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Regular points for the successive primitives of Brownian motion

By

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1. Introduction

Let $(B(t))_{t\geq 0}$ be the linear Brownian motion starting at 0. Denote by

$$X_n(t)=\frac{1}{n!}\int_0^t (t-s)^n\, dB(s)$$

its n-fold primitive, and

$$U(t)=(B(t),\, X_1(t)\, ,...,\, X_n(t)).$$

The Gaussian process X_n was first mentionned by Shepp [7]. Later, Wahba used this process to derive a correspondance between smoothing by spline and Bayesian estimation on certain stochastic models [8], [9]. See also [1] where X_n is equally introduced in describing some degenerate Gaussian diffusions.

In the study of X_n arises essentially the process U that will be shown to be in fact a Markov process. Let us point out that the process $(B(t),\, X_1(t))_{t\geq 0}$ corresponding to the case $n=1$ has been used to describe the movement of a particle driven by a Gaussian white noise ([5], [3]; *see also* [4] *for further references*).

For the process U, we write a Wiener's test which allows to decide whether a fixed point, say $\mathbf{O}$, is regular or not for a certain set B in $\mathbf{R}^{n+1}$, or in other words, if $\tau_B=\inf\{t>0\colon U(t)\in B\}$ denotes the first entrance time in B, then $\mathbf{P}_0\{\tau_B=0\}=1$ or not.

Our main result can be stated as follows. Set for any $x=(x_0,\, x_1\, ,...,\, x_n)\in\mathbf{R}^{n+1}$:

$$N(x)=\max_{0\leq i\leq n}|x_i|^{\frac{1}{2i+1}}$$
$$x^*=(x_0,\, -x_1,\, x_2\, ,...,\, (-1)^n x_n)$$

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and

$$B_k=\{x\in B: 2^{-k-1}\le N(x)\le 2^{-k}\}$$
$$B^*=\{x\in \mathbf{R}^{n+1}: x^*\in B\}$$
$$\tau_k=\tau_{B_k}.$$

Theorem 1.1. *Let B be a Borel set in $\mathbf{R}^{n+1}$ such that $B^*=B$ which closure includes the origin $\mathbf{O}$. Then, $\mathbf{O}$ is a regular point for B if and only if the series $\sum_{k=0}^{+\infty}\mathbf{P}_{\mathbf{O}}\{\tau_k<+\infty\}$ is divergent.*

The method we have adopted is classical and is based on the potential theory related to the process U (The reader may consult *e. g.* [6] for an exhibition of the classical Wiener's test for the space-valued Brownian motion). However, we will not introduce the notion of capacity, which seems difficult to be extended to that case. In contrast to this gap, an appropriate martingale and a duality property associated to U will be effective in this study.

Let us point out that replacing B_k by $\{x\in B: 2^k\le N(x)\le 2^{k+1}\}$ yields an analoguous test which tell us whether B is a recurrent set ($i.e.$ $\mathbf{P}_{\mathbf{O}}\{\exists\ (t_n)_{n\geq 0}\nearrow+\infty: U(t_n)\in B\}=1$) or not.

Next, we display two examples:

Theorem 1.2. *Let f: $(0,+\infty)\to(0,+\infty)$ a function such that $x\mapsto x^{-\frac{1}{2n+1}}f(x)$ is nondecreasing near 0 and set*

$$T=\{x\in\mathbf{R}^{n+1}: x_n\ge 0, \sum_{i=0}^{n-1}|x_i|^{\frac{2}{2i+1}}\le f(x_n)^2\}.$$

Then:

- *if $n=1$, $\mathbf{O}$ is a regular point for T;*
- *for $n\ge 2$, $\mathbf{O}$ is a regular point for T if and only if the integral $\int_{0^+}\left(\frac{f(x)}{x^{\frac{1}{2n+1}}}\right)^{n^2-2}\frac{dx}{x}$ is divergent.*

Theorem 1.3. *Let f: $(0,+\infty)\to(0,+\infty)$ a function such that $x\mapsto\frac{f(x)}{x}$ is nondecreasing near 0, and set*

$$T'=\{x\in\mathbf{R}^{n+1}: x_n\ge 0, \sum_{i=0}^{n-1}x_i^2\le f(x_n)^2\}.$$

Then:

- *if $n=1$, $\mathbf{O}$ is a regular point for T';*

● *for* $n\geq 2$, **O** *is a regular point for* T' *if and only if the integral* $\int_{0^+}\left(\frac{f(x)}{x}\right)^{\frac{n^2-2}{2n+1}}\frac{dx}{x}$ *is divergent.*

For instance, choose in theorem 1.2

$$f(x)=x^{\frac{1}{2n+1}}\left[\log\frac{1}{x}\log_2\frac{1}{x}\ldots\log_{k-1}\frac{1}{x}\left(\log_k\frac{1}{x}\right)^{1+\varepsilon}\right]^{-\frac{1}{n^2-2}}$$

where $\log_k=\log\log_{k-1}$. Then if $\varepsilon\leq 0$, **O** is a regular point for T and if $\varepsilon>0$, **O** is an irregular point for T.

It may be derived a similar test (with $\int^{+\infty}$ instead of $\int_{0^+}$) for a function f such that $x\mapsto x^{-\frac{1}{2n+1}}f(x)$ (resp. $x^{-1}f(x)$) is nonincreasing near $+\infty$ that allows us to decide whether the thorn T (resp. T') is recurrent of not.

Now, we set out a brief sketch of the content of this paper. In section 2 are introduced some notations and basic properties of the process U. In particular Markov and duality properties are explained, and some estimates of the 0-potential related to U are written. Section 3 is devoted to proving theorem 1.1. Some geometrical facts such transience and polarity for the process U are mentioned. Naturally, a classical extension of the Borel-Cantelli lemma will also be used in order to study the series written down in the theorem 1.1. Section 4 is concerned with the proofs of both theorems 1.2 and 1.3. These results require some estimates about the hitting probabilities of certain classes of parallelepipeds in $\mathbf{R}^{n+1}$. Finally, the last part contains the proofs of several intermediate lemmas.

2. Preliminaries

The first noteworthy fact is the Markov property:

Proposition 2.1. *U is a strong Markov process.*

Proof. Fix an instant $t_0\geq 0$. For each $k\in\{0,\ldots,n\}$ we have by Newton's formula:

$$\begin{aligned}X_k(t_0+t)&=\int_0^{t_0}\frac{(t+(t_0-s))^k}{k!}dB(s)+\int_{t_0}^{t_0+t}\frac{(t+(t_0-s))^k}{k!}dB(s)\\&=\sum_{j=0}^{k}\frac{t^j}{j!}X_{k-j}(t_0)+\int_0^t\frac{(t-\sigma)^k}{k!}d_\sigma B(t_0+\sigma)\end{aligned}$$

and then

$$U(t_0+t)=W(t_0,t)+U(t_0)J_t \tag{1}$$

where we set

$$J_t = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^n}{n!} \\ 0 & 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$W(t_0, t) = \left(\int_0^t \frac{(t-\sigma)^k}{k!} d_\sigma B(t_0+\sigma)\right)_{0\leq k\leq n}$$

The previous stochastic integral is to be understood as being

$$\int_0^t \frac{(t-\sigma)^{k-1}}{(k-1)!} B(t_0+\sigma)\,d\sigma - \frac{t^k}{k!} B(t_0) \qquad (\text{for } k\geq 1).$$

The classical Markov property of Brownian motion allows us to write

$$(B(t_0+t))_{t\geq 0} \overset{law}{=} (\tilde{B}(t_0, t) + B(t_0))_{t\geq 0}$$

where $\widetilde{B}(t_0,.)$ is a new Brownian motion starting at 0, which is independant of the σ-field $\sigma\{B(t), 0\leq t\leq t_0\}$. Therefore

$$W(t_0, t) = (\int_0^t \frac{(t-\sigma)^k}{k!} d_\sigma \tilde{B}(t_0, \sigma))_{0\leq k\leq n},$$

so that the process $W(t_0,.)$ is a copy of U that is independant of the σ-field $\sigma\{U(t), 0\leq t\leq t_0\} = \sigma\{B(t), 0\leq t\leq t_0\}$, and then (1) is surely a Markov decomposition related to U. By replacing t_0 with a stopping time, it is easy to see that U is in fact a strong Markov process.

Now write

$$p_t(x; y)dy = \mathbf{P}_x\{U(t) \in dy\}, \quad x = (x_0, ..., x_n), y = (y_0, ..., y_n)$$

for the transition densities of the Markov process U.

Since U is a Gaussian process, we get an explicit formula for $p_t(x; y)$:

$$p_t(x; y) = \frac{c}{t^{d+1}} \exp\left[-\sum_{0\leq i,j\leq n} \frac{a_{ij}}{t^{i+j+1}}\left(y_i - \sum_{k=0}^{i} \frac{t^k}{k!} x_{i-k}\right)\left(y_j - \sum_{k=0}^{j} \frac{t^k}{k!} x_{j-k}\right)\right] \tag{2}$$

where the double of the matrix $(a_{ij})_{0\leq i,j\leq n}$ is the inverse of the covariance matrix of the random vector U_1, namely:

$$\Gamma=\left(\frac{1}{(i+j+1)\,i!j!}\right)_{0\le i,j\le n} \text{ and } c=\frac{1}{(2\pi)^{(n+1)/2}\sqrt{\det\Gamma}}, \quad d=\frac{1}{2}(n+1)^2-1.$$

The density (2) has the following matricial representation:

$$p_t(x;\, y)=\frac{c}{t^{d+1}}\exp[-(y-xJ_t)A_t(y-xJ_t)^T] \tag{3}$$

where we put for all $x=(x_0,\dots,x_n)\in\mathbf{R}^{n+1}$:

$$x^T=\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \text{ and } A_t=\left(\frac{a_{ij}}{t^{i+j+1}}\right)_{0\le i,j\le n}.$$

Set:

$$\Phi(x;\, y)=\int_0^{+\infty} p_t(x;\, y)\,dt$$

$$\Phi_A(x;\, y)=\int_0^{+\infty} \mathbf{P}_x\{U(t)\in dy,\, t<\tau_A\}/dy.$$

These functions are respectively the 0-potential and the 0-potential of the process absorbed when U hits the set $A\subset\mathbf{R}^{n+1}$.

Since U satisfies the following stochastic differential system

$$\begin{cases} dX_0(t) &= dB(t) \\ dX_1(t) &= X_0(t)\,dt \\ \quad\vdots & \quad\vdots \\ dX_n(t) &= X_{n-1}(t)\,dt \end{cases}$$

it is easy to derive the differential generator associated to U (*see e.g.* [2]):

$$\mathscr{D}_x=\frac{1}{2}\frac{\partial^2}{\partial x_0^2}+\sum_{k=1}^{n} x_{k-1}\frac{\partial}{\partial x_k}.$$

It is well known that

$$\mathscr{D}_x\Phi(x;\, y)=-\delta(y-x). \tag{4}$$

Finally, notice that each component of U has a scaling property such that for any $\nu>0$:

$$(X_k(\nu.))_{0\le k\le n}\overset{law}{=}(\nu^{k+1/2}X_k(.))_{0\le k\le n}.$$

All scaling factors are different, so this leads us to change the Euclidean norm on $\mathbf{R}^{n+1}$ into the application N defined by:

$$N(x)=\max_{0\leq i\leq n}|x_i|^{\frac{1}{2i+1}}.$$

Proposition 2.2. 1) *Let $y\in\mathbf{R}^{n+1}$. The function $x\mapsto\Phi(x;y)$ is continuous over $\mathbf{R}^{n+1}\backslash\{y\}$.*
2) *There are some positive constants α, β such that for any $x\in\mathbf{R}^{n+1}$:*

$$\Phi(x;y)\geq\alpha\min\left(|x_n-y_n|^{-\frac{2d}{2n+1}},\ \min_{0\leq i\leq n-1}(|x_i|\vee|y_i|)^{-\frac{2d}{2i+1}}\right) \tag{5}$$

$$\Phi(x;y)\leq\beta\min_{0\leq i\leq n}\left[\frac{1}{|x_i-y_i|^{\frac{2d}{2i+1}}}+\sum_{j=1}^{i}\left(\frac{|x_{i-j}|}{|x_i-y_i|}\right)^{\frac{d}{j}}\wedge\sum_{j=1}^{i}\left(\frac{|y_{i-j}|}{|x_i-y_i|}\right)^{\frac{d}{j}}\right] \tag{6}$$

In particular:

$$\frac{\alpha}{N(x)^{2d}}\leq\Phi(x;\mathbf{O})\leq\frac{\beta}{N(x)^{2d}}. \tag{7}$$

The proof of proposition 2.2 is given in section 5.

In the proposition below is stated an important result which comes from Itô's rule as well as proposition 2.2:

Proposition 2.3. *Under $\mathbf{P}_x$, $(\Phi(U(t),y))_{t\geq0}$ is a local martingale with respect to the Brownian filtration and if A is a Borel set in $\mathbf{R}^{n+1}$ such that $x\notin\mathring{A}$ and $y\in\mathring{A}$, then $(\Phi(U(t\wedge\tau_A),y))_{t\geq0}$ is a continuous bounded martingale.*

Now, we introduce the dual process U^* of U, *i.e.* the Markov whose transition densities are defined as

$$p_t^*(x;y)=\mathbf{P}_x^*\{U^*(t)\in dy\}/dy:=p_t(y;x)$$

and we denote by Φ^* and Φ_A^* the related dual potentials similar to Φ and Φ_A. We set also

$$\tau_A^*=\inf\{t>0;\ U^*(t)\in A\}.$$

In the following proposition, we write out some explicit relationships between the processes U and U^*:

Proposition 2.4. *If $x=(x_0,\dots,x_n)\in\mathbf{R}^{n+1}$, put $x^*=(x_0,-x_1,x_2,\dots,(-1)^nx_n)$. For all $x,y\in\mathbf{R}^{n+1}$ we have:*

$$p_t^*(x;y)=p_t(x^*;y^*) \tag{8}$$

$$\Phi^*(x;y)=\Phi(x^*;y^*) \tag{9}$$

$$\Phi_A^*(x;y)=\Phi_A(x^*;y^*) \tag{10}$$

Therefore, the following identity holds:

$$(U^*(t))_{t\geq 0} \overset{law}{=} (U(t)^*)_{t\geq 0}$$

Moreover:

$$\mathbf{E}_x(\Phi(U(\tau_A);\, y)\mathbf{1}_{\{\tau_A<+\infty\}}) = \mathbf{E}^*_y(\Phi(x;\, U^*(\tau^*_A))\mathbf{1}_{\{\tau^*_A<+\infty\}}). \tag{11}$$

Proof. In view of (3) we have to verify that the following equality holds:

$$(y-xJ_t)A_t(y-xJ_t)^T = (x^*-y^*J_t)A_t(x^*-y^*J_t)^T,$$

or, equivalently:

$$J_tA_tJ_t^T = -A_{-t}. \tag{12}$$

Let us compute $(J_tA_tJ_t^T)^{-1} = J_{-t}^TA_t^{-1}J_{-t}$. We have:

$$A_t^{-1} = \left(\frac{2t^{i+j+1}}{i!j!(i+j+1)}\right)_{0\leq i,j\leq n}$$

The generic term of $J_{-t}^TA_t^{-1}J_{-t}$ is

$$\sum_{\substack{k,l\\ k\leq i\\ l\leq j}} \frac{(-t)^{i-k}}{(i-k)!}\,\frac{2t^{k+l+1}}{k!l!(k+l+1)}\,\frac{(-t)^{j-l}}{(j-l)!} = (-1)^{i+j}\frac{2t^{i+j+1}}{i!j!}\sum_{\substack{k,l\\ k\leq i\\ l\leq j}} \frac{(-1)^{k+l}}{k+l+1}\binom{i}{k}\binom{j}{l}. \tag{13}$$

It can be easily seen that the last sum actually equals $\frac{1}{i+j+1}$ so that the left-hand side of (12) is equal to $-\frac{2(-t)^{i+j+1}}{i!j!(i+j+1)}$, that is to say the generic term of $-A_{-t}^{-1}$. This proves (13) and the relations (8) and (9) are satisfied.

The equality (10) can be checked by writing

$$\Phi_A(x,y) = \lim_{N\to+\infty}\int_0^{+\infty} dt\mathbf{P}_x\{\forall k\in\{1,\dots,N-1\},\ U\left(\frac{k}{N}t\right)\notin A,\ U(t)\in dy\}/dy$$

and by using the Markov property as well as the relation (8).

Now, since U is a strong Markov process we get:

$$\begin{aligned}
\Phi(x;y)-\Phi_A(x,y) &= \int_0^{+\infty} dt\mathbf{P}_x\{U(t)\in dy,\ \tau_A\leq t\}/dy\\
&= \int_0^{+\infty} dt\mathbf{E}_x(\mathbf{1}_{\{\tau_A\leq t\}}\mathbf{P}_{U(\tau_A)}\{\omega:\ U(t-\tau_A,\omega)\in dy\}/dy)\\
&= \mathbf{E}_x\left(\mathbf{1}_{\{\tau_A<+\infty\}}\int_{\tau_A}^{+\infty} p_{t-\tau_A}(U(\tau_A);\,y)dt\right)\\
&= \mathbf{E}_x(\Phi(U(\tau_A);\,y)\mathbf{1}_{\{\tau_A<+\infty\}}).
\end{aligned}$$

Working with the dual process U^* yields through a same way for any $x, y \in \mathbf{R}^{n+1}$:

$$\begin{aligned}\Phi^*(y; x) - \Phi^*_A(y; x) &= \mathbf{E}^*_y(\Phi^*(U^*(\tau_A); x)\mathbf{1}_{\{\tau^*_A<+\infty\}}) \\ &= \mathbf{E}^*_y(\Phi(x; U^*(\tau_A))\mathbf{1}_{\{\tau^*_A<+\infty\}}).\end{aligned}$$

This proves (11).

3. Proof of theorem 1.1

Put $\mathbf{P}=\mathbf{P}_O$. The proof of theorem 1.1 hinges on the following propositions (3.1) and (3.2). See Port and Stone [6].

Proposition 3.1. $\mathbf{O}$ *is regular for* B *if and only if* $\mathbf{P}(\limsup_{k\geq 0}\{\tau_k<+\infty\})>0$.

Proposition 3.2. *There is a positive constant* C *such that for all integers* k, l, *if* $|k-l|\geq 2$, *then*

$$\mathbf{P}\{\tau_k<+\infty, \tau_l<+\infty\}\leq C\mathbf{P}\{\tau_k<+\infty\}\mathbf{P}\{\tau_l<+\infty\}.$$

To show proposition 3.1 it requires following two lemmas, proof of which is given in section 5.

Lemma 3.1. *The process* U *is transient, namely for all* $x\in\mathbf{R}^{n+1}$:

$$\mathbf{P}_x\{\lim_{t\to+\infty}\|U(t)\|=+\infty\}=1.$$

Lemma 3.2. $\{\mathbf{O}\}$ *is a polar set for* U, *i.e. for all* $x\in\mathbf{R}^{n+1}$:

$$\mathbf{P}_x\{\exists\, t>0: U(t)=\mathbf{O}\}=0.$$

Proof of proposition 3.1. ● If $\mathbf{O}$ is regular for B, then it is easy to see from the definition of a regular point that the events $\{\tau_k<+\infty\}$ occurs i.o. with probability one.

● Conversely, suppose $\mathbf{P}(\limsup_{k\geq 0}\{\tau_k<+\infty\})>0$. This implies in particular that, with a strictly positive probability, there exists a sequence $(t_k(\omega))_{k\geq 0}$ and an increasing sequence $(n_k)_{k\geq 0}$ such that

$$\forall k\geq 0,\ U(t_k)\in B_{n_k}$$

and then

$$\mathbf{P}\{\lim_{k\to+\infty}U(t_k)=\mathbf{O}\}>0.$$

Since U is transient, the sequence $(t_k)_{k\geq 0}$ is bounded with a strictly positive probability. Let $\mathbf{t}=\mathbf{t}(\omega)$ be one of its limit points. We have $\mathbf{P}\{U(\mathbf{t})=\mathbf{O}\}>0$. Seeing that $\{\mathbf{O}\}$ is polar, we get

$$\mathbf{P}\{\mathbf{t}=0\}>0.$$

Thus, with a strictly positive probability, there is a subsequence of $(t_k)_{k\geq 0}$, say $(s_k)_{k\geq 0}$, converging to 0 such that

$$\forall k\geq 0,\ U(s_k)\in B.$$

As a result:

$$\mathbf{P}\{\tau_B=0\}>0.$$

Since $\{\tau_B=0\}$ is a tail event with respect to the Brownian filtration, the Blumenthal zero-one law asserts that in fact

$$\mathbf{P}\{\tau_B=0\}=1,$$

which completes the proof.

The proof of proposition 3.2 is based on some estimates which are supplied in the following lemmas:

Lemma 3.3. *The following inequality holds for any $x, y\in\mathbf{R}^{n+1}$ and any integer $k\geq 0$:*

$$\mathbf{P}_x\{\tau_k<+\infty\}\leq\frac{\sup_{z\in B_k}\Phi(x;\,z)}{\inf_{z\in B_k}\Phi(z;\,y)}\mathbf{P}_y\{\tau_k<+\infty\}. \tag{14}$$

Lemma 3.4. *For all integers k, l, if $|k-l|\geq 2$, then*

$$\frac{\sup_{(x,y)\in B_k\times B_l}\Phi(x;\,y)}{\inf_{x\in B_l}\Phi(x;\,\mathbf{O})}\leq C \tag{15}$$

where C is the constant introduced in proposition 3.2.

The proofs of these lemmas are transferred to section 5.

Proof of proposition 3.2. By the Markov property of U we get:

$$\begin{aligned}\mathbf{P}\{\tau_k<+\infty,\ \tau_l<+\infty\}&=\\ &=\mathbf{P}\{\tau_k\leq\tau_l<+\infty\}+\mathbf{P}_x\{\tau_l<\tau_k<+\infty\}\\ &=\mathbf{E}[\mathbf{1}_{\{\tau_k<+\infty\}}\mathbf{P}_{U(\tau_k)}\{\tau_l<+\infty\}]+\mathbf{E}[\mathbf{1}_{\{\tau_l<+\infty\}}\mathbf{P}_{U(\tau_l)}\{\tau_k<+\infty\}]\\ &\leq\mathbf{P}\{\tau_k<+\infty\}\sup_{x\in B_k}\mathbf{P}_x\{\tau_l<+\infty\}+\mathbf{P}\{\tau_l<+\infty\}\sup_{x\in B_l}\mathbf{P}_x\{\tau_k<+\infty\}.\end{aligned}$$

Combining (14) with (15) yields

$$\sup_{x\in B_k}\mathbf{P}_x\{\tau_l<+\infty\}\leq C\mathbf{P}\{\tau_l<+\infty\}$$

and the same inequality holds also by exchanging k and l.

The proof of proposition 3.2 is complete.

It is now easy to achieve the proof of theorem 1.1 by invoking the following version of the Borel-Cantelli lemma (*see e. g.* [6]):
if

$$\sum_{k=0}^{+\infty}\mathbf{P}(A_k)=+\infty$$

and if there is a constant $c>0$ such that for all integers k, l:

$$\mathbf{P}(A_k\cap A_l)\ \leq c\mathbf{P}(A_k)\mathbf{P}(A_l)\quad \text{when}\quad |k-l|\geq 2,$$

then:

$$\mathbf{P}(\limsup_{k\geq 0} A_k)>0.$$

4. Proof of theorems 1.2 and 1.3

With the view of checking theorem 1.2 as well as thorem 1.3, we need some estimates about the hitting probabilities of certain parallelepipeds. These ones are provided in the follwing proposition, whose proof is transferred to the next section.

Proposition 4.1. 1) Set:

$$\begin{aligned}P(a,b,\varepsilon)&=\prod_{i=0}^{n-1}[-\varepsilon^{2i+1},\varepsilon^{2i+1}]\times[a,b]\\&=\{x\in\mathbf{R}^{n+1}\colon a\leq x_n\leq b,\ \forall i\in\{0,\dots,n-1\},\ |x_i|\leq\varepsilon^{2i+1}\}\end{aligned}$$

where a, b, ε are some positive real numbers such that $a<b$ and $L=b-a\geq 2\varepsilon^{2n+1}$. Then, there are positive constants γ, δ such that

$$\left.\begin{array}{ll}\text{if } n\geq 2, & \gamma\dfrac{L\varepsilon^{n^2-2}}{b^{\frac{2d}{2n+1}}}\\[2ex] \text{if } n=1, & \gamma\left(\dfrac{L}{b}\right)^{2/3}\end{array}\right\}\leq\mathbf{P}\{\tau_{P(a,b,\varepsilon)}<+\infty\}\leq\delta\frac{L\varepsilon^{n^2-2}}{a^{\frac{2d}{2n+1}}}.\tag{16}$$

2) *Set*:

$$\begin{aligned}P'(a,b,\varepsilon)&=[-\varepsilon,\varepsilon]^n\times[a,b]\\&=\{x\in\mathbf{R}^{n+1}\colon a\leq x_n\leq b,\ \forall_i\in\{0,\dots,n-1\},\ |x_i|\leq\varepsilon\}\end{aligned}$$

where a, b, ε are some positive real numbers such that $a<b$ and $L=b-a\geq 2\varepsilon$. Then, there are positive constants γ', δ' such that

$$\left.\begin{array}{ll} \text{if } n\geq 2, & \gamma'\dfrac{L\varepsilon^{\frac{n^2-2}{2n+1}}}{b^{\frac{2d}{2n+1}}} \\ \text{if } n=1, & \gamma'\left(\dfrac{L}{b}\right)^{2/3} \end{array}\right\} \leq \mathbf{P}\{\tau_{P'(a,b,\varepsilon)}<+\infty\}\leq \delta'\frac{L\varepsilon^{\frac{n^2-2}{2n+1}}}{a^{\frac{2d}{2n+1}}}. \tag{17}$$

Proof of theorem 1.2. ● Suppose at first that $\lim_{x\to 0^+} x^{-\frac{1}{2n+1}}f(x)>0$. In this case, the integral arising in theorem 1.2 is divergent. On the other hand, it can be found two positive constants μ, ε such that the set

$$F_{\mu,\varepsilon}=\{x\in\mathbf{R}^{n+1}: x_n\geq 0, \sum_{i=0}^{n-1}|x_i|^{\frac{2}{2i+1}}\leq \mu x_n^{\frac{2}{2n+1}}, N(x)\leq\varepsilon\}$$

is contained in T. Then

$$\mathbf{P}\{\tau_T=0\}\geq\mathbf{P}\{\tau_{F_{\mu,\varepsilon}}=0\}=\inf_{0<t<\varepsilon^2}\mathbf{P}\{\tau_{F_{\mu,\varepsilon}}\leq t\}\leq\inf_{0<t<\varepsilon^2}\mathbf{P}\{U(t)\in F_{\mu,\varepsilon}\}.$$

By scaling, we have for any $t\in(0,\varepsilon^2)$:

$$\mathbf{P}\{U(t)\in F_{\mu,\varepsilon}\}=\mathbf{P}\{U(1)\in F_{\mu,\varepsilon/\sqrt{t}}\}\geq\mathbf{P}\{U(1)\in F_{\mu,1}\}>0.$$

Whence $\mathbf{P}\{\tau_T=0\}>0$, and in fact by the zero-one law

$$\mathbf{P}\{\tau_T=0\}=1.$$

Aa a result $\mathbf{O}$ is regular for T and the claimed assertion is true.

● Assume now that $n\geq 2$ and $\lim_{x\to 0^+}x^{-\frac{1}{2n+1}}f(x)=0$. Let us introduce the parallelepipeds:

$$P_k=P(2^{-(k+1)(2n+1)}, 2^{-k(2n+1)}, \tfrac{1}{\sqrt{n}}f(2^{-(k+1)(2n+1)}))$$
$$Q_k=P(\tfrac{1}{2}2^{-(k+1)(2n+1)}, 2^{-k(2n+1)}, f(2^{-k(2n+1)}))$$

and set:

$$T_k=T\cap\{x\in\mathbf{R}^{n+1}: 2^{-k-1}\leq N(x)\leq 2^{-k}\}.$$

It may be seen that for sufficiently large k:

$$P_k\subset T_k\subset Q_k.$$

On the other hand, according to (16), there are positive constants γ', δ', γ'', δ'' which can be expressed by means of γ, δ and n such that:

$$\gamma'[2^k f(2^{-(k+1)(2n+1)})]^{n^2-2}\leq\mathbf{P}\{\tau_{P_k}<+\infty\}\leq\delta'[2^k f(2^{-(k+1)(2n+1)})]^{n^2-2}.$$
$$\gamma''[2^k f(2^{-k(2n+1)})]^{n^2-2}\leq\mathbf{P}\{\tau_{Q_k}<+\infty\}\leq\delta''[2^k f(2^{-k(2n+1)})]^{n^2-2}.$$

Thus the following series

$$\Sigma\mathbf{P}\{\tau_{P_k}<+\infty\},\ \Sigma\mathbf{P}\{\tau_{Q_k}<+\infty\},\ \Sigma\mathbf{P}\{\tau_{T_k}<+\infty\},\ \Sigma[2^k f(2^{-k(2n+1)})]^{n^2-2}$$

simultaneously converge or diverge.

Since the function $x\mapsto x^{-\frac{1}{2n+1}}f(x)$ does not decrease, the convergence of this last series is equivalent to the convergence of the integral $\int_{0^+}\left(\frac{f(x)}{x^{\frac{1}{2n+1}}}\right)^{n^2-2}\frac{dx}{x}$ and we get the desired result with the aid of theorem 1.1.

● Finally, when $n=1$ and $\lim_{x\to 0^+} x^{-\frac{1}{3}}f(x)=0$, we get from (16)

$$\mathbf{P}\{\tau_{T_k}<+\infty\}\geq\mathbf{P}\{\tau_{P_k}<+\infty\}\geq\gamma(1-2^{-2n-1})^{2/3}>0$$

so that the series $\Sigma\mathbf{P}\{\tau_{T_k}<+\infty\}$ diverges and $\mathbf{O}$ is therefore a regular point for T.

Proof of theorem 1.3. Let us introduce the parallelepipeds:

$$P'_k=P'(2^{-(k+1)(2n+1)},\ 2^{-k(2n+1)},\ \tfrac{1}{\sqrt{n}}f(2^{-(k+1)(2n+1)}))$$
$$Q'_k=P'(\mu 2^{-(k+1)(2n+1)},\ 2^{-k(2n+1)},\ f(2^{-k(2n+1)}))$$

where μ is a constant which will be explained thereafter, and set:

$$T'_k=T'\cap\{x\in\mathbf{R}^{n+1}:\ 2^{-k-1}\leq N(x)\leq 2^{-k}\}.$$

We will see that we have for large enough k:

$$P'_k\subset T'_k\subset Q'_k.$$

● Indeed, if $x\in P'_k$ then:

i) $\sum_{i=0}^{n-1}x_i^2\leq f(2^{-(k+1)(2n+1)})^2\leq f(x_n)^2$ since f does not decrease, and then $x\in T'$.

ii) $2^{-k-1}\leq|x_n|^{\frac{1}{2n+1}}\leq N(x)\leq 2^{-k}\vee\max_{0\leq i\leq n-1}[\frac{1}{\sqrt{n}}f(2^{-k(2n+1)})]^{\frac{1}{2i+1}}$.

By assumption, we know that the function $x\mapsto x^{-1}f(x)$ is bounded from above near zero. Therefore, for all large k:

$$\max_{0\leq i\leq n-1}f(2^{-k(2n+1)})^{\frac{1}{2i+1}}\leq\text{constant}\times 2^{-k\frac{2n+1}{2n-1}}=o(2^{-k})\text{ as }k\to+\infty.$$

Thus $N(x)\leq 2^{-k}$ if k is sufficiently large. As a result: $x\in T'_k$.

● Now, if $x\in T'_k$ then

i) $x_n\leq N(x)^{2n+1}\leq 2^{-k(2n+1)}$.

ii) $\sum_{i=0}^{n-1}x_i^2\leq f(x_n)^2\leq f(2^{-k(2n+1)})^2$

and this implies for each $i\in\{0,\dots,n-1\}$: $|x_i|\leq f(2^{-k(2n+1)})$.

iii) Since $N(x)\geq 2^{-k-1}$, it can be found an index $i_0\in\{0,\dots,n\}$ such that $|x_{i_0}|\geq 2^{-(k+1)(2i_0+1)}\geq 2^{-(k+1)(2n+1)}$. Hence

$$x_n^2\geq 2^{-2(k+1)(2n+1)}-\sum_{i=0}^{n-1}x_i^2\geq 2^{-2(k+1)(2n+1)}-f(x_n)^2.$$

This can be rewritten as follows:

$$x_n^2+f(x_n)^2\geq 2^{-2(k+1)(2n+1)}.$$

Since $f(x)\leq \text{constant}\times x$ when $x\to 0^+$, it is clear that there is a constant $\mu>0$ such that if k is large enough, $x_n\geq\mu 2^{-(k+1)(2n+1)}$. As a result: $x\in Q'_k$.

The proof of theorem 1.3 can now be achieved as follows: the three series of the hitting probabilities associated to the collections of sets (P'_k), (Q'_k), (T'_k) are simultaneously convergent or divergent with the series

$$\sum\left(\frac{f(2^{-2(k+1)(2n+1)})}{2^{-2(k+1)(2n+1)}}\right)^{\frac{n^2-2}{2n+1}}$$

or, equivalently, with the integral displayed in theorem 1.3 and the proof is complete.

5. Proof of lemmas 3.1, 3.2, 3.3, 3.4 and proposition 4.1

We begin to exhibit a set of general results which will be useful thereafter.

Lemma 5.1. *Let A, A_1, $A_2\subset\mathbf{R}^{n+1}$ be some Borel sets.*
1) *For any positive measure m on A and any $x\in\mathbf{R}^{n+1}\backslash\mathring{A}$ we have:*

$$\frac{\int_A\Phi(x;y)m(dy)}{\sup_{z\in\partial A}\int_A\Phi(z;y)m(dy)}\leq\mathbf{P}_x\{\tau_A<+\infty\}\leq\frac{\int_A\Phi(x;y)m(dy)}{\inf_{z\in\partial A}\int_A\Phi(z;y)m(dy)}. \tag{18}$$

In particular, if $x\in\mathbf{R}^{n+1}\backslash\mathring{A}$ and $y\in\mathring{A}$:

$$\mathbf{P}_x\{\tau_A<+\infty\}\leq\frac{\Phi(x;y)}{\inf_{z\in\partial A}\Phi(z;y)}. \tag{19}$$

2) *If $\bar{A}_1\cap\mathring{A}_2=\emptyset$, then for all $x\in\mathbf{R}^{n+1}$ and all $y\in\mathring{A}_2$:*

$$\mathbf{P}_x\{\tau_{A_1}<+\infty,\ \tau_{A_2}\circ\theta_{\tau_{A_1}}<+\infty\}\leq\frac{\sup_{z\in\bar{A}_1}\Phi(z;y)}{\inf_{z\in\partial A_2}\Phi(z;y)}. \tag{20}$$

3) *For any $x\in\mathbf{R}^{n+1}\backslash(\mathring{A}_1\cup\mathring{A}_2)$ and any $y\in\mathring{A}_1\cup\mathring{A}_2$ such that* $\inf_{z\in\partial A_1}\Phi(z;y)>\inf_{z\in\partial A_2}\Phi(z;y)$ *we have:*

$$\mathbf{P}_x\{\tau_{A_1}\leq\tau_{A_2}\}\leq\frac{\Phi(x;y)-\inf_{z\in\partial A_2}\Phi(z;y)}{\inf_{z\in\partial A_1}\Phi(z;y)-\inf_{z\in\partial A_2}\Phi(z;y)}. \tag{21}$$

Proof. 1) Proposition 2.1 and Doob's optional sampling theorem lead to the following identity, which is valid for any $y\in A$:

$$\Phi(x;y)=\mathbf{E}_x(\Phi(U(\tau_A);y)\mathbf{1}_{\{\tau_A<+\infty\}}).$$

Hence

$$\int_A \Phi(x;y)m(dy)=\mathbf{E}_x\left[\mathbf{1}_{\{\tau_A<+\infty\}}\int_A \Phi(U(\tau_A);y)m(dy)\right]$$

which makes clear (18).

2) (20) may be easily deduced from (19) thanks to the strong Markov property.

3) Apply again Doob's theorem to the continuous martingale $(\Phi(U(t\wedge\tau_{A_1}\wedge\tau_{A_2});y))_{t\geq 0}$. We get:

$$\begin{aligned}\Phi(x;y) &= \mathbf{E}_x(\Phi(U(\tau_{A_1});y)\mathbf{1}_{\{\tau_{A_1}\leq\tau_{A_2}\}})+\mathbf{E}_x(\Phi(U(\tau_{A_2});y)\mathbf{1}_{\{\tau_{A_1}>\tau_{A_2}\}})\\ &\geq \inf_{z\in\partial A_1}\Phi(z;y)\mathbf{P}\{\tau_{A_1}\leq\tau_{A_2}\}+\inf_{z\in\partial A_2}\Phi(z;y)\left(1-\mathbf{P}\{\tau_{A_1}\leq\tau_{A_2}\}\right).\end{aligned}$$

Thus (21) ensues.

Lemma 5.2. *Let $\alpha_0, \alpha_1, \ldots, \alpha_i$ be some real numbers, $\alpha_0\neq 0$. The following inequality holds (with the convention $\sum_1^0=0$):*

$$\int_0^{+\infty}\exp\left(-\frac{1}{t^{2i+1}}\left(\sum_{j=0}^{i}\alpha_j t^j\right)^2\right)\frac{dt}{t^{d+1}}\leq\text{constant}\times\left(\frac{1}{|\alpha_0|^{\frac{2d}{2i+1}}}+\sum_{j=1}^{i}\left|\frac{\alpha_j}{\alpha_0}\right|^{\frac{d}{j}}\right). \tag{22}$$

Proof. Suppose $t\leq\frac{1}{3}\min_{1\leq j\leq i}|\alpha_0/\alpha_j|^{1/j}$, so that $|\alpha_j|t^j\leq 3^{-j}|\alpha_0|$. Then:

$$\left|\sum_{j=0}^{i}\alpha_j t^j\right|\geq|\alpha_0|-\sum_{j=0}^{i}|\alpha_j|t^j\geq|\alpha_0|\,(1-\sum_{j=1}^{i}3^{-j})\geq\tfrac{1}{2}|\alpha_0|.$$

Now, the integral in question is bounded from above as follows:

$$\int_0^{+\infty}\exp\left(-\frac{1}{t^{2i+1}}\left(\sum_{j=0}^{i}\alpha_j t^j\right)^2\right)\frac{dt}{t^{d+1}}\leq\int_0^{\frac{1}{3}\min_{1\leq j\leq i}|\alpha_0/\alpha_j|^{1/j}}e^{-\frac{1}{4}\alpha_0^2/t^{2i+1}}\frac{dt}{t^{d+1}}+$$
$$+\int_{\frac{1}{3}\min_{1\leq j\leq i}|\alpha_0/\alpha_j|^{1/j}}^{+\infty}\frac{dt}{t^{d+1}}.$$

Since

$$\int_0^{\frac{1}{3}\min_{1\leq j\leq i}|\alpha_0/\alpha_j|^{1/j}}e^{-\frac{1}{4}\alpha_0^2/t^{2i+1}}\frac{dt}{t^{d+1}}\leq\int_0^{+\infty}e^{-\frac{1}{4}\alpha_0^2/t^{2i+1}}\frac{dt}{t^d+1}\leq\text{constant}\times|\alpha_0|^{-\frac{2d}{2i+1}}$$

and

$$\int_{\frac{1}{3}\min_{1\leq j\leq i}|\alpha_0/\alpha_j|^{1/j}}^{+\infty}\frac{dt}{t^{d+1}}\leq\text{constant}\times\max_{1\leq j\leq i}\left|\frac{\alpha_j}{\alpha_0}\right|^{\frac{d}{j}},$$

the inequality (22) is immediately obtained.

Lemma 5.3. *Let $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_{n-1}$ be non zero real numbers. The following inequality holds:*

$$\int_0^{+\infty} \exp\left(-\sum_{i=0}^{n} \frac{1}{t^{2i+1}}\left(\alpha_i+\sum_{j=1}^{i}\beta_{i-j}t^j\right)^2\right)\frac{dt}{t^{d+1}}$$
$$\geq \text{constant}\times\min\left(|\alpha_n|^{-\frac{2d}{2n+1}}, \min_{0\leq i\leq n-1}(|\alpha_i|\vee|\beta_i|)^{-\frac{2d}{2i+1}}\right). \tag{23}$$

Proof. Write first

$$\left(\alpha_i+\sum_{j=1}^{i}\beta_{i-j}t^j\right)^2 \leq \text{constant}\times\left(\alpha_i^2+\sum_{j=1}^{i}\beta_{i-j}^2t^{2j}\right)$$

so that

$$\begin{aligned}\sum_{i=0}^{n}\frac{1}{t^{2i+1}}\left(\alpha_i+\sum_{j=1}^{i}\beta_{i-j}t^j\right)^2 &\leq \text{constant}\times\sum_{i=0}^{n}\left(\frac{\alpha_i^2}{t^{2i+1}}+\sum_{j=1}^{i}\frac{\beta_j^2}{t^{2j+1}}\right)\\ &\leq \text{constant}\times\sum_{i=0}^{n-1}\left(\frac{\alpha_i^2+\beta_i^2}{t^{2i+1}}+\frac{\alpha_n^2}{t^{2n+1}}\right)\\ &\leq \text{constant}\times\sum_{i=0}^{n}\left(\frac{\gamma}{t}\right)^{2i+1}\end{aligned}$$

where we set

$$\gamma=\max\left(|\alpha_n|^{\frac{2}{2n+1}}, \max_{0\leq i\leq n-1}(|\alpha_i|\vee|\beta_i|)^{\frac{2}{2i+1}}\right).$$

It is then easy to derive (23).

Proof of proposition 2.2. 1) Fix $y\in\mathbf{R}^{n+1}$ and let $x_0\in\mathbf{R}^{n+1}\setminus\{y\}$. There is a small Euclidean ball $B(x_0, \varepsilon\sqrt{n+1})$ with centre x_0 and radius $\varepsilon\sqrt{n+1}>0$ not containing y. Choose $R>0$ large enough so that $B(x_0, \varepsilon\sqrt{n+1})\subset B(\mathbf{O}, R)$. We are going to prove that $\Phi(x; y)$ is in fact continuous in $B(x_0,\varepsilon\sqrt{n+1})$.

One can find an index $i_0\in\{0,\dots,n\}$ such that $|x_{i_0}-y_{i_0}|>\varepsilon$. We have also for all index $i\in\{0,\dots,n\}$: $|x_i|\leq R$.

On the other hand, since the matrix $(a_{ij})_{0\leq i,j\leq n}$ that appears in the definition (2) of the transition density $p_t(x; y)$ is positive-definite, it is clear that there exist some positive constants λ_1 and λ_2 such that:

$$\lambda_1\sum_{i=0}^{n}x_i^2\leq\sum_{0\leq i,j\leq n}a_{ij}x_ix_j\leq\lambda_2\sum_{i=0}^{n}x_i^2. \tag{24}$$

Therefore

$$\begin{aligned} p_t(x;y) &\le \frac{c}{t^{d+1}}\exp\left(-\lambda_1\sum_{i=0}^{n}\frac{1}{t^{2i+1}}\left(y_i-\sum_{k=0}^{i}\frac{t_k}{k!}x_{i-k}\right)^2\right) \\ &\le \frac{c}{t^{d+1}}\exp\left(-\lambda_1\frac{1}{t^{2i_0+1}}\left(y_{i_0}-\sum_{k=0}^{i_0}\frac{t^k}{k!}x_{i_0-k}\right)^2\right). \end{aligned}$$

Put $a'=a\left(\frac{\varepsilon}{R}\right)^{\frac{1}{2n+1}}$ where a will be chosen later. Then for $t\le a'$,

$$\forall k\in\{0,\dots,i_0\}\ t\le a\left|\frac{y_{i_0}-x_{i_0}}{x_{i_0-k}}\right|^{\frac{1}{k}}$$

and

$$\left|y_{i_0}-\sum_{k=0}^{i_0}\frac{t^k}{k!}x_{i_0-k}\right|\ge|y_{i_0}-x_{i_0}|-\sum_{k=1}^{i_0}\frac{t^k}{k!}|x_{i_0-k}|\ge\varepsilon\left(1-\sum_{k=1}^{i_0}\frac{a^k}{k!}\right)\ge a''$$

where $a''=\varepsilon(2-e^a)$ (with the convention $\sum_{k=1}^{0}=0$). So, we will pick a less than log 2 (so that $a''>0$). When $t\ge a'$, $p_t(x;y)$ is bounded from above by $\frac{c}{t^{d+1}}$.

Consequently:

$$\forall t>0,\ \forall x\in B(x_0,\varepsilon\sqrt{n+1}),\ p_t(x;y)\le q(t)$$

where

$$q(t)=\frac{c}{t^{d+1}}(\mathbf{1}_{(a',+\infty)}(t)+\mathbf{1}_{(0,a']}(t)e^{-a''/t^{2i_0+1}}).$$

The function q does not depend on x and is integrable with respect to t over $(0,+\infty)$ so that the desired result derives from the dominated convergence theorem.

2) The inequalities (5) and (6) can be easily obtained through (22), (23) and the relationship (9).

Proof of lemma 3.1. Set

$$\begin{aligned} S_R &= \{x\in\mathbf{R}^{n+1}: N(x)\le R\} \\ S_R^c &= \mathbf{R}^{n+1}\setminus S_R \\ A_R &= \{\exists\,(t_k)_{k\ge0}\nearrow+\infty: N(U(t_k))\le R\}. \end{aligned}$$

Let us apply (20) and (7):

$$\mathbf{P}_x(A_R)\le\mathbf{P}_x\{\tau_{S_{R^2}^c}<+\infty,\ \exists\,t>\tau_{S_{R^2}^c}: N(U(t))\le R\}\le\frac{\sup_{z\notin S_{R^2}}\Phi(z;\mathbf{O})}{\inf_{z\notin\partial S_R}\Phi(z;\mathbf{O})}\le\frac{\beta}{\alpha R^{2d}}.$$

Let $R\to+\infty$. We get $\mathbf{P}_x(A_R)\to0$. Notice that the family $(A_R)_{R>0}$ is increasing so that, in fact:

$$\forall R>0,\ \mathbf{P}_x(A_R)=0,$$

that is to say, for any $x \in \mathbf{R}^{n+1}$:

$$\mathbf{P}_x\{\lim_{t\to+\infty}\|U(t)\|=+\infty\}=1$$

as desired.

Proof of lemma 3.2. ● Suppose at first $x\neq\mathbf{O}$ and introduce ε, $R>0$ such that $\varepsilon<R$ and $x\in S_\varepsilon^c\cap S_R$. We have from (7) and (21) that:

$$\mathbf{P}_x\{\tau_{S_\varepsilon}\leq\tau_{S_R^c}\}\leq\frac{\Phi(x;\mathbf{O})-\frac{\alpha}{R^{2d}}}{\frac{\alpha}{\varepsilon^{2d}}-\frac{\alpha}{R^{2d}}}.$$

Let $\varepsilon\to0^+$. This yields:

$$\mathbf{P}_x\{\tau_{\{\mathbf{O}\}}\leq\tau_{S_R^c}\}=0.$$

Letting $R\to+\infty$ then leads to

$$\mathbf{P}_x\{\tau_{\{\mathbf{O}\}}<+\infty\}=0.$$

● Suppose now $x=0$. We can successively write

$$\mathbf{P}_0\{\tau_{\{\mathbf{O}\}}<+\infty\}=\lim_{\varepsilon\downarrow0^+}\mathbf{P}_0\{\exists\, t>\varepsilon:\ U(t)=0\}=\lim_{\varepsilon\downarrow0^+}\mathbf{E}_0(\mathbf{P}_{U(\varepsilon)}\{\tau_{\{\mathbf{O}\}}<+\infty\}).$$

Since $U(\varepsilon)\neq\mathbf{O}$ a.s., according to the previous case we can conclude that $\mathbf{P}_{U(\varepsilon)}\{\tau_{\{\mathbf{O}\}}<+\infty\}=0$ a.s. and then

$$\mathbf{P}_0\{\tau_{\{\mathbf{O}\}}<+\infty\}=0.$$

Proof of lemma 3.3. Lemma 3.3 is an easy consequence of (11). Indeed, the first member of (11) is bounded from below by

$$\inf_{z\in\bar{A}}\Phi(z;y)\mathbf{P}_x\{\tau_A<+\infty\}$$

and the second one is bounded from above by

$$\sup_{z\in\bar{A}}\Phi(x;z)\mathbf{P}_y^*\{\tau_A^*<+\infty\}.$$

Since $B^*=B$ (which implies $\mathbf{P}_y^*\{\tau_k^*<+\infty\}=\mathbf{P}_y\{\tau_k<+\infty\}$), the previous remarks lead to the inequality (14).

Proof of lemma 3.4. Denote by $B_{l,i}$, $i\in\{0,\dots,n\}$, the subsets of B_l defind as being

$$B_{l,i}=\{x\in B_l:\ 2^{-(l+1)(2i+1)}\leq|x_i|\leq2^{-l(2i+1)}\}.$$

We get:

$$\bigcup_{i=0}^{n} B_{l,i} = B_l.$$

Fix two integers k, l such that $k-l\geq 2$ and let $(x, y)\in B_k\times B_l$. There is an index $i\in\{0,\dots,n\}$ such that $y\in B_{l,i}$. Thus:

$$|y_i - x_i| \geq 2^{-(l+1)(2i+1)} - 2^{-k(2i+1)}$$

and for all index $j\in\{0,\dots,n\}$:

$$|y_j| \leq 2^{-l(2j+1)}.$$

These inequalities and the upper bound (6) provide the following ones:

$$\begin{aligned}\Phi(x;\, y) &\leq \beta\left(\left|2^{-(l+1)(2i+1)} - 2^{-k(2i+1)}\right|^{-\frac{2d}{2i+1}} + \sum_{j=1}^{i}\left|\frac{2^{-l(2i-2i+1)}}{2^{-(l+1)(2i+1)} - 2^{-k(2i+1)}}\right|^{\frac{d}{j}}\right)\\ &\leq \beta\left(2^{(l+1)d}\left(1-2^{-(2i+1)}\right)^{-\frac{2d}{2i+1}} + \sum_{j=1}^{i}\left(\frac{2^{2i+1}}{1-2^{-(2i+1)}}\right)^{\frac{d}{j}}\right).\end{aligned}$$

Whence:

$$\sup_{(x,y)\in B_k\times B_l} \Phi(x;\, y) \leq \beta' 2^{2ld}$$

for an appropriate constant β'. On the other hand, thanks to (7):

$$\inf_{x\in B_l} \Phi(x;\, \mathbf{O}) \geq \alpha' 2^{2ld}$$

for another constant α'. Consequently, the ratio $\dfrac{\sup_{(x,y)\in B_k\times B_l}\Phi(x;\, y)}{\inf_{x\in B_l}\Phi(x;\, \mathbf{O})}$ is bounded.

Proof of proposition 4.1. *Part one.* Put $P=P(a, b, \varepsilon)$.

● Assume at first $L/(2\varepsilon^{2n+1})$ is an integer, say $q\geq 1$: $L=2q\varepsilon^{2n+1}$, and next introduce the decomposition

$$P = \bigcup_{j=0}^{q-1} P^{(j)}$$

into q "N-cubes" $P^{(j)}$ defined as being

$$\begin{aligned}P^{(j)} &= \prod_{i=0}^{n-1}\left[-\varepsilon^{2i+1}, \varepsilon^{2i+1}\right] \times \left[a+2j\varepsilon^{2n+1}, a+2(j+1)\varepsilon^{2n+1}\right]\\ &= \{x\in\mathbf{R}^{n+1} : N(x-y^{(j)}) \leq \varepsilon\}\end{aligned}$$

where $y^{(j)} = (0,\dots,0,\, a+(2j+1)\,\varepsilon^{2n+1})$ is the middle point of the segment $\{0\}^n\times\left[a+2j\varepsilon^{2n+1}, a+2(j+1)\varepsilon^{2n+1}\right]$.

In order to get the upper bound of the probability in question we use

(18) in which m is chosen to be $\delta_{\{0\}}(x_0,\dots,x_{n-1})\otimes$the Lebesgue measure on $[a,b]$. We have:

$$\int \Phi(\mathbf{O};y)m(dy)=\int_a^b \Phi(\mathbf{O};(0,\dots,0,u))du\leq\beta\int_a^b\frac{du}{u^{\frac{2d}{2n+1}}}\leq\beta\frac{L}{a^{\frac{2d}{2n+1}}} \tag{25}$$

and if $x\in\partial P^{(j)}$:

$$\int \Phi(x;y)m(dy)\geq\alpha\int_{a+2j\varepsilon^{2n+1}}^{a+2(j+1)\varepsilon^{2n+1}}\frac{du}{N(x-(0,\dots,0,u))^{2d}}$$

(the equality $\Phi(x;y)=\Phi(x-y;\mathbf{O})$ holds when $y\in\{0\}^n\times\mathbf{R}$). But, when $x\in\partial P^{(j)}$ and $u\in[a+2j\varepsilon^{2n+1},a+2(j+1)\varepsilon^{2n+1}]$ we get:

$$|x_n-u|\leq 2\varepsilon^{2n+1} \text{ and for every } i\in\{0,\dots,n-1\}: |x_i|\leq\varepsilon^{2i+1}.$$

Then

$$N(x-(0,\dots,0,u))\leq\varepsilon 2^{\frac{1}{2n+1}}$$

and there is $\alpha''>0$ such that:

$$\int \Phi(x;y)m(dy)\geq\alpha''\varepsilon^{2n+1-2d}=\frac{\alpha''}{\varepsilon^{n^2-2}}. \tag{26}$$

The upper bound of $\mathbf{P}\{\tau_P<+\infty\}$ in (16) is obtained by dividing the last members of (25) by (26).

Now, let us check the lower bound of $\mathbf{P}\{\tau_P<+\infty\}$. Choose this time

$$m=\sum_{i=0}^{q-1}\delta_{y^{(i)}}.$$

Similar arguments yield:

$$\int \Phi(\mathbf{O};y)m(dy)\geq\alpha\sum_{i=0}^{q-1}N(y^{(i)})^{-2d}\geq\frac{\alpha q}{b^{\frac{2d}{2n+1}}} \tag{27}$$

and if $x\in\partial P^{(j)}$ then:

$$\int \Phi(x;y)m(dy)\leq\beta\sum_{i=0}^{q-1}N(x-y^{(i)})^{-2d}.$$

It can be seen that for all $i,j\in\{0,\dots,q-1\}$,

$$N(x-y^{(i)})\geq(\||j-i|-1|^{\frac{1}{2n+1}}\vee 1)\varepsilon.$$

Now, it can be found a constant κ such that

$$\int \Phi(x; y) m(dy) \leq \frac{\kappa}{\varepsilon^{2d}} \sum_{k=1}^{q} k^{-\frac{2d}{2n+1}}. \tag{28}$$

Since

$$\sum_{k=1}^{q} k^{-\frac{2d}{2n+1}} \leq \begin{cases} \text{constant} & \text{if } n \geq 2 \\ \text{constant } \times q^{1/3} & \text{if } n=1 \end{cases}$$

and $q = L/(2\varepsilon^{2n+1})$, the desired lower bounds in (16) are acquired through dividing (27) by (28).

● Consider now the general case. Let $q = [L/(2\varepsilon^{2n+1})] \geq 1$ be the greatest integer less than $L/(2\varepsilon^{2n+1})$, and set:

$$R_1 = \prod_{i=0}^{n-1} [-\varepsilon^{2i+1}, \varepsilon^{2i+1}] \times [a, a+2(q+1)\varepsilon^{2n+1}]$$
$$R_2 = \prod_{i=0}^{n-1} [-\varepsilon^{2i+1}, \varepsilon^{2i+1}] \times [b-2q\varepsilon^{2n+1}, b].$$

Since $R_2 \subset P \subset R_1$ and seeing that proposition 4.1 is true for the parallelepipeds R_1 and R_2, we get:

$$\mathbf{P}\{\tau_P < +\infty\} \leq \mathbf{P}\{\tau_{R_1} < +\infty\} \leq 2\delta(q+1)\varepsilon^{2n+1} \frac{\varepsilon^{n^2}-2}{a^{\frac{2d}{2n+1}}}$$
$$\mathbf{P}\{\tau_P < +\infty\} \geq \mathbf{P}\{\tau_{R_2} < +\infty\} \geq 2\gamma q\varepsilon^{2n+1} \frac{\varepsilon^{n^2}-2}{b^{\frac{2d}{2n+1}}}$$

so that proposition 4.1 remains true by replacing another constants since:

$$2(q+1) \leq 4q \leq \frac{2L}{\varepsilon^{2n+1}} \text{ and } 2q \geq q+1 \geq \frac{L}{2\varepsilon^{2n+1}}.$$

Proof of proposition 4.1. Part two. The proof of the second part of proposition 4.1 is quite similar to that of the first part. So we will only point out some modifications made in that of the second paart.

In the case where $L = 2q\varepsilon'$ for a certain integer $q \geq 1$, we shall decompose $P'(a, b, \varepsilon)$ as follows:

$$P'(a, b, \varepsilon) = \bigcup_{j=0}^{q-1} P'^{(j)}$$

where

$$P'^{(j)} = [-\varepsilon, \varepsilon]^n \times [a+2j\varepsilon', a+2(j+1)\varepsilon'].$$

In order to evaluate $\inf_{x \in \partial P'^{(j)}} \int_a^b \Phi(x; (0, \dots, 0, u))\, du$, we write down the following inequality which is valid for any $x \in \partial P'^{(j)}$:

$$\int_a^b \Phi(x;\ (0,\dots,0,u))\,du \geq \alpha \int_{a+2j\varepsilon'}^{a+2(j+1)\varepsilon'} \frac{du}{N(x-(0,\dots,0,u))^{2d}}.$$

When $x \in \partial P^{(j)}$ and $u \in [a+2j\varepsilon', a+2(j+1)\varepsilon']$ we have:

$$|x_n - u| \leq 2\varepsilon' \text{ and for all } i \in \{0,\dots,n-1\}: |x_i| \leq \varepsilon.$$

Then

$$N(x-(0,\dots,0,u)) \leq (\max_{0\leq i\leq n-1} \varepsilon^{\frac{1}{2i+1}}) \vee (2\varepsilon')^{\frac{1}{2n+1}} \leq (2\varepsilon')^{\frac{1}{2n+1}}$$

and that is $\alpha''' > 0$ such that:

$$\int_a^b \Phi(x;\ (0,\dots,0,u))\,du \geq \alpha''' \varepsilon'^{1-\frac{2d}{2n+1}} = \alpha''' \varepsilon'^{-\frac{n^2-2}{2n+1}}.$$

The remainder of the proof is now omitted.

Remark 1) We have not been able to extend the test in theorem 1.1 to the case where **O** is replaced by another point out of the line $\{0\}^n \times \mathbf{R}$.

2) It seems difficult to derive estimates on the hitting probabilities of parallelepipeds such $\{x \in \mathbf{R}^{n+1}: a \leq x_k \leq b,\ \forall i \neq k\, |x_i| \leq \varepsilon^{2i+1}\}$ for $k \in \{0,\dots,n-1\}$ and we do not know whether an integral test may be written for the sets $\{x \in \mathbf{R}^{n+1}: x_k \geq 0,\ \sum_{i\neq k} |x_i|^{\frac{2}{2i+1}} \leq f(x_k)^2\}$, $k \in \{0,\dots,n-1\}$.

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