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Chern classes for parabolic bundles

By

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1. Introduction

This is a continuation of our earlier works, [Bi2], [Bi3], where we studied various properties of parabolic bundles (both on curves and on higher dimensional varieties). Parabolic bundles (introduced in [MS] for curves and generalized to higher dimension in [MY]) are vector bundles (or more generally torsion-free coherent sheaves) on open varieties together with a weighted filtration at the boundary. Various results on vector bundles over projective manifolds generalize to the parabolic context.

Here we give a candidate for what should be the Chern classes of a parabolic bundle. Taking a hint from the definition of parabolic degree, which should be the first parabolic Chern class, of a parabolic bundle, in Section 3 we define parabolic Chern classes. (Indeed, the definition of the parabolic degree in higher dimensions, which is rather nontrivial (introduced in [MY]), serves as a good hint.)

Given a representation in $GL(r, \mathbf{C})$ of the fundamental group of a smooth open variety, there is a natural extension of the corresponding flat bundle to some suitable compactification (the divisor at infinity should be of normal crossing) as a parabolic bundle. We give a justification for our definition of parabolic Chern classes by pointing out that all the parabolic Chern classes of such a parabolic bundle vanish.

S. Bloch and D. Gieseker showed that the Chern classes of an ample vector bundle are numerically positive. This result was extended in [FL], and all the numerically positive characteristic polynomials for ample vector bundles were identified. In [Bi2] we defined parabolic ample bundles and showed that they exhibit various properties analogous to an ample vector bundle—for example, Hartshorne's characterization of ample vector bundles on curves, Le Potier vanishing theorem.

In Section 4a we show that the parabolic Chern classes of a parabolic ample bundle are numerically positive. The statement correpponding to the theorem of [FL] is also valid. We prove that under certain conditions on the filtration, a parabolic stable bundle with vanishing parabolic Chern classes share the characteristics of a stable vector bundle with vanishing Chern

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classes. The characteristics in question are the relations with the unitary representations of the fundamental group, and the Hodge and the Lefschetz decompositions of the cohomology groups.

2. Preliminaries

Let X be a connected smooth projective variety over $\mathbf{C}$ of complex dimension d. Let D be an effective divisor on X.

Fix an ample line bundle L on X. For a coherent $\mathcal{O}_X$ module E, define the degree

$$\deg(E) := (c_1(E) \cup c_1(L)^{d-1}) \cap [X] \in \mathbf{Z}$$

Definition 2.1. Let E be a torsion-free coherent $\mathcal{O}_X$ module. A *quasi-parabolic* structure on E (with respect to D) is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = F_1(E) \supset F_2(E) \supset \ldots \supset F_l(E) \supset F_{l+1}(E) = E(-D)$$

Where $E(-D)$ is the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$. The integer l is called the *length of the filtration*. A *parabolic structure* is a quasi-parabolic structure, as above, together with a system of *weights* $\{\alpha_1, \ldots, \alpha_l\}$ such that

$$0 \leq \alpha_1 < \alpha_2 \ldots < \alpha_{l-1} < \alpha_l < 1$$

where the weight α_i corresponds to the sheaf $F_i(E)$.

We will denote a parabolic sheaf, as above, by (E, F_*, α_*); and when there is no scope of confusion, simply by E_*. Define the following filtration, $\{E_t\}$, of coherent sheaves parametrized by $\mathbf{R}$:

$$E_t := F_i(E)(-[t]D) \tag{2.2}$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. Any coherent subsheaf V of E has an induced parabolic structure such that if $\{V_t\}$ is the corresponding filtration then $V_t = E_t \cap V$ for any $t \geq 0$.

The *parabolic degree* of E_*, denoted by par-deg E_*, is defined as:

$$\text{par-deg } E_* := \int_{-1}^{0} \deg(E_t)\, dt \tag{2.3}$$

The quotient par-degE_*/rank E is usually denoted by par-μE_*.

Definition 2.4. The parabolic sheaf E_* in (2.1) is called *parabolic semistable* (resp. *parabolic stable*) if for any subsheaf V of E, with $0 <$ rank V $<$ rank E, and E/F being torsion-free, the condition par-$\mu V_* \leq$ par-μE_* (resp. par-$\mu V_* <$ par-μE_*) is satisfied.

All the above definitions can be found in [MY].

Consider the decomposition

$$D=\sum_{i=1}^{n} n_i D_i \tag{2.5}$$

where any D_i is a reduced irreducible divisor and $n_i \geq 1$. Let

$$f_i : n_i.D_i \to X$$

denote the inclusion of the subscheme $n_i.D_i$.

Take a torsion-free coherent sheaf E on X. For $1 \leq i \leq n$, let

$$0=F^i_{l_i+1} \subset F^i_{l_i} \subset F^i_{l_i-1} \subset \cdots \subset F^i_2 \subset F^i_1 = f_i^* E \tag{2.6}$$

with $l_i \geq 1$, be a filtration of coherent sheaves on $n_i D_i$. Given strings of real numbers α^i_j, $1 \leq j \leq l_i+1$, satisfying

$$1=\alpha^i_{l_i+1} > \alpha^i_{l_i} > \alpha^i_{l_i-1} > \cdots > \alpha^i_2 > \alpha^i_1 \geq 0 \tag{2.7}$$

we may construct a parabolic structure on E which we will describe now. Define the coherent subsheaf, $\bar{F}^i_j$, where $1 \leq i \leq n$ and $1 \leq j \leq l_i$, of E using the following short exact sequence:

$$0 \to \bar{F}^i_j \to E \to (f_i^* E)/F^i_j \to 0 \tag{2.8}$$

(the surjective homomorphism is given by the restriction map). For $1 \leq i \leq n$ and $0 \leq t \leq 1$, let l^i_t be the smallest number in the set of integers

$$\{j \in \{1, \ldots, l_i+1\} \mid \alpha^i_j \geq t\}$$

Define E_t to be the following intersection of subsheaves of E :

$$E_t := \bigcap_{i=1}^{n} \bar{F}^i_{l^i_t} \subseteq E \tag{2.9}$$

The filtration $\{E_t\}$ defines a parabolic structure on E. It is easy to see that any parabolic structure on E, with D as the parabolic divisor, arises this way.

Let $f : X - D \to X$ denote the inclusion of the complement of D. For two parabolic sheaves E_* and W_* on X, with D as the parabolic divisor, and $c \in \mathbf{R}$, define M_c to be the subsheaf of the quasi-coherent sheaf $f_* f^*(E \otimes W)$ generated by all $E_s \otimes W_t$ with $s+t \geq c$. The parabolic sheaf given by the filtration $\{M_c\}_{c \in \mathbf{R}}$ is called the *parabolic tensor product* of E_* and W_*, and it is denoted by $E_* \otimes W_*$ [Bi2]. The parabolic m-fold symmetric product, $S^m(E_*)$, is the invariant subsheaf of the m-fold parabolic tensor product of E_* for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^m(E_*)$ will be denoted by $S^m(E_*)_0$.

The parabolic sheaf E_* is called *parabolic ample* if for any coherent sheaf F on X there is an integer m_0 such that for any $m \geq m_0$, the tensor product

$F \otimes S^m(E_*)_0$ is generated by its global sections ([Bi2], Definition 2.3).

We will now recall the definition of an orbifold bundle.

Let $Y/\mathbf{C}$ be a smooth projective variety, and let

$$\rho : G \to \mathrm{Aut}(Y)$$

be a finite group acting faithfully on Y.

Definition 2.10. An *orbifold sheaf* on Y, with G as the orbifold group, consists of the following data: a torsion-free coherent sheaf, V, on Y, and a lift of the action of G on Y to V, i.e., G acts on the total space of stalks of V such that for any $g \in G$ this action gives a coherent sheaf isomorphism between V and $\rho(g^{-1})^*V$. A coherent subsheaf, F, of V, with V/F being torsion-free, will be called an *orbifold subsheaf* if the action of G on V preserves F.

Let $\tilde{L}$ be an orbifold line bundle on Y which is also ample. So using $\tilde{L}$ we may define the degree of any coherent sheaf on Y.

An orbifold sheaf V on Y is called *orbifold semistable* (resp. *orbifold stable*) if for any nonzero proper orbifold subsheaf, F, of V with V/F torsion free, the following holds:

$$\deg F/\mathrm{rank}\, F \leq \deg V/\mathrm{rank}\, V \quad (\text{resp. } \deg F/\mathrm{rank}\, F < \deg V/\mathrm{rank}\, V) \tag{2.11}$$

If V is an orbifold stable bundle with $c_1(V) = 0 = c_2(V)$, then there is an unitary flat connection on V which is invariant under the action of G; moreover, such a connection is unique ([S1], page 878, Theorem 1, Proposition 3.4). This connection is irreducible in the sense that there is no proper nonzero orbifold subsheaf of V which is left invariant by the connection.

We will now recall some results proved in [Bi2], [Bi3].

Assume that D is a divisor of normal crossing, i.e., all $n_i = 1$ and D_i are smooth divisors and they interect transversally.

Let (E_*, F_*, α_*) be a parabolic bundle on X given by (2.9). Assume that all F^i_j on D_i are subbundles of f_i^*E. Also, assume that all the weights α^i_j are rational numbers; so $\alpha^i_j = m^i_j/N$, where N is a fixed integer and $m^i_j \in \{0, 1, 2, \dots, N-1\}$.

We will now recall the "Covering Lemma" of Y.Kawamata (Theorem 1.1.1 of [KMM], Theorem 17 of [K]).

With the above notation, there is a connected smooth projective variety Y and a Galois covering morphism

$$p : Y \to X$$

with Galois group $G = \mathrm{Gal}(\mathrm{Rat}(Y)/\mathrm{Rat}(X))$ such that $\tilde{D} := (p^*D)_{\mathrm{red}}$ is a divisor of normal crossing on Y, and $p^*D_i = k_i N.(p^*D_i)_{\mathrm{red}}$, $1 \leq i \leq n$, where k_i are positive integers.

Let $\tilde{D}_i$ denote the reduced divisor $(p^*D_i)_{\text{red}}$. Let Q_j^i denote the vector bundle on D_i defined by:

$$Q_j := (f_i^*E/F_j^i) \otimes f_i^* \mathcal{O}_X(D_i) = (f_i^*E/F_j^i) \otimes N_i \tag{2.12}$$

where N_i is the normal bundle to the divisor D_i.

Let U_j^i denote the kernel of the obvious projection of the pullback bundle, $p^*(E \otimes \mathcal{O}_X(D))$, onto the restriction of the pullback sheaf, $f^*Q_j^i$ (which is supported on $k_iN\tilde{D}_i$), to $k_i.(N-m_{j-1}^i)\tilde{D}_i$.

There is a natural orbifold bundle structure on $f^*(E \otimes \mathcal{O}_X(D))$ for the group of deck transformations. Since the divisor $\tilde{D}_i$ is invariant under the action of G on Y, there is an orbifold structure on $\mathcal{O}_Y(j.\tilde{D}_i)$ for any $j \in \mathbf{Z}$. Hence there is an induced orbifold structure on the sheaf U_j^i.

Define $V := \cap U_j^i$ to be the intersection of all U_j^i inside $f^*(E \otimes \mathcal{O}_X(D))$.

From the assumption on E_*, namely that any F_j^i is a subbundle of E_i, it follows that any U_j^i and V are all locally free coherent sheaves on Y.

Note that since p is a covering morphism, the direct image p_*V is locally free on X. Moreover, the orbifold structure of V gives an injective homomorphism of G into $\mathrm{Aut}(p_*V)$, the group of global automorphisms of p_*V.

The parabolic bundle E_* can be recovered from the orbifold bundle V in the following way: define

$$E_t := \left(f_*(V \otimes \mathcal{O}_Y(\sum_{i=1}^{n} [-t.k_i.N].\tilde{D}_i)) \right)^G \tag{2.13}$$

to be the invariant part of the direct imagde. Then the filtration $\{E_t\}_{t\in\mathbf{R}}$ is precisely the filtration associated to E_* as in (2.2) [Bi2], [Bi3].

Let $\beta^i : \tilde{D}_i \to Y$ denote the embedding in Y. The normal bundle on $\tilde{D}_i$ (for the embedding β^i) will be denoted by $N_{\tilde{D}_i}$. Let $K(Y)$ denote the Grothendieck group of coherent sheaves on Y. (Since Y is smooth, $K(Y)$ coincides with the Grothendieck group of locally free sheaves.) In [Bi3] ((3.15) and Lemma (3.16)) we showed that the following equality of elements of $K(Y)$ holds:

$$V = p^*E + \sum_{i=1}^{n} \sum_{j=1}^{l_i} \sum_{k=1}^{k_i m_j^i} \beta_*^i((F_j^i/F_{j+1}^i) \otimes N_{\tilde{D}_i}^k) \tag{2.14}$$

If we use $\tilde{L} = p^*L$ in order to define the degree of a coherent sheaf on Y, then (2.14) implies that

$$\deg V = \# G.\text{par\_deg} E_* \tag{2.15}$$

where $\# G$ is the cardinality of the group G.

Using the obvious identification of coherent subsheaves of E and orbifold subsheaves of V and (2.15) we get the following proposition:

Proposition 2.16. *The orbifold bundle V is orbifold semistable (resp. orbifold stable) if and only if E_* is parabolic semistable (resp. parabolic stable).*

In ([Bi2], Lemma 4.6) we proved the following lemma:

Lemma 2.17. *If E_* is parabolic ample then the vector bundle V is ample (in the usual sense).*

The symmetric tensor power $S^m(V)$, $m\geq 1$, of an orbifold bundle has an induced orbifold structure. The parabolic bundle corresponding to $S^i(V)$ (by (2.13)) is the parabolic symmetric power $S^m(E_*)$. It is easy to see that for a torsion-free coherent sheaf F on X, the equality, $(V\otimes p^*F)^G=E\otimes F$, holds. This immediately gives the following converse of Lemma 2.17:

Lemma 2.18. *If the orbifold vector bundle V is ample then the corresponding parabolic bundle E_* is parabolic ample.*

3. Parabolic Chern character

We now want to define parabolic Chern classes. We do not assume D to be a divisor of normal crossing.

For a coherent sheaf V on X, let $Ch(V)\in H^{\text{even}}(X, \mathbf{Q})$ be the Chern character of V. If V is a vector bundle of rank r, for a real number t, the t-th power of the Chern character of V, namely $Ch(V)^t$, makes sense as an element of $H^{\text{even}}(X, \mathbf{R})$. Indeed, setting $A := \sum_{j>1}Ch^j(V)$, where $Ch^j(V)$ is the component of $Ch(V)$ of degree $2j$, the power series expansion

$$Ch(V)^t=\Big(r+\sum_{j\geq 1}Ch^j(V)\Big)^t:=(r+A)^t=\sum_{j=0}^{\infty}\binom{t}{j}r^{t-j}A^j$$

where $\binom{t}{j}: = t\,(t-1)\ldots(t-j+1)/1.2..j$, is actually a finite sum, since $H^j(X, \mathbf{R})=0$ for $j>2d$.

Let E_* be a parabolic bundle given by (2.9). We define the *parabolic Chern Character* of E_*, denoted by $Ch(E_*)$, as follows:

$$Ch(E_*) :=Ch(E)\prod_{i=1}^{n}Ch(\mathcal{O}_X(n_i.\,D_i))^{\alpha_1^i}$$
$$+\sum_{i=1}^{n}\sum_{j=2}^{l_i}Ch(\bar{F}_j^i).\big(Ch(\mathcal{O}_X(n_i.D_i))^{\alpha_j^i}-Ch(\mathcal{O}_X(n_i.\,D_i))^{\alpha_{j-1}^i}\big)\in H^{\text{even}}(X, \mathbf{R}) \quad (3.1)$$

where $\bar{F}_j^i$ are as in (2.8).

There are polynomials P_k of k-variables and with rational coefficients such that for a coherent sheaf F on X, the k-th Chern class, $c_k(F)$, is $P_k(Ch^0(F), \ldots, Ch^k(F))$, where $Ch^j(F)$ is the component of $Ch(F)$ of degree $2j$.

The *k-th parabolic Chern class* of E_*, denoted by $c_k(E_*)$, is defined to be

$$c_k(E_*) :=P_k(Ch^0(E_*), Ch^1(E_*), \ldots, Ch^k(E_*))\in H^{2k}(X, \mathbf{R}) \quad (3.2)$$

where $Ch^j(E_*)$ is the component of $Ch(E_*)$ of degree $2j$.

Clearly we have that $Ch^0(E_*) = \text{rank } E$, and par_deg $E_* = (c_1(E_*) \cup c_1(L)^{d-1}) \cap [X]$.

Let $CH^*(X)$ denote the Chow ring of cycles on X modulo rational equivalence. There is a natural cycle class map

$$\psi : CH^*(X) \otimes_{\mathbf{Z}} \mathbf{R} \to H^{\text{even}}(X, \mathbf{R}) \tag{3.3}$$

Following (3.1) we may define $\overline{Ch}(E_*) \in CH^*(X) \otimes_{\mathbf{Z}} \mathbf{R}$ such that $\psi(\overline{Ch}(E_*)) = Ch(E_*)$.

Let D be a divisor of normal crossing. Take a vector bundle V on X equipped with a logarithmic connection, ∇, on V which is singular along D. (See [D], [Ka] for the definition of a logarithmic connection and its properties.) Let

$$R(\nabla, D_i) \in H^0(D_i, \text{End}(f_i^*E))$$

denote the residue of ∇ along D_i. So locally around D the connectinn ∇, with respect to some suitable trivialization of V, is of the form

$$d + \sum_{i=1}^{n} R(\nabla, D_i) \frac{dz_i}{z_i}$$

where z_i is a local defining equation for the divisor D_i. Assume that the real part of any eigenvalue, λ, of any $R(\nabla, D_i)$ satisfies the condition that

$$-1 < \text{Re}(\lambda) \leq 0$$

For example, given a flat vector bundle on $X - D$, there is a natural extension of the flat bundle as a logarithmic connection on X, known as the *Deligne extension*, satisfying the above eigenvalue condition for the residue [Ka]. The generalized eigenspace decomposition for $R(\nabla, D_i)$ gives a filtration as in (2.6), and the negative of the real part of the eigenvalues of the residue gives a string of numbers as in (2.7). Thus we have a parabolic structure on V. Let V_* denote the parabolic bundle obtained this way.

The Chern classes of V can be expressed in terms of the residues of ∇; the precise expression can be found in Theorem 3 (page 16) of [Oh].

It is an elaborate but straight-forward calculation to check that the parabolic Chern character

$$Ch(V_*) = \text{rank } V \tag{3.4}$$

The following inductive step is necessary in the computation of (3.4): The connection ∇ using local coordinates induces a logarthmic connection on f_i^*V, which is singular along $D_i \cap (D - D_i)$, and any $F^i{}_j$ is invariant under this connection. Thus F^i_j has an induced logarithmic connection, and hence any $\overline{F}^i_j$ has an induced logarithmic connection. So we may use the above mentioned result in [Oh] to calculate the Chern classes of $\bar{F}^i_j$.

4. Parabolic ample bundle and parabolic Chern classes

4a. Positivity of parabolic Chern classes. In [BG] it was proved that the (nontrivial) Chern classes of an ample vector bundle are all numerically positive. In [FL], extending the above result of [BG], the class of all numerically positive characteristic classes for ample vector bundles was identified. We will show that (2.14), Lemma 2.17, and the definition (3.1) combine together to give the generalizations of the above results to the parabolic context. For that we need to restrict the class of parabolic bundles.

Assumption 4.1. Henceforth we will always impose the following two conditions on the parabolic bundles that we will consider:
(1) the parabolic divisor is a divisor of normal crossing;
(2) all F^i_j (in (2.6)) are subbundles of f_i^*E.

Let $\mathscr{C}$ denote the collection of all pairs of the form (X, E_*) where X is as in Section 2 and E_* is a parabolic ample bundle of rank r on X satisfying the conditions in 4.1. Take a weighted homogeneous polynomial of degree d in r variables

$$P\in\mathbf{Q}[x_1, x_2..., x_r] \tag{4.2}$$

with the weight of x_i being i. Following [FL] we will call P as *numerically positive for parabolic ample bendles* if for any $(X, E_*)\in\mathscr{C}$,

$$\int_X P(c_1(E_*), c_2(E_*), ..., c_r(E_*))\in\mathbf{R} \tag{4.3}$$

is actually a strictly positive number.

Let Λ denote the space of partitions of d by nonnegative integers bounded by r.

For $\lambda\in\Lambda$, let P_λ denote the corresponding Schur polynomial [FL]. So

$$P=\sum_{\lambda\in\Lambda}c_\lambda P_\lambda \tag{4.4}$$

where c_λ are rational numbers.

Theorem 4.5. *Any parabolic Chern class is numerically positive for parabolic ample bundles. More generally, the characteristic polynomial P (in (4.2)) is numerically positive for parabolic ample pundles if and only if all c_λ (in (4.4)) are nonnegative and P is nonzero (i.e., not all c_λ are zero).*

Proof. Since the usual characteristic classes of a vector bundle are special cases of the parabolic characteristic classes (take the zero divisor as the parabolic divisor!) Theorem I (page 36) of [FL] (the result mentioned above) would imply that if P is numerically positive for parabolic ample bundles then P is nonzero with all c_λ being nonnegative. To prove the converse, take a nonzero polynomial P such that all c_λ (in (4.4)) are nonnegative. Take any $(X, E_*) \in \mathscr{C}$. We want to check that the real number in (4.3) is strictly positive.

A parabolic bundle is parabolic ample if and only if the new parabolic structures obtained from sufficiently small perturbations of the parabolic weights (keeping the qtuasi-parabolic structure fixed) are all parabolic ample. Similarly, a top cohomology class on a projective manifold is numerically (strictly) positive if and only if the cohomology classes obtained from sufficiently small perturbations of it are all numerically strictly positive classes. From these observations we conclude that in order to prove that P is numerically positive for parabolic ample bundles, it is enough to check the positivity of (4.3) only for parabolic bundles with rational parabolic weights. So we will assume that the parabolic weights of E_* are all rational numbers.

Comparing (2.14) and (3.1) it is a straight-forward calculation to check that

$$p^*Ch(E_*) = Ch(V) \tag{4.6}$$

(We will omit the computation for the above equality.) Note that the equality (4.6) is equivalent to the equality between Chern classes, namely $p^*c_i(E_*) = c_i(V)$, for all $i \geq 0$. From Lemma 2.17 we know that the vector bundle V on Y is ample. Thus, from Theorem I of [FL] we conclude that

$$c := \int_Y P(c_1(V), c_2(V), \dots, c_r(V)) > 0$$

Now the equality (4.6) implies that the real number in (4.3) is strictly positive $(= c/\#G)$. This completes the proof of the theorem.

Remark 4.7. In [F], Fulton identified the set of all positive characteristic classes for filtered ample vector bundles (filtered by subbundles). Imitating the above argument it is easy to establish the parabolic analogue of this result of Fulton.

4b. Examples of parabolic ample bundles. In [Bi2] we observed that a parabolic line bundle L_* on X is parabolic ample if the first parabolic Chern class $c_1(L_*) \in H^2(X, \mathbf{R})$ is contained in the positive cone in $NS(X) \otimes_{\mathbf{Z}} \mathbf{R}$ (i.e., the cohomology class is represented by a positive (1, 1)-form). We will show how using a certain result of E. Viehweg it is possible to construct examples of parabolic ample bundles of higher ranks. First we will describe the result of Viehweg in question.

Let $f : X \to M$ be a smooth surjective morphism between connected smooth projective varieties. The relative canonical bundle on X for the projection f will be denoted by $K_{X/M}$. Let $\mathscr{L}$ be an ample line bundle on X. The Proposition 2.43 (page 75) of [V] (which is proved using some results of T. Fujita and Y. Kawamata) implies that the direct image on M

$$W := f_*(\mathcal{L} \otimes K_{X/M})$$

is nef. (From the Kodaira vanishing theorem all the higher direct images of $\mathcal{L} \otimes K_{X/M}$ vanish, and hence W is locally free on M; now observe Remark 2.12.2 in page 59 of [V].)

We now want to show that W is actually an ample vector bundle on M. Take an ample line bundle ξ on M and a positive integer n such that the line bundle $\mathcal{L}^n \otimes f^*\xi^*$ on X is ample. Let $h : M' \to M$ be a finite surjective morphism, where M' is a connected smooth projective variety, such that there is a line bundle ζ on M' with $\zeta^n = h^*\xi$. The existence of such a morphism h is guaranteed by Lemma 2.1 of [BG].

Let X' be a component of the fiber product $X \times_M M'$. The obvious projection of X' to M' (resp. X) will be denoted by $\bar{f}$ (resp. $\bar{h}$). Replacing f and $\mathcal{L}$ by $\bar{f}$ and $\bar{h}^*\mathcal{L} \otimes \bar{f}^*\zeta^*$ respectively, in the above mentioned result of Viehweg and using the projection formula we get that the vector bundle $h^*W \otimes \zeta^*$ on M' is nef. (That $\bar{h}^*\mathcal{L} \otimes \bar{f}^*\zeta^*$ is ample on X' follows from the fact that the pullback of a line bundle by a finite morphism is ample if and only if the original line bundle is ample.) Tensor product of an ample line bundle and a nef bundle is ample. Since h is a finite morphism and ζ is ample (ζ is ample since ξ is assumed to be ample), we conclude that the vector bundle W is ample.

Let D' be a divisor of normal crossing on M. Since f is smooth, the pullback divisor $D := f^*D'$ on X is also a of normal crossing divisor. Let L_* be a parabolic ample line bundle on X with parabolic structure along the divisor D and with rational parabolic weights. Consider the decomposition

$$D' = \sum_{i=1}^{n} D'_i$$

of D' into its irreducible components. Let the parabolic weight of L_* along $f^*D'_i$ be m^i/N, where m^i and N are nonnegative integers.

Let $h : M' \to M$ be a Kawamata cover (as in Section 2) with Galois group G such that $h^*D'_i = k_i N (h^*D'_i)_{\mathrm{red}}$. Consider the fiber product

$$Y := X \times_M M'$$

The obvious projection of Y onto X is clearly a Galois cover with the same Galois group G. There is an orbifold line bundle $\mathcal{L}$ on Y (constructed as in Section 2) which corresponds to the parabolic line bundle L_*. Since L_* is assumed to be parabolic ample, from Lemma 2.17 we get that the line bundle $\mathcal{L}$ is ample on Y.

The obvious projection of Y onto M' will be denoted by $\bar{f}$. Let $K_{Y/M'}$ denote the relative canonical bundle on Y for the (smooth) projection $\bar{f}$. Earlier we saw that the amplitude of $\mathcal{L}$ implies that the direct image on M', namely $\bar{f}_*(\mathcal{L} \otimes K_{Y/M'})$, is ample. Clearly this direct image is an orbifold bundle for the Galois action of G on M'. We may apply the construction (2.14) to the orbifold bundle $\bar{f}_*(\mathcal{L} \otimes$

$K_{Y/M'}$). Now Lemma 2.18 implies that the parabolic bundle on M obtained this way (which has a parabolic structure along D') is actually parabolic ample. (This parabolic bundle can be directly constructed without using the covering h (i.e., using just L_* and f), but we will need the covering h in order to be able to conclude that the parabolic bundle is actually ample.)

4c. Unitary local systems. Let E_* be a parabolic stable bundle with rational weights (Definition 2.4). Assume the following vanishing of parabolic Chern classes:

$$c_1(E_*)=0=c_2(E_*)$$

Proposition 2.16 says that the corresponding orbifold bundle V is orbifold stable. From the equality (4.6) we conclude that

$$c_1(V)=0=c_2(V)$$

So there is a unique unitary flat connection on V, which is left invariant by the action of the orbifold group G on V (i.e., the connection operator commutes with the action of G), and it is irreducible for the action of G (i.e., there no proper nonzero orbifold subsheaf of V which is left invariant by the connection) ([S1], page 878, Theorem 1, Proposition 3.4). Let ∇ denote this flat unitary connection on V.

Since the restriction of V to $Y-\tilde{D}$ is the pullback of the restriction of E to $X-D$, the G invariance of the connection ∇ implies that ∇ would induce an unitary flat connection on the restriction of E to $X-D$. Let $\bar{\nabla}$ denote this connection on the restriction of E to $X-D$. The G-irreducibility of ∇ would imply that $\bar{\nabla}$ is actually an irreducible connection in the usual sense. Clearly, this connection $\bar{\nabla}$ does not in general extend across D; but it extends as a logarithmic connection on E.

The holonomy of $\bar{\nabla}$ around a component D_i is a k_iN-th root of the identity. Indeed, the k_iN-th multiple of the loop in $X-D_i$ around D_i lifts as a loop in $Y-\tilde{D}$ — a consequence of the fact that $p^*D_i=k_iN\tilde{D}_i$ — hence the k_iN-th power of the holonomy of $\bar{\nabla}$ along the loop around D_i must be the identity. Examining the construction of V from E_*, it is easy to deduce that the eigenvalues of the above holonomy are actually of the form $\exp(2\pi\sqrt{-1}m^i_j/N)$ (recall that $\alpha^i_j=m^i_j/N$). More precisely, the residue of $\bar{\nabla}$ (defined in Section 3) along the divisor D_i preserves the flag (2.6), and on the graded piece, F^i_j/F^i_{j+1}, this residue acts as multiplication by $-\alpha^i_j$.

Conversely, let E_* be a parabolic bundle with rational parabolic weights and let ∇' be an irreducible unitary flat connection on the restriction of E to $X-D$, such that its residue along any D_i has the above property - that it preserves the flag (2.6) and it acts on F^i_j/F^i_{j+1} as multiplication by $-\alpha^i_j$. Consider the induced connection on the restriction of V to $Y-\tilde{D}$. This connection extends across $\widetilde{D}$, and we get an unitary flat connection on V which is invariant under the action of G on V and it is G-irreducible. So V is orbifold stable, and $c_i(V)=0$, for any $i\geq 1$.

We put down the above observations in the form of the following:

Theorem 4.8. *Let E_* be a parabolic stable bundle with rational parabolic weights such that it satisfies the following two conditions: (1) Assumption 4.1 holds; and (2)*

$$c_1(E_*)=0=c_2(E_*) \tag{4.9}$$

Then E admits a unique irreducible flat unitary connection outside the parabolic divisor such that the residue along any component of the parabolic divisor satisfies the above condition. Conversely, let E_* be a parabolic bundle with rational parabolic weights such that E admits an rreducible flat unitary connection outside the parabolic divisor and the residue along any component of the parabolic divisor satisfies the above condition. Then E_* must be a parabolic stable bundle satisfying (4.9).

In the rest of this section we will always assume that the parabolic weights of E_* are rational numbers, and that E_* is parabolic stable satisfying (4.9).

Let $\Omega_X^*(\log D)$ denote the sheaf of *logarithmic forms* on X [D, page 72, Definition 3.1]. Recall that there a natural *residue* map

$$\mathcal{R} : \Omega_X^*(\log D) \to \sum_{i=1}^{n} \Omega_{D_i}^* \tag{4.10}$$

such that the kernel of $\mathcal{R}$ is Ω_X^*.

Let E_* be a parabolic bundle (as in (2.9)). For some $i\in\{1, \ldots, n\}$, the number α_i^j may be zero. Let $\alpha_1^j=0$ for $j=1, 2, \ldots, m$. Define

$$\bar{E} := \bigcap_{j=1}^{m} \bar{F}_2^j \subseteq E \tag{4.11}$$

to be the vector bundle on X (the subsheaves $\bar{F}_j^i$ were defined in (2.8)). If all the α_1^i are nonzero, then $\bar{E}=E$.

For $k\geq 0$, consider the vector bundle $\Omega_X^k(\log D)\otimes E$ on X. Let $\Omega_X^k(\log D)(\bar{E})$ denote the subsheaf of it generated together by $\Omega_X^k\otimes E$ and $\Omega_X^k(\log D)\otimes\bar{E}$. Clearly $\Omega_X^k(\log D)(\bar{E})$ is locally free on X, and it coincides with $\Omega_X^k(\log D)\otimes E$ if all α_1^i are nonzero.

In Lemma 4.11 and Corollary 4.14 of [Bi2] we proved that

$$(p_*(\Omega_Y^k\otimes V))^G=\Omega_X^k(\log D)(\bar{E}) \tag{4.12}$$

(The orbifold bundle V corresponds to E_*.) Since p is a finite covering morphism, for any coherent sheaf F on Y and any $q\geq 0$, the following equality holds:

$$H^q(Y, F)=H^q(X, p_*F)$$

The group G has acts naturally on the bundle $\Omega_Y^k \otimes V$; so it has an induced action on its cohomology $H^q(Y, \Omega_Y^k \otimes V)$.

Let $\bar{\rho} : G \to \mathrm{Aut}(p_*(\Omega_Y^k \otimes V))$ denote the natural homomorphism induced by the action of G on $\Omega^k{}_Y \otimes V$. The inclusion of the sheaf of invariants, $(p_*(\Omega_Y^k \otimes V))^G$, in $p_*(\Omega_Y^k \otimes V)$ has a natural splitting given by the kernel of the endomorphism

$$\sum_{g \in G} \bar{\rho}(g) \in H^0(X, \mathrm{End}(p_*(\Omega_Y^k \otimes V)))$$

Now using (4.12) we get that

$$H^q(Y, \Omega_Y^k \otimes V)^G = H^q(X, p_*(\Omega_X^k \otimes V)^G) = H^q(X, \Omega_X^k(\log D)(\bar{E})) \quad (4.13)$$

Let $\mathscr{E}$ denote the sheaf on X given by the kernel of the logarithmic connection

$$\bar{\nabla} : E \to E \otimes \Omega_X^1(\log D)$$

The restriction of $\mathscr{E}$ to $X-D$ is the local system given by the (restriction of the) connection $\bar{\nabla}$. Let $\mathscr{V}$ denote the local system on Y given by the connection ∇. So,

$$H^q(Y, \mathscr{V})^G = H^q(X, \mathscr{E}) \quad (4.14)$$

Since $\mathscr{V}$ is an unitary local system, its cohomology, $H^*(Y, \mathscr{V})$, has Hodge decomposition and Lefschetz decomposition. The following is the Hodge decomposition:

$$H^q(Y, \mathscr{V}) = \sum_{j+k=q} H^j(Y, \Omega_Y^k \otimes V) \quad (4.15)$$

Define $\bar{\omega} := c_1(\tilde{L}) \in H^1(Y, \Omega_Y^1)$. The cohomology class acts by multiplication on the right hand side of (4.15). Let $P^j(\mathscr{V}) \subset H^j(Y, \mathscr{V})$ denote the primitive part, i.e., it is the kernel of the operator given by multiplication by the $(d-j+1)$-th power of $\bar{\omega}$. The operator given by the multiplication by $\bar{\omega}^j$ will also be denoted by $\bar{\omega}^j$. The following is the Lefschetz decomposition :

$$H^*(Y, \mathscr{V}) = \sum_{j=0}^{[d/2]} \sum_{i=0}^{(d-j)} P^j(\mathscr{V}) \bar{\omega}^i \quad (4.16)$$

The pairing

$$\theta_1 \otimes \theta_2 \mapsto \int_Y < \theta_1, \theta_2 > \bar{\omega}^{(d-j)}$$

defines a nondegenerate bilinear form on $P^j(\mathscr{V})$. So using the decomposition (4.16) we get a nondegenerate bilinear form on $H^j(Y, \mathscr{V})$, which is symmetric or skew-symmetric depending on whether q is even or odd. Since the cohomology class $\bar{\omega}$ is an invariant for the action of G on $H^1(Y, \Omega_Y^1)$, the decompositions (4.15) and (4.16) are both equivariant for the action G. Thus we have Hodge decomposition and the Lefschetz decomposition for the invariant part $H^*(Y, \mathcal{V})^G$, along with a nondegenerate bilinear form on it.

Let $\mathcal{V}^*$ denote the dual local system. Taking conjugations, in Ω_Y^* and the one from $\mathcal{V}$ to $\mathcal{V}^*$, we get an $\mathbf{C}$-antilinear isomorphism

$$\gamma : H^q(Y, \Omega_Y^k \otimes V) \to H^k(Y, \Omega_Y^q \otimes V^*) \tag{4.17}$$

Let $\omega := c_1(L) \in H^1(X, \Omega_X^1)$ be the cohomology class on X. The above observations combine together to imply that there is a Hodge decomposition and a Lefschetz decomposition for $\mathcal{E}$. In other words,

$$H^q(Y, \mathcal{E}) = \sum_{j+k=q} H^j(X, \Omega_X^k(\log D)(\bar{E})) \tag{4.18}$$

$$H^*(Y, \mathcal{E}) = \sum_{j=0}^{[d/2]} \sum_{i=0}^{(d-j)} P^j(\mathcal{E})\,\omega^i \tag{4.19}$$

where $P^j(\mathcal{E})$ denotes the subspace of $H^j(X, \mathcal{E})$ which is the kernel of the operator given by the multiplication by $\omega^{(d-j+1)}$. (The class ω acts by multiplication on the right hand side of (4.18).)

Let E_*^* denote the dual parabolic bundle for E_*. (The underlying sheaf for E_*^* (i.e., E_0^*) is the subsheaf of E^* which maps any E_t, $0<t\leq 1$, into $\mathcal{O}_X(-D)$. The filtration of E_*^* is defined by the following rule : E_t^* maps E_s into $\mathcal{O}_X(f(t, s).D)$, where $f(t, s)$ is the smallest integer than or equal to $t+s$.)

It is easy to check that the parabolic bundle E_*^* corresponds to the orbifold bundle V^*. Replacing the parabolic bundle E_* by E_*^* in the construction of $\bar{E}$, the vector bundle on X thus obtained will be denoted by $\hat{E}$. Let $\Omega_X^i(\log D)(\hat{E})$ denote the vector bundle on X obtained by replacing E_* by E_*^* in the construction of $\Omega_X^i(\log D)(\bar{E})$.

Taking G invariants of both sides of (4.17) we get the following $\mathbf{C}$-antilinear isomorphism induced by γ :

$$\bar{\gamma} : H^q(X, \Omega_X^k(\log D)(\bar{E})) \to H^k(X, \Omega_X^q(\log D)(\hat{E})) \tag{4.20}$$

As a consequence of (4.18), (4.19) and (4.20), we obtain the result of [T] (see also Theorem 13.5, page 139 of [EV]) in the case of elliptic local systems (i.e., the finite order of the holonomy around any boundary component).

5. Parabolic Higgs bundles

Let $\mathrm{End}_{\mathrm{par}}(E) \subseteq \mathrm{End}(E)$ be the coherent subsheaf consisting of all endomorphisms which preserve the flag (2.6) (for all i). Similarly, let $\mathrm{End}_{\mathrm{par}}(f_i^*E)$ be the coherent subsheaf of $\mathrm{End}(f_i^*E)$ on D_i which preserves the flag (2.6). For a section θ of $\Omega_X^1(\log D) \otimes \mathrm{End}_{\mathrm{par}}(E)$, using the residue map in (4.10) we have

$$\text{residue}(D_i, \theta) \in H^0(D_i, \mathrm{End}_{\mathrm{par}}(f_i^* E)) \tag{5.2}$$

Using the algebra structure of $\mathrm{End}_{\mathrm{par}}(E)$ given by composition, and the exterior algebra structure of $\Omega_X^*(\log D)$, there is an algebra structure on $\Omega_X^*(\log D) \otimes \mathrm{End}_{\mathrm{par}}(E)$; the multiplication operation of this algebra will be denoted by $\wedge$.

Definition 5.3. A *parabolic Higgs bundle* is a pair of the form (E_*, θ), where E_* is a parabolic bundle (as in (2.9)) and $\theta \in H^0(X, \Omega_X^1(\log D) \otimes \mathrm{End}_{\mathrm{par}}(E))$, satisfying the following two conditions:

(1) the endomorphism, residue (D_i, θ), (defined in (5.2)) maps the subbundle F_j^i (in (2.6)) into F_{j+1}^i,

(2) the section of $\Omega_X^2(\log D) \otimes \mathrm{End}_{\mathrm{par}}(E)$, namely $\theta \wedge \theta$ is the zero section.

A *paraboljc Higgs field* on a parabolic bundle E_* is a section θ, as above, such that (E_*, θ) is a parabolic Higgs bundle.

A parabolic Higgs bundle (E_*, θ) is defined to be *stable* if for any proper nonzero subsheaf, $F \subset E$, with E/F torsion-free and $\theta(F) \subseteq F \otimes \Omega_X^1$, the following inequality holds: $\text{par-}\mu F_* < \text{par-}\mu E_*$.

The above condition 5.3(1) can be rephrased as: the residue of θ along D_i is nilpotent with respect to the flag in (2.6).

Remark 5.4. The definition of parabolic Higgs that we adapt above is slightly diffierent from the one given in [Y]. The residue of a parabolic Higgs field according to [Y] would preserve the flag in (2.6), as opposed to the stronger condition here that the Higgs field is actually nilpotent with respect to the flag.

Following [S1], we define an *orbifold Higgs field* on an orbifold bundle V on Y (as in section 2) to be a section $\phi \in H^0(Y, \Omega_Y^1 \otimes \mathrm{End}(V))$, such that $\phi \wedge \phi = 0$, and ϕ is an invariant for the action of the orbifold group G on $\Omega_Y^1 \otimes \mathrm{End}(V)$. The pair (V, ϕ) is called *orbifold stable* if for any proper nonzero orbifold subsheaf, $F \subset V$, with V/F torsion-free and $\phi(F) \subseteq F \otimes \Omega_Y^1$, the following inequality holds: $\mu(F) < \mu(V)$.

Theorem 5.5. *For E_* and V related as in Section 2, there is a natural one-to-one correspondence between parabolic Higgs field on E_* and orbifold Higgs field on V, such that the parabolic stable Higgs bundles correspond to the orbifold stable Higgs bundles.*

Proof. This again is an easy computation. Let $\mathrm{End}(E_*)$ denote the parabolic bundle given by the parabolic tensor product $E_*^* \otimes E_*$. It is a straight-forward computation to check that the underlying vector bundle for the parabolic bundle $\mathrm{End}(E_*)$ is $\mathrm{End}_{\mathrm{par}}(E)$.

We noted in (4.12) that $(p_*(\Omega_Y^1 \otimes V))^G = \Omega_X^1(\log D)(\bar{E})$. First check that if in the construction of $\Omega_X^1(\log D)(\bar{E})$ we replace E_* by $\mathrm{End}(E_*)$, then we obtain precisely the subsheaf of $\Omega^1_X(\log D)\otimes\mathrm{End}(E)$ defined by the condition that the residue along D_i is nilpotent with respect to the flag in (2.6).

Now replacing E_* by $\mathrm{End}(E_*)$ in (4.12) and setting $k=1$, and then taking the global sections of both sides of (4.12), we get an identification between the parabolic Higgs fields on E_* and the orbifold Higgs fields on V. (We used the facts that $H^0(Y, \mathrm{End}(V)\otimes\Omega^1_Y)=H^0(X, p_*(\mathrm{End}(V)\otimes\Omega^1_Y))$, and that the condition $\phi\wedge\phi=0$ translates into the condition that $\theta\wedge\theta=0$.)

Since the G-invariant subsheaves of V are in one-to-one correspondence with the subsheaves of E-that the parabolic stable Higgs bundles correspond to the orbifold stable bundles, is immediate. This complete the proof of the theorem.

Let E_* be a parabolic stable Higgs bundles satisfying (4.9). Now we may apply the Theorem 1 and Proposition 3.4 of [S1] (page 878), and obtain a G-irreducible flat Hermitian-Yang-Mills connection on V. This in turn will induce a flat Hermitian-Yang-Mills connection on the restriction of E to $X-D$ with some precies boundary conditions at the infinity. Thus we obtain the analogue of Theorem 4.8 for parabolic stable Higgs bundles.

In Section 2 of [S2] (page 23-26) Simpson proved several properties of the (hyper) cohomologies of the various complexes associated to a Higgs bundle equipped with a flat Hermitian-Yang-Mills connections. For example, the Lefschetz decomposition (Lemma 2.6), the Kodaira-Serre duality (Lemma 2.5) and the isomorphism between the (hyper) cohomologies of all the relevant complexes (Lemma 2.2) are proved. We may apply these results to the vector bundle V. Just as in Section 4c, taking G-invariants, we may derive the parabolic analogues of all the above mentioned results in [S2]. We will omit this routine work. We also observe that the parabolic analogues of the results on the Hitchin system on the moduli of Higgs bundles proved in [Bi1] are also valid.

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