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The adjoint action of the Dwyer-Wilkerson H-space on its loop space

Dedicated to Professor Seiya Sasao on his 60th birthday

By

Akira KONO and Kazumoto KOZIMA

1. Introduction

Let G be a compact, connected, simply connected Lie group and e its unit. Denote by ΛG the space of free loops on G and by ΩG the space of based loops on G the base point e. By the multiplication of G and compact open topology ΛG is a topological group and ΩG is a closed normal subgroup. We define a map. $Ad: G \times \Omega G \to \Omega G$ by $Ad(g,l)(t) = gl(t)g^{-1}$ for $g \in G$, $l \in \Omega G$. Then the following theorem holds:

Theorem (*Kono-Kozima* [10]). *Let G be a compact, connected, simply connected Lie group and p a prime. Then the follwing three conditions are equivalent*:

(1) $H^*(G;\boldsymbol{Z})$ *is p-torsion free*,
(2) $H^*(Ad;\boldsymbol{Z}/p) = H^*(p_2;\boldsymbol{Z}/p)$, *where p_2 is the second projection*,
(3) $H^*(B\Lambda G;\boldsymbol{Z}/p)$ *is isomorphic to* $H^*(BG;\boldsymbol{Z}/p) \otimes H^*(G;\boldsymbol{Z}/p)$ *as an algebra*.

The above Theorem is a good characterization of the triviality of the p-torsion part of $H^*(G, \boldsymbol{Z})$ for compact 1-connected Lie groups. In general, (1) implies (2) and (3) for a 1-connected finite associative H-space G. The purpose of this paper is to show that $H^*(Ad;\boldsymbol{Z}/2)$ is non-trivial for a finite H-space which was constructed by Dwyer and Wilkerson. Dwyer and Wilkerson proved:

Theorem (*Dwyer-Wilkerson* [5]). *There is a complex B such that*

$$H^*(B;\boldsymbol{Z}/2) = \boldsymbol{Z}/2[y_8, y_{12}, y_{14}, y_{15}], \text{ where } \deg y_j \text{ is } j.$$

Then, if we put $X = \Omega B$, one can obtain

$$H^*(X;\boldsymbol{Z}/2) = \boldsymbol{Z}/2[x_7]/(x_7^4) \otimes E(x_{11}, x_{13}) \text{ where } \deg x_j \text{ is } j,$$

and

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$H^*(\Omega X; \boldsymbol{Z}/2) = \boldsymbol{Z}/2[a_6] \otimes E(a_{10}, a_{20})$ $* \leq 21$ where deg a_j is j ,

using only the algebraic structure of $H^*(B; \boldsymbol{Z}/2)$. (See §2)

We can define Ad: $X \times \Omega X \rightarrow \Omega X$, since ΩX has a homotopy inverse. Our result is the folltwing:

Theorem. $Ad^*(a_{20}) = x_7^2 \otimes a_6 + 1 \otimes a_{20}$.

The non-triviality of the adjoint map Ad or the commutator map

$$\Gamma: G \times \Omega G \rightarrow \Omega G$$

defined by

$$\Gamma(g, l)(t) = gl(t)\, g^{-1} l(t)^{-1}$$

is a reflection of some geometrical properties of G and has connections with another invariants like Whitehead and Samelson products. In our case, the above formula for Ad^* (a_{20}) or, more directly, the formula $\Gamma^* a_{20} = x_7^2 \otimes a_6$ (See §4) says that the commutator map

$$\Gamma: X \times \Omega X \rightarrow \Omega X$$

is not trivial. From this fact, one can easily conclude that the generalized Samelson product $\langle i_X, i_{\Omega X}\rangle$ can't vanish where i_X (resp. $i_{\Omega X}$) is the inclusion of X (resp. ΩX) to the free loop space ΛX.

The above formula is similar to those obtained in [10] for the exceptional Lie groups. This shows a similarity of ΩB and the exceptional Lie groups.

This paper is organized as follows: In section 2, we compute the cohomological structures of the spaces associated with B. We construct some spaces and a diagram to get more information in section 3. The main result is deduced from these computations and the formula for Γ^* in section 4.

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2. Properties of Dwyer-Wilkerson complex

Let B be the Dwyer-Wilkerson complex and X the loop space of B. First, we recall the $\boldsymbol{Z}/2$-cohomology of B and X. We abbreviate $H^*(\ ; \boldsymbol{Z}/2)$ to $H^*(\ )$.

One can see easily that

$$\dim H^j(B) \leq 1 \quad \text{for} \quad j \leq 23 .$$

By the Adem relation $Sq^{15} = Sq^7 Sq^8$, we have $y_{15}^2 = Sq^{15} y_{15} = Sq^7 Sq^8 y_{15}$. Thus $Sq^8 y_{15}$ is non-zero and there is only one possible element $y_8 y_{15}$ in H^{23}. So we have $Sq^8 y_{15} = y_8 y_{15}$.

Since $Sq^7=Sq^1Sq^2Sq^4$ and $H^{21}(B)$ is a zero group, we obtain the equation

$$y_{15}^2=Sq^7(y_8y_{15})=(Sq^7y^8)\ y_{15}+y_8Sq^1Sq^2Sq^4y_{15}=(Sq^7y_8)\ y_{15}$$

and conclude that

$$Sq^7y_8=y_{15}$$

and

$$y_{12}=Sq^4y_8,\ y_{14}=Sq^2y_{12},\ y_{15}=Sq^1y_{14}\ .$$

Then, by the Adem relation $Sq^8Sq^1=Sq^2Sq^7+Sq^9=Sq^2Sq^7+Sq^1Sq^8$, we obtain

$$Sq^8y_{15}=Sq^8y_{14}=Sq^2Sq^7y_{14}+Sq^1Sq^8y_{14}=Sq^1Sq^8y_{14}\ ,$$

since $H^{21}(B)=0$. Thus $y_8y_{15}=Sq^1Sq^8y_{14}$ and Sq^8y_{14} must be y_8y_{14} by dimensional reasons. Similary from the Adem relation $Sq^8Sq^2=Sq^{10}+Sq^4Sq^6=Sq^2Sq^8+Sq^4Sq^6$ and $H^{18}(B)=0$, we can deduce $y_8y_{14}=Sq^2Sq^8y_{12}$ and $Sq^8y_{12}=y_8y_{12}$.

Thus we get a following lemma.

Lemma. 2.1. $Sq^8y_j=y_8y_j$ for $j=8,12,14,15$.

By using Serre spectral sequence, we see

$$H^*(X)=\Delta(\sigma(y_8),\sigma(y_{12},\sigma(y_{14}),\sigma(y_{15}))$$

and

$$Sq^7\sigma(y_8)=\sigma(Sq^7y_8)=\sigma(y_{15})\ .$$

Since X is an associative H-space, one can easily show

$$H^*(X)=\boldsymbol{Z}/2[x_7]/(x_7^4)\otimes E(x_{11},x_{13})$$

where $x_j=\sigma(y_{j+1})$ and

$$Sq^4x_7=x_{11},\ Sq^2x_{11}=x_{13}\ .$$

Applying the Serre spectral sequence one more time, we obtain

$$H^*(\Omega X)=\Delta(a_6,a_{10},a_{12},a_{20})\ \text{for}\ *\leq 23$$

where $a_j=\sigma(x_{j+1})$ for $j=6,10,12$ and

$$Sq^6a_6=Sq^6\sigma(x_7)=\sigma(Sq^6x_7)=\sigma(x_{13})=a_{12}\ ,$$
$$Sq^{10}a_{10}=Sq^{10}\sigma(x_{11})=\sigma(Sq^{10}x_{11})=0\ .$$

Since in the Eilenberg-Moore spectral sequence

$$\mathrm{Cotor}^{H^*(\Omega X;\boldsymbol{Z}/2)}(\boldsymbol{Z}/2,\boldsymbol{Z}/2)\Rightarrow H^*(X;\boldsymbol{Z}/2)\ ,$$

a_{20} corresponds to the relation $x_{11}^2=0$, we see

$$H^*(\Omega X)=\boldsymbol{Z}/2[a_6]\otimes E(a_{10}, a_{20}) \quad \text{for} \quad * \leq 23$$

and

$$\bar{\phi}(a_{20})=a_{10}\otimes a_{10}$$

where $\bar{\phi}$ is the reduced coproduct induced from the loop product λ: $\Omega X\times\Omega X\rightarrow\Omega X$.

3. A construction

We assume that the all spaces are localized at 2 in the following sequel. Let $A=X^{(14)}\cong S^7\cup S^{11}\cup S^{13}\cup S^{14}$. We denote by

$$f: \Sigma A\vee S^8\rightarrow B$$

the composition of the wedge of following two maps

$$\Sigma A\rightarrow\Sigma(\Omega B)\xrightarrow{\text{eval}} B,$$
$$S^8\rightarrow\Sigma(\Omega B)\xrightarrow{\text{eval}} B.$$

and the folding map $B\vee B\rightarrow B$. We denote also by j the inclusion

$$j: \Sigma A\vee S^8\longrightarrow\Sigma A\times S^8.$$

Le M be the double mapping cylinder of f and j. Then, we have the following diagram:

$$(*) \qquad \begin{array}{ccccc} K & \xrightarrow{h} & B & \xrightarrow{i_f} & M \\ \bar{f}\uparrow & & f\uparrow & & i_j\uparrow \\ L & \xrightarrow{h'} & \Sigma A\vee S^8 & \xrightarrow{j} & \Sigma A\times S^8 \end{array}$$

where K and L are the homotopy fibers of the inclusions and $\bar{f}$ the induced map from f on fibers.

Let

$$\alpha_j\epsilon H^j(\Sigma A) \quad (j=8,12,14,15), \quad \beta_k\epsilon H^k(S^k) \quad (k=6,7,8)$$

be the generators. Then $\sigma(\alpha_j)$ is the generator of $H^{j-1}(A)$ for $j=8,12,14$ and $(\sigma(\alpha_7))^2$ the generator of $H^{14}(A)$. Then, clearly $H^{15}(L)$ is generated by $h'^*(\alpha_{15})$ and $H^j(L)=0$ for $j<15$.

Since M is the double mapping cylinder, we obtain the following exact sequence

$$\longrightarrow H^*(M)\xrightarrow{\xi} H^*(B)\oplus H^*(\Sigma A\times S^8)\xrightarrow{\pi} H^*(\Sigma A\vee S^8)\longrightarrow$$

where $\xi(m)=(i_f^*m, i_j^*m)$ and $\pi(y,\alpha)=f^*y-j^*\alpha$. One can easily see that π

is epic, so the above exact sequence splits to the short exact sequences. Since $(y_8, \alpha_8+\beta_8)$, (y_j, α_j) for $j=12,14,15$ and $(0, \alpha_j\beta_8)$ for $j=8,12,14,15$ are in the kernel of the homomorphism π, there exist elements u_j and v_{j+8} for $j=8,12,14,15$ such that

$$\xi(u_8)=(y_8, \alpha_8+\beta_8), \ \xi(u_j)=(y_j, \alpha_j) \ \text{for } j\neq 8$$

and

$$\xi(v_{j+8})=(0, \alpha_j\beta_8) \ .$$

Then, for $j=12,14,15$,

$$\begin{aligned}\xi(Sq^8u_j+v_{j+8}) &= Sq^8\xi(u_j)+\xi(v_{j+8})\\ &= Sq^8(y_j, \alpha_j)+(0, \alpha_j\beta_8)=(y_8y_j, 0)+(0, \alpha_j\beta_8)\\ &= (y_8y_j, \alpha_j\beta_8) \ .\end{aligned}$$

On the other hand, since $\alpha_j\alpha_8=0$ in $H^*(\Sigma A)$, we obtain

$$\xi(u_ju_8)=(y_j, \alpha_j)(y_8, \alpha_8+\beta_8)=(y_8y_j, \alpha_j\beta_8) \ .$$

By using the fact that ξ is monic, we can prove (1) of the folltwing lemma, and the rest of the lemma can be proved by a quite similar manner.

Lemma 3.1. (1) $Sq^8u_j=v_{j+8}+u_ju_8$ $(j=12,14,15)$
(2) $Sq^4u_8=u_{12}$, $Sq^2u_{12}=u_{14}$, $Sq^1u_{14}=u_{15}$
(3) $Sq^4v_{16}=v_{20}$, $Sq^2v_{20}=v_{22}$, $Sq^1v_{22}=v_{23}$

Since $\xi(v_{16}u_8)=(0, \alpha_8\beta_8)(y_8, \alpha_8+\beta_8)=(0, 0)$, we have also $v_{16}u_8=0$.

To get the imformation of cohomolgies of ΩM and $\Omega^2 M$, we compute the cohomologies of K and ΩK up to some dimension.

By using the Serre spectral spectral sequence of the fibration

$$K \xrightarrow{h} B \xrightarrow{i_f} M$$

and the above lemma, one can obtain elements $\gamma_j \in H^j(K)$ so as to satisfy $\tau(\gamma_j)=v_{j+1}$ for $j=15,19,21,11$. Since $v_{16}u_8=0$, $u_8\otimes\gamma_{85}$ is a cycle in the spectral sequence and there is $\gamma'_{22}\in H^{22}(K)$ to kill this cycle. Then, we have easily

$$\widetilde{H}^*(K)=\langle\gamma_{15}, \gamma_{19}, \gamma_{21}, \gamma_{22}, \gamma'_{22}\rangle \quad *\leq 25$$

where $\langle S\rangle$ represents the $\boldsymbol{Z}/2$-vector space with the basis S, and

$$Sq^4\gamma_{15}=\gamma_{19}, \quad Sq^2\gamma_{19}=\gamma_{21}, \quad Sq\gamma_{21}=\gamma_{22} \ .$$

So

$$\widetilde{H}^*(\Omega K)=\langle\mu_{14}, \mu_{18}, \mu_{20}, \mu_{21}, \mu_{21}\rangle \quad *\leq 24$$

is turned out by the Serre spectral sequence of the path-loop fibration of K.

Proposition 3.2. *As an $H^*(K)$-module,*

$$H^*(\Omega M) \simeq H^*(K) \otimes H^*(\Omega B) = \mathbf{Z}/2[\upsilon_7, \upsilon_{11}, \upsilon_{13}, \upsilon_{15}, \upsilon_{19}, \upsilon_{21}, \upsilon_{22}]$$

for degree ≤ 25.

The Serre spectral sequence of the fibration $\Omega B \to \Omega M \to K$ is trivial in this range of degree. So we have only to show $(\sigma(y_{12}))^2 = \gamma_{22}$ in this spectral sequence. Since we can obtain easily the following equations

$$\sigma(u_j) = \upsilon_{j-1}, \quad \sigma(v_{j+8}) = \upsilon_{j+7} \quad (j=8, 12 \text{ and } 14)$$

where σ: $H^*(M) \to H^{*-1}(\Omega M)$ is the cohomology suspension. So we have also

$$\sigma(u_{15}) = \sigma(Sq^1 Sq^2 Sq^4 u_8) = Sq^7 \nu_7 = \nu_7^2$$
$$\sigma(v_{23}) = \sigma(Sq^1 Sq^2 Sq^4 Sq^8 u_{12}) = Sq^{11} \nu_{11} = \nu_{11}^2$$

and the last equation shows that $(\sigma(y_{12}))^2 = \gamma_{22}$ as required.

From (3.2), we have

$$H^*(\Omega M) = \Delta(\rho_6, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}, \rho_{20}, \rho_{21}) \quad * \leq 23$$

as a module where $\rho_{j-1} = \sigma(\upsilon_j)$.

Since $\upsilon_{13} = Sq^2 Sq^4 \nu_7$ and $\nu_{21} = Sq^2 Sq^8 \upsilon_{11} = Sq^{10} \upsilon_{11}$, we can deduce

$$\rho_{12} = \rho_6^2 \text{ and } \rho_{20} = \rho_{10}^2 .$$

Thus, as a ring,

$$H^*(\Omega^2 M) = Z/2[\rho_6, \rho_{10}, \rho_{14}, \rho_{18}, \rho_{21}] \quad * \leq 23$$

and there are operations

$$Sq^4 \rho_6 = \rho_{10}, \quad Sq^2 \rho_{10} = \rho^2{}_6, \quad Sq^4 \rho_{14} = \rho_{18}, \quad \rho^2 \rho_{18} = \rho_{10}^2 .$$

Now we turn to the Serre spectral sequence of the fibration

$$\Omega^2 K \xrightarrow{\Omega^2 h} \Omega^2 B \xrightarrow{\Omega^2 i_f} \Omega^2 M .$$

Clearly, we can show that the cohomology of the fiber and total space are following:

$$\widetilde{H}^*(\Omega^2 K) = \langle \zeta_{13}, \zeta_{17}, \zeta_{19}, \zeta_{20}, \zeta'_{20} \rangle \quad * \leq 23$$
$$H^*(\Omega^2 B) = \mathbf{Z}/2[a_6, a_{10}, a_{20}]/(a^2 H_1) \quad * \leq 20 .$$

Since $\tau(\zeta_{13}) = \rho_{14}$, we have

$$\tau(\zeta_{20}) = \tau(Sq^1 Sq^2 Sq^4 \zeta_{13}) = Sq^1 Sq^2 Sq^4 \rho_{14} = Sq^1(\rho^2{}_{10}) = 0 .$$

Thus, $\zeta \in \mathrm{Im}\ (\Omega^2 h)^*$ and the one possibility is

Lemma 3.3. $\Omega^2 h^* a_{20} = \zeta_{20}$.

By comparing the Serre spectral sequences of $\zeta^2 K \to \zeta^2 B \to \zeta^2 M$ and $\zeta^2 L \to \zeta^2(\Sigma A \vee S^8) \to \zeta^2(\Sigma A \times S^8)$, we easily obtain the following lemma.

Lemma 3.4.
$(\zeta^2 \bar{f})^* : H^{13}(\Omega^2 K) \to H^{13}(\Omega^2 L)$
is am isomorphism.

4. The main result

Let Y be an H-space with inverse. Then, we define two commutator maps $\Gamma_Y : Y \times Y \to Y$ and $\Gamma'_Y : Y \times \Omega Y \to \Omega Y$ by the following equations:

$$\Gamma_Y(a, b) = aba^{-1}b^{-1},$$
$$\Gamma'_Y(a, l)(t) = al(t)a^{-1}l(t)^{-1}.$$

Let D be the composiotion

$$D: A \times S^7 \xrightarrow{j_1 \times j_2} \Omega(\Sigma A \vee S^8) \times \Omega(\Sigma A \vee S^8) \xrightarrow{\Gamma} \Omega(\Sigma A\, VS^8)$$

and put $b_7 = \sigma(\sigma_8)$, $b'_7 = \sigma(\beta_8)$ in $H^7(\Omega(\Sigma A \vee S^8))$.

Since $\alpha_8 \beta_8 = 0$ in $H^*(\Sigma A \vee S^8)$, there is an element $b_{14} \in H^{14}(\Omega(\Sigma A \vee S^8))$ satisfying $\bar{\phi}(b_{14}) = b_7 \otimes b'_7$. We abuse the notation $\alpha_7 \epsilon H^*(A)$ for $\sigma(\alpha_8)$ where α_8 is the generator of $H^8(\Sigma A)$.

Lemma 4.1. $D^* b_{14} = \alpha_7 \otimes \beta_7$.

Proof. First we show that $\Gamma^* b_{14} = b_7 \otimes b'_7 + b'_7 \otimes b_7$. Since $\bar{\phi}(b_{14}) = b_7 \otimes b'_7$ and b_7, b'_7 are primitive, one can easily obtain

$$Ad^* b_{14} = 1 \otimes b_{14} + b_7 \otimes b'_7 + b'_7 \otimes b_7.$$

(See [10], Lemma 3.2.) We can put

$$\Gamma^* b_{14} = A \cdot b_{14} \otimes 1 + B \cdot b_7 \otimes b'_7 + C \cdot b'_7 \otimes b_7 + D \cdot 1 \otimes b_{14}$$

by the dimensional reasons. Since $Ad = \lambda(\Gamma \times id)(id \times \Delta)$, we get

$$\begin{aligned} Ad^* b_{14} &= (id \times \Delta)^* (\Gamma \times id)^* \lambda^* b_{14} \\ &= (id \otimes \Delta^*)(\Gamma^* \otimes id)\phi b_{14} \\ &= (id \otimes \Delta^*)(\Gamma^* \otimes id)(1 \otimes b_{14} + b_7 \otimes b'_7 + b_{14} \otimes 1) \\ &= 1 \otimes b_{14} + (id \otimes \Delta^*)(\Gamma^* b_7 \otimes b'_7) + \Gamma^* b_{14}\ . \end{aligned}$$

On the other hand, since $Ad^* b_7 = 1 \otimes b_7$ and

$$Ad^*b_7=(id\otimes\Delta^*)(\Gamma^*\otimes id)\phi b_7=\Gamma^*b_7+1\otimes b_7\ ,$$

We have $\Gamma^*b_7=0$. Thus it follows that $A=D=0$ and $B=C=1$. So, we have

$$\begin{aligned}D^*b_{14}&=(j_1^*\otimes j_2^*)\Gamma^*b_{14}\\&=(j_1^*\otimes j_2^*)(b_7\otimes b'_7+b'_7\otimes b_7)\\&=\alpha_7\otimes\beta_7.\end{aligned}$$

Now, since the composition

$$A\vee S^7\subset A\times S^8\to\Omega(\Sigma\vee S^8)\ ,$$

is homotopic to the constant map, D induces the map from $A\wedge S^7$. Let us denote this map by $\bar{D}$. We denote also the composition of $j_1\times\sigma(j_2)$ and Γ' by

$$D'\colon A\times S^6\to\Omega^2(\Sigma A\vee S^8)\ .$$

Since D' is homotopic to the constant map on $A\vee S^6$, we get an induced map

$$\overline{D'}\colon A\wedge S^6\to\Omega^2(\Sigma A\vee S^8)$$

whose adjoint map is $\bar{D}$.

Proposition 4.2. $\Omega_j\circ D\simeq$ *constant map.*

Proof. By the definition of D, we obtain the equation

$$\begin{aligned}(\Omega_j\circ D)(a,s))(t)&=j[(a,t)\vee *,\ *\vee(s,t)]\\&=[((a,t),*),(*,(s,t))]\end{aligned}$$

where $a\in A$, $s\in S^7$ and $t\in I$. Since $\Omega_j\circ D$ can be deformed to the map $\hat{D}$ defined by the formula

$$(\widehat{D}(a,s))(t)=[((a,t),(s,t)),((a,t),(s,t))]$$

in $\Omega(\Sigma A\times S^8)$ which is clearly homotopic to the constant map, the result follows.

We can prove also $\Omega^2 j\circ D'\simeq$ constant map by the very similar way. So there exists a lift $\widetilde{D}'\colon A\times S^6\to\Omega^2 L$ satisfying $(\Omega^2h')\widetilde{D}'\simeq D'$. Then we get the following homotopy commutative diagram:

$$\begin{array}{ccccc}\Omega^2K & \xrightarrow{\Omega^2h} & \Omega^2B & \longrightarrow & \Omega^2M\\ \Omega^2\bar{f}\uparrow & & \Omega^2f\uparrow & & \uparrow\\ \Omega^2L & \xrightarrow{\Omega^2h'} & \Omega^2(\Sigma A\vee S^8) & \xrightarrow{\Omega^2j} & \Sigma A\times S^8\\ \widetilde{D}'\uparrow & & D'\uparrow & & \\ A\times S^6 & = & A\times S^6 & & \end{array}$$

We denote the composition

$$A\times S^6 \longrightarrow \Omega B\times\Omega^2 B \xrightarrow{\Gamma_{\Omega B}} \Omega^2 B$$

by D'' where the first map is the product of the inclusion $A\subset\Omega B$ and the adjoint of the inclusion $S^7\subset\Omega B$. Then

$$D''\simeq\Omega^2 f\circ D'\simeq\Omega^2 f\circ\Omega^2\bar{f}\circ\widetilde{D}' ,$$

and we obtain the equation

$$\begin{aligned} D''^*a_{20} &= \widetilde{D}'\Omega^2\bar{f}^*Sq^7\tau_{13} \\ &= Sq^7\widetilde{D}'^*\Omega^2\bar{f}^*\tau_{13} \\ &= Sq^7\widetilde{D}'^*b_{13} . \end{aligned}$$

Since $\Gamma'^*\sigma=(id\otimes\sigma)\Gamma^*$ and $b_{13}=\sigma(b_{14})$, we conclude that

$$\widetilde{D}'^*b_{13}=\alpha_7\otimes\beta_6$$

by Lemma 4.1. So we have

$$D''^*a_{20}=Sq^7\alpha_7\otimes\beta_6=\alpha_7^2\otimes\beta_6 .$$

which implies

$$\Gamma'^*_{\Omega B}a_{20}=x_7^2\otimes a_6 .$$

Thus the result follows.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
DEPARTMENT OF INFORMATION SCIENCES
FACULTY OF SCIENCE
KANAGAWA UNIVERSITY

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