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The Artin invariant of supersingular weighted Delsarte K3 surfaces

By

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1. Introduction

Let k be an algebraically closed field of positive characteristic p. Let X_k be a K3 surface defined over k. Denote by $\mathrm{NS}(X_k)$ the Néron-Severi group of X_k. It is known that $\mathrm{NS}(X_k)$ is a finitely generated abelian group with $\mathbf{Z}$-rank at most 22; put $\rho(X_k) = \mathrm{rank}_{\mathbf{Z}} \mathrm{NS}(X_k)$. As in [10], we call X_k a *supersingular K3 surface* if $\rho(X_k) = 22$. Write disc $\mathrm{NS}(X_k)$ for the determinant of the intersection matrix of $\mathrm{NS}(X_k)$. If X_k is supersingular, then

$$\mathrm{disc}\,\mathrm{NS}(X_k) = -p^{2\sigma_0(X_k)}$$

for some integer $\sigma_0 = \sigma_0(X_k)$ satisfying $1 \le \sigma_0 \le 10$ (cf. [1]). The integer σ_0 may be called the *Artin invariant* of X_k. In [8], Shioda showed that σ_0 takes all the 10 possible values; furthermore, in [10], he gave concrete examples of K3 surfaces for all values of σ_0 except for $\sigma_0 = 7$ and 10. In this paper, we apply Shioda's ethod (which is based on Ekedahl's algorithm of computing σ_0) to weighted Delsarte surfaces and construct supersingular K3 surfaces with Artin invariant 10.

Let $Q = (q_0, q_1, q_2, q_3)$ be a quadruplet of positive integers such that $p \nmid q_i$ $(0 \le i \le 3)$ and $\gcd(q_\alpha, q_\beta, q_\gamma) = 1$ for every triple $\{\alpha, \beta, \gamma\} \subset \{0, 1, 2, 3\}$. The weighted projective 3-space over k of type Q is the projective variety $\mathbf{P}_k^3(Q) := \mathrm{Proj}\, k[x_0, x_1, x_2, x_3]$ where the polynomial algebra is graded by the condition $\deg(x_i) = q_i$ $(0 \le i \le 3)$ (cf. [4]). Let μ_{q_i} be the group of q_i-th roots of unity in $k^\times$. Put $\boldsymbol{\mu} = \mu_{q_0} \times \mu_{q_1} \times \mu_{q_2} \times \mu_{q_3}$. Then $\boldsymbol{\mu}$ acts on $\mathbf{P}_k^3$ diagonally and we have $\mathbf{P}_k^3/\boldsymbol{\mu} \cong \mathbf{P}_k^3(Q)$ (cf. [4], §1.2.2).

Let m be a positive integer such that $p \nmid m$. Let $A = (a_{ij})$ be a 4×4 matrix of integer entries satisfying the conditions

$$\begin{cases} \text{(i)} & a_{ij} > 0 \text{ and } p \nmid a_{ij} \text{ for every } (i,j) \\ \text{(ii)} & p \nmid \det A \\ \text{(iii)} & \sum_{j=0}^{3} q_j a_{ij} = m \text{ for } 0 \le i \le 3 \\ \text{(iv)} & \text{given } j,\ a_{ij} = 0 \text{ for some } i\,. \end{cases}$$

We define a *weighted Delsarte surface in* $\mathbf{P}_k^3(Q)$ *of degree* m *with matrix* A (cf. [2], [9]) to be the surface

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$$X_A : \sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} = 0 \subset \mathbf{P}_k^3(Q).$$

Weighted Delsarte surfaces are, in general, singular surfaces. We write $\tilde{X}_A$ for the minimal resolution (of singularities) of X_A. The minimal resolution $\tilde{X}_A$ may be called a *supersingular weighted Delsarte K3 surface* if it is supersingular and K3.

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2. The Artin invariant of supersingular weighted Delsarte K3 surfaces

Let X_A be a weighted Delsarte surface in $\mathbf{P}_k^3(Q)$ of degree m with matrix A. Put $d = \det A$. Define Y_k to be the Fermat surface in $\mathbf{P}_k^3$ of degree d:

$$Y_k : y_0^d + y_1^d + y_2^d + y_3^d = 0.$$

As in the case of Delsarte surfaces in $\mathbf{P}_k^3$ (cf. [9]), X_A is a finite quotient of Y_k. In fact, put $\Gamma = \mu_d \times \mu_d \times \mu_d \times \mu_d/(\text{diagonal elements})$. Let

$$\Gamma_A = \left\{ \gamma = \left(\prod_{j=0}^{3} \lambda_j^{a_{0j}}, \prod_{j=0}^{3} \lambda_j^{a_{1j}}, \prod_{j=0}^{3} \lambda_j^{a_{2j}}, \prod_{j=0}^{3} \lambda_j^{a_{3j}} \right) \in \Gamma \,\middle|\, (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \Gamma \right\} \subset \Gamma.$$

Then Γ_A acts on Y_k by

$$\gamma \cdot (y_0 : y_1 : y_2 : y_3) = \left(\left(\prod_{i=0}^{3} \lambda_j^{a_{0j}} \right) y_0 : \left(\prod_{j=0}^{3} \lambda_j^{a_{1j}} \right) y_1 : \left(\prod_{j=0}^{3} \lambda_j^{a_{2j}} \right) y_2 : \left(\prod_{j=0}^{3} \lambda_j^{a_{3j}} \right) y_3 \right)$$

for $\gamma \in \Gamma_A$ and $(y_0 : y_1 : y_2 : y_3) \in Y_k$, and X_A is birational to the quotient Y_k/Γ_A.

Let W be the ring of Witt vectors over k. Denote by $H_{cris}^2(\tilde{X}_A/W)$ the second crystalline cohomology of $\tilde{X}_A$. It is known that $\sigma_0(\tilde{X}_A)$ is equal to the p-rank of the cokernel of the Chern class map $c_1 : \mathrm{NS}(\tilde{X}_A) \otimes W \to H_{cris}^2(\tilde{X}_A/W)$ (cf. [6]). Further, $\sigma_0(\tilde{X}_A)$ is a birational invariant ([9], Proposition 5). Hence to compute $\sigma_0(\tilde{X}_A)$, it suffices to look into the cohomology of Y_k/Γ_A. Recall (see [9]) that $H_{cris}^2(Y_k/W)$ is decomposed as:

$$H_{cris}^2(Y_k/W) \cong V(0) \oplus \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{U}(Y_k)} V(\boldsymbol{\alpha})$$

$$\mathfrak{U}(Y_k) = \left\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \mid \alpha_i \in \mathbf{Z}/d\mathbf{Z},\ \alpha_i \neq 0 \ (0 \le i \le 3),\ \sum_{i=0}^{3} \alpha_i = 0 \right\}$$

$$V(\boldsymbol{\alpha}) = \{ v \in H_{prim}^2(Y_k/W) \mid \gamma^*(v) = \gamma_0^{\alpha_0} \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \gamma_3^{\alpha_3} \cdot v,\ \forall \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in \Gamma \}.$$

Proposition 2.1. *Let X_A be a weighted Delsarte surface in $\mathbf{P}_k^3(Q)$ of degree m with matrix A. Let Y_k be the Fermat surface in $\mathbf{P}_k^3$ of degree $d = \det A$. Put $Y_k' := Y_k/\Gamma_A$. Define*

$$\mathfrak{U}(X_A) = \left\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{U}(Y_k) \,\middle|\, \sum_{i=0}^{3} a_{ij}\alpha_i = 0 \ (0 \le j \le 3) \right\}$$

Then

$$H^2_{cris}(Y_k'/W) \cong V(0) \oplus \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{U}(X_A)} V(\boldsymbol{\alpha}).$$

Proof. The Hochschild-Serre spectral sequence yields $H^2_{cris}(Y_k'/W) \cong H^2_{cris}(Y_k/W)^{\Gamma_A}$. Choose an arbitrary $\boldsymbol{\alpha} \in \mathfrak{U}(Y_k)$. Then $V(\boldsymbol{\alpha})$ is fixed by Γ_A if and only if

$$\left(\prod_{j=0}^{3} \lambda_j^{a_{0j}}\right)^{\alpha_0} \left(\prod_{j=0}^{3} \lambda_j^{a_{1j}}\right)^{\alpha_1} \left(\prod_{j=0}^{3} \lambda_j^{a_{2j}}\right)^{\alpha_2} \left(\prod_{j=0}^{3} \lambda_j^{a_{3j}}\right)^{\alpha_3} = 1 \qquad \text{for all } (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \Gamma.$$

This gives rise to the assertion.

The main reason for looking at weighted projective surfaces is that we can determine various properties of them very naturally. We assume that X_A is quasi-smooth (cf. [4], §3) and that $\operatorname{codim}_{X_A}(X_A \cap \mathbf{P}_k^3(Q)_{sing}) \ge 2$. Then X_A has only cyclic quotient singularities of type A (cf. [4], [5]). Furthermore, $(X_k)_{sing} = X_k \cap \mathbf{P}_k^3(Q)_{sing}$ ([3]) and the dualizing sheaf of X_A is calculated by $\omega_{X_A} \cong \mathcal{O}_{X_A}(m - q_0 - q_1 - q_2 - q_3)$ (cf. [4]). In particular, $\tilde{X}_A$ is K3 if and only if $m = q_0 + q_1 + q_2 + q_3$. There are exactly 95 pairs of m and Q which produce K3 surfaces in $\mathbf{P}_k^3(Q)$ (cf. [7]). If $\tilde{X}_A$ is K3, there exists a unique $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) =: \boldsymbol{\alpha}_{ss} \in \mathfrak{U}(X_A)$ such that $V(\boldsymbol{\alpha}_{ss})$ is of type $(2, 0)$ (in the Hodge decomposition of $H^2_{cris}(\tilde{X}_A/W)$); if we assume $1 \le \alpha_i < d$ for $0 \le i \le 3$, then this is equivalent to $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = d$. Given $\boldsymbol{\alpha}_{ss} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, we define

$$e_A = d/\gcd(\alpha_0, \alpha_1, \alpha_2, \alpha_3, d). \tag{1}$$

Lemma 2.2. *Let X_A be a weighted Delsarte surface with matrix A. Assume that $\tilde{X}_A$ is K3. Then $\tilde{X}_A$ is supersingular if and only if $p^\mu \equiv -1 \pmod{e_A}$ for some integer $\mu \ge 1$.*

Using Shioda's method of computing the Artin invariant of Fermat surfaces, we can now generalize Theorem 4 of [10] to supersingular weighted Delsarte K3 surfaces.

Theorem 2.3. *Let X_A be a quasi-smooth weighted Delsarte surface in $\mathbf{P}_k^3(q_0, q_1, q_2, q_3)$ of degree m with matrix A. Write $\tilde{X}_A$ for the minimal resolution of X_A. Assume that $p^\mu \equiv -1 \pmod{e_A}$ for some positive integer μ, where e_A is the integer defined in (1); let μ_0 be the smallest integer among such μ's. Assume also that $m = q_0 + q_1 + q_2 + q_3$. Then $\tilde{X}_A$ is a supersingular K3 surface and the Artin invariant of $\tilde{X}_A$ is equal to μ_0.*

3. Supersingular K3 surfaces with Artin invariant 10

We give 2 examples of supersingular K3 surfaces with Artin invariant 10.

Example 3.1. Assume $p \neq 2, 3, 5$. Let $Q = (1, 1, 1, 3)$ and $m = 6$. Let X_A be a weighted Delsarte surface in $\mathbf{P}_k^3(1, 1, 1, 3)$ defined by the equation:

$$x_0^5 x_1 + x_1^5 x_2 + x_2^6 + x_3^2 = 0 .$$

Since X_A is quasi-smooth and $(X_k)_{sing} = X_k \cap \mathbf{P}_k^3(Q)_{sing} = \emptyset$, X_A is smooth. As $m = q_0 + q_1 + q_2 + q_3$, X_A is a K3 surface. We find $d = 2^2 \cdot 3 \cdot 5^2$, $\boldsymbol{\alpha}_{ss} = (90, 48, 42, 150)$ and $e_A = 2 \cdot 5^2$. Therefore

$$\rho(X_A) = \begin{cases} 2 & \text{if } p \equiv 1,\ 11,\ 21,\ 31,\ 41 \pmod{50} \\ 22 & \text{otherwise} . \end{cases}$$

When X_A is supersingular, we obtain

$$\sigma_0 = \begin{cases} 10 & \text{if } p \equiv \pm 3,\ \pm 27,\ \pm 33,\ \pm 37 \pmod{50} \\ 5 & \text{if } p \equiv 9,\ 19,\ 29,\ 39 \pmod{50} \\ 2 & \text{if } p \equiv \pm 43 \pmod{50} \\ 1 & \text{if } p \equiv -1 \pmod{50} \end{cases}$$

Example 3.2. Assume $p \neq 2, 3, 5$. Let $Q = (1, 1, 1, 3)$ and $m = 6$. Let X_A be a weighted Delsarte surface in $\mathbf{P}_k^3(1, 1, 1, 3)$ defined by the equation:

$$x_0^5 x_1 + x_1^5 x_2 + x_2^3 x_3 + x_3^2 = 0 .$$

For the same reason as above, X_A is a K3 surface. We find $d = 2 \cdot 3 \cdot 5^2$, $\boldsymbol{\alpha}_{ss} = (30, 24, 42, 54)$ and $e_A = 5^2$. Therefore,

$$\rho(X_A) = \begin{cases} 2 & \text{if } p \equiv 1,\ 6,\ 11,\ 16,\ 21 \pmod{25} \\ 22 & \text{otherwise} . \end{cases}$$

When X_A is supersingular, we obtain

$$\sigma_0 = \begin{cases} 10 & \text{if } p \equiv \pm 2,\ \pm 3,\ \pm 8,\ \pm 12 \pmod{25} \\ 5 & \text{if } p \equiv 4,\ 9,\ 14,\ 19 \pmod{25} \\ 2 & \text{if } p \equiv \pm 7 \pmod{25} \\ 1 & \text{if } p \equiv -1 \pmod{25} . \end{cases}$$

Remark 3.3. We must modify our method to realize $\sigma_0 = 7$ since there is no integer d such that the maximal order of the elements in $\{x \in \mathbf{Z}/d\mathbf{Z} \mid \gcd(x, d) = 1\}$ is equal to 14.

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