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Decomposition of the canonical representation of $W(1)\times \mathrm{End}[m]$ on $\Lambda(m)$

Delicated to Professor Takashi Hirai on his 60th birthday

By

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Introduction. There are several ways to obtain finite dimensional irreducible representations of the general linear group $GL(n, \mathbf{C})$, or equivalently of its Lie algebra $\mathfrak{gl}(n, \mathbf{C})$. Among them the method introduced by Schur and Weyl is classical. They considered the m-fold tensor product $V^{\otimes m}$ of the defining representation of $GL(V)$ $(V=\mathbf{C}^n)$, and the action of the permutation group $\mathfrak{S}_m$ on it. These actions are commutative each other, and $V^{\otimes m}$ decomposes multiplicity freely as a $GL(V)\times\mathfrak{S}_m$-module:

$$V^{\otimes m}=\sum_D \rho_D\otimes\sigma_D,$$

where D runs over Young diagrams of size m with depth at most n,and ρ_D (resp. σ_D) is an irreducible representation of $GL(n)$ (resp. $\mathfrak{S}_m$). Furthermore, through the decomposition, ρ_D determines σ_D and vice versa (e.g., see [1]). If m varies in the set of non-negative integers, each irreducible representation of $GL(n, \mathbf{C})$ appears in this decomposition up to multiplication by a suitable power of the determinant character.

The Cartan-type Lie algebras are $\mathbf{Z}$-graded, simple, infinite-dimensional Lie algebras, whose properties and representations have been discussed extensively. I.A.Kostrikin ([6]) proved that all the finite type graded representations are either representations of height 1 (in the sense of A. N. Rudakov [11]) or their conjugate except the algebra $W(1)$. In the case of $W(1)$, Kostrikin gave models of all the irreducible graded modules of finite type with one-dimensional homogeneous components. On the other hand, K. Nishiyama ([8]) considered Schur-Weyl duality for the natural representation of Cartan-type Lie algebra $W(n)$. In particular, he suggested to use End $[m]$ insted of $\mathfrak{S}_m$, which is a semigroup consisted of all the mappings from a finite set $[m]=\{1, 2, \cdots, m\}$ into itself.

For Cartan-type Lie *superalgebras*, which is a "superanalogue" of Cartan-type Lie algebras, we also want to have an analogue of the Schur-Weyl duality. As a first step, we consider one of Cartan-type Lie superalgebras: Lie superalgebra of all the superderivations on the Grassmann algebra $\Lambda(n)$, which is denoted by $W(n)$ (see, e.g., [4]). In [9] and [10], using the semigroup End $[m]$, the author and Nishiyama have determined the

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commutant algebra of $W(n)$ in the m-fold tensor product of the natural representation under the condition $m \leq n$.

Let us explain it more precisely. Let ψ be the natural representation of $W(n)$ on $\Lambda(n)$ and φ be a representation of End$[m]$ on $\otimes^m \Lambda(n)$ (the definition is given in Section 1). If $m \leq n$, then the commutant algebra of $\psi^{\otimes m}(W(n))$ in End$(\otimes^m \Lambda(n))$ is the algebra generated by $\varphi(\mathrm{End}[m])$ in End$(\otimes^m \Lambda(n))$ ([10]). Moreover, if $n=1$ or $(n, m)=(2, 3)$, the same conclusion also holds (see [9]).

Along the idea of Schur and Weyl, we want to decompose the space $\otimes^m \Lambda(n)$ as $W(n) \times \mathrm{End}[m]$-module. However, in our case, the representation of $W(n)$ on $\otimes^m \Lambda(n)$ is not semisimple, and it seems difficult to decompose it. Therefore we are forced to consider the quotient representations. Nishiyama suggests the following conjectures:

Conjecture 1. Let $\rho \otimes \sigma$ be a finite dimensional irreducible representation of $W(n) \times \mathrm{End}[m]$. Then we have

$$\dim \mathrm{Hom}_{W(n) \times \mathrm{End}[m]}(\otimes^m \Lambda(n), \rho \otimes \sigma) \leq 1.$$

Conjecture 2. Put $\mathfrak{R}_{W(n)}(U, \rho) := \{\rho \in W(n)^\wedge | \mathrm{Hom}_{W(n)}(U, \rho) \neq 0\}$ and $\mathfrak{R}_{\mathrm{End}[m]}(U) := \{\sigma \in \mathrm{End}[m]^\wedge | \mathrm{Hom}_{\mathrm{End}[m]}(U, \sigma) \neq 0\}$. Then for any $\rho \in \mathfrak{R}_{W(n)}(\otimes^m \Lambda(n))$, there is one and only one $\sigma \in \mathfrak{R}_{\mathrm{End}[m]}(\otimes^m \Lambda(n))$ such that

$$\dim \mathrm{Hom}_{W(n) \times \mathrm{End}[m]}(\otimes^m \Lambda(n), \rho \otimes \sigma) \neq 0.$$

Therefore, the following mapping is a bijection:

$$W(n)^\wedge \supseteq \mathfrak{R}_{W(n)}(\otimes^m \Lambda(n)) \ni \rho \longleftrightarrow \sigma \in \mathfrak{R}_{\mathrm{End}[m]}(\otimes^m \Lambda(n)) \subseteq \mathrm{End}[m]^\wedge,$$

where $W(n)^\wedge$ (resp. End$[m]^\wedge$) is the set of all the irreducible modules of $W(n)$ (resp. End$[m]$).

For general integer n, we could not prove the above conjectures, but in the simplest case $n=1$, we have affirmative results (Theorems 3.1, 3.2 and 3.3). These are main results of this article.

Let us describe the contents of each section briefly. In the first section, we give the basic notations and preliminary results. In the second section, we give the decomposition of $\otimes^m \Lambda(1)$ as $W(1) \times \mathfrak{S}_m$-module (Proposition 2.3). In the third section, we give the decomposition of $\otimes^m \Lambda(1)$ as $W(1) \times \mathfrak{S}_m$-module (Theorem 3.1); and the decomposition of $\otimes^m \Lambda(1)$ as $\mathfrak{gl}(1) \times \mathrm{End}[m]$-module (Theorem 3.2). In Theorem 3.3 we give affirmative answers to Conjectures 1 and 2 in the case of $n=1$. In the last section, we give models of some indecomposable modules of $W(1)$.

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article, they have given many suggestions.

1. Notations and Preliminaries

1.1 Symmetric group $\mathfrak{S}_m$. In this article, we denote by D a Young diagram, by $(\lambda_1, \lambda_2, \cdots \lambda_k)$ its associated partition and by $B(D)$ or by B one of Young tableaux of shape D. We denote by $\mathfrak{S}_m$ the symmetric group of degree m. For a Young tableau B of size m, we put

$$P_B := \{g \in \mathfrak{S}_m : g \text{ preserves each row of } B\},$$

$$Q_B := \{g \in \mathfrak{S}_m : g \text{ preserves each column of } B\},$$

$$a_B := \sum_{g \in P_B} g ; \qquad b_B := \sum_{g \in Q_B} \operatorname{sgn}(g) g.$$

The Young symmetrizer c_B of B is defined by $c_B := a_B b_B$. Denote the group ring of $\mathfrak{S}_m$ by $\mathfrak{R} := \mathbf{C}\mathfrak{S}_m$. It is well-known that $\mathfrak{R}_B := \mathfrak{R} \cdot c_B$ is an irreducible $\mathfrak{S}_m$-module, Further, $\mathfrak{R}_B \cong \mathfrak{R}_{B'}$ as $\mathfrak{S}_m$-modules if and only if Young diagrams of B and B' coincide. We denote this $\mathfrak{S}_m$-module by σ_D.

A Young tableau B is called *standard* if the numbers in each row of B are increasing from left to right and the ones in each column are increasing from top to bottom. For example, for Young diagram $D = (3, 2)$, we list all the standard tableaux of shape D below.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}$$

Denote the set of all Young tableaux of shape D by $\mathfrak{B}(D)$ and the set of all standard Young tableaux by $\mathfrak{B}^s(D)$.

Lemma 1.1. (1) *The number of standard tableaux of shape D is equal to the dimension of the irreducible representation σ_D of $\mathfrak{S}_m$ corresponding to D.*

(2) *If B and B' are two different standard tableaux of shape D, then $a_B b_{B'} = b_{B'} a_B = c_B c_{B'} = 0$.*

(3) *Let $\mathfrak{B}^s(D) = \{B_1, B_2, \cdots, B_r\}$ be the set of all the standard tableaux of shape D, and put $\mathfrak{a}(D) := \sum_{B \in \mathfrak{B}(D)} \mathfrak{R} c_B$; then $\mathfrak{a}(D) = \mathfrak{R} c_{B_1} \oplus \cdots \oplus \mathfrak{R} c_{B_r}$.*

For proof of the above lemma, see [2], [3], $\cdots$, for example.

1.2. Cartan-type Lie superalgebra $W(n)$ and its natural representation. Let $\Lambda(n)$ be a Grassmann algebra over $\mathbf{C}$ in n variables $\{\xi_1, \xi_2, \cdots, \xi_n\}$, and Λ^k be the subspace of k-homogeneous elements of $\Lambda(n)$. Let $W(n)$ be the set of all the superderivations of $\Lambda(n)$. It becomes naturally a Lie superalgebra. Every superderivation $D \in W(n)$ can be written in the form

$D=\sum_{i=1}^{n} P_i \partial/\partial\xi_i$ with $P_i \in \Lambda(n)$ $(1 \leq i \leq n)$, where $\partial/\partial\xi_i$ is a superderivation of degree 1 defined by $(\partial/\partial\xi_i)\xi_j=\delta_{ij}$. By definition, the Lie superalgebra $W(n)$ acts on Grassmann algebra $\Lambda(n)$ as follows: for any homogeneous $D \in W(n)$ and $\forall\, \xi_{i1}\wedge\cdots\wedge\xi_{ir}$,

$$D(\xi_{i1}\wedge\cdots\wedge\xi_{ir})=\sum_{s=1}^{r}(-1)^{(s-1)\deg D}\xi_{i1}\wedge\cdots\wedge D(\xi_{is})\wedge\cdots\wedge\xi_{ir}.$$

We call it the *natural representation* of $W(n)$, and denote it by $(\psi, \Lambda(n))$ or simply by ψ.

Let us consider the m-fold tensor product $\otimes^m \Lambda(n)$ of $(\psi, \Lambda(n))$. We have a natural isomorphism as $W(n)$-modules:

$$\otimes^m \Lambda(n) \simeq \Lambda[\xi_{ij} | 1 \leq i \leq n,\ 1 \leq j \leq m] =: \Lambda(n, m),$$

where $\Lambda[\xi_{ij}|1 \leq i \leq n, 1 \leq j \leq m]$ is a Grassmann algebra generated by $\{\xi_{ij}(1 \leq i \leq n, 1 \leq j \leq m)\}$. In the following, we identify $\otimes^m\Lambda(n)$ with $\Lambda(n, m)$. By means of tensor product, $W(n)$ is imbedded into $\mathrm{End}(\otimes^m\Lambda(n)) \cong \mathrm{End}(\Lambda(n, m))$. More precisely, an element

$$D=\sum_{i=1}^{n} P_i(\xi_1, \cdots, \xi_n)\frac{\partial}{\partial\xi_i} \in W(n)$$

corresponds to a superderivation

$$\psi^{\otimes m}(D)=\sum_{i=1}^{n}\sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha})\frac{\partial}{\partial\xi_{i\alpha}} \in \mathrm{Der}\Lambda(n, m)$$

via the m-fold tensor product $\psi^{\otimes m}$ of ψ.

In particular, in the case of $n=1$, we have

$$W(1)=\left\langle \frac{\partial}{\partial\xi}, \xi\frac{\partial}{\partial\xi} \right\rangle, \ \deg\left(\frac{\partial}{\partial\xi}\right)=1,\ \deg\left(\xi\frac{\partial}{\partial\xi}\right)=0.$$

For convenience, we use the isomorphism $\Lambda(1, m) \cong \Lambda[\xi_1, \xi_2, \cdots, \xi_m]$. Then we have

$$D_{-1}:=\psi^{\otimes m}\left(\frac{\partial}{\partial\xi}\right)=\sum_{i=1}^{m}\frac{\partial}{\partial\xi_i}, \quad D_0:=\psi^{\otimes m}\left(\xi\frac{\partial}{\partial\xi}\right)=\sum_{i=1}^{m}\xi_i\frac{\partial}{\partial\xi_i}.$$

Obviously, $D_{-1}(\Lambda^k) \subseteq \Lambda^{k-1}$, $D_0(\Lambda^k) \subseteq \Lambda^k$ for any k.

Denote by $[m]$ the set $\{1, 2, \cdots, m\}$ of integers, and put $\mathrm{End}[m]=\{\varphi : [m] \to [m]\}$ the set of all the maps from $[m]$ into itself. By composition of maps, $\mathrm{End}[m]$ becomes a semigroup with unit, whose group elements form the permutation group $\mathfrak{S}_m$ of degree m. We call $\mathrm{End}[m]$ the *permutation semigroup*. Denote its semigroup ring by $\mathfrak{C}_m$. An element $\varphi \in \mathrm{End}[m]$ acts on $\Lambda(n, m)$ as $(\varphi P)(\xi_{i,j})=P(\xi_{i,\varphi(j)})$ $(P \in \Lambda(n, m))$ and we extend it to $\mathfrak{C}_m$ by linearity (see [10]). Thus, we have a representation of $\mathfrak{C}_m$ on $\Lambda(n, m)$. Denote the image algebra of this representtion by $\mathscr{E}_m \subset \mathrm{End}\,\Lambda(n, m)$.

Let $\mathscr{C}_m$ denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in End $\Lambda(n, m)$. We have the following fundamental results.

Theorem 1.2. ([9], [10]). *Assume $m \leq n$, or $n=1$, or $(n, m) = (2, 3)$. The commutant algebra $\mathscr{C}_m$ of $\psi^{\otimes m}(W(n))$ coincides with the representation image $\mathscr{E}_m$ of the semigroup ring $\mathfrak{E}_m$ of the permutation semigroup* End$[m]$:

$$\mathscr{E}_m = \mathscr{C}_m.$$

Theorem 1.3. ([10]). *The bicommutant algebra of the m-fold tensor product $\psi^{\otimes m}$ of the natural representation ψ of $W(1)$ is equal to the image $\psi^{\otimes m}(U(W(1)))$ of the universal enveloping algebra $U(W(1))$.*

1.3. Composition series. Let R be an arbitrary ring with unit and V an R-module. A descending chain

$$V = V_1 \supset V_2 \supset \cdots \supset V_k \supset V_{k+1} = (0)$$

of submodules of V is called a composition series of V if all the quotient modules V_i/V_{i+1} are irreducible. These quotient modules V_i/V_{i+1} are called the composition factors. Two composition series are said to be equivalent if they have the same length and if the factors can be paired off in such a way that the corresponding factors are R-isomorphic.

Lemma 1.4. (Jordan-Hölder). *If an R-module V possesses a composition series, any two composition series of V are equivalent.*

An R-module $V \neq \{0\}$ is said to be indecomposable if it is impossible to express V as a direct sum of two non-trival submodules.

2. The Decomposition of Λ(m) as $W(1) \times \mathfrak{S}_m$-module

In this section, we give the decomposition of $\Lambda(m)$ as $W(1) \times \mathfrak{S}_m$-module explicitly. The decomposition of $\Lambda(m)$ according to the action of $\mathfrak{S}_m$ is well-known (e.g., see [5, p. 55]). However, we give the explicit structure of the decomposition here because it is used in the subsequent sections. Let us begin with the determination of the subspace $c_B\Lambda(m)$.

Lemma 2.1. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of m corresponding to a Young diagram D.*

(1) *If D is not of hook type, then we have $c_B\Lambda(m) = 0$ for $B \in \mathscr{B}(D)$.*

(2) *Let D be a hook type diagram and $\lambda = (\lambda_1, 1, \cdots, 1)$. We have $c_B\Lambda^k(m) \neq 0$ $(B \in \mathscr{B}(D))$ if and only if $k = \mu_1$ or $k = \mu_1 - 1$, where $\mu_1 + \lambda_1 = m+1$.*

(3) *Under the same assumption as in (2), we have*

$$c_B\Lambda(m) = c_B\Lambda^{\mu_1}(m) \oplus c_B\Lambda^{\mu_1-1}(m)\ (B \in \mathscr{B}(D)).$$

The both subspaces are of one-dimension. Moreover, we have an explicit description of each subspace as follows:

$$c_B\Lambda^{\mu_1}(m) = \mathbf{C}a_B(\xi_{b(1,1)} \wedge \cdots \wedge \xi_{b(\mu_1,1)}),\ c_B\Lambda^{\mu_1-1} = \mathbf{C}a_B(D_{-1}(\xi_{b(1,1)} \wedge \cdots \wedge \xi_{b(\mu_1,1)})$$

where $b(i, j)$ *denotes the number in the* (i, j) *-box in the Young tableau* $B \in \mathscr{B}(D)$.

Proof. Take $B \in \mathscr{B}(D)$ and put

$$a_i := \sum_{g \in P_{B,i}} g,\ b_j := \sum_{g \in Q_{B,j}} \operatorname{sgn}(g) g,$$

where

$P_{B,i} := \{g | g \in \mathfrak{S}_m,\ g$ preserves the i-th row and fixes the other letters$\}$,
$Q_{B,j} := \{g | g \in \mathfrak{S}_m,\ g$ preserves the j-th column and fixes the other letters$\}$.

Then we can write down a_B and b_B as follows:

$$a_B = a_k a_{k-1} \cdots a_2 a_1, \quad b_B = b_s b_{s-1} \cdots b_2 b_1,$$

where s is the number of the column of the diagram D.

Note that, for any $\varphi \in Q_B$ and $\Xi \in \Lambda(m)$, we have $c_B\varphi(\Xi)$. Therefore, in order to calculate $c_B\Lambda(m)$, we only need to consider $c_B(\Xi)$ for the following type of Ξ with appropriate integers $h_1, h_2, \cdots, h_s$ after translation by a certain $\varphi \in Q_B$:

$$\Xi = (\xi_{b(1,1)} \wedge \cdots \wedge \xi_{b(h_1,1)}) \wedge (\xi_{b(1,2)} \wedge \cdots \wedge (\xi_{b(h_2,2)}) \wedge \cdots.$$

Let us consider (1). Since D is not of hook type, $\lambda_2 \geq 2$. Let $\mu = (\mu_1, \mu_2, \cdots, \mu_s)$ be the transposition of λ (see. e.g., [7], § 1). Note that $\mu_2 \geq 2$. If $h_i \leq \mu_i - 2$ for some i, then it is easy to see that $b_i(\Xi) = 0$, hence $c_B(\Xi) = 0$. So we assume that $\mu_i \geq h_i \geq \mu_i - 2$. In particular, we assume that $h_2 \geq \mu_2 - 2 \geq 1$. We consider the expansion of $b_B(\Xi)$. For any term of this expansion (e.g., $\xi_{iu} \wedge \cdots \wedge \xi_{iv}$), if there are two subscripts (e.g., j_u, j_v) from the same row, then the result of the operator a_B acting on this term is zero. Hence we only consider the terms which do not have two subscripts from the same row. Therefore we can assume that $h_3 = h_4 = \cdots = 0$, hence $\mu_3 \leq 1$.

If $h_2 \geq 2$, then $h_1 + h_2 \geq \mu_1 - 1 + 2 = \mu_1 + 1$. This means that, in each term, there are at least two subscripts from the same row, and so $c_B(\Xi) = 0$. Therefore we can assume $\mu_2 - 1 \leq h_2 \leq 2$, which means $\mu_2 < 3$. Combining with the assumpion $\lambda_2 \geq 2$, we need only to consider the situation $\mu_2 = 2$ and $h_2 = 1$.

(a) If $h_1 = \mu_1$, then in each term in the expansion of $b_B(\xi_{b(1,1)} \wedge \cdots \wedge \xi_{b(\mu_1,1)} \wedge \xi_{b(1,2)}$, there are at least two subscripts from the same row, hence $c_B(\Xi) = 0$.

(b) If $h_1 = \mu_1 - 1$, then $b_1(\xi_{b(1,1)} \wedge \cdots \wedge \xi_{b(\mu_1-1,1)} \wedge \xi_{b(1,2)}) = D_{-1}(\xi_{b(1,1)} \wedge \cdots \wedge$

$\xi_{b(\mu_1,1)})\wedge\xi_{b(1,2)})$, hence $c_B(\Xi)=a_Bb_B(\Xi)=D_{-1}a_Bb_2(\xi_{b(1,1)}\wedge\cdots\wedge\xi_{b(\mu_1,1)}\wedge\xi_{b(1,2)})$. By the same reason as in (a), we have $c_B(\Xi)=0$.

For (2) and (3), we can do the similar calculation and easily get the desired results.

Next we consider the $\mathfrak{S}_m$-module strucure.

Lemma 2.2. *Let $\lambda=(\lambda_1, 1\cdots, 1)$ be a partition of m and D the corresponding Young diagram of hook-type. The subspace*

$$\sum_{r=1}^{s} c_{B_r}(\Lambda^{\mu_1}(m)) \qquad (\lambda_1+\mu_1=m+1)$$

is irreducible under $\mathfrak{S}_m$, where $\mathscr{B}^s(D)=\{B_1, B_2\cdots, B_s\}$ are the set of all the standard Young tableaux of shape D.

Proof. We use the notations in Lemma 2.1. By Lemma 2.1, we have

$$c_B\Lambda^{\mu_1}(m)=\mathbf{C}a_B(\xi_{b(1,1)}\wedge\cdots\wedge\xi_{b(h_1,1)}).$$

Since $c_{B_i}c_{B_j}=\text{const}\cdot\delta_{ij}c_{B_i}$ (const$\neq 0$), $\{c_{B_i}(\xi_{b_i(1,1)}\wedge\cdots\wedge\xi_{b_i(\mu_1,1)})\,|i=1, 2, \cdots, s\}$ are linearly independent. Put

$$\mathscr{H}_D:=\langle c_{B_i}(\xi_{b_i(1,1)}\wedge\cdots\wedge\xi_{b_i(\mu_1,1)}|\mathrm{i}=1, 2, \cdots, \mathrm{s}\rangle.$$

Then $\{c_{B_i}(\xi_{b_i(1,1)}\wedge\cdots\wedge\xi_{b_i(\mu_1,1)})\,|i=1, 2, \cdots, \mathrm{s}\}$ is a basis of $\mathscr{H}_D$.

From $\sum_{i=1}^{s}\mathfrak{R}c_{B_i}=\sum_{i=1}^{s}c_{B_i}\mathfrak{R}$, we know that $\mathscr{H}_D$ is an invariant space of $\mathfrak{S}_m$. We prove the space $\mathscr{H}_D$ is irreducible under $\mathfrak{S}_m$. Take an element $x=\sum_i k_i c_{B_i}(\xi_{b_i(\mu_1,1)}\wedge\cdots\wedge\xi_{b_i(\mu_1,1)})\in\mathscr{H}_D$. If $k_h\neq 0$, then

$$c_{Bh}(x)=c_{B_h}(\sum_i k_i c_{B_i}(\xi_{b_i(1,1)}\wedge\cdots\wedge\xi_{b_i(\mu_1,1)})=k_h c_{B_h}(\xi_{b_h(1,1)}\wedge\cdots\wedge(\xi_{b_h(\mu_1,1)})\neq 0.$$

Clearly $c_{B_h}(\xi_{b_h(1,1)}\wedge\cdots\cdots\xi_{b_h(\mu_1,1)})$ generates $\mathscr{H}_D$ as an $\mathfrak{S}_m$-module, and we have done.

Using the above lemma, we can decompose $\Lambda(m)$ as a $W(1)\times\mathfrak{S}_m$-module.

Proposition 2.3. *Let $W(1)$ and $\mathfrak{S}_m$ act on $\Lambda(\mathrm{m})$ naturally. We have the followng decomposition of $\Lambda(m)$:*

$$\Lambda(m)=\bigoplus_D\bigoplus_{B\in\mathscr{B}^s(D)} c_B\Lambda(m),$$

where D runs over the Young diagrams of hook type corresponding to the partition $(k, 1, \cdots, 1)=(k, 1^{m-k})$ $(k=1, 2, \cdots m)$ and $\mathscr{B}^s(D)$ is the set of all the standard tableaus of shape D.

For any Young tableau B, $c_B\Lambda(m)$ is an indecomposable module of $W(1)$ with composition series

$$c_B\Lambda(m) \supseteq c_B\Lambda^{m-k}(m) \supseteq \{0\}; \quad \dim c_B\Lambda(m)=2, \quad \dim c_B\Lambda^{m-k}(m)=1.$$

For any D, the sum $\sum^{\oplus}_{B\in\mathscr{B}^s(D)} c_B\Lambda^{m-k+1}(m)$ *is an irreducile module of* $\mathfrak{S}_m$ *of type D. Furthermore,* $D_{-1}(\sum^{\oplus}_{B\in\mathscr{B}^s(D)} c_B\Lambda^{m-k+1}(m)) = \sum^{\oplus}_{B\in\mathscr{B}^s(D)} c_B\Lambda^{m-k}(m)$ *is also an irreducible module of the same type D.*

Proof. Let D be a Young diagram of hook type with partition $(k, 1, \cdots, 1)$. By the above lemmas, for any $B\in B(D)$, we have

$$c_B\Lambda(m) = c_B\Lambda^{m-k}(m) \oplus c_B\Lambda^{m-k+1}(m).$$

Let $V_1(B) := c_B\Lambda(m)$ and $V_2(B) := c_B\Lambda^{m-k}(m)$. By a simple calculation we have $D_{-1}(V_1) = V_2$, and $D_{-1}(V_2) = 0$ because $D^2_{-1} = 0$. Clearly, $D_0(V_i) \subseteq V_i (i=1, 2)$, so V_1 and V_2 are invariant spaces of $W(1)$. From the proof of Lemma 2.2, we can deduce that $\dim(V_2(B))=1$, therefore V_2 is an irreducible module of $W(1)$. On the other hand, $\dim \ker D_{-1}|_{v_1(B)} = \dim V_2(B) = 1$, we have $V_1(B)$ is an indecomposable module of $W(1)$. Again from the proof of Lemma 2.2, $\dim(V_1(B)/V_2(B))=1$, hence $\{0\} \subseteq V_2(B) \subseteq V_1(B)$ is a composition series of $W(1)$.

The rest of the proof is clear from Lemmas 2.1 and 2.2.

3. The duality between $W(1)$ and $\mathrm{End}[m]$

Along the idea of Schur and Weyl, we want to obtain the decomposition of $\otimes^m\Lambda(n)$ as a $W(n)\times\mathrm{End}[m]$-module. Because the representation of $W(n)$ on $\otimes^m\Lambda(n)$ is not semisimple, it seems difficult to do it. However, for the case $n=1$, thanks to Proposition 2.3, we can obtain the decompositions of $\otimes^m\Lambda(1)$ as a $W(1)\times\mathfrak{S}_m$-module, and as a $W_0(1)\times\mathrm{End}[m]=\mathfrak{gl}(1)\times\mathrm{End}[m]$ -module. Using these two decompositions we can get the further result which is similar to the classical duality between $GL(n, \mathbf{C})$ and $\mathfrak{S}_m$. To be more precise, for any $\rho\otimes\sigma\in W(1)^\wedge\otimes\mathrm{End}[m]^\wedge$, where $W(1)^\wedge$ (resp. $\mathrm{End}[m]^\wedge$) is the set of equivalence classes of irreducible finite dimensional representations of $W(1)$ (resp. $\mathrm{End}[m]$), we have

$$\dim \mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m), \rho\otimes\sigma) \leq 1.$$

If the equality holds,

$$\dim \mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m), \rho\otimes\sigma) = 1,$$

then $\rho\in W(1)^\wedge$ uniquely determines $\sigma\in\mathrm{End}[m]^\wedge$ satisfying the above equality. Based on these explicit calculations and results, Conjectures 1 and 2 are suggested by Nishiyama for general n and m.

Let us begin to state the main results of this paper.

Let D be a Young diagram corresponding to the partition $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ and B be one of the Young tableaux of shpae D. Put $V_1(B) = V_2(B) = 0$ if D

is not of hook type. If D is one of the hook type diagrams corresponding to the partition $(\lambda_1, 1, \cdots, 1)$, then $V_1(B) = c_B\Lambda^{k-1}(m)$ $(B \in B^s(D))$ and $\dim c_B\Lambda^k(m) = \dim c_B\Lambda^{k-1}(m) = 1$. Using these facts, we have

Theorem 3.1. (1) *As $W(1)$-modules, we have isomorphisms:*

$$V_1(B) \cong \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \Lambda(m)), \quad V_2(B) \cong \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \Lambda^{k-1}(m)),$$

where B is a Young tableaux of shape D with partition $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ and σ_D is the irreducible representation of $\mathfrak{S}_m$ with diagram D.

(2) *Let $\tilde{\rho}_{D_{m,k}} = \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_{D_{m,k}}, \Lambda(m))$. As a $W(1)$-module, $\tilde{\rho}_{D_{m,k}}$ is an indecomposable module and has a composition series:*

$$\tilde{\rho}_{D_{m,k}} \cong V_1(B) \supsetneq V_2(B) \supset \{0\},$$

where $D_{m,k}$ denotes the hook type Young diagram with partition $(m-k+1, 1, \cdots, 1)$ and $B \in \mathfrak{B}(D_{m,k})$. Furthermore, the composition factors are 1-dimensional modules.

(3) *The $W(1) \times \mathfrak{S}_m$-module $\Lambda(m)$ has the following decomposition:*

$$\Lambda(m) \cong \bigoplus_{k=1}^{m} \tilde{\rho}_{D_{m,k}} \otimes \sigma_{D_{m,k}}.$$

Proof. We want to prove (1). As a realization of σ_D, we take $\mathfrak{R}c_B$ $(B \in \mathfrak{B}(D))$. If D is not of hook type, then the both hands side are zero. Therefore we can assume that $D = D_{m,k}$. We define $W(1)$-equivariant map $G:\mathrm{Hom}_{\mathfrak{S}_m}(\mathfrak{R}c_B, \Lambda(m)) \to V_1(B)$ by $G(f) = f(c_B)$ $(\forall f \in \mathrm{Hom}_{\mathfrak{S}_m}(\mathfrak{R}c_B, \Lambda(m))$. It is clear that G is an injection. Take a non-zero element $x_1 \in c_B\Lambda(m)$ and define $f_1 \in \mathrm{Hom}_{\mathfrak{S}_m}(\mathfrak{R}c_B, \Lambda(m))$ by $f_1(\varphi c_B) = \varphi \cdot x_1$ $(\forall \varphi \in \mathfrak{R})$. By definition, we have $G(f_1) = x_1$, hence G is surjective. This proves the first isomorphism in (1). For the second isomorphism in (1), it is enough to note the following facts:

$$D_{-1}\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \Lambda^{k-1}(m)) = 0, \; D_{-1}(V_2(B)) = 0.$$

Combining Proposition 2.3 and (1), we can easily complete the proof of (2) and (3).

The commutant algebra of $W(1)$ in the representation $\Lambda(m) = \otimes^m\Lambda(1)$ is $\mathscr{E}_m$, which is the representation image of the semigroup ring of $\mathrm{End}[m]$. Following the idea of the classical situation, we want to get the decomposition of $\Lambda(m)$ as a module of $W(1) \times \mathscr{E}_m$. As an intermediate step, we describe the decomposition of $\Lambda(m)$ as a module of $W(1) \times \mathrm{End}[m] \cong \mathfrak{gl}(1) \times \mathrm{End}[m]$-module.

Put $U_1(k) := \Lambda^k(m)$, and $U_2(k) := \mathrm{Ker}(D_{-1}|_{\Lambda^k(m)})$ $(0 \leq k \leq m)$. Then $U_1(k)$ is an indecomposable $\mathrm{End}[m]$-module which we denote by $\sigma_{m,k}$. We denote the irreducible representation of $\mathfrak{gl}(1)$ with weight k by ρ_k.

Theorem 3.2. *We use the same notations as in Theorem* 3.1. *As a* $\mathfrak{gl}(1)\times\mathrm{End}[m]$*-module, we have the decomposition:*

$$\Lambda(m)=\bigoplus_{k=0}^{m}\rho_k\otimes\sigma_{m,k}.$$

If $1\leq k\leq m-1$, $\sigma_{m,k}$ *has a composition series* $U_1(\mathrm{k})\supseteq U_2(k)\supset\{0\}$. *For* $k=0$, m, $\sigma_{m,k}$ *is irreducible,*

Proof. Assume that $1\leq k\leq m$. Because D_{-1} and $\mathrm{End}[m]$ commutes with each other, $U_2(k)=\ker(D_{-1}|_{\Lambda^k(m)})$ is an invariant subspace of $\mathrm{End}[m]$. By the relation $\ker(D_{-1}|_{\Lambda^k})=\mathrm{Im}(D_{-1}|_{\Lambda^{k+1}})$ and Proposition 2.3, $U_2(k)$ is irreducible under $\mathfrak{S}_m$, hence it is irreducible under $\mathrm{End}[m]$. They only thing to proved is that $U_1(k)/U_2(k)$ is irreducible under $\mathrm{End}[m]$. By the definition of $U_1(k)$ and $U_2(k)$, $U_1(k)/U_2(k)$ is linearly isomorphic to $U_2(k-1)$ by operator D_{-1} (see [10], Lemma 3.1), furthermore, it is an intertwining operator between $\mathrm{End}[m]$-module $U_1(k)/U_2(k)$ and $\mathrm{End}[m]$-module $U_2(k-1)$. Using the irreducibility of $\mathrm{End}[m]$-module $U_2(k-1)$, we complete the proof of the theorem.

We denote the irreducible representation of $\mathrm{End}[m]$ on $\Lambda^k(m)/U_2(k)$ by $\sigma_{D_{m,k}}(k\geq 1)$ and the irreducible representation of $W(1)$ on $V_1(B(D_{m,k}))/V_2(B(D_{m,k}))$ by $\rho_{D_{m,k}}(k\geq 1)$.
Then we have the following duality.

Theorem 3.3. (1) *For any* $\rho\otimes\sigma\in W(1)^\wedge\times\mathrm{End}[m]^\wedge$, *we have*

$$\dim\mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m),\rho\otimes\sigma)\leq 1.$$

The equality holds if and only if $\rho\otimes\sigma=\rho_{D_{m,k}}\otimes\sigma_{D_{m,k}}$ *for some* $1\leq k\leq m$.

(2) *Put*

$$\mathfrak{R}_{W(1)}(\Lambda(m)):=\{\rho\in W(1)^\wedge|\mathrm{Hom}_{W(1)}(\Lambda(m),\rho)\neq(0)\},$$
$$\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(m)):=\{\sigma\in\mathrm{End}[m]^\wedge|\mathrm{Hom}_{\mathrm{End}[m]}(\Lambda(m),\sigma)\neq(0)\}.$$

For any $\rho\in\mathfrak{R}_{W(1)}(\Lambda(m))$, *there is one and only one* $\sigma\in\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(m))$ *such that*

$$\dim\mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m),\rho\otimes\sigma)=1.$$

Therefore, the following mapping is a bijection:

$$W(1)^\wedge\supseteq\mathfrak{R}_{W(1)}(\Lambda(m))\ni\rho\longleftrightarrow\sigma\in\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(m))\subseteq\mathrm{End}[m]^\wedge.$$

Furthermore, this correspondence is explicitly realized by $\rho_{D_{m,k}}\longleftrightarrow\sigma_{D_{m,k}}$.

Proof. We keep to the notation in § 3. From Propoisition 2.3, we have $\Lambda(m)=\sum_{k=0}^{m}(\mathcal{H}_{D_{m,k}}\oplus U_2(k))$ (for the definition of $\mathcal{H}_{D_{m,k}}(k\geq 1)$, see the proof of Lemma 2.2 and we set $\mathcal{H}_{D_{m,0}}=\{0\}$). Note that $D_{-1}(\mathcal{H}_{D_{m,k+1}})=U_2(k)$.

Assume that $\dim\mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m),\rho\otimes\sigma)\neq 0$. We take a non-zero element $f \in \mathrm{How}_{W(1)\times\mathrm{End}[m]}(\Lambda(m), \rho\otimes\sigma)$. As $W_0(1) \cong \mathfrak{gl}(1)$-module, $\Lambda(m)$ decomposed into the direct sum of one-dimensional irreducible representations of weight $k(0 \leq k \leq m)$. Note that $\rho \in W(1)^{\wedge}$ is one-dimensional and completely determined by the weight of $W_0(1)$. Since $\Lambda(m)/\ker f \cong \rho\otimes\sigma \cong (\dim\sigma)\rho$ as a $W(1)$-module, ρ must be an irreducible module of $W(1)$ with highest weight $k(0\leq k\leq m)$. As a consequence, there exists a unique k such that $f(\Lambda^k(m)) \neq (0)$. We put $k=k(\rho)$.

Assume that $1\leq k(\rho)\leq \mathrm{m}-1$. Since f is $W(1)\times\mathrm{End}[m]$-equivariant, $\ker f$ is invariant under the action of $W(1)\times\mathrm{End}[m]$. Hence we have

$$U_2(k(\rho)) = D_{-1}(\mathcal{H}_{D_{m,k(\rho)+1}}) \subset D_{-1}(\Lambda^{k(\rho)+1}) \subset \ker f.$$

On the other hand, by Lemma 2.2, $\mathcal{H}_{D_{m,k(\rho)}}$ is an irreducible module of $\mathfrak{S}_m$, therefore, $\ker f \cap \mathcal{H}_{D_{m,k(\rho)}} = \{0\}$. Summarizing above, we know that for any $f \in \mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m), \rho\otimes\sigma)$,

$$\ker f = \sum_{k\neq k(\rho)} (\Lambda^k(m) \oplus D_{-1}(\mathcal{H}_{D_{m,k(\rho)+1}})),$$

hence,

$$\dim \mathrm{Hom}_{W(1)\times\mathrm{End}[m]}(\Lambda(m), \rho\otimes\sigma) = 1.$$

If $k(\rho)=0$, then by the same argument as above, we can conclude $\Lambda^0 = U_2(0) \subset \ker f$. This means $f=0$ and we get a contradiction.

Finally, assume $k(\rho)=m$. Since $\sum_{i=0}^{m-1}\Lambda^i(m)$ is invariant under $W(1)\times\mathrm{End}[m]$, $\Lambda^m \cong \Lambda(m)/\sum_{i=0}^{m-1}\Lambda^i(m)$ is irreducible under $W(1)\times\mathrm{End}[m]$. Therefore f is uniquely determined up to constant multiplication.

Note taht $\Lambda(m)/\ker f \cong U_2(k(\rho)-1)$ as $\mathrm{End}[m]$-module. This proves that $\rho\otimes\sigma \cong \rho_{D_{m,k}} \otimes \sigma_{D_{m,k}}$. (2) is obvious from the above arguments.

We call the correspondence in (2) *Howe's correspondence* for $W(1)\times\mathrm{End}[m]$.

Appendix: Indecomposable Representations of $W(1)$

In this appendix, for the sake of completeness, we prove that the representations of $W(1)$ on $\Lambda(m)$ give a model of one principal class of representations of $W(1)$. Let us begin with a general representation theory of $W(1)$.

Take a finite-dimensional indecomposable graded representation ρ of $W(1)$ on a representation space $V=V_0\oplus V_1$. Put $\mathrm{T}_\lambda := \lambda - \rho(D_0)$ and $V_\lambda := \{v \mid T_\lambda^k v = 0, \exists k \in \mathbf{Z}_{\geq 0}\}$, the generalized eigenvector space with eigenvalue λ. Then we have the generalized eigenspace decomposition of V:

$$V = \bigoplus_\lambda V_\lambda$$

Obviously, $\rho(D_0)(V_\lambda) \subseteq V_\lambda$. Let us consider the action of $\rho(D_{-1})$ on V_λ.

Since we have the relation $[D_0, D_{-1}] = -D_{-1}$ and $2D_{-1}^2 = [D_{-1}, D_{-1}] = 0$, so $T_\lambda\rho(D_{-1}) = \rho(D_{-1})T_{\lambda+1}$ Using induction, we get $\rho(D_{-1})T_\lambda^k = T_{\lambda-1}^k\rho(D_{-1})$.

Let λ be any generalized eigenvalue of $\rho(D_0)$. Clearly $\mathscr{H}_\lambda := \oplus_{k\in\mathbf{Z}}V_{\lambda+k}$ is invariant under $W(1)$. By the finite-dimensionality of V, we can assume that $\mathscr{H}_\lambda = V_\lambda \oplus V_{\lambda-1} \oplus \cdots \oplus V_{\lambda-\mu}$ with some $\mu \in \mathbf{Z}_{\geq 0}$. Again by the finite-dimensionality of V and the generalized eigenspace decomposition of V, we have $V = \mathscr{H}_{\lambda_1} \oplus \mathscr{H}_{\lambda_2} \cdots \oplus \mathscr{H}_{\lambda_r}$ By the indecomposability of the representation, we see that $r=1$ and conclude $V = \mathscr{H}_\lambda$.

Now, we assume $\rho(D_0)$ is semisimple. Then $V_{\lambda-s}$ $(0 \leq s \leq \mu)$ is the eigenspace with eigenvalue $\lambda - s$. In this case, we get a $W(1)$-invariant decomposition $V = (V_\lambda \oplus D_{-1}(V_\lambda)) \oplus (V'_{\lambda-1} \oplus V_{\lambda-2} \oplus \cdots V_{\lambda-\mu})$, where $V'_{\lambda-1}$ is any complement space of $D_{-1}(V_\lambda)$ in $V_{\lambda-1}$. By the indecomposability of $W(1)$, we conclude that $V = V_\lambda \oplus D_{-1}(V_\lambda)$. Using the similar argument, we can deduce that

$$V = \mathbf{C}v \oplus \mathbf{C}D_{-1}v.$$

Summarizing the above results, we have the following proposition.

Proposition A. *Let (ρ, V) be an indecomposable graded representation of $W(1)$, then*

$$V = V_\lambda \oplus V_{\lambda-1} \oplus \cdots \oplus V_{\lambda-\mu},$$

where μ is an integer. If moreover $\rho(D_0)$ acts semisimply on V, then V has the following decomposition:

$$V = \mathbf{C}_v \oplus \mathbf{C}\rho(D_{-1})v,$$

where v is an eigenvector of $\rho(D_0)$ with eigenvalue λ. If $\rho(D_{-1})v = 0$, then $V = \mathbf{C}v$, and so the representation is irreducible. If $\rho(D_{-1})\ v \neq 0$, then the representation is an indecomposable representation with composition series $V \supseteq V_1 \supseteq \{0\}$, where $V_1 = \mathbf{C}\rho(D_{-1})v$.

Combining the above proposition with results in § 3, we conclude that any indecomposable representation (ρ, V) of $W(1)$ for which $\rho(D_0)$ is semisimple with positive integral eigenvalue is realized in the tensor product $\Lambda(m)$ of the natural representation of $W(1)$ for a certain m.

For an arbitrary integer μ in the proposition, there really exists an indecomposable $W(1)$-module (ρ, V) such that $\rho(D_0)$ is not semisimple and $V = V_\lambda \oplus V_{\lambda-1} \oplus \cdots \oplus V_{\lambda-\mu}$.

Example. Let $V = \mathbf{C}^{2\mu+1}$ and $n(\lambda) = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$. Note that $n(\lambda)$ is not semisimple. An action of $W(1)$ on V is given as follows:

$$\rho(D_0)=\begin{bmatrix} n(\lambda) & 0 & 0 & \cdots & 0 & 0 \\ 0 & n(\lambda-1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & n(\lambda-2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & & \cdots & n(\lambda-\mu) \end{bmatrix},$$

$$\rho(D_{-1})=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ n(0) & 0 & 0 & \cdots & 0 & 0 \\ 0 & n(0) & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & & n(0) & 0 \end{bmatrix},$$

where $\rho(D_0)$ and $\rho(D_{-1})$ are $(2\mu+2)\times(2\mu+2)$ matrices. It is easy to check by calculation that (ρ, V) is an indecomposable module of $W(1)$ and has the properties just stated as above.

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