Φ_i and φ. Therefore if we identify every manifold with a standard model via a fixed homeomorphism and if we fix, for each manifold L and each normal subgroup N of $\pi_1(L, *)$, a covering map Φ_N: $(L_0, \tilde{*}) \to (L, *)$ corresponding to N, then each epimorphism φ: $\pi_1(L, *) \to G$ defines a foliated neighborhood $U = L_0 \times D^3/G$ defined in Proposition 1, which is diffeomorphic to $U(L)$. For each surface L we choose a fixed set of canonical generators for $\pi_1(L, *)$, i.e., a set of generators $(a_1, b_1, \ldots, a_r, b_r)$ if L is orientable of genus r, or $(d_1, d_2, \ldots, d_r)$ if L is non-orientable of genus r, satisfying $\prod_{i=1}^{r}[a_i, b_i]=1$ or $d_1^2 d_2^2 \ldots d_r^2 = 1$ respectively. For given L and G, $U(L)$ is completely determined by a vector $(g_1, \ldots, g_{2r})$ with $g_{2i-1} = \varphi(a_i)$, $g_{2i} = \varphi(b_i)$ or $(g_1, \ldots, g_r)$ with $g_i = \varphi(d_i)$ respectively (Vogt [V]). We say that $U(L)$ is a foliated neighborhood of type $(g_1, \ldots, g_{2r})$ or $(g_1, \ldots, g_r)$ and L is of type $(g_1, \ldots, g_{2r})$ or $(g_1, \ldots, g_r)$.

We consider a Hausdorff foliation F of a closed 5-manifold M by Klein bottles and investigate the type of singular leaves of F. Let L be a singular leaf of F. We take generators $a(=d_1)$, $b(=d_1 d_2)$ of $\pi_1(L, *)$ instead of d_1, d_2. The generators a and b have the relation $aba^{-1}b = 1$. Note that a foliated neighborhood $U(L)$ is determined by a vector $(\varphi(a), \varphi(b))$. Then we have the following.

Theorem 5. *Let F be a Hausdorff foliation of a closed 5-manifold M by Klein bottles. Then the following singular leaves can appear in F:*

Name of a singular leaf	*Structure of G*	*Type*
$G_{\mathrm{I}}(\boldsymbol{Z}_n)$ -*leaf*	$\boldsymbol{Z}_n$ $(n$: *odd*$)$	$(u^l, 1)$, $(l, n)=1$
	$\boldsymbol{Z}_2$ $(n=2)$	$(1, u)$
$G_{\mathrm{II}}(\boldsymbol{Z}_n)$ -*leaf*	$\boldsymbol{Z}_n \times \boldsymbol{Z}_2$ $(n$: *odd*$)$	(u^l, J), $(l, n)=1$
	$\boldsymbol{Z}_2$ $(n=1)$	$(1, J)$
$G_{\mathrm{III}}(\boldsymbol{Z}_n)$ -*leaf*	$\boldsymbol{Z}_2$ $(n=2)$	$(1, JA)$

$$,\text{where } u=\begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Proof of Theorem 5

(1) Case $G=G_{\mathrm{I}}(\boldsymbol{Z}_n)$. We define an epimorphism $\varphi: \pi_1(L, *)\rightarrow G$ by $\varphi(a)=u^l$, $\varphi(b)=1$, where $(n, l)=1$. Then Kerφ is abelian or non-abelian according to that n is even or odd. Therefore the singular leaf L can appear as a singular leaf of type $(u^l, 1)$ if n is odd. For $n=2$, we define an epimorphism $\varphi: \pi_1(L, *)\rightarrow G$ by $\varphi(a)=1$, $\varphi(b)=u$. Then Kerφ is non-abelian, so the singular leaf L can appear as a singular leaf of type $(1, u)$. It is easy to see that the kernels of any other epimorphisms are abelian. For example, we define an epimorphism $\varphi: \pi_1(L, *)\rightarrow G$ by $\varphi(a)=u^l$, $\varphi(b)=u^{n/2}$, where n is even and g.c.m.$(l, n/2, n)=1$. In this case, Kerφ is abelian.

(2) Case $G=G_{\mathrm{I}}(\boldsymbol{D}_{2n})$. For $n=1$, We define an epimorphism φ: $\pi_1(L, *)\rightarrow G$ by $\varphi(a)=1$, $\varphi(b)=v$. Then Kerφ is non-abelian, so the singular leaf L can appear as a singular leaf of type $(1, v)$. This leaf is identified with the leaf of type $(1, u)$ in Case (1), $n=2$. It is easy to see that the kernels of any other epimorphisms are abelian.

(3) Case $G=G_{\mathrm{I}}(A_4)$, $G_{\mathrm{I}}(S_4)$, $G_{\mathrm{I}}(A_5)$. There can not appear any singular leaves with holonomy group G from the following proposition.

Proposition 6. *Let G be as above. There does not exist an epimorphism*

$\varphi: \pi_1(L,*) \to G$.

Proof. We suppose that there exists an epimorphism $\varphi: \pi_1(L,*) \to G$. Let H denote the subgroup of $\pi_1(L,*)$ generated by a^2 and b. H is an abelian normal subgroup of $\pi_1(L,*)$ and is isomorphic to $\boldsymbol{Z} \times \boldsymbol{Z}$. Thus $\varphi(H)$ is also an abelian normal subgroup of G. When $G = G_{\mathrm{I}}(A_4)$, $G_{\mathrm{I}}(S_4)$, there is a composition sequence $G_{\mathrm{I}}(S_4) \supset G_{\mathrm{I}}(A_4) \supset V_4 \supset \{1, (1, 2)(3, 4)\} \supset \{1\}$, where V_4 is the Kleinian group and isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. Thus $\varphi(H)$ is isomorphic to V_4, $\{1, (1, 2)(3, 4)\}$ or $\{1\}$. Then we have the quotient epimorphism $\bar{\varphi}$: $\pi_1(L,*)/H \to G/\varphi(H)$. Since the order of $\pi_1(L,*)/H$ is two and the order of $G/\varphi(H)$ is greater than two, this is impossible. When $G = G_{\mathrm{I}}(A_5)$, G is simple. Thus $\varphi(H) = \{1\}$. We have the quotient epimorphism $\bar{\varphi}$: $\pi_1(L,*)/H \to G$. Since the order of G is 60, this is impossible. This completes the proof.

(4) Case $G = G_{\mathrm{II}}(\boldsymbol{Z}_n)$. We define an epimorphism $\varphi: \pi_1(L,*) \to G$ by $\varphi(a) = u^l$, $\varphi(b) = J$, where $(n, l) = 1$. Then $\mathrm{Ker}\varphi$ is abelian or non-abelian according to that n is even or odd. Therefore the singular leaf L can appear as a singular leaf of type (u^l, J) if n is odd. For $n = 1$, we define an epimorphism $\varphi: \pi_1(L,*) \to G$ by $\varphi(a) = 1$, $\varphi(b) = J$. Then $\mathrm{Ker}\varphi$ is non-abelian, so the singular leaf L can appear as a singular leaf of type $(1, J)$. It is easy to see that the kernels of any other epimorphisms are abelian.

(5) Case $G = G_{\mathrm{II}}(\boldsymbol{D}_{2n})$. For $n \geq 2$, there does not exist an epimorphism $\varphi: \pi_1(L,*) \to G$ since the number of generators of G is greater than two. For $n = 1$, we see that the kernels of any epimorphisms are abelian. For $n = 0$, this leaf is identified with the leaf of type $(1, J)$ in Case (4), $n = 0$.

(6) Case $G = G_{\mathrm{II}}(A_4)$, $G_{\mathrm{II}}(S_4)$, $G_{\mathrm{II}}(A_5)$. There can not appear any singular leaves with holonomy group G from Proposition 6 because that these groups contain the groups in Case (3) respectively.

(7) Case $G = G_{\mathrm{III}}(\boldsymbol{Z}_n)$ (n : even). For $n = 2$, we define an epimorphism $\varphi: \pi_1(L,*) \to G$ by $\varphi(a) = 1$, $\varphi(b) = JA$. Since $\mathrm{Ker}\varphi$ is non-abelian, the singular leaf L can appear as a singular leaf of type $(1, JA)$.

(8) Case $G = G_{\mathrm{III}}(S_4)$. There can not appear any singular leaves with holonomy group G from Proposition 6.

(9) Case $G = G^{\mathrm{Z}}_{\mathrm{III}}(\boldsymbol{D}_{2n})$. For $n = 1$, we define an epimorphism φ: $\pi_1(L,*) \to G$ by $\varphi(a) = 1$, $\varphi(b) = Jv$. Since $\mathrm{Ker}\varphi$ is non-abelian, the singular leaf L can appear as a singular leaf of type $(1, Jv)$. This leaf is identified with the leaf of type $(1, JA)$ in Case (7). It is easy to see that the kernels of any other epimorphisms are abelian.

(10) Case $G = G^{\mathrm{D}}_{\mathrm{III}}(\boldsymbol{D}_{2n})$ (n : even). We easily see that the kernels of any epimorphisms are abelian.

We complete the proof.

3. Stability of Hausdorff foliations of 5-manifolds by Klein bottles

In this section we consider the stability of Hausdorff foliations of closed

5-manifolds by Klein bottles. First we have the following.

Proposition 7. *Let F be a Hausdorff foliation of a closed 5-manifold M by Klein bottles. Suppose that F has a $G_{II}(\boldsymbol{Z}_n)$-leaf ($n=1$ or odd). Then F is C^1-stable.*

Proof. Since a $G_{II}(\boldsymbol{Z}_n)$-leaf is of type (u', J) or $(1, J)$, the proof follows from (ii) of Theorem A of [F2].

Theorem 8. *Let F be a Hausdorff foliation of a closed 5-manifold M by Klein bottles. Suppose that F has no leaves with holonomy group isomorphic to $\boldsymbol{Z}_n (n\equiv 2 \bmod 4)$. If $\chi(M/F)\neq 0$, then F is C^1-stable.*

Proof. From the assumption, F can have $G_I(\boldsymbol{Z}_n)$-leaves and $G_{II}(\boldsymbol{Z}_n)$-leaves (n: odd) as singular leaves. If F has a $G_{II}(\boldsymbol{Z}_n)$-leaf, F is C^1-stable from Proposition 7. We suppose that F has not any $G_{II}(\boldsymbol{Z}_n)$-leaves. Since every $G_I(\boldsymbol{Z}_n)$-leaf is of type $(u', 1)$, we can apply the theorem of C. Bonatti and A. Haefliger [B-H, Theorem of II.5] to the foliated manifold (M, F). Then we have the following diagram:

$$\begin{array}{ccc} M & \stackrel{f}{\longrightarrow} & M_1 \\ \downarrow p & & \downarrow p_1 \\ M/F & = & M_1/F_1 , \end{array}$$

where F_1 is a Hausdorff foliation of a closed 4-manifold M_1 by circles and p, p_1 are the quotient maps.

Note that 1) the differential map f is a submersion (cf. Proposition 2 of [F5]), 2) a $G_I(\boldsymbol{Z}_n)$-leaf of F is mapped to a leaf of type I of F_1 by f (see [F4] for the definition of a leaf of type I) and 3) the leaf space $M/F=M_1/F_1$ is a compact topological 3-manifold without boundary. By Theorem 4 of [F4], we have that if $\chi(M_1/F_1)=\chi(M/F)\neq 0$, then F_1 is C^0-stable, hence C^1-stable. Then by following the proof of Theorem 1 of [B], we can see that F is C^1-stable. This completes the proof.

Theorem 9. *Let F be a Hausdorff foliation of a closed 5-manifold M by Klein bottles. Suppose that 1) F has no leaves with holonomy group isomorphic to $\boldsymbol{Z}_n (n\equiv 2 \bmod 4)$ or $\boldsymbol{Z}_n\times\boldsymbol{Z}_2$ and 2) the associated fibre bundle over M/F whose fibre over $x\in M/F$ is $H_1(p^{-1}(x); \boldsymbol{R})$ is trivial. If $\chi(M/F)=0$, then F is C^r-unstable ($r\geq 0$).*

Proof. From the assumption 1), we have the following diagram as in the proof of Theorem 8:

$$\begin{array}{ccc} M & \stackrel{f}{\longrightarrow} & M_1 \\ \downarrow p & & \downarrow p_1 \\ M/F & = & M_1/F_1 . \end{array}$$

From the assumption 2), it follows that τF_1 is trivial, where τF_1 denotes the subbundle of the tangent bundle τM_1, which consists the vectors tangent to the foliation F_1. By Theorem 5 of [F4], we have that if $\chi(M_1/F_1)=\chi(M/F)=0$, then F_1 is C^r-unstable $(r\geq 0)$. Hence $F=f^*F_1$ is C^r-unstable $(r\geq 0)$. This completes the proof.

4. Stability of foliations with $G_{\mathrm{I}}(\boldsymbol{Z}_n)$-leaves $(n>2)$

In this section we consider a Hausdorff foliation F of a closed 5-manifold M by Klein bottles with $G_{\mathrm{I}}(\boldsymbol{Z}_n)$-leaves of type $(u^l, 1)$. We denote by $\mathrm{rot}(F)$ a connected component of the union of $G_{\mathrm{I}}(\boldsymbol{Z}_n)$-leaves of F. Then the quotient map $p: \mathrm{rot}(F)\rightarrow S^1$ is a fibre bundle with Klein bottle K as a fibre (see [F4]). Thus $\mathrm{rot}(F)$ is considered as $K\times[0,1]/h$, where $h: K\rightarrow K$ is a diffeomorphism and $(x, 0)$ and $(h(x), 1)$ $(x\in K)$ are identified. Let $h_*: H_1(K;\boldsymbol{R})\rightarrow H_1(K;\boldsymbol{R})$ be its automorphism. In this case, h_* is a non-zero real number.

Theorem 10. *Let F be as avove. If $h_*<0$, then F is C^1-stable. Indeed, every foliation which is sufficiently C^1-close to F has a compact leaf near* $\mathrm{rot}(F)$.

Proof. Let $U(\mathrm{rot}(F))$ denote an open saturated tubular neighborhood of $\mathrm{rot}(F)$ in M. By applying the theorem of C. Bonatti and A. Haefliger [B-H, Theorem of II.5] to $(U(\mathrm{rot}(F)), F)$ as in the proof of Theorem 8, we have the following diagram:

$$\begin{array}{ccc} U(\mathrm{rot}(F)) & \xrightarrow{f} & U(\mathrm{rot}(F_1)) \\ \downarrow p & & \downarrow p_1 \\ U(\mathrm{rot}(F))/F & = & U(\mathrm{rot}(F_1))/F_1 , \end{array}$$

where F_1 is a Hausdorff foliation of $U(\mathrm{rot}(F_1))$ by circles and $\mathrm{rot}(F_1)$ is the union of leaves of type I of F_1. The assumption $h_*<0$ implies that $\mathrm{rot}(F_1)$ is homeomorphic to a Klein bottle. Thus by the following theorem, we have that every foliation which is sufficiently C^1-close to F_1 has a compact leaf near $\mathrm{rot}(F_1)$. Hence by following the proof of Theorem 1 of [B], we can see that every foliation of M which is sufficiently C^1-close to F has a compact leaf. This completes the proof.

Theorem 11. *Let F_1 be a Hausdorff foliation of $U_1=U(\mathrm{rot}(F_1))$ by circles and* $\mathrm{rot}(F_1)$ *the union of leaves of type* I *of F_1. If* $\mathrm{rot}(F_1)$ *is homeomorphic to a Klein bottle, every foliation which is sufficiently C^1-close to F_1 has a compact leaf near* $\mathrm{rot}(F_1)$.

Proof. Note that $\mathrm{rot}(F_1)$ is identified with $S^1\times[0,1]/h$, where h is a diffeomorphism of S^1. We foliate $S^1\times D^2$ with leaves of the form $S^1\times\{\mathrm{pt}\}$. This foliation is preserved by the diagonal action of $\boldsymbol{Z}_n(\subset \mathrm{SO}(2))$, defined by

$g(x, y)=(x\cdot g^{-1}, g\cdot y)$ for $g\in \boldsymbol{Z}_n$, $x\in S^1$ and $y\in D^2$, where $\boldsymbol{Z}_n$ acts linearly on D^2 and freely on S^1 on the right. So we have a foliation F_2 induced on $S^1\times D^2/\boldsymbol{Z}_n$. We define a foliation F_3 on $(S^1\times D^2/\boldsymbol{Z}_n)\times[0, 1]$ with leaves of the form $L\times\{pt\}$, $L\in F_2$. It follows from Proposition 1 and 2 (i) of [F4] that a saturated tubular neighborhood of a leaf of type I is diffeomorphic to such a foliation F_3 on $(S^1\times D^2/\boldsymbol{Z}_n)\times[0, 1]$. Let N be a saturated tubular neighborhood of rot(F_1) in U_1. Then N is diffeomorphic to $((S^1\times D^2/\boldsymbol{Z}_n)\times[0, 1], F_3)/H$, where H: $S^1\times D^2/\boldsymbol{Z}_n\rightarrow S^1\times D^2/\boldsymbol{Z}_n$ is a foliation preserving diffeomorphism extended from h. Let $(p, 0)\in S^1\times D^2$ be a fixed point of H. Since $S^1\times D^2/\boldsymbol{Z}_n$ is diffeomorphic to $S^1\times D^2$, they are identified. We may assume that $\{p\}\times D^2$ is left invariant by H. Then $\{p\}\times D^2\times[0, 1]$ is a disk transverse to F_3. We abbreviate $\{p\}\times D^2\times[0, 1]$ by $D^2\times[0, 1]$. Let π: $[-\varepsilon, 1+\varepsilon]\rightarrow S^1=\boldsymbol{R}/\boldsymbol{Z}$ be the map defined by $\pi(t)=t$ (mod 1), $t\in[-\varepsilon, 1+\varepsilon]$, for small $\varepsilon>0$. Let F' be a foliation which is C^1-close to F_1. Then the perturbed holonomy map $H(F')$: $D^2(\delta)\times[0, 1]\rightarrow D^2\times[-\varepsilon, 1+\varepsilon]$ is defined for small $\delta>0$, where $D^2(\delta)$ denotes the disk of radius δ (see [F1]). Note that $H(F')$ is an imbedding and C^1-close to the map $R(x, t)=(x\cdot g^{-1}, t)$ because the holonomy group of every leaf in rot(F_1) is isomorphic to $\boldsymbol{Z}_n$, where g $(\in\boldsymbol{Z}_n)$ is a generator of the holonomy group. We put $H(F')(x, t)=(f_1(x, t), t+f_2(x, t))$ using the coordinate (x, t) of $D^2\times[-\varepsilon, 1+\varepsilon]$ $(x\in D^2, t\in[0, 1])$. Then there exists a unique $x(t)\in D^2(\delta)$ for each t with $f_1(x(t), t)=x(t)$ because the map R has the fixed points $(0, t)$. The set $l=\{(x(t), t); t\in[0, 1]\}$ is a continuous curve in $D^2(\delta)\times[0, 1]$. We may assume that $f_2(x(0), 0)>0$.

(Case 1) If $f_2(x(1), 1)<0$, then there is a $t_0\in(0, 1)$ such that $f_2(x(t_0), t_0)=0$ because that $f_2(x, t)$ is continuous on l. That is, $(x(t_0), t_0)$ is a fixed point of $H(F')$. Then the leaf L' of F' through $(x(t_0), t_0)$ is compact.

(Case 2) Suppose that $t_1=f_2(x(1), 1)>0$. Since $H(F')(x(1), 1)=(x(1), 1+t_1)$, we have $H(F')(x(1), t_1))=(x(1), t_1+f_2(x(1), t_1))=(x(1), 0)$ from the assumption that rot(F_1) is homeomorphic to a Klein bottle. Thus we have $(x(1), t_1)\in l$ and $f_2(x(1), t_1)<0$. By the similar argument in Case 1, we complete the proof.

5. Stability of foliations with $G_{\mathrm{I}}(Z_2)$-leaves

In this section we consider a Hausdorff foliation F of a closed 5-manifold M by Klein bottles with $G_{\mathrm{I}}(Z_2)$-leaves of type $(1, u)$. A connected component rot(F) of the union of $G_{\mathrm{I}}(Z_2)$-leaves is consider as $K\times[0, 1]/h$ as in **4**. Let h_*: $\pi_1(K, *)\rightarrow\pi_1(K, *)$ be its automorphism. Then we have the following.

Theorem 12. *Let F be as avove. Suppose that $h_*(a)=a^{-1}$ and $h_*(b)=b^{-1}$, where a and b are generators of $\pi_1(K, *)$ with $aba^{-1}b=1$. Then F is C^1-stable. Indeed, every foliation of M which is sufficiently C^1-close to F has a compact leaf near* rot(F).

Proof. Let U be a saturated tubular neighborhood of $\mathrm{rot}(F)$ in M. Take an appropriate double cover $\widetilde{U}$ of U such that the induced foliation $\widetilde{F}$ on $\widetilde{U}$ is a foliation satisfying the following: 1) all leaves of $\widetilde{F}$ are homeomorphic to the torus T^2 and 2) for each singular leaf L of $\widetilde{F}$, a saturated tubular neighborhood $U(L)$ is completely determined by the vector $(1, u)$, where φ: $\pi_1(L,*)\to \boldsymbol{Z}_2$ is an epimorphism as in **1**, $\varphi((1, 0))=1$ and $\varphi((0, 1))=u$, $(1, 0)$ and $(0, 1)$ are generators of $\pi_1(L,*)\cong\boldsymbol{Z}\times\boldsymbol{Z}$ such that $\pi_*((1, 0))=a$ and $\pi_*((0, 1))=b$ for the covering map $\pi: L\to K$. We denote by $\mathrm{rot}(\widetilde{F})$ the union of singular leaves of type $(1, u)$ of $\widetilde{F}$. Note that $\mathrm{rot}(\widetilde{F})$ is concidered as $T^2\times[0, 1]/\widetilde{h}$, where $\widetilde{h}$ is a diffeomorphism of the torus T^2 which covers h. Then we have $\widetilde{h}_*=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ for the induced automorphism $\widetilde{h}_*$: $H_1(T^2; \boldsymbol{Z})\to H_1(T^2; \boldsymbol{Z})$. If every foliation of $\widetilde{U}$ which is C^1-close to $\widetilde{F}$ has a compact leaf near $\mathrm{rot}(\widetilde{F})$, then every foliation of M which is C^1-close to F has a compact leaf near $\mathrm{rot}(F)$. Thus we investigate the stability for $\widetilde{F}$.

We foliate $S^1\times D^2$ with leaves of the form $S^1\times\{\mathrm{pt}\}$. This foliation is preserved by the diagonal action of $\boldsymbol{Z}_2$ $(\subset SO(2))$, defined by $\iota(x, y)=(x\cdot\iota, -y)$ for $\iota\in\boldsymbol{Z}_2$, $x\in S^1$ and $y\in D^2$, where $\boldsymbol{Z}_2$ acts freely on S^1 on the right. So we have a foliation $\widetilde{F}_1$ on $S^1\times D^2$ $(\cong S^1\times D^2/\boldsymbol{Z}_2)$. So we define a foliation $\widetilde{F}_2$ on $T^2\times D^2\times[0, 1]$ $(=S^1\times S^1\times D^2\times[0, 1])$ with leaves of the form $S^1\times L\times\{\mathrm{pt}\}$, $L\in\widetilde{F}_1$. Then $(\widetilde{U}, \widetilde{F})$ is diffeomorphic to $(T^2\times D^2\times[0, 1], \widetilde{F}_2)/\widetilde{H}$, where $\widetilde{H}$ is a foliation preserving deffeomorphism of $T^2\times D^2$ extended from $\widetilde{h}$. Let $(p, 0)\in T^2\times D^2 (p\in T^2)$ be a fixed point of $\widetilde{H}$. We may assume that $\{p\}\times D^2$ is left invariant by $\widetilde{H}$. Then $\{p\}\times D^2\times[0, 1]$ is a disk transverse to $\widetilde{F}_2$. We abbreviate $\{p\}\times D^2\times[0, 1]$ by $D^2\times[0, 1]$. $D^2\times[0, 1]/\widetilde{H}$ can be considered to be $D^2\times S^1$, if necessary, by taking an appropriate double covering of $\widetilde{U}$.

Let α and β be loops in $L_{(p,0)}$ with base point $(p, 0)$ such that α and β represent the generators $(1, 0)$ and $(0, 1)$ of $\pi_1(L_{(p,0)},*)\cong\boldsymbol{Z}\times\boldsymbol{Z}$ respectively. Note that the holonomy along α (resp. β) is trivial (resp. non-trivial). Let $\alpha(t)$ and $\beta(t)$ be translations of α and β along the curve $(p, 0)\times\{t\}$, $t\in[0, 1]$. Let $\widetilde{F}'$ be a foliation of $\widetilde{U}$ which is sufficiently C^1-close to $\widetilde{F}$. Then we can define perturbed holonomy maps $H(\widetilde{F}', \alpha(t))$, $H(\widetilde{F}', \beta(t))$: $D_\delta^2\times\{t\}=\{y\in D^2; \|y\|<\delta\}\times\{t\}\to D^2\times S^1$ for each t and some $\delta>0$, which are imbeddings (cf. [H] and [F1]). Note that 1) $H(\widetilde{F}', \alpha(t_0))$ and $H(\widetilde{F}', \beta(t_0))$ are extended to maps $H(\widetilde{F}', \alpha_{t_0})$ and $H(\widetilde{F}', \beta_{t_0})$: $D_\delta^2\times(t_0-r, t_0+r)\to D^2\times S^1$ for some small r, which are local diffeomorphisms, 2) the extended map $H(\widetilde{F}', \alpha_{t_0})$ and $H(\widetilde{F}', \alpha_{t_1})$, $H(\widetilde{F}', \beta_{t_0})$ and $H(\widetilde{F}', \beta_{t_1})$ coincide on the intersections of their domains respectively if t_0 and t_1 are close and 3) $H(\widetilde{F}', \alpha(t))$ and $H(\widetilde{F}', \beta(t))$ are C^1-close to $id(y, t)=(y, t)$ and the map $R(y, t)=(-y, t)$ respectively because $\tilde{F}$ and $\tilde{F}'$ are C^1-close. We put $\tilde{S}^1=\boldsymbol{R}/2\boldsymbol{Z}$ and let $\pi:\tilde{S}^1\to S^1$ be the double covering map defined by $\pi(\tilde{t})=\tilde{t}$ (mod 1), $\tilde{t}\in\tilde{S}^1$. Then there exist the maps $H_\alpha(\tilde{F}')$ and $H_\beta(\tilde{F}')$: $D^2_\delta\times\tilde{S}^1\to D^2\times\tilde{S}^1$ extended from $H(\tilde{F}',\alpha(t))$ and $H(\tilde{F}',\beta(t))$ (cf. [F1]) respectively, such that the following diagram commutes;

$$\begin{array}{ccc} D^2_\delta\times\tilde{S}^1 & \xrightarrow{H_\alpha(\tilde{F}')\ (\text{resp. } H_\beta(\tilde{F}'))} & D^2\times\tilde{S}^1 \\ i\uparrow & & 1\times\pi\downarrow \\ D^2_\delta\times\{t\} & \xrightarrow{H(\tilde{F}',\alpha(t))\ (\text{resp. } H(\tilde{F}',\beta(t)))} & D^2\times S^1 , \end{array}$$

where $i(y,t)=(y,t)$ and $(1\times\pi)(y,\tilde{t})=(y,\pi(\tilde{t}))$. We put $H_\beta(\tilde{F}')(y,\tilde{t})=(f_1(y,\tilde{t}),f_2(y,\tilde{t}))$ using the coordinate $(y,\tilde{t})$ of $D^2_\delta\times\tilde{S}^1$. Then there exists a unique $y(\tilde{t})$ for each $\tilde{t}\in\tilde{S}^1$ such that $y(\tilde{t})=f_1(y(\tilde{t}),\tilde{t})$, because the map R has the fixed point $(0,t)$ for each t. The set $\tilde{l}=\{(y(\tilde{t}),\tilde{t});\tilde{t}\in\tilde{S}^1\}$ is a loop in $D^2\times\tilde{S}^1$. By the same argument as in the proof of Theorem 11, there exists a point $q=(y(t_1),t_1)\in\tilde{l}$ such that $H(\tilde{F}',\beta(t_1))(q)=q$, that is, q is a fixed point of $H_\beta(\tilde{F}')$.

We consider the behavior of $H^n_\alpha(F')(q)$ $(n\in\boldsymbol{Z})$ for a fixed point q of $H_\beta(\tilde{F}')$. Following the argument in [F1, p.1162-1163], we can see that there exists a point $\tilde{q}$ in $\tilde{l}$ such that $\tilde{q}$ is a fixed point of $H_\beta(\tilde{F}')$ and $H^n_\alpha(\tilde{F}')$ for some n. Thus by the standard argument we see that the leaf of $\tilde{F}'$ through $\tilde{q}$ is compact. This completes the proof.

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Added in proof

We should add the following in the table in Theorem 5.

Name of a singular leaf	*Structure of G*	*Type*
$G_{\mathrm{I}}(\boldsymbol{Z}_n)$-*leaf*	$\boldsymbol{Z}_n\,(n\equiv 2\,(\mathrm{mod}4))$	$(u^l, u^{\frac{n}{2}})\ (l, n)=1$
$G_{\mathrm{III}}(\boldsymbol{Z}_n)$-*leaf*	$\boldsymbol{Z}_n\,(n\equiv 2\,(\mathrm{mod}4))$	$(u^l, Ju^{\frac{n}{2}})\ (l: \textit{even})$ $(Ju^l, u^{\frac{n}{2}})\ (l, n)=1$

Proof. (1) Case $G=G_{\mathrm{I}}(\boldsymbol{Z}_n)$. For $n\equiv 2\,(\mathrm{mod}4)$, we define an epimorphism $\varphi: \pi_1(L,*)\to G$ by $\varphi(a)=u^l$, $\varphi(b)=u^{\frac{n}{2}}$, $(l, n)=1$. Then $\mathrm{Ker}\varphi$ is non-abelian.

(2) Case $G=G_{\mathrm{III}}(\boldsymbol{Z}_n)$. For $n\equiv 2\,(\mathrm{mod}4)$, we define an epimorphism $\varphi: \pi_1(L,*)\to G$ by $\varphi(a)=u^l$, $\varphi(b)=Ju^{\frac{n}{2}}(l: \textit{even})$, or $\varphi(a)=Ju^l$, $\varphi(b)=u^{\frac{n}{2}}$, $(l, n)=1$. Then $\mathrm{Ker}\varphi$ is non-abelian.