J. Math. Kyoto Univ. (JMKYAZ)
37-3 (1997) 531-546

Whittaker functions of generalized principal series on $SU(2,2)$

By

Takahiro HAYATA

1. INTRODUCTION

This is a continuation of the previous paper [1].

For the calculation of Whittaker functions of irreducible admissible representation on real semisimple Lie groups, the usage of the shift operator provides us a way to obtain their differential equations. In the case of the discrete series, many works have been carried out on various groups such as $Sp(2,\mathbf{R})$, $SU(2,1)$ $SU(2,2)$ and $SU(n,1)$ in [9, 13, 2, 11, 4], using the fact that the Whittaker vectors can be characterized by the kernel of some differential operators coming from the Schmid operator. Also in the case of principal series, differential equations have obtained in several cases: $Sp(2,\mathbf{R})$ in [7,8], $SU(2,2)$ in[1].

In this paper, we treat the irreducible generalized principal series representation π induced from a representation of the standard maximal cuspidal parabolic subgroup of $SU(2,2)$. This representation is large in the sense of Vogan ([12]), and the dimension of algebraic Whittaker vectors becomes four, half of the order of the little Weyl group. Utilizing the Schmid operator, we obtained the differential equations of Whittaker functions of π with its "corner" K-type (Theorem 4.7). This system becomes holonomic, of rank 4, hence it characterizes the Whittaker vectors. Furthermore we can also find an integral expression of the rapidly decreasing Whittaker function under a parity condition of a nondegenerate character of N (Theorem 5.1). This kind of expressions could be seen in [2, 8, 9].

The author would like to thank Dr. Takuya Miyazaki for helpful communication, Professor Harutaka Koseki, for pointing out errors on §4, and Professor Takayuki Oda, for valuable comments and constant encouragement.

2. Brief summary: $SU(2,2)$ and its Lie algebra

Let us review fundamental facts on the structure of $SU(2,2)$ and its Lie algebra briefly. The notation is same as in [1, §2].

Communicated by Prof. T. Hirai, September 27, 1996

2.1. Restricted root system of $\mathfrak{su}(2,2)$**.** Let $G=SU(2,2)$ be the subgroup of $SL_4(\mathbf{C})$ leaving the hermitian form defined by $I_{2,2}=\mathrm{diag}(1,1,-1,-1)$ unchanged. A maximal compact subgroup $K=S(U(2)\times U(2))$ consists of the elements in two copies of $U(2)$ whose determinant is one. The Cartan involution is written by $\theta(g)={}^t\bar{g}^{-1}$. We make the Cartan decomposition along θ:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}, \tag{1}$$

where, $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{k}=\mathrm{Lie}(K)$ and

$$\mathfrak{p}=\left\{X=\begin{pmatrix} & X \\ {}^t\bar{X} & \end{pmatrix}\middle| X_{12}\in M_2(\mathbf{C})\right\}. \tag{2}$$

Hereafter, the blank entries are understood to be zero. We denote by $\mathfrak{a}$ the maximal abelian subalgebra in $\mathfrak{p}$ and $A=\exp\mathfrak{a}$. Then the restricted root system $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ can be described as:

$$\begin{aligned} \Delta&=\{\pm\lambda_1\pm\lambda_2,\ \pm2\lambda_1,\ \pm2\lambda_2\}, \\ \Delta_+&=\{\lambda_1\pm\lambda_2,\ 2\lambda_1,\ 2\lambda_2\}\ \text{(positive roots)}, \\ \Delta_{\text{fund}}&=\{\lambda_1-\lambda_2,\ 2\lambda_2\}\ \text{(fundamental roots)}. \end{aligned} \tag{3}$$

where $\lambda_i(H_j)=\delta_{ij}$, $H_1=X_{13}+X_{31}$, $H_2=X_{24}+X_{42}$ for $X_{kl}=(\delta_{ki}\delta_{lj})_{ij}$, the (i,j)-matrix elements in $M_4(\mathbf{C})$.

For the root decomposition with respect to $(\mathfrak{g},\mathfrak{a})$, we begin with defining some elements of $\mathfrak{g}$. Put,

$$E_1=\sqrt{-1}\left(\begin{array}{cc|cc} 1 & & -1 & \\ & 0 & & 0 \\ \hline 1 & & -1 & \\ & 0 & & 0 \end{array}\right),\quad E_2=\sqrt{-1}\left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & -1 \\ \hline 0 & & 0 & \\ & 1 & & -1 \end{array}\right),$$

$$E_3=\frac{1}{2}\left(\begin{array}{cc|cc} & 1 & & -1 \\ -1 & & 1 & \\ \hline & 1 & & -1 \\ -1 & & 1 & \end{array}\right),\quad E_4=\frac{\sqrt{-1}}{2}\left(\begin{array}{cc|cc} & 1 & & -1 \\ 1 & & -1 & \\ \hline & 1 & & -1 \\ 1 & & -1 & \end{array}\right),$$

$$E_5=\frac{1}{2}\left(\begin{array}{cc|cc} & 1 & & 1 \\ -1 & & 1 & \\ \hline & 1 & & 1 \\ 1 & & -1 & \end{array}\right),\quad E_6=\frac{\sqrt{-1}}{2}\left(\begin{array}{cc|cc} & 1 & & 1 \\ 1 & & -1 & \\ \hline & 1 & & 1 \\ -1 & & 1 & \end{array}\right),$$

We also denote by H_{ij} the matrix whose (i, i)-entry is $\sqrt{-1}$ and (j, j)-entry is $-\sqrt{-1}$.

These elements describe the root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{c}(\mathfrak{a})+\sum_{\lambda\in\Delta}\mathfrak{g}_\lambda$$

where,

$$\mathfrak{c}(\mathfrak{a})=\text{the centralizer of } \mathfrak{a}=\{H_1, H_2, H_{12}+H_{34}\}_{\mathbf{R}},$$

$$\mathfrak{g}_{2\lambda_1}=\{E_1\}_{\mathbf{R}},\ \mathfrak{g}_{2\lambda_2}=\{E_2\}_{\mathbf{R}},\ \mathfrak{g}_{\lambda_1+\lambda_2}=\{E_3, E_4\}_{\mathbf{R}},$$

$$\mathfrak{g}_{\lambda_1-\lambda_2}=\{E_5, E_6\}_{\mathbf{R}},\ \mathfrak{g}_{-\mu}={}^t\mathfrak{g}_\mu=\{{}^tX|X\in\mathfrak{g}_\mu\}.$$

Note that dim $\mathfrak{g}_{\pm\lambda_1\pm\lambda_2}=2$ and that the little Weyl group W is of order 8.

Fix a basis of $\mathfrak{k}_{\mathbf{C}}$:

$$h^1=\left(\begin{array}{c|c} h & \\ \hline & \end{array}\right),\ h^2=\left(\begin{array}{c|c} & \\ \hline & h\end{array}\right),\ e^1_\pm=\left(\begin{array}{c|c} e_\pm & \\ \hline & \end{array}\right),\ e^2_\pm=\left(\begin{array}{c|c} & \\ \hline & e_\pm\end{array}\right),\ I_{2,2}, \tag{4}$$

where $h=\begin{pmatrix} 1 & \\ & -1\end{pmatrix}$, $e_+=\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$, $e_-=\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$ are 2×2 matrices. Then,

$\mathfrak{h}_{\mathbf{C}}=\{h^1, h^2, I_{2,2}\}_{\mathbf{C}}$ is a compact Cartan subalgebra in $\mathfrak{k}_{\mathbf{C}}$.

2.2. Cuspidal parabolic subgroups of $SU(2,2)$**.** In this subsection, we consider a minimal parabolic subgroup $P=P_m$, and a Jacobi parabolic subgroup P_J of G with Langlands decomposition $P_m=MAN=M_mA_mN_m$, and $P_J=M_JA_JN_J$, respectively.

Let $A_*=\exp\mathfrak{a}_*$ is a split component of P_* ($*$ means either "m" or "J") with

$$\mathfrak{a}=\mathfrak{a}_\emptyset=\{H_1, H_2\}_{\mathbf{R}}, \tag{5}$$

$$\mathfrak{a}_J=\mathfrak{a}_{\{2\lambda_2\}}=\{H_1\}_{\mathbf{R}}. \tag{6}$$

In the following, we identify A with $(\mathbf{R}_{>0})^2$ by

$$(a_1, a_2)=\exp((\log a_1)H_1+(\log a_2)H_2).$$

Their unipotent radicals $N_*=\exp(\mathfrak{n}_*)$ can be described as follows:

$$\mathfrak{n}=\mathfrak{g}_{2\lambda_1}+\mathfrak{g}_{2\lambda_2}+\mathfrak{g}_{\lambda_1+\lambda_2}+\mathfrak{g}_{\lambda_1-\lambda_2}=\{E_j|j=1,\dots,6\}_{\mathbf{R}} \tag{7}$$

$$\mathfrak{n}_J=\mathfrak{g}_{2\lambda_1}+\mathfrak{g}_{\lambda_1+\lambda_2}+\mathfrak{g}_{\lambda_1-\lambda_2}=\{E_j|j=1,3,\dots,6\}_{\mathbf{R}} \tag{8}$$

By definition, the Levi parts are $M_*=Z_K(\mathfrak{a}_*)\exp\mathfrak{m}_*$ with Lie algebras:

$$\mathfrak{m}=\mathbf{R}\sqrt{-1}I_0,\qquad I_0=\mathrm{diag}(1, -1, 1, -1), \tag{9}$$

$$\mathfrak{m}_J=\{H_2,\ E_2,\ \sqrt{-1}I_0,\ H_{24}=\frac{\sqrt{-1}}{2}(I_{2,2}-h^1+h^2)\}_{\mathbf{R}}, \tag{10}$$

and therefore,

$$M=\{\exp(\theta I_0)\cdot\gamma^j|\theta\in\mathbf{R},\ j=0,1\},\qquad \gamma=\mathrm{diag}(1,\ -1,\ 1,\ -1), \tag{11}$$

$$M_J=\left\{\exp(\theta I_0)\cdot\left(\begin{array}{cc|cc} 1 & & & \\ & \alpha & & \beta \\ \hline & & 1 & \\ & \bar{\beta} & & \bar{\alpha}\end{array}\right)\ \middle|\ \begin{array}{c}\theta\in\mathbf{R},\ \alpha,\ \beta\in\mathbf{C},\\ |\alpha|^2-|\beta|^2=1\end{array}\right\} \tag{12}$$

$$(=:\mathbf{T}\cdot G^0)\simeq\mathbf{C}^{(1)}\times SU(1,1).$$

3. Generalized principal series of $SU(2,2)$

3.1. Discrete series of $SU(1,1)$. Let $G_0=SU(1,1)$ and $K_0\simeq\mathbf{C}^{(1)}$ be a maximal compact subgroup. We regard these groups as subgroups of M_J (cf. (12)). Let $\chi_m(e^{\sqrt{-1}\theta})=e^{m\sqrt{-1}\theta}$ be a character of $\mathbf{C}^{(1)}$. The weight lattice of $\mathfrak{g}_0=\mathrm{Lie}(G_0)$ can be identified with $\mathbf{Z}$ with property:

$$\chi_m(\mathrm{diag}(1,\ e^{\sqrt{-1}\theta},\ 1,\ e^{-\sqrt{-1}\theta}))=\chi_m(e^{\sqrt{-1}\theta})=e^{m\sqrt{-1}\theta}\ \text{for } m\in\mathbf{Z}.$$

Let $D_k^{\pm}$ be the discrete series representation with Blattner parameter $\pm k$. Namely, the minimal K_0-type of D_k^+ (resp. D_k^-) is χ_k, $(k\geq 2)$, (resp. χ_{-k}, $(k\leq -2)$) and the other K_0-types are in the form χ_{k+2j}, (resp. χ_{-k-2j}), with non-negative integers j. We say that the suffix $\pm$ is the signature of $D_k^{\pm}$ and denote it by $\mathrm{sgn}(D_k^{\pm})$. We note that the contragredient representation of D_k^+ is isomorphic to D_k^-

3.2. Generalized principal series of $SU(2,2)$. Let $\sigma=(\chi_m,\ D_k^{\pm})$ be a discrete series representation of M_J. Choose $\nu\in\mathfrak{a}_{J,\mathbf{C}}^*$. By the symbol e^{ν}, we denote the character defined by $e^{\nu}(a_1)=e^{\nu(\log a_1)}$.

Define $\pi_J=\mathrm{ind}_{P_J}^{G}(\sigma_1\otimes e^{\nu+\rho_J}\otimes 1)$ acting by right translation where $\rho_J=3\lambda_1$. We say π_J the generalized principal (P_J-) series representation of $SU(2,2)$.

3.3. Multiplicity of K-types. We briefly review the irreducible representations of K. According to [1, Prop. 3.1], $\hat{K}$ is parametrized by

$$\{d=[d_1,\ d_2;\ d_3]\in\mathbf{Z}_{\geq 0}\times\mathbf{Z}_{\geq 0}\times\mathbf{Z}|d_1+d_2+d_3\in 2\mathbf{Z}\}.$$

The representation having parameter d is denoted as $(\tau_d,\ V_d)$.

Let $V_d = \{f^{(d)}_{k_1k_2} | 0 \le k_j \le d_j\}_{\mathbf{C}}$ be the standard basis; the action can be expressed as follows:

$$\begin{aligned} &\tau_d(h^j) f^{(d)}_{k_1k_2} = (2k_j - d_j) f^{(d)}_{k_1k_2}, \qquad (j=1,2)\\ &\tau_d(e^j_+) f^{(d)}_{k_1k_2} = (d_j - k_j) f^{(d)}_{k_1+\delta_{1j}, k_2+\delta_{2j}},\\ &\tau_d(e^j_-) f^{(d)}_{k_1k_2} = k_j f^{(d)}_{k_1-\delta_{1j}, k_2-\delta_{2j}},\\ &\tau_d(I_{2,2}) f^{(d)}_{k_1k_2} = d_3 f^{(d)}_{k_1k_2}. \end{aligned} \tag{13}$$

Let $\tau \in \widehat{K}$. By Frobenius reciprocity, one sees,

$$[\pi_J|_K : \tau] = \sum_{\omega \in (K \cap M_J)^\wedge} [\sigma|_{K\cap M_J} : \omega]\,[\tau|_{K\cap M_J} : \omega]. \tag{14}$$

We prepare several lemmas to compute the multiplicity of γ in $\pi|_K$. First, we see that

$$K \cap M_J = \{(e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\zeta}) = \exp(\theta I_0)\exp(\zeta H_{24}) \,|\, \theta, \zeta \in \mathbf{R}\}.$$

Thus the characters of $K \cap M_J$ can by parametrized along the following:

$$\omega_{(l_1,l_2)}(e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\zeta}) = e^{\sqrt{-1}(l_1\theta + l_2\zeta)}.$$

Clearly we have

Lemma 3.1. *Let $\sigma = (\chi^m, D^\pm_k)$. Then,*

$$[\sigma|_{K\cap M_J} : \omega_{(l_1,l_2)}] = \begin{cases} 1 & \text{if } m = l_1,\ \operatorname{sgn}(D^\pm_k)\, l_2 \ge k \text{ and } l_2 \equiv k \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since V_τ decomposes into 1-dimensional $K \cap M_J$-modules, we have the following:

Lemma 3.2. *Let $\tau = \tau_{[r,s;u]}$ be an irreducible representation of K. Then,*

$$[\tau|_{K\cap M_J} : \omega_{(l_1,l_2)}] = \begin{cases} 1, & \begin{array}{c} -u-2r \le l_1 - 2l_2 \le 2r - u, \\ u - 2s \le l_1 + 2l_2 \le 2s + u, \\ l_1 - 2l_2 + u + 2r \equiv l_1 + 2l_2 - u + 2s \equiv 0 \pmod 4, \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

Summing up, the multiplicity is given by,

Proposition 3.3. *$[\pi_J|_K : \tau]$ equals the number of integers l_2 satisfying the following*:

(i) $l_2 \equiv k \pmod 2$,

(ii) $\operatorname{sgn}(D_k^{\pm}) l_2 \geq k$,

(iii) $2l_2 - u \equiv m + 2r \equiv -m + 2s \pmod 4$,

(iv) $\max(m-2r, -m-2s) \leq 2l_2 - u \leq \min(m+2r, -m+2s)$.

In particular the necessary and sufficient condition for multiplicity-one can be described as follows: (see Figure 1 as an example.)

Theorem 3.4. *Let* $\pi_J = \operatorname{ind}_{P_J}^{G}((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$. *Put* $\delta_D = \operatorname{sgn}(D_k^{\pm})$, $\delta_m = \operatorname{sgn}(m)$, $(\delta_0 = 0)$. *Then the multiplicity* $[\pi_J|_K : \tau_{[r,s;u]}] = 1$ *if and only if the pereameter* $[r, s; u]$ *satisfies* $2k - u \equiv m + 2r \pmod 4$ *and one of the following*:

(1) $r = 0$, $s \geq |m|$ *and* $\delta_D u \geq 2k - \delta_D m$.

(2) $0 \leq r \leq s - |m|$ *and* $u = 2\delta_D(-r+k) - m$.

(3) $r + s = |m|$ *and* $\delta_D u \geq -2\delta_D \delta_m s + 2k + \delta_D m$.

(4) $|r-s| \leq |m|$, $r + s \geq |m|$ *and*

$$u = \begin{cases} -2\delta_D s + 2\delta_m k + m \ (\textit{if } \delta_D \delta_m \geq 0), \\ -2\delta_D r - 2\delta_m k - m \ (\textit{if } \delta_D \delta_m < 0). \end{cases}$$

(5) $r - |m| \geq s \geq 0$ *and* $u = 2\delta_D(-s+k) + m$.

(6) $r \geq |m|$, $s = 0$ *and* $\delta_D u \geq 2k + \delta_D m$.

Proof. If $\tau_{[r,s;u]}$ satisfies one of these conditions, we can easily check that its multiplicity in $\pi|_K$ is one.

Let $\tau_{[r,s;u]}$ be a multiplicity-one K-type. We assume that $\operatorname{sgn}(D_k^{\pm}) > 0$ and $m > 0$ for clarity. If $r \leq s - m$, then Proposition 3.3 says that there is a unique l_2 which satisfies

$$l_2 \equiv k \pmod 2 \text{ and } \max(m-2r, 2k-u) \leq 2l_2 - u \leq m + 2r.$$

By the congruence property, $l_2 s - u$ attains $m + 2r$. In order that exactly one l_2 satisfies the above conditions, it is necessary that $m - 2r \leq 2k - u = m + 2r$ or $ks - u \leq m - sr \leq m + 2r$, equivalently

$$\begin{cases} r \geq 0, \\ 2r + u = 2k - m, \end{cases} \quad \text{or} \quad \begin{cases} r = 0, \\ u \geq 2k - m, \end{cases}$$

so we have (1) and (2). Next, if $|s-r| \leq m$, we see that $m - 2r \leq 2k - u = -m + 2s$ or $2k - u \leq m - 2r = -m + 2s$, which is equivalent to

$$\begin{cases} 2s + u = 2k + m, \\ r + s \geq m, \end{cases} \quad \text{or} \quad \begin{cases} r + s = m, \\ u \geq 2k - m + 2r. \end{cases}$$

So we have (3) and (4). If $r \geq s + m$, then we have $-m - 2s \leq 2k - u = -m + 2s$, $s \geq 0$ or $2k - u \leq -m - 2s = -m + 2s$. This is equivalent to

$$\begin{cases} 2s+u=2k+m, \\ s\geq 0, \end{cases} \quad \text{or} \quad \begin{cases} s=0, \\ u\geq 2k+m. \end{cases}$$

Hence (5) and (6) follow.

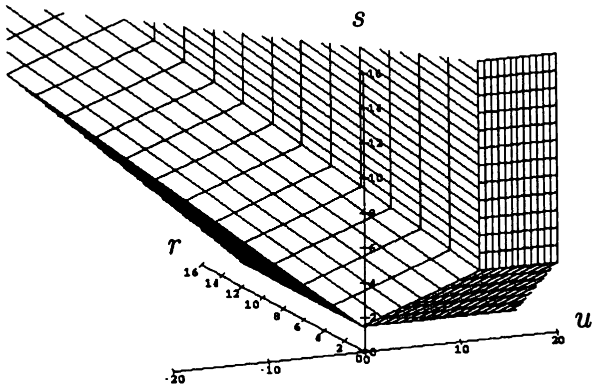

FIGURE 1. Multiplicity-one K-types of $\mathrm{ind}_{P_J}^G((\chi_{-4}, D_4^+)\otimes e^{\nu+\rho_J}\otimes 1)$.

3.4. Infinitesimal character of π_J. According to [3, Prop. 8.22], the infinitesimal character χ_{π_J} of π_J becomes $m+\mathrm{sgn}(D_k^\pm)(k-1)+\nu$ considered as the element of $(\mathfrak{t}+\mathfrak{k}_0+\mathfrak{a}_J)_{\mathbb{C}}^*$ under the Harish-Chandra homomorphism ($\mathfrak{t}=\mathrm{Lie}(\mathbf{T})$). Thus, for the Casimir operator Ω of G,

$$\Omega=H_1^2+6H_1+I_0^2/2+{}^t\bar{E}_1E_1+2\sum_{j=3}^{6}{}^t\bar{E}_jE_j+\Omega_D, \tag{15}$$

where,

$$\begin{aligned}\Omega_D&=H_2^2+2H_2+{}^t\bar{E}_2E_2\\ &=-H_{24}^2-2\sqrt{-1}H_{24}+4X_{42}X_{24}-H_{24}^2+2\sqrt{-1}H_{24}+4X_{24}X_{42}\end{aligned}$$

Then the value $\chi_{\pi_J}(\Omega)$ becomes,

$$\begin{aligned}&\chi_{\pi_J}(\Omega)\\ &\quad=\chi_{\pi_J}((H_1-3)^2+6(H_1-3)+I_0^2/2-(H_{24}\mp\sqrt{-1})^2\mp 2\sqrt{-1}(H_{24}\mp\sqrt{-1}))\\ &\quad=(m+\mathrm{sgn}(D_k^\pm)(k-1)+\nu)(H_1^2-9-H_{24}^2-1+I_0^2/2)\\ &\quad=\nu^2+(k-1)^2-10+m^2/2.\end{aligned}$$

Proposition 3.5. *Let Ω be the Casimir element defined by* (15) *and let χ_{π_J} be the infinitesimal character of π_J. Then,*

$$\chi_{\pi_J}(\Omega)=\nu^2+(k-1)^2-10+m^2/2. \tag{16}$$

4. Differential equations for Whittaker functions

4.1. Shift operators. First of all, we define Whittaker functions. For a moment, let G be a real semisimple Lie group, K a maximal compact subgroup and P a parabolic subgroup of G. Let (π, H_π) be an irreducible principal P-series representation. For the maximal unipotent subgroup N of G and its unitary character η, consider the space of intertwining operators,

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N\backslash G)),$$

where H_π^K is the K-finite vectors. We call its element Φ_π an algebraic Whittaker vector. Choose (τ, V_τ) be an irreducible representation of K such that its contragredient τ^* appears in $\pi|_K$. We fix the K-injection ι_{τ^*}: $V_{\tau^*}\mapsto H_\pi^K$. Define $\Phi_{\pi,\tau}\in C_{\eta,\tau}^\infty(N\backslash G/K)$ by

$$\Phi_\pi(\iota_{\tau^*}(v^*))(g)=\langle v^*, \Phi_{\pi,\tau}(g)\rangle$$

for any $v^*\in V_\tau^*$. We call $\Phi_{\pi,\tau}$ a Whittaker function of π with K-type τ^*. We say $\Phi_{\pi,\tau}|_A$, the radial part of the Whittaker function; it is fully characterized by the restriction of $\Phi_{\pi,\tau}$ to A by vitue of the Iwasawa decomposition.

Next, we define shift operators. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K respectively, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition. Define a K-equivariant operator ∇, called the Schmid operator,

$$\begin{aligned}\nabla: C_{\eta,\tau}^\infty(N\backslash G/K) &\to C_{\eta,\tau\otimes Ad}^\infty(N\backslash G/K)\\ F &\mapsto \nabla F=\sum_j R_{X_j}F(\cdot)\otimes X_j\end{aligned}$$

where $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}$ and that $\mathrm{Ad}=\mathrm{Ad}_{\mathfrak{p}_\mathbf{C}}$ is the adjoint representation of K on $\mathfrak{p}_\mathbf{C}$. Let τ' be an irreducible component of $\tau\otimes\mathrm{Ad}$ and $P_{\tau'}$ be its projection to τ'. Then, $P_{\tau'}\circ\nabla$ or their compositions are called shift operators. Let $P_{\tau'}^*$ be the canonical K-injection defined by

$$\langle P_{\tau'}^*(w^*), v\otimes X\rangle=\langle w^*, P_{\tau'}(v\otimes X)\rangle$$

for $w^*\in V_{\tau'}^*$, $v\in V_\tau$, $X\in\mathfrak{p}_\mathbf{C}$. Note that Ad is self-dual. Considering

$$\mathrm{mul}: V_{\tau^*}\otimes\mathfrak{p}_\mathbf{C}\ni v^*\otimes X\mapsto \pi(X)\iota_{\tau^*}(v^*)\in H_\pi$$

and a composition $\mathrm{mul}\circ P_{\tau'}^*$, we see that there is an constant $c=c(\tau, \iota_{\tau^*}; \tau', \iota_{(\tau')^*})$ such that

$$\mathrm{mul}\circ P_{\tau'}^*=c\cdot\iota_{(\tau')^*}$$

by vitue of irreducibility of τ'. Here we use the convention that if $\iota_{(\tau')^*}$ is meaningless (*i.e.* $(\tau')^*$ is not a K-type of π), then the constant c is equal to 0.

From this equation, we have, by using $(\mathfrak{g}, K)$-homomorphism Φ_π,

$$\sum_k \Phi_\pi(\mathrm{mul}\circ P_{\tau'}^*((v_k')^*))v_k' = c\sum_k \Phi_\pi(\iota_{(\tau')^*}(v_k'))v_k' = c\phi_{\pi,\tau'}, \tag{17}$$

where $\{v_k'\}$ is a basis of $V_{\tau'}$ and $\{(v_k')^*\}$, its dual basis. The left-hand side of (17) turns out to be equal to $P_{\tau'}\circ\nabla\Phi_{\pi,\tau}$. This indicates that the shift operators have Whittaker functions as "eigenfunctions".

Proposition 4.1. *Let (π, H_π) be an irreducible admissible representation of a real semisimple Lie group G and (τ, V_τ) be an irreducible representation of a maximal compact subgroup K such that $[\pi|_K: \tau^*]\neq 0$. Let $(\tau', V_{\tau'})$ be an irreducible component of $\tau\otimes\mathrm{Ad}$ and $K_{\tau'}$ be its projector to $V_{\tau'}$. Fix a K-injection ι_{τ^*}, (resp. $\iota_{(\tau')^*}$) of V_{τ^*} (resp. $V_{(\tau')^*}$) to H_π. Then there exists a constant $c=c(\tau, \iota_{\tau^*}; \tau', \iota_{(\tau')^*})$ such that*

$$P_{\tau'}\circ\nabla\Phi_{\pi,\tau}=c\cdot\Phi_{\pi,\tau'}.$$

Here if $\iota_{(\tau')^*}$ does not exist, c is understood to be zero.

From now on, we let $G=SU(2,2)$. Since it is of hermitian type, we can define the K-equivariant maps $\nabla^\pm: C^\infty_{\eta,\tau}(N\backslash G/K)\to C^\infty_{\eta,\tau\otimes Ad_\pm}(N\backslash G/K)$ by

$$\begin{aligned}\nabla^+F&=\sum_{i=1,2,j=3,4} R_{X_{ji}}F(\cdot)\otimes X_{ij},\\ \nabla^-F&=\sum_{i=1,2,j=3,4} R_{X_{ij}}F(\cdot)\otimes X_{ji}.\end{aligned}\tag{18}$$

Here, we put $\mathrm{Ad}_\pm=\mathrm{Ad}_{\mathfrak{p}_\pm}$ for the canonical decomposition $\mathfrak{p}_\mathbf{C}=\mathfrak{p}_++\mathfrak{p}_-$.

Let $P_\tau^{(\pm,\pm)}$, (resp. $\overline{P}_\tau^{(\pm,\pm)}$ be projectors to $\tau_{[r\pm1,s\pm1;\,u+2]}$ (resp. $\tau_{[r\pm1,s\pm1;\,u-2]}$) defined in [1, Lemma 3.12]. We define the following shift operators,

$$\begin{aligned}\mathscr{D}^{\mathrm{up}}&=P^{(-,-)}\circ\nabla^+\circ P^{(+,+)}\circ\nabla^+,\\ \mathscr{D}^{\mathrm{down}}&=\overline{P}^{(-,-)}\circ\nabla^-\circ\overline{P}^{(+,+)}\circ\nabla^-,\\ \mathscr{E}^{(\pm,\mp)}&=P^{(\pm,\mp)}\circ\nabla^+,\qquad \overline{\mathscr{E}}^{(\pm,\mp)}=\overline{P}^{(\pm,\mp)}\circ\nabla^-.\end{aligned}\tag{19}$$

4.2. Dimension of the space of the Whittaker vectors. The dimension of the space of Whittaker vectors can be found as follows. Let

$$F^\#=\exp(\mathfrak{a}_\mathbf{C})=\langle\gamma,\ \alpha=\sqrt{-1}\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1\\ \hline 1 & & & \\ & 1 & & \end{array}\right)\rangle. \tag{20}$$

It is known that $F^{\#}G=\{g\in G_{\mathbf{C}}|(\mathrm{Ad}\,g)\mathfrak{g}=\mathfrak{g}\}$ where $G_{\mathbf{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let (π, H_π) be an admissible G-module. For $a\in F^{\#}$, define another G-module $(\pi^{(a)}, H_\pi^{(a)})$ by,

$$\pi^{(a)}(g)v=\pi(a^{-1}ga)v,\ H_\pi^{(a)}=H_\pi\ \ (g\in G,\ v\in H_\pi).$$

If π has a $(\mathfrak{g}, K)$-module structure, so does $\pi^{(a)}$. Choose $\{a_1,\dots,a_p\}$ so that $\{\pi^{(a_i)}\}$ is a complete system of mutually infinitesimally non-isomorphic classes of $\{\pi^{(a)}|a\in F^{\#}\}$.

If $\pi=\mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm})\otimes e^{\nu+\rho_J}\otimes 1)$, then we find that $p=2$, i.e., $\{a_1, a_2\}=\{1, \alpha\}$ with $\pi^{(\alpha)}\simeq\mathrm{ind}_{P_J}^G((\chi_m, D_k^{\mp})\otimes e^{\nu+\rho_J}\otimes 1)$. We have the following:

Theorem 4.2. *Assume π_J is irreducible. Then,*

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_J}^K, C_\eta^\infty(N\backslash G))=4.$$

Proof. If π_J is irreducible, then it is large in the sense of [12, Th. 6.2, f)]. Thus we have, from [5, Th. 6.8.1],

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, C_\eta^\infty(N\backslash G))+\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J^{(\alpha)}, C_\eta^\infty(N\backslash G))=8.$$

On the other hand, the Whittaker models with η of $\pi^{(\alpha)}$ is isomorphic to those with $\eta^{(\alpha)}$ of π. But the dimension of the space of algebraic Whittaker vectors is determined independently of the choice of η, whence the dimension is 4.

$\mathrm{sgn}(D_k^{\pm})$	m	the parameter of τ
$+$	$+$	$d_1=[0, m;\ -2k+m]$
	$-$	$d_2=[-m, 0;\ -2k-m]$
	0	$d_0^+=[0, 0;\ -2k]$
$-$	$-$	$d_{-1}=[0, -\mathrm{m};\ 2k+m]$
	$+$	$d_{-2}=[\mathrm{m}, 0;\ 2k-m]$
	0	$d_0^-=[0, 0;\ 2k]$

TABLE 1. Corner K-types τ^* of $\mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm})\otimes e^{\nu+\rho_J}\otimes 1)$

Remark 4.3. One has $\eta^{(\alpha)}(E_2)=-\eta(E_2)$, which will also explain the relation (33).

4.3. Differential equations for Whittaker functions. As in §3.2, let $\pi_J=\mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm})\otimes e^{\nu+\rho_J}\otimes 1)$ be a generalized principal series representation. The corner K-type τ_d^* is characterized by the following property:

(1) $\dim \tau_d^*$ is minimum in $\pi|_K$.

(2) $\tau_d^*|_{K_0}$ has the minimal K_0-type of $D_k^\pm$.

(3) If $m \neq 0$, there exists a non-compact root δ with respect to $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ such that $\tau_{d+\delta} \in \widehat{K}$ and $[\pi|_K : \tau_{d+\delta}^*] = 0$.

Choose τ so as that its contragredient representation τ^* becomes the corner K-type of π_J, which is eventually determined uniquely as in Table 1.

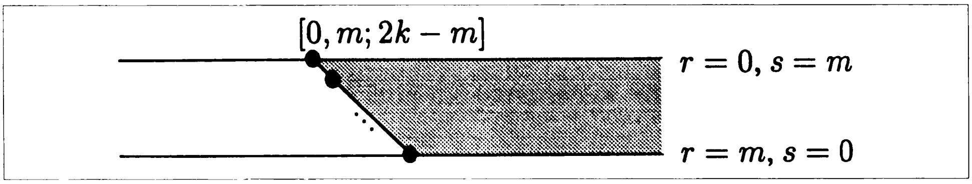

FIGURE 2. The corner K-type in the plane: $r+s=m>0$ consisting of K-types of $\mathrm{ind}_{P_J}^{G}((\chi_m, D_k^\pm)\otimes e^{\nu+\rho_J}\otimes 1)$

In the following we write $\pi = \pi_J$, $\tau_j^{(\pm)} = \tau_{d_j^{(\pm)}}$ for simplicity. For a nondegenerate unitary character $\eta \in \widehat{N}$, we put (see [1, §4.1]),

$$\eta_2 = \sqrt{-1}\,\eta(E_2),\ \eta_0 = \xi\xi',\ \xi = \eta(E_5) + \sqrt{-1}\,\eta(E_6),\ \xi' = \eta(E_5) - \sqrt{-1}\,\eta(E_6).$$

Consider the Whittaker function $\Phi_{\pi,\tau}$. Then,

Proposition 4.4. (1) *If* $\mathrm{sgn}(D_k^\pm) > 0$, *we have, for* $j = 1, 2$,

$$\mathscr{E}^{(\mathrm{sgn}(m),-\mathrm{sgn}(m))}\Phi_{\pi,\tau_j} = 0 \quad (m \neq 0), \tag{21}$$

$$\mathscr{D}^{\mathrm{up}}\Phi_{\pi,\tau_0^+} = 0 \quad (m = 0). \tag{22}$$

(2) *If* $\mathrm{sgn}(D_k^\pm) < 0$, *we have, for* $j = -1, -2$,

$$\bar{\mathscr{E}}^{(-\mathrm{sgn}(m),\mathrm{sgn}(m))}\Phi_{\pi,\tau_j} = 0 \quad (m \neq 0), \tag{23}$$

$$D^{\mathrm{down}}\Phi_{\pi,\tau_0^-} = 0 \quad (m = 0). \tag{24}$$

Proof. If $m \neq 0$, Table 1 tells us that what the parameter of the corner K-type is and that the unique direction to make the Whittaker function vanish.
If $m = 0$, we can easily find that the result of the action of $\mathscr{D}^{\mathrm{up/down}}$ becomes zero by Propositions 3.3 and 4.1. Thus the proposition follows.

In [1, Lemma 6.5], the radial part of such shift operators were calculated in the most general way. Therefore, specializing the parameters, we obtain the following:

Prposition 4.5. *Let* $\Phi_{\pi,\tau_j}(a)=\sum_{kl}c_{kl}^{(j)}(a)f_{kl}^{(j)}$ *for* $j=\pm 1, \pm 2$ *and let* $\Phi_{\pi,\tau_0^\pm}(a)=c_{00}^{(0,\pm)}(a)f_{00}^{(0,\pm)}$. *Then we have,*

(1) $\operatorname{sgn}(D_k^\pm)>0$, $m=0$ *case*:

$$\left((\partial_1-k-2)(\partial_2-a_2^2\eta_2-k)-\eta_0\left(\frac{a_1}{a_2}\right)^2\right)c_{00}^{(0,+)}=0. \tag{25}$$

(2) $\operatorname{sgn}(D_k^\pm)>0$, $m>0$ *case*:

$$(m-j)(\partial_2-a_2^2\eta_2-k+j)c_{0j}^{(1)}+(j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{0,j+1}^{(1)}=0, \tag{26}$$

$$(m-j)\left(\frac{a_1}{a_2}\right)\xi\, c_{0j}^{(1)}+(j+1)(\partial_1-m-k-1+j)c_{0,j+1}^{(1)}=0. \tag{27}$$

(3) $\operatorname{sgn}(D_k^\pm)>0$, $m<0$ *case*:

$$(m+j)(\partial_1-k-j-2)c_{j0}^{(2)}+(j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{j+1,0}^{(2)}=0,$$

$$(m+j)\left(\frac{a_1}{a_2}\right)\xi\, c_{j0}^{(2)}+(j+1)(\partial_2-a_2^2\eta_2-m-k-1-j)c_{j+1,0}^{(2)}=0.$$

(4) $\operatorname{sgn}(D_k^\pm)<0$, $m=0$ *case*:

$$\left((\partial_1-k-2)(\partial_2+a_2^2\eta_2-k)-\eta_0\left(\frac{a_1}{a_2}\right)^2\right)c_{00}^{(0,-)}=0.$$

(5) $\operatorname{sgn}(D_k^\pm)<0$, $m>0$ *case*:

$$(m-j)(\partial_2+a_2^2\eta_2-k+j)c_{j0}^{(-2)}+(j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{j+1,0}^{(-2)}=0,$$

$$(m-j)\left(\frac{a_1}{a_2}\right)\xi\, c_{j0}^{(2)}+(j+1)(\partial_1-m-k-1+j)c_{j+1,0}^{(-2)}=0.$$

(6) $\operatorname{sgn}(D_k^\pm)<0$, $m<0$ *case*:

$$(m+j)(\partial_1-k-j-2)c_{0j}^{(-1)}+(j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{0,j+1}^{(-1)}=0,$$

$$(m+j)\left(\frac{a_1}{a_2}\right)\xi\, c_{0j}^{(-1)}+(j+1)(\partial_2+a_2^2\eta_2-m-k-1-j)c_{0,j+1}^{(-1)}=0.$$

From the Casimir operator, we get another proposition by [1, Lemma 5.1].

Proposition 4.6. *Let* $\Phi_{\pi,\tau_j}(a)=\sum_{kl}c_{kl}^{(j)}(a)f_{kl}^{(j)}$, *and so* $\Phi_{\pi,\tau_0^\pm}(a)=c_{00}^{(0,\pm)}(a)f_{00}^{(0,\pm)}$. *Then,*

(1) $\operatorname{sgn}(D_k^\pm)>0$, $m>0$ *case*:

$$(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2-a_2^4\eta_2^2+2\eta_0(a_1/a_2)^2+2(j-k)\eta_2a_2^2 \\ +(2j-m)^2/2)c_{0j}^{(1)}+2(m-j+1)\xi(a_1/a_2)c_{0,j-1}^{(1)} \\ -2(j+1)\xi'(a_1/a_2)c_{0,j+1}^{(1)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(1)}. \tag{28}$$

When $m=0$, $c_{00}^{(0,+)}$ *also satisfies* (28).

(2) $\operatorname{sgn}(D_k^{\pm})>0$, $m<0$ *case*:

$$(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2-a_2^4\eta_2^2+2\eta_0(a_1/a_2)^2-2(m+k+j)\eta_2a_2^2 \\ +(2j+m)^2/2)c_{j0}^{(2)}+2(1-m-j)\xi(a_1/a_2)c_{j-1,0}^{(2)} \\ -2(j+1)\xi'(a_1/a_2)c_{j+1,0}^{(2)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(2)}. \tag{29}$$

(3) $\operatorname{sgn}(D_k^{\pm})<0$, $m>0$ *case*:

$$(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2+a_2^4\eta(E_2)^2+2\eta_0(a_1/a_2)^2+2(k-j)\eta_2a_2^2 \\ +(2j-m)^2/2)c_{j0}^{(-2)}+2(m-j+1)\xi(a_1/a_2)c_{j-1,0}^{(-2)} \\ -2(j+1)\xi'(a_1/a_2)c_{j+1,0}^{(-2)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(-2)}. \tag{30}$$

When $m=0$, $c_{00}^{(0,-)}$ *also satisfies* (30).

(4) $\operatorname{sgn}(D_k^{\pm})<0$, $m<0$ *case*:

$$(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2+a_2^4\eta(E_2)^2+2\eta_0(a_1/a_2)^2+2(m+k+j)\eta_2a_2^2 \\ +(m+2j)^2/2)c_{0j}^{(-1)}+2(1-m-j)\xi(a_1/a_2)c_{0,j-1}^{(-1)} \\ -2(j+1)\xi'(a_1/a_2)c_{0,j+1}^{(-1)}=(\nu^2+(k-1)^2-10+m^2/2)c_{0j}^{(-1)}. \tag{31}$$

From these propositions, we find that

$$(c_{j0}^{(2)},m)=(\overline{c_{0,|m|-j}^{(1)},-m}),\ (c_{j0}^{(-2)},m)=(\overline{c_{0,|m|-j}^{(-1)},-m}), \tag{32}$$

$$(c_{0j}^{(1)},\eta_2)=(c_{j0}^{(-2)},-\eta_2),\ (c_{0j}^{(-1)},\eta_2)=(c_{j0}^{(2)},-\eta_2). \tag{33}$$

Therefore we mainly treat the typical case: $\operatorname{sgn}(D_k^{\pm})>0$, $m\geq 0$.

Theorem 4.7. *Let* $\pi_J=\operatorname{ind}_{P_J}^G((\chi_m,D_k^{\pm})\otimes e^{\nu+\rho_J}\otimes 1)$. *Assume* $\operatorname{sgn}(D_k^{\pm})>0$, $m\geq 0$. *For* $\tau=\tau_{[0,m;\,m-2k]}$, *consider the Whittaker function* $\Phi_{\pi_J,\tau}=\sum_j c_{0j}^{(1)}f_{0j}^{(1)}$. *Put* $c_{0j}^{(1)}(a)=\exp(a_2^2\eta_2/2)a_1^{m+k+2-j}a_2^{k-j}h_j^{(1)}(a)$. *Then,* $h_j^{(1)}$, $(j=0,\dots,m)$ *satisfy the following*:

$$(\partial_1\partial_2-(a_1/a_2)^2\eta_0)h_j^{(1)}=0, \tag{34}$$

$$(\partial_1^2+\partial_2^2+2(m+k-2j-1)(\partial_1+\partial_2)+2\eta_2a_2^2\partial_2 \\ +(m+k-2j-1)^2-\nu^2)h_j^{(1)}=0. \tag{35}$$

These two differential equations become a holonomic system of rank 4.

Proof. When $m=0$, each equation is a direct consequence of Equations (25) and (28). When $m>0$, Equation (35) is a consequence of (28), while Equation (34) is obtained from Equations (26) and (27). To show the holonomicity, we find that the left-hand sides of Equations (34), (35) and their Poisson bracket become a Gröbner basis. Then, calculating its characteristic variety, we find the dimension is two, and the rank is four on the plane without the singular locus.

5. Integral expression of Whittaker functions

In this section, we obtain an integral expression of the rapidly decreasing solution of differential equations in Theorem 4.7.

Put formally,

$$\mathcal{W}(a)=\int_0^\infty \phi(t)\exp\Big(\frac{\eta_0 a_1^2}{t}-\frac{t}{4a_2^2}\Big)\frac{dt}{t}$$

for $\phi\in C^\infty(\mathbf{R}_{>0})$. This is a general solution of Equation (34). The formal relation

$$(\partial_1+\partial_2)\mathcal{W}=\int_0^\infty 2\partial_t\phi(t)\exp\Big(\frac{\eta_0 a_1^2}{t}-\frac{t}{4a_2^2}\Big)\frac{dt}{t}$$

tells us, from (35), that

$$(4\partial_t^2+4(m+k-2j-1)\partial_t+\eta_2 t+(m+k-2j-1)^2-\nu^2)\phi=0,$$

with $\partial_t=t(d/dt)$. Put $v=\sqrt{t}$ and $\phi(t)=v^{-(m+k-2j-1)-1/2}\psi(v)$ to get

$$\frac{d^2\psi}{dv^2}+\Big(-\frac{1}{4}(-4\eta_2)+\frac{1/4-\nu^2}{v^2}\Big)\psi=0. \tag{36}$$

This turns out to be Whittaker's differential equation. If $\eta_2<0$, we denote by $W_{0,\nu}(2\sqrt{-\eta_2}v)$ the rapidly decreasing solution of (36).

Theorem 5.1. *Let $\pi_J=\mathrm{ind}_{P_J}^G((\chi_m, D_k^\pm)\otimes e^{\nu+\rho_J}\otimes 1)$ be an irreducible generalized principal series representation of G. Let $\Phi_{\pi_J,\tau_d}=\sum c_{ij}^{(d)}(a)f_{ij}^{(d)}$ be a rapidly decreasing Whittaker function of π_J with the corner K-type τ_d^* given by Table 1. Then,*

(1) $\mathrm{sgn}(D_k^\pm)>0$, $m\geq 0$ *case: if* $\eta_2<0$,

$$\begin{aligned} c_{0j}^{(1)}(a)=&\,C_1(8\eta_2)^{m-j}\xi'^{j}e^{\eta_2 a_2^2/2}a_1^{m+k+2-j}a_2^{k-j} \\ &\times\int_0^\infty t^{2j-m-k+1/2}W_{0,\nu}(t)\exp\Big(-\frac{4\eta_2\eta_0 a_1^2}{t^2}+\frac{t^2}{16\eta_2 a_2^2}\Big)\frac{dt}{t}. \end{aligned} \tag{37}$$

(2) $\mathrm{sgn}(D_k^\pm)>0$, $m<0$ *case: if* $\eta_2<0$,

$$c_{j0}^{(2)}(a)=C_2(8\eta_2)^{m-j}(-\xi)^j e^{\eta_2 a_2^2/2} a_1^{k+2+j} a_2^{k+m+j} \times\int_0^\infty t^{-2j-m-k+1/2} W_{0,\bar{\nu}}(t)\exp\left(-\frac{4\eta_2\eta_0 a_1^2}{t^2}+\frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{38}$$

(3) $\mathrm{sgn}(D_k^{\pm})<0$, $m\geq 0$ *case*: *if* $\eta_2>0$,

$$c_{j0}^{(-2)}(a)=C_{-2}(-8\eta_2)^{m-j}\xi'^j e^{-\eta_2 a_2^2/2} a_1^{m+k+2-j} a_2^{k-j} \times\int_0^\infty t^{2j-m-k+1/2} W_{0,\nu}(t)\exp\left(\frac{4\eta_2\eta_0 a_1^2}{t^2}-\frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{39}$$

(4) $\mathrm{sgn}(D_k^{\pm})<0$, $m<0$ *case*: *if* $\eta_2>0$,

$$c_{0j}^{(-1)}(a)=C_{-1}(8\eta_2)^{m-j}\xi^j e^{-\eta_2 a_2^2/2} a_1^{k+2+j} a_2^{k+m+j} \times\int_0^\infty t^{-2j-m-k+1/2} W_{0,\bar{\nu}}(t)\exp\left(\frac{4\eta_2\eta_0 a_1^2}{t^2}-\frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{40}$$

Here, $C_{\pm 1}$, $C_{\pm 2}$ *are comstant multiples determined independently of the choice of* j.

Proof. Nondegeneracy of η says that η_0 is a nonzero negative number. So the parity condition of η_2 implies the convergence of integrals in the right-hand side. Once it converges, it clearly satisfies Equations (34) and (35). Keeping in mind the convention for other cases, we can readily deduce the other equations. The proof of the rapid decrease of the functions when $a_1/a_2\to\infty$ and $a_2\to\infty$ is completely same as [9, Theorem (9.1)].

Remark 5.2. These expressions are very similar to those in the case of $Sp(2;\mathbf{R})$ ([8, Theorems (9.1), (9.2)]).

Division of Intelligence Science,
Graduate School of Science and
Technology, Kobe University
E-mail address: hayata@math. s. kobe-u. ac. jp

References

[1] T. Hayata, Differential equations of principal series Whittaker functions on $SU(2,2)$, to appear in Indag. Math.

[2] T. Hayata and T. Oda, An explicit integral representation of Whittaker functions for the representation of the discrete series - the case of $SU(2,2)$ -, J. Math. kyoto Univ., this issue.

[3] A. W. Knapp, Representation theory of semisimple groups, —an overview based on examples—, Princeton Mathematical Series 36, Princeton University Press, Princeton, New Jersey, 1986.

[4] H. Koseki and T. Oda, Whittaker functions for the large discrete series representation of $SU(2,1)$ and related zeta integrals, Publ. Res. Inst. Math. Sci., **31** (1995), 959-999.

[5] B. Kostant, On Whittaker vectors and representation theory, Invent. Math., **48** (1978), 101-184.

[6] H. Matumoto, $C^{-\infty}$-Whittaker vectors corresponding to a principal nilpotent orbit of a real reductive linear Lie group, and wave front sets, Compositio Math., **82** (1992), 189-244.

[7] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2; \mathbf{R})$, - explicit formulae of differential equations -, Proceedings of the 1993 Workshop, Automorphic Forms and Related Topics, The Pyngsan Institute for Mathematical Sciences, 1993, pp. 59-92.

[8] ——, Principal series Whittaker functions on $Sp(2; \mathbf{R})$ II, Seminar note RIMS-992, 1993.

[9] T. Oda, An explicit integral representation of Whittaker functions of $Sp(2; \mathbf{R})$ for the large discrete series representations, Tôhoku Math. J., **46** (1994), 261-279.

[10] W. Schmid, On the realization of the discrete series of a semisimple Lie groups, Rice Univ. Stud., **56**-1 (1970), 99-108.

[11] K. Taniguchi, Discrete series Whittaker function of $SU(n, 1)$, preprint, 1995.

[12] D. A. Vogan Jr., Gelfand-Kirillov dimension for Harish-Chandra modules, Invent. Math., **48** (1978), 75-98.

[13] H. Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups I, - general theory and the case of $SU(2,2)$ -, Japan. J. Math. (N.S.), **16**-1 (1990), 31-95.