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A control problem in biconvective flow

By

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1. Introduction

An important biological problem is biconvective flow, "biconvection" being a convection caused by the concentration of upward swimming microorganisms in culture fluid. A model for this problem was introduced in [4] and [5] independently. They discuss biological and physical aspects related to this problem. This model, consisting of the equations for the motion of the culture fluid assumed to be viscous and incompressible and for the concentration of microorganisms, was studied from a mathematical point of vue in [3]. The authors prove the existence of a solution and the positivity of the concentration for the stationary problem and they also study the nonstationary case.

The purpose of this paper is to introduce and study a control problem related to biconvective flow. Our aim is to characterize the mean values α of the concentrations which lead us to a given field of concentration c.

We begin by establishing an existence and uniqueness result for the stationary biconvective flow (Section 2). Our mathematical approach is different from that of [3] and allows us to obtain the existence of a solution for a less restrictive assumption.

In Section 3 we introduce a control problem associated to the stationary biconvective flow. The existence of an optimal control is proved. When the relation mean value α - concentration c is multi-valued, the derivation of the necessary conditions of optimality is perfomed by introducing an approximate family of control problems. In the uniqueness case, these conditions are obtained directly, as in [1].

2. Model of biconvective flow. Existence and uniqueness results

The stationary flow of a culture viscous, incompressible fluid is considered. We suppose that the flow region is a bounded domain $\Omega\subset\boldsymbol{R}^3$ with Lipschitz boundary, $\partial\Omega$.

We seek for a vector function u representing the velocity of the culture fluid and two scalar functions c and p representing the concentration of microorganisms and the pressure of the culture fluid, respectively, which are

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defined in Ω and satisfy the following system of equations and boundary conditions:

$$-\nu\Delta \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=-g(1+\gamma c)\boldsymbol{i}_3+\boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u}=0 \quad \text{in } \Omega, \tag{2.2}$$

$$-\theta\Delta c+\boldsymbol{u}\cdot\nabla c+U\frac{\partial c}{\partial x_3}=0 \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{u}=0 \quad \text{on } \partial\Omega, \tag{2.4}$$

$$\theta\frac{\partial c}{\partial n}-Ucn_3=0 \quad \text{on } \partial\Omega, \tag{2.5}$$

where $\boldsymbol{f}$ is a given external force, g the acceleration of gravity , $\nu>0$ the kinematic viscosity of the culture fluid, $\theta>0$ the diffusion rate of microorganisms, $U>0$ the mean speed of upward swimming of microorganisms, $\gamma=\frac{\rho_0}{\rho_m}-1>0$, ρ_0, ρ_m being the density of an individual organism and of the culture fluid, respectively, $\boldsymbol{i}_3$ the unit vector in the vertical direction, $\boldsymbol{n}$ the outward unit normal to $\partial\Omega$ and $n_3=\boldsymbol{n}\cdot\boldsymbol{i}_3$.

We introduce the new function:

$$q=p+gx_3. \tag{2.6}$$

Therefore, the problem (2.1) - (2.5) is equivalent to:

$$-\nu\Delta \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla q=-\kappa c\boldsymbol{i}_3+\boldsymbol{f} \quad \text{in } \Omega, \tag{2.7}$$

$$\operatorname{div} \boldsymbol{u}=0 \quad \text{in } \Omega, \tag{2.8}$$

$$-\theta\Delta c+\boldsymbol{u}\cdot\nabla c+U\frac{\partial c}{\partial x_3}=0 \quad \text{in } \Omega, \tag{2.9}$$

$$\boldsymbol{u}=0 \quad \text{on } \partial\Omega, \tag{2.10}$$

$$\theta\frac{\partial c}{\partial n}-Ucn_3=0 \quad \text{on } \partial\Omega, \tag{2.11}$$

where $\kappa=g\gamma$.

It can be easily proved that a variational formulation of (2.7) - (2.11) is the following:

$$\begin{aligned}
&(\boldsymbol{u}, c)\in Y_0\times H^1(\Omega),\\
&\nu a_0(\boldsymbol{u}, \boldsymbol{z})+b_0(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{z})=-\kappa\int_\Omega c\boldsymbol{i}_3\cdot \boldsymbol{z}dx+\langle \boldsymbol{f}, \boldsymbol{z}\rangle \qquad \forall \boldsymbol{z}\in Y_0,\\
&\theta a(c, r)+b(\boldsymbol{u}, c, r)-U\int_\Omega c\frac{\partial r}{\partial x_3}dx=0 \qquad \forall r\in H^1(\Omega),
\end{aligned} \tag{2.12}$$

where:

$$a_0(\boldsymbol{u}, \boldsymbol{z})=\int_\Omega \nabla \boldsymbol{u}\cdot\nabla \boldsymbol{z}dx,$$

$$b_0(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}) = \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \cdot \boldsymbol{z} dx, \tag{2.13}$$

$$a(c, r) = \int_\Omega \nabla c \cdot \nabla r dx,$$

$$b(\boldsymbol{u}, c, r) = \int_\Omega (\boldsymbol{u} \cdot \nabla c) r dx,$$

the symbol $\langle \cdot , \cdot \rangle$ denotes the duality pairing between $(H^{-1}(\Omega))^3$ and $(H_0^1(\Omega))^3$ and Y_0 is the separable Hilbert space (see [6]):

$$Y_0 = \{\boldsymbol{v} \in (H_0^1(\Omega))^3 / \operatorname{div} \boldsymbol{v} = 0\},$$

embedded with the scalar product: $(\boldsymbol{v}, \boldsymbol{w})_{Y_0} = a_0(\boldsymbol{v}, \boldsymbol{w})$. The known function $\boldsymbol{f}$ has been taken in $(H^{-1}(\Omega))^3$.

We remark that $(\boldsymbol{u}_0, 0)$ is a solution of (2.12), where $\boldsymbol{u}_0$ satisfies the Navier-Stokes problem for incompressible fluid. Since this solution does not describe the biconvective flow, we shall study the following problem:

$$\begin{aligned}
&(\boldsymbol{u}_\alpha, c_\alpha) \in Y_0 \times H^1(\Omega), \\
&\nu a_0(\boldsymbol{u}_\alpha, \boldsymbol{z}) + b_0(\boldsymbol{u}_\alpha, \boldsymbol{u}_\alpha, \boldsymbol{z}) = -\kappa \int_\Omega c_\alpha \boldsymbol{i}_3 \cdot \boldsymbol{z} dx + \langle \boldsymbol{f}, \boldsymbol{z} \rangle \qquad \forall \boldsymbol{z} \in Y_0, \\
&\theta a(c_\alpha, r) + b(\boldsymbol{u}_\alpha, c_\alpha, r) - U \int_\Omega c_\alpha \frac{\partial r}{\partial x_3} = 0 \qquad \forall r \in H^1(\Omega), \\
&\int_\Omega c_\alpha dx = \alpha,
\end{aligned} \tag{2.14}$$

where α is a positive constant.

In the sequel we shall prove (with some assumptions about θ, U, ν and α) the existence and uniqueness of the solution of (2.14). For this purpose, we define:

$$\tilde{c}_\alpha = c_\alpha - \frac{\alpha}{|\Omega|} \tag{2.15}$$

and we obtain the following equivalent problem to (2.14):

$$\begin{aligned}
&(\boldsymbol{u}_\alpha, \tilde{c}_\alpha) \in Y_0 \times \tilde{H}^1, \\
&\nu a_0(\boldsymbol{u}_\alpha, \boldsymbol{z}) + b_0(\boldsymbol{u}_\alpha, \boldsymbol{u}_\alpha, \boldsymbol{z}) = -\kappa \int_\Omega \tilde{c}_\alpha \boldsymbol{i}_3 \cdot \boldsymbol{z} dx + \langle \boldsymbol{f}, \boldsymbol{z} \rangle \qquad \forall \boldsymbol{z} \in Y_0, \\
&\theta a(\tilde{c}_\alpha, r) + b(\boldsymbol{u}_\alpha, \tilde{c}_\alpha, r) - U \int_\Omega \tilde{c}_\alpha \frac{\partial r}{\partial x_3} dx = \frac{U\alpha}{|\Omega|} \int_\Omega \frac{\partial r}{\partial x_3} dx \qquad \forall r \in \tilde{H}^1,
\end{aligned} \tag{2.16}$$

where $\tilde{H}^1$ is the following Hilbert space:

$$\tilde{H}^1 = \{r \in H^1(\Omega) / \int_\Omega r dx = 0\},$$

embedded with the scalar product $(c, r)_{\tilde{H}^1} = a(c, r)$. We shall use the following estimates:

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{(L^2(\Omega))^3}\leq C\|\boldsymbol{v}\|_{Y_0} \qquad \forall \boldsymbol{v}\in Y_0,\\
&\|c\|_{L^2(\Omega)}\leq C\|c\|_{\widetilde{H}^1} \qquad \forall c\in \widetilde{H}^1,\\
&|b_0(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z})|\leq C_1\|\boldsymbol{u}\|_{Y_0}\|\boldsymbol{w}\|_{Y_0}\|\boldsymbol{z}\|_{Y_0} \qquad \forall \boldsymbol{u},\boldsymbol{w},\boldsymbol{z}\in Y_0,\\
&|b(\boldsymbol{u},c,r)|\leq C_1\|\boldsymbol{u}\|_{Y_0}\|c\|_{\widetilde{H}^1}\|r\|_{\widetilde{H}^1} \qquad \forall \boldsymbol{u}\in Y_0,\ \forall c,r\in\widetilde{H}^1,
\end{aligned}
\tag{2.17}
$$

where C, C_1 are positive constants depending only on Ω.

In the following we shall suppose (see [3]):

$$U<\frac{\theta}{C}. \tag{2.18}$$

Proposition 2.1. *If* $(\boldsymbol{u}_\alpha, \widetilde{c}_\alpha)$ *is a solution of* (2.16), *then we have the following estimates:*

$$\|\widetilde{c}_\alpha\|_{\widetilde{H}^1}\leq\frac{U\alpha}{\sqrt{|\Omega|}\,(\theta-UC)}, \tag{2.19}$$

$$\|\boldsymbol{u}_\alpha\|_{Y_0}\leq\frac{1}{\nu}\left(\|\boldsymbol{f}\|_{(H^{-1}(\Omega))^3}+\frac{\kappa C^2U\alpha}{\sqrt{|\Omega|}\,(\theta-UC)}\right). \tag{2.20}$$

Proof. For obtaining (2.19) and (2.20) we take $r=\widetilde{c}_\alpha$ and $\boldsymbol{z}=\boldsymbol{u}_\alpha$ in (2.16) and we use (2.17), (2.18) and the equalities:

$$
\begin{aligned}
&b_0(\boldsymbol{u},\boldsymbol{w},\boldsymbol{w})=0 \qquad \forall \boldsymbol{u},\boldsymbol{w}\in Y_0,\\
&b(\boldsymbol{u},c,c)=0 \qquad \forall \boldsymbol{u}\in Y_0,\ \forall c\in\widetilde{H}^1.
\end{aligned}
$$

In the sequel we shall establish the main result of this Section:

Theorem 2.2. *For every* $\alpha\geq 0$, *the problem* (2.16) *has at least one solution* $(\boldsymbol{u}_\alpha, \widetilde{c}_\alpha)$. *Moreover, for*

$$\nu^2>C_1\|\boldsymbol{f}\|_{(H^{-1}(\Omega))^3}, \tag{2.21}$$

there exists $\alpha'>0$ *such that for every* $\alpha\in[0,\alpha']$, (2.16) *has a unique solution.*

Proof. For obtaining the existence we define the mapping
$G\colon Y_0\times\widetilde{H}^1\mapsto(Y_0\times\widetilde{H}^1)'$,

$$
\begin{aligned}
&\langle\!\langle G(\boldsymbol{u},c),(\boldsymbol{z},r)\rangle\!\rangle=\lambda\left(\nu a_0(\boldsymbol{u},\boldsymbol{z})+b_0(\boldsymbol{u},\boldsymbol{u},\boldsymbol{z})+\kappa\int_\Omega c\boldsymbol{i}_3\cdot\boldsymbol{z}dx-\langle\boldsymbol{f},\boldsymbol{z}\rangle\right)+\\
&+\theta a(c,r)+b(\boldsymbol{u},c,r)-U\int_\Omega c\frac{\partial r}{\partial x_3}dx-\frac{U\alpha}{|\Omega|}\int_\Omega\frac{\partial r}{\partial x_3}dx\\
&\forall\,(\boldsymbol{u},c),(\boldsymbol{z},r)\in Y_0\times\widetilde{H}^1,
\end{aligned}
\tag{2.22}
$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the duality pairing between $(Y_0\times\widetilde{H}^1)'$ and $Y_0\times\widetilde{H}^1$ and λ is a positive fixed constant, $\lambda<\dfrac{4\nu(\theta-UC)}{\kappa^2C^4}$. It can be easily proved that, if

(2.18) holds, there exists $r>0$ such that $\forall\ (\boldsymbol{u}, c)\in Y_0\times\tilde{H}^1$ with $\|(\boldsymbol{u}, c)\|_{Y_0\times\tilde{H}^1}=r$ we have:

$$\ll G(\boldsymbol{u}, c),\ (\boldsymbol{u}, c)\gg\ \geq 0. \tag{2.23}$$

Moreover G is a continuous mapping with respect to the weak topologies of $Y_0\times\tilde{H}^1$ and $(Y_0\times\tilde{H}^1)'$.

Therefore, from the Gossez' theorem (see [2]) it follows the existence of an element $(\boldsymbol{u}_\alpha, \tilde{c}_\alpha)\in\bar{B}_r(0)\subset Y_0\times\tilde{H}^1$ such that $G(\boldsymbol{u}_\alpha, \tilde{c}_\alpha)=0$, and, hence, the first assertion of the Theorem holds.

For proving the uniqueness, we assume that (2.16) has two solutions $(\boldsymbol{u}_\alpha^1, \tilde{c}_\alpha^1)$ and $(\boldsymbol{u}_\alpha^2, \tilde{c}_\alpha^2)$. By subtracting the corresponding equations for $\boldsymbol{z}=\boldsymbol{u}_\alpha^1-\boldsymbol{u}_\alpha^2$ and $r=\tilde{c}_\alpha^1-\tilde{c}_\alpha^2$ and by using (2.17)-(2.20) we get:

$$\|\tilde{c}_\alpha^1-\tilde{c}_\alpha^2\|_{\tilde{H}^1}\leq\frac{C_1U\alpha}{\sqrt{|\Omega|}\,(\theta-UC)^2}\|\boldsymbol{u}_\alpha^1-\boldsymbol{u}_\alpha^2\|_{Y_0},$$

$$\|\boldsymbol{u}_\alpha^1-\boldsymbol{u}_\alpha^2\|_{Y_0}\leq C(\alpha)\,\|\boldsymbol{u}_\alpha^1-\boldsymbol{u}_\alpha^2\|_{Y_0},$$

where:

$$C(\alpha)=\frac{C_1}{\nu^2}\|\boldsymbol{f}\|_{(H^{-1}(\Omega))^3}+\alpha\frac{\kappa C^2C_1U}{\sqrt{|\Omega|}\,(\theta-UC)\,\nu}\left(\frac{1}{\nu}+\frac{1}{\theta-UC}\right). \tag{2.24}$$

It follows that, if (2.21) holds, then we obtain $C(\alpha)<1$ for all $\alpha\in[0, \alpha']$ where:

$$\alpha'<\frac{(\nu^2-C_1\|\boldsymbol{f}\|_{(H^{-1}(\Omega))^3})\sqrt{|\Omega|}\,(\theta-UC)^2}{\kappa C^2C_1U(\theta-UC+\nu)} \tag{2.25}$$

and the proof of the Theorem is achieved.

We remark that we have obtained the uniqueness of the solution of the coupled system (2.16) for the same condition (2.21) as in the case of Navier-Stokes problem (see [6]).

3. A control problem

In this Section, we suppose that (2.18) is satisfied.

We consider the functional $J\colon K\times H^1(\Omega)\mapsto\boldsymbol{R}$,

$$J(\alpha, c)=\frac{1}{2}\int_\Omega(c-c_d)^2dx+\frac{N}{2}\alpha^2, \tag{3.1}$$

where $K\subset[0, +\infty)$ is a closed, non empty interval, N is a nonnegative constant and $c_d\in L^2(\Omega)$ is a given function.

We formulate the optimal control problem as follows:

(3.2) $$\min\{J(\alpha, c)/(\alpha, c)\in T\},$$

where T is the nonempty, weakly closed set:

(3.3) $$T=\{(\alpha, c)\in K\times H^1(\Omega)/\exists\, \boldsymbol{u}\in Y_0 \text{ such that } (\boldsymbol{u}, c) \text{ satisfies } (2.14)\}$$

We remark that the minimum problem (3.2) may be also written:

$$\min\{\min\{J(\alpha, c)/c\in T_\alpha\}/\alpha\in K\},$$

where $T_\alpha=\{c\in H^1(\Omega)/\exists\, \boldsymbol{u}\in Y_0$ such that $(\boldsymbol{u}, c)$ satisfies (2.14)$\}$.

The physical relevant term in (3.1) is $\frac{1}{2}\int_\Omega (c-c_d)^2 dx$ which provides an estimate of the difference between the component c of an element $(\boldsymbol{u}, c)$ satisfying (2.14) and a given configuration c_d of concentration.

In the sequel, we suppose that K is bounded or $N>0$.

The first result to prove is the existence of a solution of (3.2).

Proposition 3.1. *The optimal control problem* (3.2) *has at least one solution.*

Proof. It can be easily proved that any minimizing sequence $\{(\alpha_n, c_n)\}_n\subset T$ of J is bounded in $K\times H^1(\Omega)$. Moreover, J is weakly continuous on $K\times H^1(\Omega)$ and T is weakly closed. Hence the assertion of Proposition holds.

We remark that, in general, if $(\alpha, c)\in T$, the correspondence $\alpha\mapsto c$ is multi-valued. Hence the derivation of the necessary conditions of optimality is not obvious.

In order to obtain these conditions we approximate J (in the sense that we make precise in Proposition 3.3) by a family of functionals $\{J_\varepsilon\}_{\varepsilon>0}$, J_ε: $Y_0\times K\mapsto \boldsymbol{R}$,

(3.4) $$J_\varepsilon(\boldsymbol{w}, \alpha)=J(\alpha, c(\boldsymbol{w}, \alpha))+\frac{1}{2\varepsilon}\|\boldsymbol{u}(\boldsymbol{w}, \alpha)-\boldsymbol{w}\|^2_{Y_0},$$

where $(\boldsymbol{u}(\boldsymbol{w}, \alpha), c(\boldsymbol{w}, \alpha))$ is the unique solution of:

(3.5) $$\begin{aligned}&(\boldsymbol{u}, c)\in Y_0\times H^1(\Omega),\\ &\nu a_0(\boldsymbol{u}, \boldsymbol{z})+b_0(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{z})=-\kappa\int_\Omega c\boldsymbol{i}_3\cdot \boldsymbol{z}dx+\langle \boldsymbol{f}, \boldsymbol{z}\rangle \qquad \forall \boldsymbol{z}\in Y_0,\\ &\theta a(c, r)+b(\boldsymbol{w}, c, r)-U\int_\Omega c\frac{\partial r}{\partial x_3}dx=0 \qquad \forall r\in H^1(\Omega),\\ &\int_\Omega c dx=\alpha.\end{aligned}$$

We remark that the correspodence $(\boldsymbol{w}, \alpha)\mapsto(\boldsymbol{u}(\boldsymbol{w}, \alpha), c(\boldsymbol{w}, \alpha))$ is uni-valued. This allows us to obtain the necessary conditions of optimality for the following control problem:

(3.6) $$\min\{J_\varepsilon(\boldsymbol{w},\alpha)/(\boldsymbol{w},\alpha)\in Y_0\times K\}.$$

Then, by passing to the limit, we derive the desired conditions for a solution of the control problem (3.2).

We begin by proving the existence of an optimal control for (3.6).

Proposition 3.2. *There exists at least one solution of* (3.6).

Proof. The proof relies on two properties: the boundedness of any minimizing sequence $\{(\boldsymbol{w}_\varepsilon^n, \alpha_\varepsilon^n)\}_n$ of J_ε on $Y_0\times K$ and the lower weak semicontinuity of J_ε.

The next result establishes the relation between the problems (3.2) and (3.6).

Proposition 3.3. *For any $\varepsilon>0$, let $(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)$ be a minimum point of J_ε. Then there exists $(\alpha^*, c^*, \boldsymbol{u}^*)\in K\times H^1(\Omega)\times Y_0$ such that on a subsequence we have, when $p\to\infty$:*

(3.7) $$\begin{aligned} &\alpha_{\varepsilon_p}\to\alpha^*, \\ &c_{\varepsilon_p}\to c^* \quad \textit{weakly in } H^1(\Omega), \\ &\boldsymbol{u}_{\varepsilon_p}\to\boldsymbol{u}^* \quad \textit{weakly in } Y_0, \\ &\boldsymbol{w}_{\varepsilon_p}\to\boldsymbol{u}^* \quad \textit{weakly in } Y_0, \end{aligned}$$

where $(\boldsymbol{u}_{\varepsilon_p}, c_{\varepsilon_p}) = (\boldsymbol{u}(\boldsymbol{w}_{\varepsilon_p}, \alpha_{\varepsilon_p}), c(\boldsymbol{w}_{\varepsilon_p}, \alpha_{\varepsilon_p}))$. Moreover, $(\boldsymbol{u}^*, c^*)$ is a solution of (2.14), *corresponding to $\alpha=\alpha^*$ and:*

(3.8) $$\lim_{\varepsilon\to 0} J_\varepsilon(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon) = J(\alpha^*, c^*) = \min\{J(\alpha, c)/(\alpha, c)\in T\}.$$

Proof. We first prove that $\{\alpha_\varepsilon\}_{\varepsilon>0}$ is bounded. Indeed, this is obvious if K is bounded. If K is not bounded, then we have:

$$\frac{N}{2}\alpha_\varepsilon^2 \le J(\alpha_\varepsilon, c_\varepsilon) \le J_\varepsilon(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon) \le J_\varepsilon(\boldsymbol{u}_0, \alpha_0) = J(\alpha_0, c_0),$$

where (α_0, c_0) is an optimal control for (3.2) and $(\boldsymbol{u}_0, c_0)$ verifies (2.14) for $\alpha=\alpha_0$; hence $\boldsymbol{u}_0=\boldsymbol{u}(\boldsymbol{u}_0, \alpha_0)$. For $\boldsymbol{u}_\varepsilon$ and $c_\varepsilon - \dfrac{\alpha_\varepsilon}{|\Omega|}$ we obtain from (3.5) the same estimates (2.19), (2.20), with $\alpha=\alpha_\varepsilon$. Therefore, the sequence $\{(\boldsymbol{u}_\varepsilon, c_\varepsilon)\}_{\varepsilon>0}$ is bounded in $Y_0\times H^1(\Omega)$.

On the other hand, we have:

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{w}_\varepsilon\|_{Y_0}^2 \le 2\varepsilon J_\varepsilon(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon) \le 2\varepsilon J(\alpha_0, c_0).$$

For the previous observations we deduce that, there exists $(\boldsymbol{u}^*, c^*)\in Y_0\times H^1(\Omega)$, $\alpha^*\in K$ such that (3.7) holds.

By passing to the limit when $p\to\infty$ in (3.5) corresponding to $(\boldsymbol{w}, \alpha) =$

$(\boldsymbol{w}_{\varepsilon_p}, \alpha_{\varepsilon_p})$, we obtain that $(\boldsymbol{u}^*, c^*)$ satisfies (2.14) for $\alpha=\alpha^*$ and hence $(\alpha^*, c^*)\in T$.

Finally, (3.8) is a consequence of the facts that $\{J_\varepsilon(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)\}_{\varepsilon>0}$ is bounded and has only one limit point.

For obtaining the necessary conditions of optimality for (3.6) we proceed as follows: for any $t>0$, $\lambda\in(0,1)$, $\boldsymbol{w}_0$, $\boldsymbol{w}\in Y_0$ and α_0, $\alpha\in K$, we denote by:

$$\begin{aligned} \boldsymbol{u}_{tw} &= \boldsymbol{u}(\boldsymbol{w}_0+t\boldsymbol{w}, \alpha_0), \\ c_{tw} &= c(\boldsymbol{w}_0+t\boldsymbol{w}, \alpha_0), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \boldsymbol{u}_{\lambda\alpha} &= \boldsymbol{u}(\boldsymbol{w}_0, \alpha_0+\lambda(\alpha-\alpha_0)), \\ c_{\lambda\alpha} &= c(\boldsymbol{w}_0, \alpha_0+\lambda(\alpha-\alpha_0)). \end{aligned} \tag{3.10}$$

Lemma 3.4. *Let $(\boldsymbol{u}_{tw}, c_{tw})$ and $(\boldsymbol{u}_{\lambda\alpha}, c_{\lambda\alpha})$ be defined by* (3.9) *and* (3.10). *Then, when $t\searrow 0$ and $\lambda\searrow 0$, we obtain:*

$$\left(\frac{\boldsymbol{u}_{tw}-\boldsymbol{u}_0}{t}, \frac{c_{tw}-c_0}{t}\right)\rightarrow(\boldsymbol{u}_w, c_w), \text{ weakly in } Y_0\times\widetilde{H}^1, \tag{3.11}$$

$$\left(\frac{\boldsymbol{u}_{\lambda\alpha}-\boldsymbol{u}_0}{\lambda}, \frac{c_{\lambda\alpha}-c_0}{\lambda}\right)\rightarrow(\boldsymbol{u}_\alpha, c_\alpha), \text{ weakly in } Y_0\times H^1(\Omega), \tag{3.12}$$

where $\boldsymbol{u}_0=\boldsymbol{u}(\boldsymbol{w}_0, \alpha_0)$, $c_0=c(\boldsymbol{w}_0, \alpha_0)$, $(\boldsymbol{u}_w, c_w)$ is the unique solution of:

$$\begin{aligned} &\nu a_0(\boldsymbol{u}_w, \boldsymbol{z})+b_0(\boldsymbol{w}_0, \boldsymbol{u}_w, \boldsymbol{z})= \\ &-\kappa\int_\Omega c_w\boldsymbol{i}_3\cdot\boldsymbol{z}dx-b_0(\boldsymbol{w}, \boldsymbol{u}_0, \boldsymbol{z}) \qquad \forall\boldsymbol{z}\in Y_0, \end{aligned} \tag{3.13}$$

$$\theta a(c_w, r)+b(\boldsymbol{w}_0, c_w, r)-U\int_\Omega c_w\frac{\partial r}{\partial x_3}dx=-b(\boldsymbol{w}, c_0, r) \qquad \forall r\in\widetilde{H}^1,$$

and $(\boldsymbol{u}_\alpha, c_\alpha)$ is the unique solution of (3.5) *corresponding to $\boldsymbol{f}=\boldsymbol{0}$, $\boldsymbol{w}=\boldsymbol{w}_0$ and $\alpha=\alpha-\alpha_0$.*

Proof. It can be easily proved that the sequences from (3.11) and (3.12) are bounded in $Y_0\times\widetilde{H}^1$ and $Y_0\times H^1(\Omega)$, respectively. By passing to the limit on subsequences in the problems satisfied by $\left(\frac{\boldsymbol{u}_{tw}-\boldsymbol{u}_0}{t}, \frac{c_{tw}-c_0}{t}\right)$ and $\left(\frac{\boldsymbol{u}_{\lambda\alpha}-\boldsymbol{u}_0}{\lambda}, \frac{c_{\lambda\alpha}-c_0}{\lambda}\right)$ we obtain that their weak limits $(\boldsymbol{u}_w, c_w)$ and $(\boldsymbol{u}_\alpha, c_\alpha)$ are solutions for (3.13) and, respectively, for (3.5) with $\boldsymbol{f}=\boldsymbol{0}$, $\boldsymbol{w}=\boldsymbol{w}_0$ and $\alpha=\alpha-\alpha_0$. The uniqueness of the solution of (3.13) is a consequence of Lax-Milgram's theorem, hence the proof is achieved.

It can be proved that J_ε is differentiable and a direct computation gives:

$$\frac{\partial J_\varepsilon}{\partial\boldsymbol{w}}(\boldsymbol{w}_0, \alpha_0)\cdot\boldsymbol{w}=\int_\Omega c_w(c_0-c_d)dx+\frac{1}{\varepsilon}a_0(\boldsymbol{u}_w-\boldsymbol{w}, \boldsymbol{u}_0-\boldsymbol{w}_0), \tag{3.14}$$

$$(3.15)\quad \frac{\partial J_\varepsilon}{\partial \alpha}(\boldsymbol{w}_0, \alpha_0)(\alpha-\alpha_0)=\int_\Omega c_\alpha(c_0-c_d)dx+\frac{1}{\varepsilon}a_0(\boldsymbol{u}_\alpha, \boldsymbol{u}_0-\boldsymbol{w}_0)+N\alpha_0(\alpha-\alpha_0).$$

We are now in a position to derive the necessary conditions of optimality for a solution of (3.6).

Theorem 3.5. *Let* $(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)$ *be an optimal control for* (3.6). *Then, there exists the unique elements* $(\boldsymbol{u}_\varepsilon, c_\varepsilon)\in Y_0\times H^1(\Omega)$ *and* $(\boldsymbol{p}_\varepsilon, q_\varepsilon)\in Y_0\times \widetilde{H}^1$ *which satisfy:*

$$(3.16)\quad \begin{aligned}&\nu a_0(\boldsymbol{u}_\varepsilon, \boldsymbol{z})+b_0(\boldsymbol{w}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z})=-\kappa\int_\Omega c_\varepsilon \boldsymbol{i}_3\cdot \boldsymbol{z}dx+\langle \boldsymbol{f}, \boldsymbol{z}\rangle \qquad \forall \boldsymbol{z}\in Y_0,\\ &\theta a(c_\varepsilon, r)+b(\boldsymbol{w}_\varepsilon, c_\varepsilon, r)-U\int_\Omega c_\varepsilon\frac{\partial r}{\partial x_3}dx=0 \qquad \forall r\in H^1(\Omega),\\ &\int_\Omega c_\varepsilon dx=\alpha_\varepsilon,\end{aligned}$$

$$(3.17)\quad \begin{aligned}&\nu a_0(\boldsymbol{p}_\varepsilon, \boldsymbol{z})-b_0(\boldsymbol{w}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z})+b_0(\boldsymbol{z}, \boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon)=b(\boldsymbol{z}, q_\varepsilon, c_\varepsilon) \qquad \forall \boldsymbol{z}\in Y_0,\\ &\theta a(q_\varepsilon, r)-b(\boldsymbol{w}_\varepsilon, q_\varepsilon, r)-U\int_\Omega r\frac{\partial q_\varepsilon}{\partial x_3}dx=-\kappa\int_\Omega r\boldsymbol{i}_3\cdot \boldsymbol{p}_\varepsilon dx+\int_\Omega r(c_\varepsilon-c_d)dx\\ &\hspace{30em}\forall r\in\widetilde{H}^1,\end{aligned}$$

$$(3.18)\quad \left(U\int_\Omega \frac{\partial q_\varepsilon}{\partial x_3}dx+(|\Omega|N+1)\alpha_\varepsilon-\int_\Omega c_d dx\right)(\alpha-\alpha_\varepsilon)\geq 0 \qquad \forall \alpha\in K.$$

Proof. It is obvious that (3.16) has a unique solution. We denote by $(\boldsymbol{p}_\varepsilon, q_\varepsilon)\in Y_0\times\widetilde{H}^1$ the unique solution (by Lax-Milgram's theorem) of:

$$(3.19)\quad \begin{aligned}&\nu a_0(\boldsymbol{p}_\varepsilon, \boldsymbol{z})-b_0(\boldsymbol{w}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z})=\frac{1}{\varepsilon}a_0(\boldsymbol{u}_\varepsilon-\boldsymbol{w}_\varepsilon, \boldsymbol{z}) \qquad \forall \boldsymbol{z}\in Y_0,\\ &\theta a(q_\varepsilon, r)-b(\boldsymbol{w}_\varepsilon, q_\varepsilon, r)-U\int_\Omega r\frac{\partial q_\varepsilon}{\partial x_3}dx=\\ &-\kappa\int_\Omega r\boldsymbol{i}_3\cdot \boldsymbol{p}_\varepsilon dx+\int_\Omega r(c_\varepsilon-c_d)dx \qquad \forall r\in\widetilde{H}^1.\end{aligned}$$

From (3.13), (3.14) and (3.19) we obtain:

$$(3.20)\quad \frac{\partial J_\varepsilon}{\partial \boldsymbol{w}}(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)\cdot \boldsymbol{w}=b(\boldsymbol{w}, q_\varepsilon, c_\varepsilon)-b_0(\boldsymbol{w}, \boldsymbol{u}_\varepsilon, \boldsymbol{p}_\varepsilon)-\frac{1}{\varepsilon}a_0(\boldsymbol{u}_\varepsilon-\boldsymbol{w}_\varepsilon, \boldsymbol{w}),$$

and from (3.5) for $\boldsymbol{f}=\boldsymbol{0}$, $\boldsymbol{w}=\boldsymbol{w}_0$ and $\alpha=\alpha-\alpha_0$, (3.15) and (3.19) we obtain:

$$(3.21)\quad \frac{\partial J_\varepsilon}{\partial \alpha}(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)\cdot(\alpha-\alpha_\varepsilon)=\left(U\int_\Omega\frac{\partial q_\varepsilon}{\partial x_3}dx+(|\Omega|N+1)\alpha_\varepsilon-\int_\Omega c_d dx\right)\frac{\alpha-\alpha_\varepsilon}{|\Omega|}.$$

From (3.19), (3.20), (3.21) and the fact that $(\boldsymbol{w}_\varepsilon, \alpha_\varepsilon)$ is a minimum point for J_ε on $Y_0\times K$, we deduce (3.17) and (3.18).

The main result of this Section is a consequence of the above Theorem.

Corollary 3.6. *There exists an optimal control* (α^*, c^*) *for* (3.2) *and there exists the elements* $\boldsymbol{u}^* \in Y_0$, $(\boldsymbol{p}^*, q^*) \in Y_0 \times \tilde{H}^1$ and $\lambda \in \{0, 1\}$ *such that:*

$$(3.22)\qquad \begin{aligned} &\nu a_0(\boldsymbol{u}^*, \boldsymbol{z}) + b_0(\boldsymbol{u}^*, \boldsymbol{u}^*, \boldsymbol{z}) = -\kappa \int_\Omega c^* \boldsymbol{i}_3 \cdot \boldsymbol{z} dx + \langle \boldsymbol{f}, \boldsymbol{z} \rangle \qquad \forall \boldsymbol{z} \in Y_0, \\ &\theta a(c^*, r) + b(\boldsymbol{u}^*, c^*, r) - U \int_\Omega c^* \frac{\partial r}{\partial x_3} dx = 0 \qquad \forall r \in H^1(\Omega), \\ &\int_\Omega c^* dx = \alpha^*, \end{aligned}$$

$$\nu a_0(\boldsymbol{p}^*, \boldsymbol{z}) - b_0(\boldsymbol{u}^*, \boldsymbol{p}^*, \boldsymbol{z}) + b_0(\boldsymbol{z}, \boldsymbol{u}^*, \boldsymbol{p}^*) = b(\boldsymbol{z}, q^*, c^*) \qquad \forall \boldsymbol{z} \in Y_0,$$

$$(3.23)\qquad \begin{aligned} &\theta a(q^*, r) - b(\boldsymbol{u}^*, q^*, r) - U \int_\Omega r \frac{\partial q^*}{\partial x_3} dx = \\ &-\kappa \int_\Omega r \boldsymbol{i}_3 \cdot \boldsymbol{p}^* dx + \lambda \int_\Omega r(c^* - c_d) dx \qquad \forall r \in \tilde{H}^1, \end{aligned}$$

$$(3.24)\qquad \left(U \int_\Omega \frac{\partial q^*}{\partial x_3} dx + \lambda \left((|\Omega| N + 1) \alpha^* - \int_\Omega c_d dx \right) \right) (\alpha - \alpha^*) \geq 0 \qquad \forall \alpha \in K,$$

$$(3.25)\qquad \lambda + \| q^* \|_{\tilde{H}^1} > 0.$$

Proof. It is obvious that (3.22) follows by passing to the limit in (3.16) on the subsequence obtained in Proposition 3.3.

If $\{\boldsymbol{p}_\varepsilon\}_{\varepsilon>0}$ is bounded in $(L^2(\Omega))^3$ then, from $(3.17)_2$ and the boundedness of $\{c_\varepsilon\}_{\varepsilon>0}$ in $H^1(\Omega)$ we deduce that $\{q_\varepsilon\}_{\varepsilon>0}$ is bounded in $\tilde{H}^1$. Moreover, from $(3.17)_1$ and the inequality (see [6]):

$$\| \boldsymbol{p}_\varepsilon \|^2_{(L^4(\Omega))^3} \leq 2 \| \boldsymbol{p}_\varepsilon \|^{1/2}_{(L^2(\Omega))^3} \| \boldsymbol{p}_\varepsilon \|^{3/2}_{(H^1(\Omega))^3},$$

the boundedness of $\{\boldsymbol{p}_\varepsilon\}_{\varepsilon>0}$ in Y_0 follows.

Therefore, there exists $(\boldsymbol{p}^*, q^*)$ a weak limit point of $\{(\boldsymbol{p}_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ in $Y_0 \times \tilde{H}^1$.

By using Proposition 3.3 and by passing to the limit on a subsequence in (3.17)-(3.18) we obtain (3.23)-(3.25) with $\lambda = 1$.

If $\{\boldsymbol{p}_\varepsilon\}_{\varepsilon>0}$ is not bounded in $(L^2(\Omega))^3$ we define the following sequences: $\{\boldsymbol{P}_\varepsilon\}_{\varepsilon>0} = \left\{ \frac{\boldsymbol{p}_\varepsilon}{\|\boldsymbol{p}_\varepsilon\|_{(L^2(\Omega))^3}} \right\}_{\varepsilon>0}$ and $\{Q_\varepsilon\}_{\varepsilon>0} = \left\{ \frac{q_\varepsilon}{\|\boldsymbol{p}_\varepsilon\|_{(L^2(\Omega))^3}} \right\}_{\varepsilon>0}$.

Dividing (3.17)-(3.18) by $\|\boldsymbol{p}_\varepsilon\|_{(L^2(\Omega))^3}$ we obtain, as in the previous case, the boundedness of the sequences $\{(\boldsymbol{P}_\varepsilon, Q_\varepsilon)\}_{\varepsilon>0}$ in $Y_0 \times \tilde{H}^1$. Hence we get (3.23)-(3.24) for $\lambda = 0$ and $(\boldsymbol{p}^*, q^*)$ a weak limit point of $\{(\boldsymbol{P}_\varepsilon, Q_\varepsilon)\}_{\varepsilon>0}$.

Finally, (3.25) is a consequence of the fact that, if $\| q^* \|_{\tilde{H}^1} = 0$ then, from (3.23) it follows that $\boldsymbol{p}^* = \boldsymbol{0}$ which is in contradiction with $\| \boldsymbol{p}^* \|_{(L^2(\Omega))^3} = 1$.

In the sequel we discuss the control problem (3.2) when (2.21) is satisfied; hence, for any $\alpha \in K = [0, \alpha']$, with α' as in (2.25), the problem (2.14) has a unique solution, $(\boldsymbol{u}_\alpha, c_\alpha)$. Since the correspondence $\alpha \mapsto c_\alpha$ is uni-valued, we can write the functional $J: K \mapsto \boldsymbol{R}$ as follows:

$$J(\alpha) = \frac{1}{2}\int_{\Omega}(c_{\alpha} - c_{d})^{2}dx + \frac{N}{2}\alpha^{2}, \tag{3.26}$$

and the control problem (3.2) becomes:

$$\min\{J(\alpha) / \alpha \in K\}. \tag{3.27}$$

In this case, the necessary conditions of optimality, obtained directly (as in [1]) from the differentiability of J on K, are:

Proposition 3.7. *Let $\alpha^* \in K$ be an optimal control of* (3.27). *Then, there exists the unique elements* $(\boldsymbol{u}^*, c^*) \in Y_0 \times H^1(\Omega)$ *and* $(\boldsymbol{p}^*, q^*) \in Y_0 \times \widetilde{H}^1$ *which satisfy* (3.22)-(3.25) *for* $\lambda = 1$.

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