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On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms

By

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Introduction

The purpose of this paper is twofold. The first one is to discuss various group topologies on inductive limits of topological groups, and unitary representations of inductive limit groups in a certain case, and the second one is to treat group topologies in the case of the group of diffeomorphisms.

Contrary to the affirmative statement in [1] or in [5], the inductive limit of topologies of an inductive system of topological groups does not always give a group topology, or more exactly, the multiplication is not necessarily continuous with respect to the inductive limit topology (denoted by τ_{ind}). In Part I of this paper, we show this by a simple example in the case of abelian groups, and then discuss in general which kinds of group topologies can be chosen on an inductive limit group under the condition that they are weaker than τ_{ind}.

We study in particular the case where inductive system is countable and essentially consists of locally compact groups. (For exact definition, see §2.3, and such a system is called a *countable LCG inductive system* in short). Then we prove that the inductive limit topology τ_{ind} gives a group topology in this case (Theorem 2.7), and also that it is essentially a unique one under a mild condition (Theorem 5.6).

Further, for a countable LCG inductive system, we discuss in a certain extent unitary representations and continuous positive definite functions of the inductive limit group $G = \varinjlim G_j$, and prove that, under the same condition as for Theorem 5.6, there exist sufficiently many of them so that the points of G can be separated (Theorem 5.7). Since there does not exist in general a Haar measure on G, the important point of the discussion is the limiting process from the case of locally compact groups G_j to G.

In Part II, we discuss the case of the group $G = \mathrm{Diff}_0(M)$ of diffeomorphisms with compact supports on a connected, non-compact, C^r-manifold M, $1 \leq r \leq \infty$, and prove that the inductive limit topology τ_{ind} never gives a group topology (Theorem 6.1).

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The paper is organized as follows. Part I consists of § 1 to § 5. In § 1 we give a counter example and discuss some generalities of group topologies, especially for inductive limit groups. In § 2, we introduce a notion of a PTA-group, and construct in the category of PTA-groups, a group topology τ_{BS}, called Bamboo-Shoot (=BS) topology, on the limit group of a countable inductive system, and show that τ_{BS} is the strongest among the group topologies weaker than τ_{ind}. In § 3, properties of BS-topology, and in § 4, a generalization of BS-topology to uncountable inductive systems are discussed. In § 5, positive definite functions and unitary representations of the limit groups of countable LCG inductive systems are treated.

Part II consists of § 6 to § 9. After giving preliminaries and the main theorem in § 6, we give in § 7 a lemma for a C^r-map on an open ball in $\mathbf{R}^d$ to be a diffeomorphism. In § 8, we study local properties of diffeomorphisms and give a key lemma, Lemma 8.2. Then, in § 9, we prove that the multiplication on $G = \mathrm{Diff}_0(M)$ is not continuous in the inductive limit topology τ_{ind} as a result of the non-compactness of M.

Part I. Group topologies and unitary representations for an inductive limit of topological groups

§ 1. The inductive limit topology and possible group topologies

1.1. A counter example. An inductive system of topological groups is given as follows. We have a family of topological groups (G_α, τ_α), $\alpha \in A$, indexed by a directed set A, where τ_α denotes the topology on G_α, and a system of continuous homomorphisms $\phi_{\beta,\alpha} : G_\alpha \to G_\beta$, for $\alpha, \beta \in A, \alpha \preceq \beta$, which satisfies the consistency condition: $\phi_{\gamma,\beta} \cdot \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$ for any $\alpha \preceq \beta \preceq \gamma$. Recall the notion of the inductive limit group $G = \varinjlim G_\alpha$, since it is essential here. Consider a disjoint union $S := \bigsqcup_{\alpha \in A} G_\alpha$ and introduce the equivalence relation as: $g_\alpha \sim g_\beta$ for $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$ if $\phi_{\gamma,\alpha}(g_\alpha) = \phi_{\gamma,\beta}(g_\beta)$ for some $\gamma \succeq \alpha, \beta$. Then G is the quotient $S/\sim$ as set, and the multiplication in G is defined in a standard way. The natural projection from G_α to G is denoted by ϕ_α, then, $\phi_\beta \cdot \phi_{\beta,\alpha} = \phi_\alpha$ for $\alpha, \beta \in A$, $\alpha \preceq \beta$. The unit elements of G and G_α are denoted respectively by e and e_α.

Note that in the case where all the homomorphisms $\phi_{\beta,\alpha}$ are injective, by identifying through $\phi_{\beta,\alpha}$, we may consider inclusions, $G_\alpha \hookrightarrow G_\beta$, and then, $G = \bigcup_{\alpha \in A} G_\alpha$ as an abstract group.

The inductive limit of topologies $\tau_{ind} = \varinjlim \tau_\alpha$ on $G = \varinjlim G_\alpha$ is given in such a way that a subset of G is open if and only if its inverse image in G_α is open with respect to τ_α (τ_α-open in short) for any $\alpha \in A$.

We see easily the following fact on τ_{ind}.

Proposition 1.1. *On the inductive limit group $G = \varinjlim G_\alpha$, the following maps are continuous with respect to $\tau_{ind} = \varinjlim \tau_\alpha$:*

(i) *the inverse*: $G \ni g \mapsto g^{-1} \in G$;

(ii) *the left and right translations*: *for a fixed $h \in G$,*

$$G \ni g \mapsto hg \in G, \quad G \ni g \mapsto gh \in G.$$

However the multiplication $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \ni G$ is not necessarily τ_{ind}-continuous as the following elementary example shows.

Example 1.2. Let $G_n = \mathbf{Q} \times F^n$, $F = \mathbf{R}$ or $\mathbf{Q}$ with the usual non-discrete topology, and imbed G_n into G_{n+1} as $x \mapsto (x, 0)$. Then, $G = \lim_{n\to\infty} G_n = \mathbf{Q} \times \prod' F$ (restricted direct product), and the multiplication is not τ_{ind}-continuous. Or, there exists an τ_{ind}-open neighbourhood U of $e \in G$ such that $V^2 \not\subset U$ for any τ_{ind}-open neighbourhood V of e. In fact, put

$$U = \{x = (x_0, x_1, \dots, x_n, \dots);\ |x_j| < |\cos(jx_0)|\ (1 \leq j)\}.$$

Then, since $x_0 \in \mathbf{Q}$, we have always $\cos(jx_0) \neq 0$, and so

$$U \cap G_n = \{x = (x_0, x_1, \dots, x_n);\ |x_j| < |\cos(jx_0)|\ (1 \leq j \leq n)\}$$

is open in G_n for any $n \geq 1$, and so U is τ_{ind}-open. Assume that there exists a τ_{ind}-open neighbourhood V of the neutral element $e \in G$ such that $V^2 \subset U$. Then, $V \cap G_j$ contains an open interval $(-\varepsilon_j, \varepsilon_j)_F$ in F with $\varepsilon_j > 0$ such that

$$(-\varepsilon_0, \varepsilon_0)_{\mathbf{Q}} \times (-\varepsilon_j, \varepsilon_j)_F \subset \{(x_0, x_j) \in \mathbf{Q} \times F;\ |x_j| < |\cos(jx_0)|\}.$$

This is impossible if $2j\varepsilon_0 > \pi$. A contradiction.

1.2. The group topology defined by positive definite functions.

Generally speaking, why τ_{ind} does not give a group topology is that τ_{ind} has too many open neighbourhoods of e. So we should have some criterion to decrease the number of these neighbourhoods. In this context, we can refer the case of locally convex topological vector spaces. In that case the criterion is the convexity of neighbourhoods.

As a group topology on G weaker than τ_{ind}, one can propose at first the topology $\tau_{p.d.}$ defined by means of the set $\mathscr{P}(\tau_{ind})$ of all positive definite functions on G continuous with respect to τ_{ind}. Note that a positive definite function f is τ_{ind}-continuous on G if it is τ_{ind}-continuous at e, because the topology τ_{ind} is translation-invariant (by Proposition 1.1(ii)), and the positive definiteness of f gives $f(e) \geq |f(g)|$, $f(g^{-1}) = \mathrm{Conj}\{f(g)\}$, and Krein's inequality [7]

$$|f(g) - f(h)|^2 \leq 2f(e)\{f(e) - \Re(f(gh^{-1}))\} \quad (g, h \in G).$$

By definition, an open neighbourhood of e with respect to $\tau_{p.d.}$ is given as follows: take a finite number of $f_j \in \mathscr{P}(\tau_{ind})$, $1 \leq j \leq N$, and an $\varepsilon > 0$, then

$$U(f_1, f_2, \dots, f_N; \varepsilon) = \{g \in G; |f_j(g) - f_j(e)| < \varepsilon(\forall j)\}.$$

The topology $\tau_{p.d.}$ is also defined as the weakest topology on G which makes all τ_{ind}-continuous unitary representations continuous.

Finally we note that $\mathscr{P}(\tau_{ind}) = \mathscr{P}(\tau_{p.d.})$.

1.3. Group topologies weaker than τ_{ind}.

Let us now discuss what kind of group topologies can be chosen on the inductive limit group G, in between τ_{ind} and $\tau_{p.d.}$.

A fundamental system $\mathscr{U}$ of neighbourhoods of the unit element e of G for a group topology is a family of subsets of G satisfying

(GT1) for any $U \in \mathscr{U}$, $U \ni e$;

(GT2) for any $U_1, U_2 \in \mathscr{U}$, $U_1 \cap U_2$ contains a $V \in \mathscr{U}$;

(GT3) for any $U \in \mathscr{U}$, $U^{-1} = \{u^{-1}; u \in U\}$ contains a $V \in \mathscr{U}$;

(GT4) for any $U \in \mathscr{U}$ and $g \in G$, gUg^{-1} contains a $V \in \mathscr{U}$;

(GT5) for any $U \in \mathscr{U}$, there exists a $V \in \mathscr{U}$ such that $V^2 \subset U$.

Starting from a family $\mathscr{S}$ of subsets of G containing e, we want to construct a family $\mathscr{U}$ satisfying (GT1)–(GT5). To satisfy the conditions (GT1)–(GT4), it is enough to enlarge $\mathscr{S}$ by applying repeatedly the following processes:

(a) for any two U_1, U_2, add $U_1 \cap U_2$;

(b) for any U, add U^{-1};

(c) for any U and $g \in G$, add gUg^{-1}.

However to satisfy the condition (GT5), we should assume a condition (GT5*) on $\mathscr{S}$ from the beginning:

(GT5*) for any $B \in \mathscr{S}$, there exists a $C \in \mathscr{S}$ such that $C^2 \subset B$.

We call a family $\mathscr{S}$ a *seed of neighbourhood system* if it satisfies the condition (GT5*). Thus, we see that introducing a group topology in G is equivalent to giving a seed $\mathscr{S}$ of neighbourhood system.

The introduced topology τ is weaker than the one τ_{ind} if the condition (GTind) holds for $\mathscr{S}$:

(GTind) every $B \in \mathscr{S}$ is a τ_{ind}-neighbourhood of e.

Lemma 1.3. *For the inductive limit $G = \varinjlim G_\alpha$, assume that a family of subsets $\mathscr{V} = \{V(\alpha, k); \alpha \in A, k = 1, 2, \dots\}$ satisfies the following conditions:*

(1) *for any α, $V(\alpha, k)$'s are τ_α-open neighbourhoods of e_α in G_α;*

(2) *for any $\alpha \prec \beta$, $\phi_{\beta,\alpha}(V(\alpha, k)) \subset V(\beta, k)$ $(\forall k)$;*

(3) *for any α, β, k, there exists a γ such that $\phi_{\gamma,\alpha}(V(\alpha, k+1)) \cdot \phi_{\gamma,\beta}(V(\beta, k+1)) \subset V(\gamma, k)$.*

Then, the family $\mathscr{S} = \{U_k = \bigcup_{\alpha \in A} \phi_\alpha(V(\alpha, k)); k = 1, 2, \dots\}$ gives a seed of neighbourhood system satisfying the condition (GTind)*.*

Proof. Let us first prove that U_k is τ_{ind}-open. Take a $\beta \in A$ and check if $W := \phi_\beta^{-1}(U_k)$ is τ_β-open in G_β. For any α, there exists a $\gamma \in A$ such that $\gamma \succ \alpha$, β and so $V(\gamma, k) \supset \phi_{\gamma,\alpha}(V(\alpha, k))$, and $\phi_\gamma(V(\gamma, k)) \supset \phi_\alpha(V(\alpha, k))$. Therefore $W = \bigcup_{\gamma; \gamma \succ \beta} \phi_\beta^{-1}(\phi_\gamma(V(\gamma, k)))$. Since $\phi_\beta = \phi_\gamma \cdot \phi_{\gamma,\beta}$, we have $\phi_\beta^{-1} = \phi_{\gamma,\beta}^{-1} \cdot \phi_\gamma^{-1}$. Let $N_\gamma = \phi_\gamma^{-1}(\{e_\gamma\})$ be the kernel of $\phi_\gamma : G_\gamma \to G$. Then, $\phi_\gamma^{-1}\phi_\gamma(V(\gamma, k)) = V(\gamma, k)N_\gamma =: W_\gamma$ (put). Then W_γ is τ_γ-open and so $\phi_{\gamma,\beta}^{-1}(W_\gamma)$ is τ_β-open in G_β. Thus $W = \bigcup_{\gamma \succ \beta} \phi_{\gamma,\beta}^{-1}(W_\gamma)$ is τ_β-open, as is desired.

The fact $(U_{k+1})^2 \subset U_k$, for $k = 1, 2, \ldots$, can be seen easily and it guarantees the condition (GT5*) for $\mathscr{S}$. Q.E.D.

We denote by $\tau(\mathscr{V})$ the group topology on G generated from $\mathscr{V}$ through $\mathscr{S}$ above by the process stated before. Then we have the following result which shows that our process is standard.

Proposition 1.4. *Let τ be a group topology on $G = \varinjlim G_\alpha$, weaker than τ_{ind}. Then there exists a family of $\mathscr{V}$'s satisfying* (1)–(3) *in Lemma* 1.3 *such that τ is the upper bound of the topologies $\tau(\mathscr{V})$'s.*

Proof. Take an arbitrary τ-open neighbourhood U of $e \in G$. Then we can find a series V_k, $k = 1, 2, \ldots$, of τ-open neighbourhoods of $e \in G$ such that $V_1 = U$, $(V_{k+1})^2 \subset V_k$. Put $V(\alpha, k) = \phi_\alpha^{-1}(V_k) = \phi_\alpha^{-1}(V_k \cap \phi_\alpha(G_\alpha)) \subset G_\alpha$, and $\mathscr{S} = \{V(\alpha, k);\ \alpha \in A,\ k = 1, 2, \ldots\}$. Then $\mathscr{S}$ satisfies the conditions (1)–(3) in Lemma 1.3. Noting that $V_k = \bigcup_{\alpha \in A} \phi_\alpha(V(\alpha, k))$, the assertions of the proposition is easy to prove. Q.E.D.

Corollary 1.5. *Let τ' be a group topology on an inductive limit group $G = \varinjlim G_\alpha$. On each G_α, take the inverse image τ'_α through ϕ_α of τ' as its group topology (which is not Hausdorff if ϕ_α is not injective). Then τ' is recovered by the process in the proof of the proposition.*

1.4. A note on inductive systems. In an inductive system $\{(G_\alpha, \tau_\alpha)_{\alpha \in A};\ \phi_{\beta,\alpha}\}$, the homomorphisms $\phi_{\beta,\alpha}$ are not necessarily assumed to be injective. Accordingly, $\phi_\alpha : G_\alpha \to G$ need not be injective, and the inductive limit topology needs not be Hausdorff.

However, ϕ_α is $(\tau_\alpha, \tau_{ind})$-continuous. Take the quotient group $(G_\alpha^\sim, \tau_\alpha^\sim)$ of the topological group (G_α, τ_α) by the kernel $N_\alpha = \mathrm{Ker}(\phi_\alpha) = \phi_\alpha^{-1}(e)$, which is τ_α-closed if τ_{ind} on G is Hausdorff. Then $(G_\alpha^\sim, \phi_\alpha^\sim)$ is isomorphic to $(\phi_\alpha(G_\alpha), \phi_\alpha(\tau_\alpha))$, where $\phi_\alpha(\tau_\alpha)$ is the image of τ_α through ϕ_α. We have a natural injective homomorphism $\phi_{\beta,\alpha}^\sim : G_\alpha^\sim \to G_\beta^\sim$, which turns out to be continuous in $(\tau_\alpha^\sim, \tau_\beta^\sim)$. In this way, we get an inductive system $\{(G_\alpha^\sim, \tau_\alpha^\sim)_{\alpha \in A};\ \phi_{\beta,\alpha}^\sim\}$ with injective homomorphisms $\phi_{\beta,\alpha}^\sim$.

Lemma 1.6. *The inductive limit $\varinjlim G_\alpha^\sim$ of the system $\{(G_\alpha^\sim, \tau_\alpha^\sim)_{\alpha \in A};\ \phi_{\beta,\alpha}^\sim\}$ is canonically isomorphic to the one $G = \varinjlim G_\alpha$ of the original system. In particular, the inductive limits of topologies of $\{\tau_\alpha^\sim\}$ and $\{\tau_\alpha\}$ are homeomorphic.*

Thus we have also a natural injective homomorphisms $\phi_\alpha^\sim : G_\alpha^\sim \to G$, which gives the natural identification of $(G_\alpha^\sim, \phi_\alpha^\sim)$ with $(\phi_\alpha(G_\alpha), \phi_\alpha(\tau_\alpha))$.

As we see above, any inductive system of topological groups can be reduced to such a system of *injective type* (that is, with injective homomorphisms). But we keep to the general case some how and to the notations there, since it has certain merits for clarifying the situations.

§ 2. The system of Bamboo-Shoot neighbourhoods

Here we prove that, under a mild assumption on $G = \varinjlim G_\alpha$ or rather on the system $\{(G_\alpha, \tau_\alpha)_{\alpha \in A}, \phi_{\beta,\alpha}\}$, we can construct the strongest group topology between τ_{ind} and $\tau_{p.d.}$.

2.1. PTA-group. A subset E of a group is called *symmetric* if $E^{-1} = E$, and a symmetric subset E of $G = \varinjlim G_\alpha$ is called a *PTA-set* if it has the following property for any $\alpha \in A$, $\alpha \succ \gamma$ with some fixed γ:

(P) for any τ_α-neighbourhood $W_\alpha \subset G_\alpha$ of e_α, there exists a τ_α-neighbouhood W'_α of $e_\alpha \in G_\alpha$ such that $\phi_\alpha(W'_\alpha) \cdot E \subset E \cdot \phi_\alpha(W_\alpha)$.

Since E is symmetric, the condition above is equivalent to the existence of τ_α-neighbourhood W''_α such that $E \cdot \phi_\alpha(W''_\alpha) \subset \phi_\alpha(W_\alpha) \cdot E$. A neighbourhood in G is called *PTA-neighbourhood* if it is a PTA-set, and a τ_α-neighbourhood in G_α is called PTA if its image in G is a PTA-set. An inductive limit group $G = \varinjlim G_\alpha$ is called a *PTA-group* if, for every $\alpha \in A$, (G_α, τ_α) has a fundamental system of neighbourhoods of e_α consisting of open PTA-sets. (*PTA* is an abriviation of *Passing Through Assumption.*) This condition is equivalent to that, for any $\alpha \in A$, the topological group $(\phi_\alpha(G_\alpha), \phi_\alpha(\tau_\alpha))$ as the quotient of (G_α, τ_α) has the same property.

We see easily that $G = \varinjlim G_\alpha$ is a PTA-group if, for any $\alpha \prec \beta$, the group $(\phi_\alpha(G_\alpha), \phi_\alpha(\tau_\alpha)) \cong (G_\alpha^\sim, \tau_\alpha^\sim)$ is a direct product of a central subgroup of $\phi_\beta(G_\beta)$ with a locally compact group.

Lemma 2.1. *For $\alpha \prec \beta$, let V_α be a PTA-neighbourhood of $e_\alpha \in G_\alpha$ in τ_α, and V_β a PTA-neighbourhood of $e_\beta \in G_\beta$ in τ_β. Then $V_\beta \phi_{\beta,\alpha}(V_\alpha)\phi_{\beta,\alpha}(V_\alpha)V_\beta$ is a PTA-neighbourhood of $e_\beta \in G_\beta$ in τ_β.*

2.2. Bamboo-Shoot neighbourhoods. Hereafter in this section, we assume that $G = \varinjlim G_\alpha$ is a PTA-group. We also assume the index set A be countable and so put $A = \mathbf{N}$ as a set.

Case 1 (monotone increasing). We first study the case where $j \prec j+1$ $(\forall j)$ in A, or $\exists \phi_{j+1,j} : G_j \to G_{j+1}$. For a system $\{U_j\}_{j \in \mathbf{N}}$ of symmetric neighbourhood U_j of $e_j \in G_j$ in τ_j, put

$$U(n,k) := \phi_n(U_n)\phi_{n-1}(U_{n-1}) \cdots \phi_k(U_k)\phi_k(U_k)\phi_{k+1}(U_{k+1}) \cdots \phi_n(U_n) \ (n \geq k)$$

$$U[k] := \bigcup\nolimits_{n=k}^{\infty} U(n,k),$$

then $U[k]$ is τ_{ind}-open as is proved below, and is called a *Bamboo-Shoot* (or *BS* in short) neighbourhood of G. Denote by $\mathscr{U}_{BS}$ the collection of all such $U[k]$ for $\{U_j\}_{j \geq k}$ with U_j's each running over symmetric neighbouhoods of $e_j \in G_j$ in τ_j, $j \geq k$ and $k = 1, 2, \ldots$.

Lemma 2.2. *Every $U[k]$ is τ_{ind}-open, and the system $\mathscr{U}_{BS}$ is a fundamental system of neighbourhoods of $e \in G$ for a group topology weaker than τ_{ind} on a PTA-group G.*

Proof. (i) Let us prove that $U[k]$ is τ_{ind}-open. For a system $\{O_j\}_{j\in\mathbf{N}}$ of τ_j-open symmetric neighbourhood of $e_j \in G_j$, put

$$O_0(n,k) := \phi_n(O_n)\phi_{n-1}(O_{n-1})\cdots\phi_k(O_k)\ (n \geq k), \quad O_0[k] := \bigcup_{n=k}^{\infty} O_0(n,k).$$

Then $O_0[k]$ is τ_{ind}-open. We fix O_j's in such a way that $O_j \subset U_j$ $(\forall j)$.

Now take an arbitrary element $g \in U[k]$. Then there exists a $p \geq k$ such that $g \in U(p,k)$, and so, for $n \geq p+1$,

$$\begin{aligned} U(n,k) &\supset \phi_n(U_n)\phi_{n-1}(U_{n-1})\cdots\phi_{p+1}(U_{p+1})U(p,k) \\ &\supset \phi_n(O_n)\phi_{n-1}(O_{n-1})\cdots\phi_{p+1}(O_{p+1})U(p,k) \supset O_0(n,p+1)g. \end{aligned}$$

Therefore we have

$$U[k] = \bigcup_{n=k}^{\infty} U(n,k) = \bigcup_{n=p+1}^{\infty} U(n,k) \supset \bigcup_{n=p+1}^{\infty} O_0(n,p+1)g = O_0[p+1]g.$$

By Proposition 1.1(ii), this proves that g is a τ_{ind}-inner point of $U[k]$, or $U[k]$ is τ_{ind}-open.

(ii) The conditions (GT1)–(GT4) is easily proved for $\mathscr{U}_{BS}$. So we prove only the condition (GT5).

Take a $U[k]$. Choose, for every $j \geq k$, τ_j-neighbourhood W_j of $e_j \in G_j$ such that $W_j^2 \subset U_j$. Further, for this k, choose symmetric PTA-neighbourhood $V_k \subset W_k$ of $e_k \in G_k$. Then, choose inductively symmetric PTA-neighbourhood $V_j \subset W_j$ of $e_j \in G_j$ for which $V(n,k)^2 \subset U(n,k)$ holds for $k \leq \forall n$. For this, it suffices to choose V_{n+1} so that $V(n,k)\phi_{n+1}(V_{n+1}) \subset \phi_{n+1}(W_{n+1})V(n,k)$ and $\phi_{n+1}(V_{n+1})V(n,k) \subset V(n,k)\phi_{n+1}(W_{n+1})$. Then we have $V(n+1,k)^2 \subset U(n+1,k)$, by the assumption of induction and by using Lemma 2.1. From this, $(V[k])^2 \subset U[k]$. Q.E.D.

We call the above topology *BS-topology* and denote it by τ_{BS}. The group G equipped with τ_{BS} is also denoted by $\tau_{BS}\text{-}\varinjlim G_j$.

Case 2 (non-monotonic). We consider the general case where G_j's are not monotone increasing. Take an increasing sequence of integers $1 \leq n(1) < n(2) < \cdots < n(j) < \cdots$ such that $n(j) \prec n(j+1)$ and $\{n(j)\}_{j\in\mathbf{N}}$ is cofinal with A, for instance, such as $n(j+1) \succ n(j), j$. Then, $\varinjlim G_{n(j)} = \varinjlim G_j = G$, and so we can apply the result in the monotone increasing case, Case 1. The topology τ_{BS} on G thus defined does not depend on the choice of the cofinal sequence $\{n(j)\}_{j\in\mathbf{N}}$, and Lemma 2.2 holds in this case too.

The topology τ_{BS} is characterized by the following property.

Proposition 2.3. *Assume that the index set A is countable and $G = \varinjlim G_j$ is PTA. Then, the BS-topology τ_{BS} is the strongest one among group topologies on G weaker than τ_{ind}.*

Proof. We prove the assertion only in the monotone increasing case, Case 1, then Case 2 is similar.

Let τ be a group topology on G weaker than τ_{ind}. Take an arbitrary neighbourhood U of $e \in G$ in τ. Then there exists a symmetric neighbourhood V_1

in τ such that $(V_1)^4 \subset U$. Inductively choose symmetric τ-neighbourhoods V_j of e in such a way that $(V_{j+1})^2 \subset V_j$. Put $W_j = \phi_j^{-1}(V_j) \subset G_j$ and, for $\{W_j\}_{j \in \mathbf{N}}$, take a BS-neighbourhood $W[1]$. Then, we see easily $U \supset W[1]$.

Corollary 2.4. *Let (H, τ) be a topological group and ϕ an algebraic homomorphism of $G = \varinjlim G_j$ into H. Then the following two assertions are mutually equivalent*:

(1) *ϕ is continuous as a map from (G, τ_{ind}) to (H, τ);*

(2) *ϕ is continuous as a map from (G, τ_{BS}) to (H, τ).*

Without any additional assumption, the topology τ_{BS} is not necessarily Hausdorff, and we remark the following facts.

Lemma 2.5. *Let M be the intersection of all $U \in \mathscr{U}_{BS}$.*

(i) *M is a normal subgroup of G and τ_{BS}-closed and so τ_{ind}-closed.*

(ii) *$M_j = \phi_j^{-1}(M) = \phi_j^{-1}(M \cap \phi_j(G_j))$ (resp. $M_j^\sim = M_j/N_j$) is a τ_j-closed (resp. $\tau_j^\sim$-closed) normal subgroup of G_j (resp. $G_j^\sim$), and $G_j/M_j \cong G_j^\sim/M_j^\sim$ as topological groups.*

Proof. For (i), the τ_{BS}-closedness of M comes from the following general fact for a topological group:

($*$) *Let E be a subset and U a neighbourhood of e, then EU contains the closure of E.*

The rests of the assertions are easy to prove. Q.E.D.

Note that $M_j \supset N_j = \mathrm{Ker}(\phi_j)$ from (i), and that, if G_j is locally compact, then so is $G_j/M_j \cong G_j^\sim/M_j^\sim$. So it is interesting to ask what is the difference between M_j and N_j. We can also ask if the quotient topologies of τ_{ind} and τ_{BS} on G/M coincide with each other.

Lemma 2.6. *The quotient G/M with the quotient topology of τ_{ind} (resp. τ_{BS}) is isomorphic to τ_{ind}-$\varinjlim G_j/M_j \cong \tau_{ind}$-$\varinjlim G_j^\sim/M_j^\sim$ (resp. τ_{BS}-$\varinjlim G_j/M_j \cong \tau_{BS}$-$\varinjlim G_j^\sim/M_j^\sim$).*

2.3. Case of countable LCG inductive system. The case where A is countable and all the groups $(G_j^\sim, \tau_j^\sim)$ are locally compact (which are Hausdorff by definition), is especially interesting and will be studied later on. An inductive system in such a case is called a *countable LCG inductive system*.

Theorem 2.7. *For any countable LCG inductive system, the inductive limit topology τ_{ind} on $G = \varinjlim G_j$ gives a group topology and it coincides with BS-topology τ_{BS}.*

Proof. It is enough to prove $\tau_{ind} = \tau_{BS}$. Since τ_{ind} is stronger than τ_{BS}, we prove the converse. It may be assumed that we are in the monotone increasing case. In this proof, we denote $G_j^\sim$, $\phi_{j',j}^\sim$, $\phi_j^\sim$ and $\tau_j^\sim$ simply by G_j, $\phi_{j',j}$, ϕ_j and τ_j respectively, omitting tildes.

Take an arbitrary τ_{ind}-open neighbourhood U. Then $U_j = \phi_j^{-1}(U) \subset G_j$ is τ_j-open neighbourhood of $e_j \in G_j$, and $\phi_{j+1,j}(U_j) \subset U_{j+1}$. With respect to τ_1, choose a relatively compact, open symmetric neighbourhood W_1 of $e_1 \in G_1$ so that $\mathrm{Cl}(W_1^2) \subset W_1^3 \subset U_1$, where $\mathrm{Cl}(\cdot)$ denotes the closure. Then, $\phi_{2,1}(\mathrm{Cl}(W_1^2))$ is τ_2-compact, and so there exists a relatively compact, open symmetric neighbourhood W_2 of $e_2 \in G_2$ (in τ_2) such that $\mathrm{Cl}(W_2\phi_{2,1}(W_1)^2W_2) \subset U_2$. Inductively we take a relatively compact, open symmetric neighbourhood W_j of $e_j \in G_j$ (in τ_j) so that $\mathrm{Cl}(W_j\phi_{j,j-1}(W_{j-1})\cdots\phi_{j,2}(W_2)\phi_{j,1}(W_1)^2\phi_{j,2}(W_2)\cdots\phi_{j,j-1}(W_{j-1})W_j) \subset U_j$. Then the original U contains a BS-neighbourhood $W[1]$ for $\{W_j\}_{j\in\mathbf{N}}$. Q.E.D.

As a criterion to get a countable LCG inductive system, we have the following simple one for the moment.

Proposition 2.8. *For a countable inductive system $\{(G_j, \tau_j)\}_{j\in\mathbf{N}}$, assume that every (G_j, τ_j) is locally compact, and that, with the topology τ_{ind} (not necessarily a group topology), the limit group G is a T_0-space. Then N_j coincides with M_j and is τ_j-closed for any j. So the topological groups $(G_j^\sim, \tau_j^\sim) \cong (\phi_j(G_j), \phi_j(\tau_j))$ are all locally compact, and the system $\{(G_j^\sim, \tau_j^\sim)\}_{j\in\mathbf{N}} \cong \{(\phi_j(G_j), \phi_j(\tau_j))\}_{j\in\mathbf{N}}$ is a countable LCG inductive system. This system is of injective type and gives as its limits the same G and $\tau_{ind} = \tau_{BS}$.*

To get the above T_0-property, it is sufficient for example that $G_j^\sim$ is closed in $G_{j+1}^\sim$ for $j \geq 1$.

Remark 2.9. We will see in §5 that there exist sufficiently many τ_{ind}-continuous positive definite functions on $G = \varinjlim G_\alpha$ for a countable LCG inductive system, under a mild condition (Theorem 5.7). Using this fact, we can prove $\tau_{ind} = \tau_{BS} = \tau_{p.d.}$ in that case in Theorem 5.6.

Example 2.10. Let $G = GL(\infty, F)$, with $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{Q}_p$, be the inductive limit group of $G_n = GL(n, F)$, $n = 1, 2, \ldots$, where G_n is imbedded into G_{n+1} as

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, by the above proposition, τ_{ind} is a group topology on G. A basis for τ_{ind}-neighbourhoods of e is given by A. Yamasaki. We give here another basis as follows. For $g \in G$, put $g = 1 + x$, $x = (x_{ij})_{i,j=1}^\infty$. Take $\kappa = (\kappa_{ij})_{i,j=1}^\infty$, with $\kappa_{ij} > 0$, and put

$$V(\kappa) := \{g = 1 + x;\ |x_{ij}| < \kappa_{ij}\ (\forall i, j)\}.$$

Here we prove only that this topology τ gives actually a group topology on G. For a matrix $z = (z_{ij})_{i,j=1}^\infty$, put

$$\|z\|_{HS} = \left(\sum_{i,j=1}^\infty |z_{ij}|^2\right)^{1/2}, \quad \|z_{i\bullet}\| = \left(\sum_{j=1}^\infty |z_{ij}|^2\right)^{1/2}, \quad \|z_{\bullet j}\| = \left(\sum_{i=1}^\infty |z_{ij}|^2\right)^{1/2},$$

with $z_{i\bullet} = (z_{ij})_{j=1}^\infty$, $z_{\bullet j} = (z_{ij})_{i=1}^\infty$. Then, $\|zz'\|_{HS} \leq \|z\|_{HS}\|z'\|_{HS}$.

For $x = (x_{ij})_{i,j=1}^{\infty}$, $x' = (x'_{ij})_{i,j=1}^{\infty}$, let $(1+x)(1+x') = 1+y$. Then, $y = x + x' + xx'$ and $|y_{ij}| \leq |x_{ij}| + |x'_{ij}| + \|x_{i\bullet}\| \, \|x'_{\bullet j}\|$.

For an x with $\|x\|_{HS} < 1$, let $(1+x)^{-1} = 1+y$. Then $y = \sum_{n=1}^{\infty}(-1)^n x^n$, and $|y_{ij}| \leq |x_{ij}| + \|x_{i\bullet}\| \, \|x_{\bullet j}\|(1 - \|x\|_{HS})^{-1}$.

These evaluations show that the multiplication and the inverse in G are continuous in τ, if the following fact is taken into account: *For any* $\varepsilon = (\varepsilon_{ij})_{i,j=1}^{\infty}$ *with* $\varepsilon_{ij} > 0$, *there exists a matrix* $\kappa = (\kappa_{ij})_{i,j=1}^{\infty}$ *with* $\kappa_{ij} > 0$ *such that* $\|\kappa\|_{HS} < 1$, *and* $\varepsilon_{ij} \geq \|\kappa_{i\bullet}\| \, \|\kappa_{\bullet j}\|$ *for any* $i, j \geq 1$.

Unitary representations of this kind of groups for $F = \mathbf{R}$ or $\mathbf{C}$ are studied in [10], and those for $U(\infty) = \varinjlim U(n)$ etc. are also studied by many mathematicians, see also §5.

§3. Properties of BS-topology and extended BS-topology

In this section, we treat the case where A is countable and $G = \varinjlim G_j$ is PTA. Here we consider the image $\phi_j(G_j) \subset G$ together with the topology $\phi_j(\tau_j)$ given as the quotient of (G_j, τ_j) by $N_j = \phi_j^{-1}(e)$, and identify them: $(\phi_j(G_j), \phi_j(\tau_j)) \cong (G_j^{\sim}, \tau_j^{\sim})$, in the notation in §1.4. The restriction of $\tau_{j'}^{\sim}$ onto $G_j^{\sim}$ is denoted by $\tau_{j'}^{\sim} \mid G_j^{\sim}$.

3.1. Properties of BS-topology. Let us give some important properties of BS-topology τ_{BS} or of τ_{BS}-$\varinjlim G_j$.

Proposition 3.1. *Assume that, for any* $j \prec j'$, *the image* $\phi_j(G_j)$ *is closed in* $(\phi_{j'}(G_{j'}), \phi_{j'}(\tau_{j'}))$, *and that the injective homomorphism* $(\phi_j(G_j), \phi_j(\tau_j)) \to (\phi_{j'}(G_{j'}), \phi_{j'}(\tau_{j'}))$ *is an isomorphism. If a filter* $\mathscr{F}$ *with a countable base is* τ_{BS}-*convergent in* $G = \varinjlim G_j$, *then there exists an* n *and an* $F \in \mathscr{F}$ *such that* $F \subset \phi_n(G_n)$.

Proof. We apply repeatedly the next fundamental property of a topological group:

(†) *Let* S *be a subset of a topological group* H, *and* $h \notin \mathrm{Cl}(S)$. *Then there exists a neighbourhood* U *of the unit element of* H *such that* $h \notin \mathrm{Cl}(USU)$.

Now it may be assumed that we are in a monotone increasing case and that $\mathscr{F}$ converges to e. Take a countable base $\{F_k\}_{k \in \mathbf{N}}$ of $\mathscr{F}$ such that $F_k \supset F_{k+1}$. Assume that there exists no such n, or, $F_k \not\subset \phi_j(G_j)$ for any k, j. Then, there exist an increasing sequence $\{n_k\}_{k \geq 0}$ of natural numbers and $g_k \in G$, $k \geq 1$, satisfying for $k \geq 1$, $g_k \in (\phi_{n_k}(G_{n_k}) \backslash \phi_{n_{k-1}}(G_{n_{k-1}})) \cap F_k$.

By applying (†), from $g_1 \notin \phi_{n_0}(G_{n_0})$ closed in $(\phi_{n_1}(G_{n_1}), \phi_{n_1}(\tau_{n_1}))$, we have an open neighbourhood $V_1 \subset \phi_{n_1}(G_{n_1})$ in $\phi_{n_1}(\tau_{n_1})$ of the unit element such that $g_1 \notin \mathrm{Cl}(V_1^2)^\wedge \phi_{n_1}(\tau_{n_1})$, where $\mathrm{Cl}(\cdot)^\wedge \tau$ means the closure in the topology τ. Put $W_1 := \phi_{n_1}^{-1}(V_1)$.

By assumption, $\mathrm{Cl}(V_1^2)^\wedge \phi_{n_1}(\tau_{n_1}) = \phi_{n_1}(G_{n_1}) \cap \mathrm{Cl}(V_1^2)^\wedge \phi_{n_2}(\tau_{n_2})$, and $\phi_{n_1}(G_{n_1})$ is closed in $(\phi_{n_2}(G_{n_2}), \phi_{n_2}(\tau_{n_2}))$, and therefore $\mathrm{Cl}(V_1^2)^\wedge \phi_{n_1}(\tau_{n_1}) = \mathrm{Cl}(V_1^2)^\wedge \phi_{n_2}(\tau_{n_2})$. Hence, from $g_2, g_1 \notin \mathrm{Cl}(V_1^2)^\wedge \phi_{n_2}(\tau_{n_2})$, we have by (†) a neighbourhood V_2 in

$(\phi_{n_2}(G_{n_2}), \phi_{n_2}(\tau_{n_2}))$ of the unit element such that $g_2, g_1 \notin \mathrm{Cl}(V_2V_1^2V_2)^\wedge \phi_{n_2}(\tau_{n_2})$. Put $W_2 := \phi_{n_2}^{-1}(V_2)$.

In this way, at the k-th step, we have by (†) a neighbourhood V_k in $(\phi_{n_k}(G_{n_k}), \phi_{n_k}(\tau_{n_k}))$ of the unit element such that

$$g_k, g_{k-1}, \dots, g_2, g_1 \notin \mathrm{Cl}(V_kV_{k-1}\cdots V_2V_1^2V_2\cdots V_{k-1}V_k)^\wedge \phi_{n_k}(\tau_{n_k}).$$

Then put $W_k := \phi_{n_k}^{-1}(V_k)$.

Thus we get a τ_{BS}-neighbourhood of $e \in G$ as

$$W[1] = \bigcup_{k=1}^{\infty} \phi_{n_k}(W_k)\cdots\phi_{n_2}(W_2)\phi_{n_1}(W_1)\phi_{n_1}(W_1)\phi_{n_2}(W_2)\cdots\phi_{n_k}(W_k),$$

which does not contain all of g_k's. This contradicts to the convergence of $\mathscr{F}$ to the unit element e. Q.E.D.

Under another simpler assumption, the same assertion holds as seen below. We denote by $\tau_{BS} \mid \phi_{j'}(G_{j'})$ the restriction of τ_{BS} onto $\phi_{j'}(G_{j'})$.

Proposition 3.1′. *Assume that, for any $j \prec j'$, $\phi_j(G_j)$ is closed in $(\phi_{j'}(G_{j'}), \tau_{BS} \mid \phi_{j'}(G_{j'}))$. If a filter $\mathscr{F}$ with a countable base is τ_{BS}-convergent in $G = \varinjlim G_j$, then, there exists an n and an $F \in \mathscr{F}$ such that $F \subset \phi_n(G_n)$.*

The proof of this proposition goes similarly as above. Only the difference is to replace the topologies $\phi_{n_k}(\tau_{n_k})$ by $\tau_{BS,n_k} := \tau_{BS} \mid \phi_{n_k}(G_{n_k})$.

Proposition 3.2. *Assumptions are the same as in Proposition* 3.1. *Then, the image $\phi_k(G_k)$ of G_k is closed in τ_{BS}-$\varinjlim G_j$, for any k. Furthermore, any $\phi_k(\tau_k)$-closed subset of $\phi_k(G_k) \subset G = \varinjlim G_j$ is also τ_{BS}-closed in G, and so the topologies $\phi_k(\tau_k)$ on $\phi_k(G_k)$ and the restriction $\tau_{BS} \mid \phi_k(G_k)$ coincide with each other.*

Proof. Assume we are in the monotone case. For the first assertion, it is enough for us to prove that, for any $g \notin \phi_k(G_k)$, there exists a BS-neighbourhood V' such that $gV' \cap \phi_k(G_k) = \varnothing$ or $V' \cap E = \varnothing$ with $E = g^{-1}\phi_k(G_k)$. Let $g \in \phi_n(G_n)$ for an $n > k$. Then $E \subset \phi_n(G_n)$ and is $\phi_n(\tau_n)$-closed by assumption. Further, for any $j \geq n$, $E \subset \phi_j(G_j)$ is $\phi_j(\tau_j)$-closed too. Choosing appropriately τ_j-neighbourhood V_j of $e_j \in G_j$ for $j > n$, as seen below, we give V' as $V' = \bigcup_{j=n+1}^{\infty} \phi_{n+1}(V_{n+1})\phi_{n+2}(V_{n+2})\cdots\phi_{j-1}(V_{j-1})\phi_j(V_j)^2$, which is a BS-neighbourhood of $e \in G$ by Lemma 3.3 below since $G = \varinjlim G_j$ is assumed to be PTA.

First take $V_{n+1} \subset G_{n+1}$ in such a way that $\phi_{n+1}(V_{n+1})^2 \cap E = \varnothing$. By assumption, $(\phi_{n+1}(G_{n+1}), \phi_{n+1}(\tau_{n+1}))$ is embedded homeomorphically into $(\phi_{n+2}(G_{n+2}), \phi_{n+2}(\tau_{n+2}))$, and so there exists a $V_{n+2} \subset G_{n+2}$ such that $\phi_{n+2}(V_{n+2})^2 \cap \phi_{n+1}(G_{n+1}) \subset \phi_{n+1}(V_{n+1})$. Then $\phi_{n+1}(V_{n+1})\phi_{n+2}(V_{n+2})^2 \cap \phi_{n+1}(G_{n+1}) \subset \phi_{n+1}(V_{n+1})^2$. In a similar way we proceed by induction on $j = n+2, n+3, \dots$. Take $V_{j+1} \subset G_{j+1}$ as $\phi_{j+1}(V_{j+1})^2 \cap \phi_j(G_j) \subset \phi_j(V_j)$. Then,

$$\phi_{n+1}(V_{n+1})\phi_{n+2}(V_{n+2})\cdots\phi_j(V_j)\phi_{j+1}(V_{j+1})^2 \cap \phi_j(G_j)$$

$$\subset \phi_{n+1}(V_{n+1})\phi_{n+2}(V_{n+2})\cdots\phi_{j-1}(V_{j-1})\phi_j(V_j)^2.$$

Finally we define V' as above, then $V' \cap \phi_{n+1}(G_{n+1}) \subset \phi_{n+1}(V_{n+1})^2$, and so V' does not touch the subset $E = g^{-1}\phi_k(G_k) \subset \phi_n(G_n)$.

For the proof of the second assertion, we start from a $\phi_k(\tau_k)$-closed subset D of $\phi_k(G_k)$. Then, replacing $\phi_k(G_k)$ by D and $E = g^{-1}\phi_k(G_k)$ by $E = g^{-1}D$, we see that the above discussion proves without any change the existence of a τ_{BS}-neighbourhood V' of $e \in G$ such that $V' \cap E = \varnothing$. This proves that $G \backslash E$ is τ_{BS}-open and so E is τ_{BS}-closed. Q.E.D.

Lemma 3.3. *Let the index set $A = \mathbf{N}$ as ordered set and $G = \varinjlim G_j$ be PTA. Then, for any sequence of τ_j-neighbourhood V_j of $e_j \in G_j$ for $j \geq k$, the union $\bigcup_{j=k}^{\infty} \phi_k(V_k)\phi_{k+1}(V_{k+1}) \cdots \phi_j(V_j)$ is a τ_{BS}-neighbourhood of $e \in G$.*

Proof. Firstly take, for every j, τ_j-neighbourhood W_j of $e_j \in G_j$ such that $W_j^2 \subset V_j$. Then we choose inductively a symmetric τ_j-neighbourhood $U_j \subset W_j$ of $e_j \in G_j$ in such a way that $\phi_j(U_j) \subset G$ is a PTA-subset and $\phi_j(U_j)U(j-1,k) \subset U(j-1,k)\phi_j(W_j)$ with $U(n,k) = \phi_n(U_n) \cdots \phi_k(U_k)^2 \cdots \phi_n(U_n)$ as in §2.2. If this is done, we have $U(n,k) \subset \phi_k(V_k)\phi_{k+1}(V_{k+1}) \cdots \phi_n(V_n)$ and so a τ_{BS}-neighbourhood $U[k] = \bigcup_{n=k}^{\infty} U(n,k)$ is contained in the union of the latter sets. Hence the proof is finished.

Now, for $n = k$, choose simply a $U_n \subset W_n$. Assume the choice is done until $n = j-1$. Then, the symmetric set $U(j-1,k)$ is PTA as a product of PTA-sets $\phi_p(U_p)$, $k \leq p \leq j-1$. Hence, for $n = j$, there exists such a $U_n \subset W_n$ that $\phi_n(U_n)U(n-1,k) \subset U(n-1,k)\phi_n(W_n)$ and $\phi_n(U_n)$ is PTA. Q.E.D.

Proposition 3.4. *Assume that A is countable and $G = \varinjlim G_j$ is PTA. Let $\bar{G}_j$ be τ_{BS}-closure of $\phi_j(G_j)$ in τ_{BS}-$\varinjlim G_j$ topologized with the restriction of τ_{BS}. If $\bar{G}_j$ are PTA-groups for $j \geq 1$, then τ_{BS}-$\varinjlim G_j = \tau_{BS}$-$\varinjlim \bar{G}_j$.*

The proof can be given by applying Corollary 1.5.

3.2. Extended BS-topology. We discuss here other group topologies weaker than τ_{BS}.

A subgroup H with a group topology τ_H is called a PTA-subgroup of G if τ_H has a fundamental system $\mathscr{Y} = \{Y_\gamma\}_{\gamma \in \Gamma}$ of neighbourhoods whose images are PTA-sets in G. Each Y_γ is symmetric by definition and, for any Y_γ, there exists a Y_δ such that $(Y_\delta)^2 \subset Y_\gamma$.

For a family $\{U_j\}_{j \geq k}$ of symmetric open neighbourhood of $e_j \in G_j$ in τ_j, we put $U(\gamma, n, k)$, $n \geq k$, as equal to

$$\phi_n(U_n)\phi_{n-1}(U_{n-1}) \cdots \phi_k(U_k) Y_\gamma \phi_k(U_k)\phi_{k+1}(U_{k+1}) \cdots \phi_{n-1}(U_{n-1})\phi_n(U_n),$$

and define $U[\gamma, k] := \bigcup_{n=k}^{\infty} U(\gamma, n, k)$. Let $\mathscr{U}'$ be the set of all such $U[\gamma, k]$'s, where $\gamma \in \Gamma$ and $\{U_j\}_{j \geq k}$ runs over all possible families for $k = 1, 2, \ldots$. Then the conditions (GT1)–(GT5) in §1.3 are clear for $\mathscr{U}'$ except (GT4).

Proposition 3.5. *Assume that (H, τ_H) is a PTA-subgroup of $G = \varinjlim G_j$. Then, in case H is normal, the family $\mathscr{U}'$ gives a fundamental system of neighbourhoods of $e \in G$ for a group topology (called an extended BS-topology). With respect to this topology the natural homomorphism from H into G is continuous.*

§4. The case of uncountable inductive systems

Here we discuss when we can reduce an inductive system $\{(G_\alpha, \tau_\alpha), \alpha \in A; \phi_{\beta,\alpha}, \alpha, \beta \in A, \alpha \preceq \beta\}$ to a certain equivalent one with a countable directed set as its index set. Then we know when the results in the preceeding sections can be applied in uncountable cases.

A directed set A is called of *fish-bone type* if it has a cofinal, totally ordered subset A_{tot}. Then, as a totally ordered set, A_{tot} itself has a cofinal, well-ordered subset $A_{w.o.}$, as is easily seen by using Zorn's lemma. We see easily the following fact.

Lemma 4.1. *Assume that A is of fish-bone type. Then*

$$\operatorname*{ind\,lim}_{\alpha \in A} G_\alpha = \operatorname*{ind\,lim}_{\beta \in A_{tot}} G_\beta = \operatorname*{ind\,lim}_{\gamma \in A_{w.o.}} G_\gamma.$$

For a well-ordered set B and $\beta \in B$, put $\beta_+ = \min\{\gamma \in B; \gamma \succ \beta\}$, and define β_- as α for which $\alpha_+ = \beta$, if exists. Put further

$$B^+ = \{\beta \in B; \beta_- \text{ does not exist}\}, \quad B^- = \{\beta \in B; \alpha \prec \beta \ (\forall \alpha \in B^+)\}.$$

Lemma 4.2. (i) *For a well-ordered set B, the subsets B^+ and B^- are well-ordered.* (ii) *The subset B^- is at most countable.* (iii) *If $B^- \neq \emptyset$, it is cofinal with B, otherwise B^+ is cofinal with B.*

Now, take $B = A_{w.o.}$ in place of the directed set A, then we have two cases depending on $B^- \neq \emptyset$ or $= \emptyset$.

Case 1 ($B^- \neq \emptyset$). By Lemma 4.2, B^- is countable and cofinal with B, hence with A. Thanks to Lemma 4.1, we have

$$\operatorname*{ind\,lim}_{\alpha \in A} G_\alpha = \operatorname*{ind\,lim}_{\beta \in B} G_\beta = \operatorname*{ind\,lim}_{\beta \in B^-} G_\beta.$$

Thus we come to the case of countable inductive system.

Case 2 ($B^- = \emptyset$). In this case, the set B^+ is cofinal with B and so with A itself, and $\operatorname*{ind\,lim}_{\alpha \in A} G_\alpha = \operatorname*{ind\,lim}_{\beta \in B^+} G_\beta$. Therefore we start again from B^+ and consider $(B^+)^-$. If $(B^+)^- \neq \emptyset$, the situation is reduced to a case of countable inductive system. If $(B^+)^- = \emptyset$, we repeat the process again.

There may exist the case where we cannot arrive at a countable case even by such an induction on the well ordered sets.

§5. Positive definite functions and unitary representations

Let $\{(G_j, \tau_j), j \in \mathbf{N}; \phi_{j',j}\}$ be a countable LCG inductive system and put $G = \varinjlim G_j$. Then, τ_{ind} coincides with τ_{BS} and is a group topology for G as is proved in Theorem 2.7. In this section, we study positive definite functions and unitary representations of G. Although every G_j are assumed to be locally compact, G itself is in general no longer locally compact, and so the following problem is interesting.

Problem 5.1. *Construct sufficiently many unitary representations G so that elements of G can be separated.*

However the above problem has a natural limitation coming from the topology on G. In fact, the normal subgroup $M = \bigcap_{U \in \mathscr{U}_{BS}} U$ in Lemmas 2.5 and 2.6 is a natural bound, that is, the points in M cannot be separated by continuous positive definite functions.

Therefore we assume from the beginning that the induced topology τ_{ind} on G is T_0 and so Hausdorff (since $\tau_{ind} = \tau_{BS}$ is now a group topology). Under this assumption, we can replace the original inductive system by $\{(\phi_j(G_j),\ \phi_j(\tau_j))\}_{j \in \mathbf{N}}$ to get the same G, as seen in Proposition 2.8. Here $\phi_j(G_j) \subset G$ is actually a subgroup of G with topology $\phi_j(\tau_j)$. Thus we can assume that the inductive system is of injective type, that is, all the homomorphisms $\phi_{j',j}$ are injective. Furthermore as is discussed in Case 2 in §2.2, we may assume without loss of generality that G_j's are monotone increasing or $A = \mathbf{N}$ as directed sets. For simplicity, we omit the notations $\phi_{j',j}$ and ϕ_j, and consider imbeddings $G_j \hookrightarrow G_{j'}$ and $G_j \hookrightarrow G$. Then $G = \bigcup_{j \in \mathbf{N}} G_j$.

To get an affirmative answer to Problem 5.1, we assume further that the above injective homomorphisms $G_j \hookrightarrow G_{j'}$ are all homeomorphisms into.

5.1. Positive definite functions. To solve the above problem we construct sufficiently many τ_{ind}-continuous positive definite functions.

Take a τ_{ind}-neighbourhood O of $e \in G$. We construct a τ_{ind}-continuous positive definite function F with $\mathrm{supp}(F) \subset O$. Choose inductively a symmetric, relatively compact open neighbourhood U_j in τ_j of $e \in G_j$ (so, $e \in U_j \subset G_j$), for $j = 1, 2, \dots$ in such a way that

(1) $(U_1)^4 \subset G_1 \cap O$, and

(2) for $j > 1$, $U_j^2 \cap G_{j-1} \subset U_{j-1}$, $U(j,1) := U_j \tilde{U}_j (\tilde{U}_j)^{-1} U_j \subset G_j \cap O$ and $U(j,1) \cap G_k \subset U(k,1)$ $(1 \le k \le j)$ with $\tilde{U}_j := U_j \tilde{U}_{j-1} = U_j U_{j-1} \cdots U_1$.

Then, taking into account the process in the proof of Theorem 2.7, we see that such a system $\{U_j\}_{j \in \mathbf{N}}$ exists. Furthermore $U[1] := \bigcup_{j=1}^{\infty} U(j,1)$ is an open neighbourhood in τ_{BS} contained in O.

Denote by $C_0(G_j;\tau_j)$ (resp. $C_0^+(G_j;\tau_j)$, $C_0^{+s}(G_j;\tau_j)$) the space of all τ_j-continuous functions φ on G_j, with compact supports (resp. non-negative, and symmetric (i.e., $\varphi(g) = \varphi(g^{-1})$) in addition). Further let μ_j be a right Haar measure on G_j normalized later (in abbreviation, $d\mu_j(g) = d_j g$), and $\Delta_j(g) = d\mu_j(g^{-1})/d\mu_j(g)$ $(g \in G_j)$ be the modular function, or $\mu_j(g^{-1}E) = \Delta_j(g)\mu_j(E)$ for any Borel subset E of G_j. Put,

$$p_j := \max\{1, \mu_k(U(k,1))\ (k \le j)\};$$

and for functions $\varphi, \varphi_1, \varphi_2$ on G or on G_m with $m \ge j$,

$$\|\varphi\|_{L^q(G_j)} := \|\varphi|_{G_j}\|_{L^q(G_j)}, \quad \langle\varphi_1,\varphi_2\rangle_j := \int_{G_j} \varphi_1(g)\overline{\varphi_2(g)}\, d_jg;$$

and for functions $f_j \in C_0(G_j;\tau_j)$, $j = 1,2,\dots,$

$$(f_n *_n f_{n-1})(g) := \int_{h\in G_{n-1}} f_n(gh^{-1})f_{n-1}(h)\, d_{n-1}h \quad (g \in G_n),$$

$$\tilde{f}_n := f_n *_n \tilde{f}_{n-1} = f_n *_n f_{n-1} *_{n-1} f_{n-2} *_{n-2} \cdots *_2 f_1.$$

Then, $f_n *_n f_{n-1}, \tilde{f}_n \in C_0(G_n;\tau_n)$. For $f_j \in C_0(G_j;\tau_j)$, put $(f_j) := \{g \in G_j; f_j(g) \neq 0\}$ and $[f_j] := \operatorname{supp}(f_j)$.

Lemma 5.2. *For* $f_j \in C_0^+(G_j;\tau_j)$, $j = 1,2,\dots,$

$$(\tilde{f}_n) = (f_n)(f_{n-1})(f_{n-2})\cdots(f_2)(f_1),$$

$$[\tilde{f}_n] = [f_n][f_{n-1}][f_{n-2}]\cdots[f_2][f_1].$$

Now choose triplets $\{V_n, f_n, \mu_n\}$ inductively as follows, where $V_n \subset U_n$ is a symmetric, relatively compact open neighbourhood (in τ_n) of $e \in G_n$ and $f_n \in C_0^{+s}(G_n;\tau_n)$.

Step 1. $\Delta_1(g) < 1 + 2^{-1}$ $(g \in V_1)$, $[f_1][f_1] \subset V_1$, $e \in (f_1)$, and normalize μ_1 as $\|f_1\|_{L^2(G_1)} = 1$.

Step 2. Assume that $\{V_j, f_j, \mu_j\}$ have been chosen for $j = 1,2,\dots,n$, and choose V_{n+1} in such a way that

$$\Delta_{n+1}(g) < 1 + 2^{-n-1} \ (g \in V_{n+1}), \quad \text{and}$$

$$\|L(g)\tilde{f}_n - \tilde{f}_n\|_{L^\infty(G_n)} < 2^{-2n-4}/p_n \quad (\forall g \in V_{n+1} \cap G_n),$$

where $L(g)\tilde{f}_n(h) = \tilde{f}_n(g^{-1}h)$ $(g,h \in G_n)$.

Step 3. Choose f_{n+1} as $[f_{n+1}][f_{n+1}] \subset V_{n+1}$, $e \in (f_{n+1})$, and

$$\int_{G_n} f_{n+1}(h^{-1})\, d_nh = 1.$$

Then, normalize μ_{n+1} as $\|\tilde{f}_{n+1}\|_{L^2(G_{n+1})} = 1$.

Proposition 5.3. *There exists a τ_{ind}-continuous function $\tilde{f} \in C^+(G;\tau_{ind})$ such that, for any fixed j, $\tilde{f}_n$ converges to $\tilde{f}$ uniformly on G_j, as $n \to \infty$.*

The τ_{ind}-open subset $(\tilde{f}) = \{g \in G; \tilde{f}(g) \neq 0\}$ is contained in $\bigcup_{n=1}^{\infty} \tilde{U}_n \subset O$ and $(\tilde{f}) \cap G_j \subset U_j\tilde{U}_j$ $(j \geq 1)$.

Proof.

$$\|\tilde{f}_{n+1} - \tilde{f}_n\|_{L^\infty(G_n)} = \sup_{g\in G_n} \left| \int_{G_n} (f_{n+1}(gh^{-1})\tilde{f}_n(h) - f_{n+1}(h^{-1})\tilde{f}_n(g))\, d_nh \right|$$

$$\leq \int_{G_n} f_{n+1}(h^{-1})\|L(h^{-1})\tilde{f}_n - \tilde{f}_n\|_{L^\infty(G_n)}\, d_nh < 2^{-2n-4}/p_n,$$

because $\mathrm{supp}(f_{n+1}) \subset V_{n+1}$, and so $h \in V_{n+1} \cap G_n$ in the last integration. On the other hand, $\|\tilde{f}_{n+1} - \tilde{f}_n\|_{L^\infty(G_m)} \geq \|\tilde{f}_{n+1} - \tilde{f}_n\|_{L^\infty(G_k)}$ for $n \geq m \geq k$. Therefore we see that $\tilde{f}_n|_{G_j}$ converge uniformly to an $\tilde{f}|_{G_j}$ for any j. Since $\tilde{f}|_{G_j}$ is τ_j-continuous for any j, the limit function $\tilde{f}$ is τ_{ind}-continuous.

Assertions for the support of $\tilde{f}$ are easy to see. Q.E.D.

Lemma 5.4. *For any fixed j, $\tilde{f}_n|_{G_j}$ converges to $\tilde{f}|_{G_j}$ in $L^2(G_j; \mu_j)$ and*

$$\lim_{j\to\infty} \|\tilde{f}|_{G_j}\|_{L^2(G_j)} = 1.$$

Proof. For $k > j$, put

$$I_k := \|\tilde{f}_k - \tilde{f}_{k-1}\|^2_{L^2(G_j)} = \int_{G_j} |\tilde{f}_k(g) - \tilde{f}_{k-1}(g)|^2 \, d_j g.$$

Since $g \in (\tilde{f}_k) \cap G_j \subset (\tilde{f}) \cap G_j \subset U_j \tilde{U}_j$, I_k is majolized by

$$\int_{U_j\tilde{U}_j} \|\tilde{f}_k - \tilde{f}_{k-1}\|^2_{L^\infty(G_j)} \, d_j g \leq \mu_j(U_j \tilde{U}_j) \cdot \|\tilde{f}_k - \tilde{f}_{k-1}\|^2_{L^\infty(G_{k-1})}$$
$$\leq p_j \cdot 2^{-4k-4}/(p_{k-1})^2 \leq 2^{-4k-4},$$

and so, $\|\tilde{f}_k - \tilde{f}_{k-1}\|_{L^2(G_j)} \leq 2^{-2k-2}$ for $k > j$. This proves the convergence in $L^2(G_j; \mu_j)$. Furthermore we have

$$| \, \|\tilde{f}\|_{L^2(G_j)} - 1| = | \, \|\tilde{f}\|_{L^2(G_j)} - \|\tilde{f}_j\|_{L^2(G_j)}| \leq \sum_{k=j}^{\infty} \|\tilde{f}_{k+1} - \tilde{f}_k\|_{L^2(G_j)}$$
$$\leq \sum_{k=j}^{\infty} 2^{-2k-2} = 2^{-2j}/3 \to 0 \quad (j \to \infty).$$

Proposition 5.5. *Put $F_n(g) := \langle R(g)\tilde{f}_n, \tilde{f}_n \rangle_{L^2(G_n)}$ for $g \in G_n$, where $R(g)\tilde{f}_n(h) = \tilde{f}_n(hg)$, $h \in G_n$. Then the series of positive definite functions F_n on G_n converges, as $n \to \infty$, to a non-zero function F on G, uniformly on each fixed G_j. The function F is positive definite on G, continuous in $\tau_{ind} = \tau_{BS}$, and* $\mathrm{supp}\,(F) \subset U[1] \subset O$.

The proof of this proposition needs rather lengthy calculations and will be essentially carried out in the proof of Lemma 5.9 below. Note that the function F here is nothing but the limit function $\langle R(g)\tilde{f}, \tilde{f} \rangle$ in Lemma 5.9.

5.2. Coincidence of topologies, τ_{ind}, τ_{BS}, and $\tau_{p.d.}$. As is proved above we have $F \in \mathscr{P}(\tau_{ind})$ with $\mathrm{supp}(F) \subset U[1]$ for a previously chosen τ_{BS}-neighbourhood $U[1]$. This means that $\tau_{p.d.}$ is not weaker than τ_{BS}. Since $\tau_{ind} = \tau_{BS}$ is already known, we get the coincidence of all these three topologies.

Theorem 5.6. *For a countable LCG inductive system $\{G_j\}_{j\in\mathbf{N}}$, assume that the topology τ_{ind} on $G = \varinjlim G_j$ is T_0 (and so Hausdorff), and that the injective homomorphism of $\phi_j(G_j)$ to $\phi_{j'}(G_{j'})$ is homeomorphism into for any $j \prec j'$. Then,* *the topologies τ_{ind}, τ_{BS} and $\tau_{p.d.}$ on G coincide with each other: $\tau_{ind} = \tau_{BS} = \tau_{p.d.}$, and there exist sufficiently many continuous positive definite functions on G.*

From the result in Proposition 5.5, we have thus

Theorem 5.7. *For a countable LCG inductive system $\{G_j\}_{j\in\mathbf{N}}$, let assumptions be as in Theorem 5.6. Then, there exist on $G = \varinjlim G_j$ sufficiently many continuous positive definite functions with respect to the unique group topology $\tau_{ind} = \tau_{BS} = \tau_{p.d.}$.*

5.3. Unitary representations. Let us construct a continuous unitary representation on a space generated by the function $\tilde{f} = \lim_{n\to\infty} \tilde{f}_n$.

Let $\varphi, \varphi_1, \varphi_2 \in C^+(G)$. Under the assumption of existence, put

$$\|\varphi\|_2 := \lim_{j\to\infty} \|\varphi\|_{L^2(G_j)}, \quad \langle \varphi_1, \varphi_2\rangle := \lim_{j\to\infty} \langle \varphi_1, \varphi_2\rangle_j.$$

Then, for any $g \in G$,

$$\|R(g)\varphi\|_2 = \|\varphi\|_2, \quad \langle R(g)\varphi_1, R(g)\varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle,$$

and Lemma 5.4 gives $\|\tilde{f}\|_2 = 1$.

Lemma 5.8.

$$\int_{G_n} d_n g \int_{G_{n-1}} f_n(gh) f_n(g)\, d_{n-1}h \le (1 - 2^{-2n-2})^{-1}.$$

Proof.

$$1 = \|\tilde{f}_n\|^2_{L^2(G_n)} = \int_{G_n} d_n g \left| \int_{G_{n-1}} f_n(gh^{-1}) \tilde{f}_{n-1}(h)\, d_{n-1}h \right|^2$$

$$= \int_{G_n} d_n g \int_{G_{n-1}} \int_{G_{n-1}} f_n(gh^{-1}) f_n(gh_1^{-1}) \tilde{f}_{n-1}(h) \tilde{f}_{n-1}(h_1)\, d_{n-1}h\, d_{n-1}h_1$$

$$= \int_{G_{n-1}} \left(\int_{G_n} f_n(gh_1) f_n(g)\, d_n g \right) \langle L(h_1^{-1})\tilde{f}_{n-1}, \tilde{f}_{n-1}\rangle_{n-1}\, d_{n-1}h_1,$$

where $\langle L(h_1^{-1})\tilde{f}_{n-1}, \tilde{f}_{n-1}\rangle_{n-1} = \int_{G_{n-1}} \tilde{f}_{n-1}(h_1 h)\tilde{f}_{n-1}(h)\, d_{n-1}h$.

In the last two integarations, $gh_1 \in (f_n)$, $g \in (f_n)$, and so, $h_1 \in (f_n)(f_n) \cap G_{n-1}$ $=: C_n$(put) $\subset V_n \cap G_{n-1}$. Further,

$$|\langle L(h_1^{-1})\tilde{f}_{n-1}, \tilde{f}_{n-1}\rangle_{n-1} - \|\tilde{f}_{n-1}\|^2_{L^2(G_{n-1})}|$$

$$\le \|L(h_1^{-1})\tilde{f}_{n-1} - \tilde{f}_{n-1}\|_{L^2(G_{n-1})} \cdot \|\tilde{f}_{n-1}\|_{L^2(G_{n-1})}$$

$$\le \mu_{n-1}(C_n \cdot (\tilde{f}_{n-1}))^{1/2} \|L(h_1^{-1})\tilde{f}_{n-1} - \tilde{f}_{n-1}\|_{L^\infty(G_{n-1})}$$

$$\le (p_{n-1})^{1/2} 2^{-2n-2}/p_{n-1} \le 2^{-2n-2} \quad (\text{since } \|\tilde{f}_j\|_{L^2(G_j)} = 1,\ p_j \ge 1).$$

Hence we get successively,

$$\langle L(h_1^{-1})\tilde{f}_{n-1}, \tilde{f}_{n-1}\rangle_{n-1} \geq \|\tilde{f}_{n-1}\|^2_{L^2(G_{n-1})} - 2^{-2n-2} = 1 - 2^{-2n-2},$$

$$1 = \|\tilde{f}_n\|^2_{L^2(G_n)} \geq (1 - 2^{-2n-2}) \int_{G_{n-1}} \int_{G_n} f_n(gh_1) f_n(g)\, d_n g\, d_{n-1} h_1,$$

$$\int_{G_n} \int_{G_{n-1}} f_n(gh) f_n(g)\, d_{n-1} h\, d_n g \leq (1 - 2^{-2n-2})^{-1}.$$

Lemma 5.9. *Let $\tilde{f} = \lim_{n\to\infty} \tilde{f}_n$. Then, both of $\|R(g)\tilde{f} - \tilde{f}\|_2$ and $\langle R(g)\tilde{f}, \tilde{f}\rangle$ exist for $g \in G$, and continuous on G, where $R(g)\tilde{f}(h) = \tilde{f}(hg)$. Furthermore the convergences of $\lim_{j\to\infty} \|R(g)\tilde{f} - \tilde{f}\|_{L^2(G_j)}$ and $\lim_{j\to\infty} \langle R(g)\tilde{f}, \tilde{f}\rangle_j$ are uniform on each G_k.*

Proof. It is enough to prove the assertion for $\|R(g)\tilde{f} - \tilde{f}\|_2$.

(i) CONVERGENCE AS $j \to \infty$. Let $g \in G_k$ and take any $j \geq k$, then

$$\begin{aligned}
&|\, \|R(g)\tilde{f} - \tilde{f}\|_{L^2(G_j)} - \|R(g)\tilde{f}_j - \tilde{f}_j\|_{L^2(G_j)}| \\
&\leq \sum_{s=j}^{\infty} \|(R(g)\tilde{f}_{s+1} - \tilde{f}_{s+1}) - (R(g)\tilde{f}_s - \tilde{f}_s)\|_{L^2(G_j)} \\
&\leq \sum_{s=j}^{\infty} (\|R(g)(\tilde{f}_{s+1} - \tilde{f}_s)\|_{L^2(G_j)} + \|\tilde{f}_{s+1} - \tilde{f}_s\|_{L^2(G_j)}) \leq 2^{-2j-1}/3.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|R(g)\tilde{f}_j - \tilde{f}_j\|^2_{L^2(G_j)} &= \int_{G_j} d_j g_1 \int_{G_{j-1}} \int_{G_{j-1}} f_j(g_1 h^{-1}) f_j(g_1 h_1^{-1}) \\
&\quad \cdot (\tilde{f}_{j-1}(hg) - \tilde{f}_{j-1}(h))(\tilde{f}_{j-1}(h_1 g) - \tilde{f}_{j-1}(h_1))\, d_{j-1} h\, d_{j-1} h_1 \\
&= \int_{G_{j-1}} d_{j-1} h_1 \left(\int_{G_j} f_j(g_1 h_1) f_j(g_1)\, d_j g_1 \right) \\
&\quad \cdot \left\{ \int_{G_{j-1}} (\tilde{f}_{j-1}(hg) - \tilde{f}_{j-1}(h))(\tilde{f}_{j-1}(h_1 hg) - \tilde{f}_{j-1}(h_1 h))\, d_{j-1} h \right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\{\cdots\} &= \langle R(g)\tilde{f}_{j-1} - \tilde{f}_{j-1}, L(h_1^{-1})(R(g)\tilde{f}_{j-1} - \tilde{f}_{j-1})\rangle_{j-1} \\
&\leq \Delta_{j-1}(h_1) \|R(g)\tilde{f}_{j-1} - \tilde{f}_{j-1}\|^2_{L^2(G_{j-1})}.
\end{aligned}$$

Here, $g_1 h_1 \in (f_j)$, $g_1 \in (f_j) \Rightarrow h_1 \in (f_j)(f_j) \cap G_{j-1} \Rightarrow \Delta_{j-1}(h_1) < 1 + 2^{-j+1}$.

Hence by Lemma 5.8,

$$\begin{aligned}
\|R(g)\tilde{f}_j - \tilde{f}_j\|^2_{L^2(G_j)} &\leq (1 - 2^{-2j-2})^{-1}(1 + 2^{-j+1}) \|R(g)\tilde{f}_{j-1} - \tilde{f}_{j-1}\|^2_{L^2(G_{j-1})} \\
&\leq (1 + 2^{-j+2}) \|R(g)\tilde{f}_{j-1} - \tilde{f}_{j-1}\|^2_{L^2(G_{j-1})}.
\end{aligned}$$

Put $I_j(g) = \prod_{s=k}^{j}(1+2^{-s+2})^{-1} \cdot \|R(g)\tilde{f}_j - \tilde{f}_j\|^2_{L^2(G_j)}$. Then the above inequality shows that $0 \le I_j(g) \le I_{j-1}(g)$. So $I_j(g)$ converges as $j \to \infty$. On the other hand, $\prod_{s=k}^{j}(1+2^{-s+2})^{-1}$ is also convergent, whence we have the existence of $\lim_{j\to\infty} \|R(g)\tilde{f}_j - \tilde{f}_j\|_{L^2(G_j)}$.

Thus the existence of $\lim_{j\to\infty} \|R(g)\tilde{f} - \tilde{f}\|_{L^2(G_j)} = \|R(g)\tilde{f} - \tilde{f}\|_2$ is now guaranteed.

(ii) CONTINUITY. Choose a symmetric, relatively compact open neighbourhood $W_n \subset V_n$ (in τ_n) of $e \in G_n$ as

$$\|R(g)\tilde{f}_n - \tilde{f}_n\|_{L^2(G_n)} < 2^{-n-4} \quad (g \in W_n)$$

and define a system of of neighbourhoods of $e \in G$ in τ_{BS} as

$$\tilde{W}^r := \bigcup_{s=r}^{\infty} \tilde{W}^r_s \quad \text{with } \tilde{W}^r_s = W_r W_{r+1} \cdots W_s \quad (s \ge r).$$

Let us prove that, for a fixed $\varepsilon > 0$, there exists an r such that $\|R(h)\tilde{f} - \tilde{f}\|_2 < \varepsilon$ $(\forall h \in \tilde{W}^r)$. In fact, for $h \in \tilde{W}^r$, take $s \ge r$ such that $h \in \tilde{W}^r_s$. Then, $h = h_r h_{r+1} \cdots h_s$ with some $h_k \in W_k$ $(s \ge k \ge r)$. Hence

$$\begin{aligned}\|R(h)\tilde{f} - \tilde{f}\|_2 &\le \sum_{k=r}^{s} \|R(h_r h_{r+1} \cdots h_{k-1} h_k)\tilde{f} - R(h_r h_{r+1} \cdots h_{k-1})\tilde{f}\|_2 \\ &= \sum_{k=r}^{s} \|R(h_k)\tilde{f} - \tilde{f}\|_2\end{aligned}$$

From the inequality at the end of (i), we get, for $h_k \in W_k$, $k \le j$,

$$\begin{aligned}\|R(h_k)\tilde{f}_j - \tilde{f}_j\|^2_{L^2(G_j)} &\le \prod_{m=k+1}^{j}(1+2^{-m+2}) \cdot \|R(h_k)\tilde{f}_k - \tilde{f}_k\|^2_{L^2(G_k)} \\ &\le 2^{-2k-8}C^2 \quad \text{with } C = \left(\prod_{m=1}^{\infty}(1+2^{-m+2})\right)^{1/2}.\end{aligned}$$

Further, from the inequality proved at the beginning of (i), we get

$$\begin{aligned}\|R(h_k)\tilde{f} - \tilde{f}\|_{L^2(G_j)} &\le \|R(h_k)\tilde{f}_j - \tilde{f}_j\|_{L^2(G_j)} + 2^{-2j-1}/3 \\ &\le 2^{-k-4}C + 2^{-2j-1}/3.\end{aligned}$$

Letting $j \to \infty$, we obtain $\|R(h_k)\tilde{f} - \tilde{f}\|_2 \le 2^{-k-4}C$.

Take r such that $2^{-r-3}C < \varepsilon$, then finally we have, for $h \in \tilde{W}^r$,

$$\begin{aligned}\|R(h)\tilde{f} - \tilde{f}\|_2 &\le \sum_{k=r}^{s} \|R(h_k)\tilde{f} - \tilde{f}\|_2 \\ &\le C \cdot \sum_{k=r}^{\infty} 2^{-k-4} = C \cdot 2^{-r-3} < \varepsilon.\end{aligned}$$

(iii) Uniform convergence on G_k. Note that the limit function $J(g) := \|R(g)\tilde{f} - \tilde{f}\|_2$ is continuous in $g \in G$, and that the series of continuous functions $I_j(g) > 0$ in (i) is decreasing and pointwise convergent to a continuous function $\prod_{s=k}^{\infty}(1+2^{-s+2})^{-1} \cdot J(g)$. Then we see, by Dini's Theorem on a decreasing sequence of non-negative continuous functions on a compact, that the convergence is uniform on every G_k.

Lemma 5.9 is now completely proved.

Let $\mathscr{H}$ be the space spanned by $\{R(g)\tilde{f}; g \in G\}$. Then, by the results above, we can introduce on it a G-invariant inner product $\langle \cdot, \cdot \rangle$. Let $\mathscr{N}$ be the kernel of this inner product, then $\mathscr{H}/\mathscr{N}$ becomes a pre-Hilbert space. Denote by $\mathscr{H}(\tilde{f})$ its completion. Then the right translations $R(g)$, $g \in G$ on it give a unitary representation of G as follows.

Theorem 5.10. *For a countable LCG inductive system* $\{(G_j, \tau_j)\}_{j \in \mathbf{N}}$, *let assumptions be as in Theorem* 5.6. *Then, there exists a continuous unitary representation* $\{R(g), \mathscr{H}(\tilde{f})\}$ *with a unit cyclic vector* $\tilde{f}$. *Its spherical function* $\langle R(g)\tilde{f}, \tilde{f}\rangle$ *is equal to the positive definite function* $F(g) = \lim_{n\to\infty} \langle R(g)\tilde{f}_n, \tilde{f}_n\rangle_{L^2(G_n)}$ *in Proposition* 5.5.

5.4. Induced representations. In the case where the limit group $G = \lim_{j\to\infty} G_j$ is no longer locally compact, there does not exist an invariant measure on G similar to Haar measures. However, we show here, in a certain circumstance, the notion of induced representations can be carried over for G and its subgroup G_m.

At first we remark the following fact [13]. Recall that a Borel measure ν on a locally compact group H is called *positive definite* if, for any $\varphi \in C_0(H)$,

$$\int_H \left(\int_H \psi(hh')\overline{\psi(h')}\, dh' \right) d\nu(h) \geq 0.$$

A measure ν of the form $d\nu(h) = f(h)\, dh$ $(h \in H)$ with a continuous function f and a right Haar measure dh on H is positive definite if and only if the function f is positive definite, and the unitary representation of H associated to ν through the GNS construction is equivalent to the one associated to f.

Lemma 5.11. *Let* ν_n *be a positive definite measure on* G_n, *and define a measure* ν_{n+1} *on* G_{n+1} *by*

$$d\nu_{n+1}(h) = (\Delta_n(h)/\Delta_{n+1}(h))^{1/2}\, d\nu_n(h) \quad (h \in G_n).$$

Then, ν_{n+1} *is positive definite on* G_{n+1}. *The unitary representation of* G_{n+1} *associated to* ν_{n+1} *through the GNS construction is equivalent to the induced representation* (*from* G_n *to* G_{n+1}) *of the unitary representation of* G_n *associated to* ν_n.

Assumption Δ. The sequence $\Delta_n, n = 1, 2, \ldots,$ converges on G in such a way that on each G_j it converges uniformly in the wider sense.

Theorem 5.12. *Let $\{G_j\}_{j\in\mathbf{N}}$ be a countable LCG inductive system, and the assumptions be as in Theorem 5.6. Take any subgroup G_m of $G = \varinjlim G_j$. Then, under Assumption Δ, the unitary representation of G_m associated to a positive definite Borel measure ν on G_m can be induced up to a unitary representation of G.*

Proof. Put $\nu_m = \nu$. Then, for any $n \geq m$, the measure $d\nu_n(h) = (\Delta_m(h)/\Delta_n(h))^{1/2} \cdot d\nu_m(h)$ $(h \in G_n)$ is positive definite on G_n by Lemma 5.11, and it is supported on G_m. By Assumption Δ, the series of measures ν_n $(n = m, m+1, \ldots)$, considered as measures on G_m, converges to a measure μ on G_m.

The following function is a continuous positive definite function on G and the cyclic representation associated to it is the induced representation looked for:

$$\begin{aligned}\langle\!\langle U(g)\tilde{f}, \tilde{f} \rangle\!\rangle &:= \int_{G_m} \langle R(hg)\tilde{f}, \tilde{f} \rangle \, d\mu(h) \\ &= \lim_{j\to\infty} \int_{G_m} \langle R(hg)\tilde{f}_j, \tilde{f}_j \rangle_j (\Delta_m(h)/\Delta_j(h))^{1/2} \, d\nu_m(h).\end{aligned}$$

The representation U is realized on the space spanned by $\tilde{f}$, similarly as for Lemma 5.11. This proves our assertion.

Part II. Group topologies for the group of diffeomorphisms $\mathrm{Diff}_0(M)$

§ 6. Preliminaries and main result

Let M be a connected, non-compact, σ-compact C^r-manifold with $1 \leq r \leq \infty$. Denote by $\mathrm{Diff}(M)$ the group of all diffeomorphisms and by $\mathrm{Diff}_0(M)$ its subgroup consisting of diffeomorphisms with compact supports. Here we study group topologies on the group $G = \mathrm{Diff}_0(M)$.

Usually, as seen in the beginning of [6], we have been considering on G the topology τ given by the following way of convergence: *a sequence g_k, $k = 1, 2, \ldots$, converges to g if supports of g and of all g_k are contained in a compact subset K and $g_k \to g$ on K uniformly together with all derivatives.*

This topology τ is normally understood as an inductive limit of topologies of canonical subgroups $G_n \nearrow G$, $n \to \infty$, as follows. First take an increasing sequence $M_0 \subset M_1 \subset M_2 \subset \cdots$ of relatively compact open subsets so that $\bigcup_{n=0}^{\infty} M_n = M$ and that each $K_n := \overline{M}_n$, the closure of M_n, is a manifold with boundary. Put

$$G_n = \mathrm{Diff}(K_n) := \{g \in G; \operatorname{supp}(g) \subset K_n\}.$$

Then we have an increasing sequence of subgroups as

$$G_0 \subset G_1 \subset G_2 \subset \cdots, \quad \bigcup_{n=0}^{\infty} G_n = G.$$

The topology τ_n on G_n is given by considering G_n as a topological subgroup of the Fréchet Lie group $\mathrm{Diff}(M_n'')$, where M_n'' is the compact manifold obtained by patching M_n and its mirror image M_n' through the boundary. For the Lie group structure of the group $\mathrm{Diff}(N)$ of a compact manifold N, we refer [8] or [11].

In an algebraic sense, $G = \varinjlim G_n$, and as a topology on G, we have $\tau_{ind} = \varinjlim \tau_n$. On the other hand, as suggested by the results in Part I, the phenomenon that τ_{ind} does not give a group topology seems to be rather general for the case of non-locally-compact groups. The purpose of this part is to prove that, when M is non-compact, this is actually the case for $G = \mathrm{Diff}_0(M)$ with the inductive system consisting of highly non-locally-compact groups G_n.

Thus our main theorem here is the following (cf. Proposition 1.1).

Theorem 6.1. *Let M be a connected, non-compact, σ-compact C^r-manifold, $1 \le r \le \infty$. For the group $G = \mathrm{Diff}_0(M)$, the multiplication $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$, is not continuous with respect to the inductive limit topology τ_{ind}.*

This fact does not affect so much the theory of unitary representations of the group G (for instance, in [2], [9] and [14] etc.), because we can take, as our background, the group topology $\tau_{p.d.}$ on G which is defined by means of the set $\mathscr{P}(\tau_{ind})$ of τ_{ind}-continuous positive definite functions (cf. §1). However it has certainly some effects, for instance, for determining *continuous* 1-cocycles $\alpha(g, p)$, $(g, p) \in G \times M$, depending on which continuity we choose (cf. [3], [12]).

Note that if a *sequence* $g_k \in G$, $k = 1, 2, \dots$, is τ_{ind}-convergent to $g \in G$, then there exists a compact subset K of M such that $\mathrm{supp}(g_k)$ and $\mathrm{supp}(g)$ are contained in K, and the convergence is as in [6]. To see this, we remark that the restriction on $G_n = \mathrm{Diff}(K_n)$ of the inductive limit τ_{ind} on G is exactly the original τ_n. In fact, let O_n be a τ_n-open subset of G_n, then, for $k > n$, we can choose inductively a τ_k-open subset O_k of G_k such that $O_k \cap G_{k-1} = O_{k-1}$, since the restriction of τ_k onto G_{k-1} is equal to τ_{k-1}. Put $O = \bigcup_{k=n}^{\infty} O_k$, then O is τ_{ind}-open in G and $O \cap G_n = O_n$.

§7. Preparation for the proof of the theorem

Let $d = \dim M$. To express $G = \mathrm{Diff}_0(M)$ as an inductive limit, we choose $M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ under the following additional condition: There exists a coordinate neighbourhood $(V_M, \imath_M)$ containing the closure $\overline{M}_1$ such that, with respect to a C^r-class Riemannian structure on M, the subsets M_0 and M_1 are open balls with the common center, and that, under the coordinate map $\imath_M$, the Riemannian structure is of the canonical form on M_1:

$$ds^2 = dp_1^2 + dp_2^2 + \cdots + dp_d^2 \quad \text{for } p = (p_i)_{i=1}^d \in M_1 \overset{\imath_M}{\hookrightarrow} \mathbf{R}^d.$$

Denote by $\rho(p, q)$ the distance of two points $p, q \in M$. We fix the origin $\mathbf{O}$ of the coordinates on the boundary ∂M_0 of M_0, and put $\rho(p) = \rho(p, \mathbf{O})$.

Let $C^r(\overline{M}_0, M_1)$ denote the set of all maps from $\overline{M}_0$ into M_1 which are restrictions on $\overline{M}_0$ of C^r-maps from some open sets containing $\overline{M}_0$ into M_1. Take $\phi \in C^r(\overline{M}_0, M_1)$. For $1 \le k \le r$, finite, and $p \in \overline{M}_0$, put alike a jet at p

$$j_p^k \phi = (\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p))_{|\alpha| \le k},$$

$$\text{with } \partial_i = \frac{\partial}{\partial p_i}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

Considering this value as an element of a Euclidean space $(\mathbf{R}^d)^{N_k}$ for an appropriate N_k, we take its norm:

$$\|j_p^k \phi\| := \left(\sum_{|\alpha| \le k} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p)\|^2 \right)^{1/2},$$

and put for $\phi, \psi \in C^r(\overline{M}_0, M_1) \subset C^r(\overline{M}_0, \mathbf{R}^d)$,

$$d^k(\phi, \psi) := \sup_{p \in \overline{M}_0} \|j_p^k(\phi - \psi)\|.$$

We put also, taking the k-th homogeneous part,

$$j_p^{(k)} \phi := (\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p))_{|\alpha|=k}, \quad d^{(k)}(\phi, \psi) := \sup_{p \in \overline{M}_0} \|j_p^{(k)}(\phi - \psi)\|.$$

The next lemma is a key of our proof of Theorem 6.1. Let $D_1, D_2 \subset \mathbf{R}^d$ be connected open sets, and $C^r(D_1, D_2)$ be the set of all C^r-class maps ϕ from D_1 to D_2. For $\phi = (\phi_i)_{i=1}^d \in C^r(D_1, D_2)$, we have $j_p^{(1)} \phi = (\partial_j \phi_i)_{1 \le i,j \le d}$. Considering it as a linear map on $\mathbf{R}^d$ canonically, we denote its operator norm by $\|j_p^{(1)} \phi\|_{op}$, where we take $\|x\| = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$ as the norm of $x = (x_i)_{i=1}^d \in \mathbf{R}^d$.

Lemma 7.1. *Let $D \subset \mathbf{R}^d$ be an open ball and denote by* id *the identity map on D. Assume for $\phi \in C^r(D, D)$, the support* $\operatorname{supp}(\phi) := \operatorname{Cl}\{p \in D; \phi(p) \ne p = \operatorname{id}(p)\}$ *is compact, and*

$$\|j_p^{(1)}(\phi - \mathrm{id})\|_{op} = \|j_p^{(1)} \phi - 1_d\|_{op} < 1 \quad (\forall p \in D),$$

where 1_d denotes the $d \times d$ identity matrix. Then ϕ is a diffeomorphism on D.

Proof. Since $\det(j_p^{(1)} \phi) \ne 0$ $(\forall p \in D)$, by the theorem of implicit functions, we see that ϕ is an open map and locally diffeomorphic.

On the other hand, ϕ is globally 1-1. In fact, for $p, q \in D \subset \mathbf{R}^d$, $p \ne q$, take $p - q \in \mathbf{R}^d$ and put $p_t = q + t(p - q)$ $(0 \le t \le 1)$, then

$$\phi(p) - \phi(q) = \int_0^1 \frac{d}{dt} \phi(p_t)\, dt = \int_0^1 (j_{p_t}^{(1)} \phi)(p - q)\, dt.$$

From the similar formula for $\psi = \phi - \mathrm{id}$, we have

$$\|\psi(p) - \psi(q)\| \le \int_0^1 \|j_{p_t} \psi\|_{op} \|p - q\|\, dt < \|p - q\|.$$

Hence $\|\phi(p) - \phi(q)\| \ge \|p - q\| - \|\psi(p) - \psi(q)\| > 0$.

Now let us prove that ϕ is onto. To do so, it is enough to prove that $\phi(D)$ is relatively closed, i.e., $D \cap \overline{\phi(D)} = \phi(D)$, because we know already that $\phi(D)$ is open. Take a $p \in D \cap \overline{\phi(D)}$. Then there exists a sequence $q_n \in D$ such that $\phi(q_n) \to p$ as $n \to \infty$. Since ϕ is 1-1 and $=$ id near the boundary $\partial(D)$, q_n has an accumulation point q inside D. Thus we get $p = \phi(q)$. Q.E.D.

§ 8. Behavior of a diffeomorphism on M_0 and $\overline{M}_0$.

8.1. A basis of neighbourhoods of $e \in G_0$. We denote the identity map id on M also by e as the unit element of G. Put

$$\Omega := \{g \in G;\, g\overline{M}_0 \subset M_1\} \subset G.$$

Then Ω is τ_{ind}-open in G, as is easily seen. Note that, for $g \in \Omega$, its restriction $g|_{\overline{M}_0}$ on $\overline{M}_0$ belongs to $C^r(\overline{M}_0, M_1)$.

We define subsets W_k of Ω as follows depending on the class C^r:

$$W_k := \{g \in \Omega; d^k(g, e) \leq 1/k\} \text{ in Case } r = \infty,$$

$$W_k := \{g \in \Omega; d^r(g, e) \leq 1/k\} \text{ in Case } r < \infty.$$

Then we have the following lemma.

Lemma 8.1. *Put $W_{k,0} := W_k \cap G_0$ for $k = 1, 2, \ldots$. Then they form a basis of neighbourhoods of the unit element $e \in G_0$ with respect to the topology τ_0.*

8.2. Convex combination of maps. Take $g \in \Omega$. For $0 \leq s \leq 1$, we can put

$$g_s := s \cdot \mathrm{id}_{\overline{M}_0} + (1 - s) \cdot g|_{\overline{M}_0} \in C^r(\overline{M}_0, M_1). \tag{8.1}$$

More generally we put, for $\phi \in C^r(\overline{M}_0, M_1)$,

$$\phi_s := s \cdot \mathrm{id}_{\overline{M}_0} + (1 - s) \cdot \phi \in C^r(\overline{M}_0, M_1).$$

Further put

$$\alpha_k(\phi) := \inf\{s; 0 \leq s \leq 1, d^k(\phi_s, \mathrm{id}) \leq 1/k\} \quad \text{in Case } r = \infty,$$

$$\alpha_k(\phi) := \inf\{s; 0 \leq s \leq 1, d^r(\phi_s, \mathrm{id}) \leq 1/k\} \quad \text{in Case } r < \infty.$$

Since $d^k(\phi_s, \mathrm{id}) = \sup_{p \in \bar{M}_0} \|j_p^k(\phi_s - \mathrm{id})\| = (1 - s) \cdot d^k(\phi, \mathrm{id})$, we have according as $r = \infty$ or $r < \infty$,

$$\alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^k(\phi, \mathrm{id})}\right) \quad \text{in Case } r = \infty, \tag{8.2}$$

$$\alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^r(\phi, \mathrm{id})}\right) \quad \text{in Case } r < \infty. \tag{8.2$'$}$$

Define further, for $\phi \in C^r(\overline{M}_0, M_1)$,

$$P_k\phi = \phi_{\alpha_k(\phi)} = \alpha_k(\phi) \cdot \mathrm{id}_{\overline{M}_0} + (1 - \alpha_k(\phi)) \cdot \phi \in C^r(\overline{M}_0, M_1).$$

Then we have the following facts.

(a) *Let $g \in W_k \subset \Omega$. Then $\alpha_k(g) = 0$, whence $P_k g = g|_{\overline{M}_0}$.*

(b) *Let $g \in G_0 \subset \Omega$. Assume $g \in W_{k,0} = W_k \cap G_0$ with $k \geq 2$. Then, for any $s, 0 \leq s \leq 1$, we can extend g_s outside of M_0 as $g_s = \mathrm{id}$, and get $g_s \in G_0 \subset G$.*

Proof. Since M_0 is an open ball, we have $g_s \in C_0^r\ (M_0, M_0)$. Moreover, for any $p \in M_0$,

$$\|j_p^{(1)}(g_s - \mathrm{id})\|_{op} \leq d^{(1)}(g_s, \mathrm{id}) \leq d^1(g_s, \mathrm{id}) \leq 1/k < 1.$$

By Lemma 7.1 applied to $D = M_0$, we see $g_s \in \mathrm{Diff}_0(M_0) \subset G_0 \subset G$. Q.E.D.

8.3. A crucial inequality on M_0. Now put for $g \in \Omega$

$$\beta_k := \inf_{g \in W_{k,0}} \int_{\overline{M}_0} \rho(g(p))\, dp = \inf_{g \in W_{k,0}} \int_{\overline{M}_0} \|g(p)\|\, dp_1\, dp_2 \cdots dp_d, \tag{8.3}$$

where $p = (p_i)_{i=1}^d$, $dp = dp_1 dp_2 \cdots dp_d$, and $\|g(p)\| = \left(\sum_{i=1}^d g_i(p)^2\right)^{1/2}$ with $g(p) = (g_i(p))_{i=1}^d$. The inequality in the following lemma reflects the fact that G_0 is not locally compact and is crucial for our proof of Theorem 6.1.

Lemma 8.2. *Let $k \geq 2$. Then, for any $g \in W_{k,0} = W_k \cap G_0$, we have*

$$\int_{\overline{M}_0} \rho(g(p))\, dp > \beta_k.$$

Proof. Step 1. Since $g \in G_0$, $\mathrm{supp}(g) \subset \overline{M}_0$ and so g and the identity map id have, at the origin $\mathbf{O}$, C^r-class contact. Hence

$$j_{\mathbf{O}}^{k'}(g) = j_{\mathbf{O}}^{k'}(\mathrm{id}) \quad (\forall k' \leq r, \text{finite}).$$

We can consider $g - \mathrm{id}$ as an element of $C^r\ (M_1, \mathbf{R}^d)$, then

$$j_{\mathbf{O}}^{k'}(g - \mathrm{id}) = 0 \quad (\forall k' \leq r, \text{finite}).$$

Fix $k \geq 2$, and take $k' = k$ in Case $r = \infty$, and $k' = r$ in Case $r < \infty$. Then there exists an open neighbourhood U_M of $\mathbf{O}$ in M such that

$$\|j_p^{k'}(g - \mathrm{id})\| < \frac{1}{2k} \quad (\forall p \in U_M \cap M_0),$$

$$j_p^{k'}(g - \mathrm{id}) = \mathbf{0} \quad (\forall p \notin M_0).$$

Now take an $\eta = (\eta_i)_{i=1}^d \in C_0^r(U_M \cap M_0, \mathbf{R}^d)$ satisfying

$$\|j_p^{k'}\eta\| < \frac{1}{2k} \quad \text{and} \quad \|j_p^0\eta\| = \|\eta\| < \frac{1}{2}\{\mathrm{diam}(M_1) - \mathrm{diam}(M_0)\}, \tag{8.4}$$

where $\mathrm{diam}(M_1)$ denotes the diameter of M_1. Put $\phi = g - \eta$. Then,

$$\phi(\overline{M}_0) \subset M_1 \quad \text{and} \quad \phi = \mathrm{id} \quad \text{on } M_1 \backslash M_0,$$

$$\|j_p^{k'}(\phi - \mathrm{id})\| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \quad (\forall p \in U_M \cap M_0).$$

Hence $\phi \in C^r(M_1, M_1)$ and, for any $p \in M_1$,

$$\|j_p^{(1)}(\phi - \mathrm{id})\|_{op} \leq \|j_p^{k'}(\phi - \mathrm{id})\| \leq \frac{1}{k} < 1.$$

Therefore we can apply Lemma 7.1 to ϕ and $D = M_1$, and see that $\phi \in \mathrm{Diff}_0(M_1)$. Since $\mathrm{supp}(\phi) \subset \overline{M}_0$, we get $\phi \in G_0 = \mathrm{Diff}(\overline{M}_0)$ and so $\phi \in W_{k,0} = W_k \cap G_0$.

STEP 2. Let us compare the following two values:

$$A := \int_{\overline{M}_0} \rho(g(p))\, dp = \int_{\overline{M}_0} \left(\sum_{i=1}^{d} g_i(p)^2 \right)^{1/2} dp,$$

$$B := \int_{\overline{M}_0} \rho(\phi(p))\, dp = \int_{\overline{M}_0} \left(\sum_{i=1}^{d} (g_i(p) - \eta_i(p))^2 \right)^{1/2} dp.$$

It is enough for us to prove $A > B\ (\geq \beta_k)$. For this, it is sufficient to have

$$|g_i(p)| \geq |g_i(p) - \eta_i(p)| \qquad (\forall i, \forall p \in \overline{M}_0),$$
$$|g_{i_0}(p_0)| > |g_{i_0}(p_0) - \eta_{i_0}(p_0)| \quad (\exists i_0, \exists p_0 \in \overline{M}_0).$$

On the other hand, since the maps g and id are sufficiently near to each other on $U_M \cap M_0$, there certainly exist i_0 and $p_0 \in U_M \cap M_0$ such that $g_{i_0}(p_0) \neq 0$. Then there exists a small neighbourhood $U(p_0)$ of p_0 such that, for $\varepsilon = 1$ or -1 and some $\kappa > 0$, $\varepsilon \cdot g_{i_0}(p) > \kappa$ $(\forall p \in U(p_0))$.

We can choose $\eta = (\eta_i)_{i=1}^d \in C_0^r(U(p_0) \cap U_M \cap M_0, \mathbf{R}^d)$ satisfying the condition (8.4) in such a way that $\eta_i = 0$ for $i \neq i_0$, and

$$\varepsilon \cdot \eta_{i_0}(p_0) > 0, \quad \kappa \geq \varepsilon \cdot \eta_{i_0}(p) \geq 0 \quad (\forall p).$$

Under this choice of η the above sufficient condition for $A > B$ holds. Q.E.D.

§9. A τ_{ind}-neighbourhood of $e \in G$.

9.1. Neighbourhood U. We define a τ_{ind}-neighbourhood U of $e \in G$, for which it will be proved that $V^2 \not\subset U$ for any τ_{ind}-neighbourhood V of $e \in G$.

Let $M_0^c = M \backslash M_0$, and put, for $g \in \Omega \subset G$,

$$F_k(g) := \left| \int_{\overline{M}_0} \rho((P_k g)(p))\, dp - \beta_k \right| + \int_{M_0^c} \rho(g(p), \mathrm{id}(p))\, dp. \tag{9.1}$$

where $\mathrm{id}(p) = p$. Then the following fact is a consequence of Lemma 8.2.

Lemma 9.1. *Let $k \geq 2$. Then, $F_k(g) > 0$ $(\forall g \in \Omega)$.*

Proof. Assume that the 2nd term in $F_k(g)$ is equal to zero. Then, $g = \mathrm{id}$ on M_0^c, and so $\mathrm{supp}(g) \subset \overline{M}_0$ whence $g \in G_0 \subset C^r(\overline{M}_0, M_1)$. Then,

$$P_k g \in C^r(\overline{M}_0, M_1) \subset C^r(M_1, M_1),$$
$$\mathrm{supp}(P_k g) \subset \mathrm{supp}(g) \subset \overline{M}_0 \quad \text{and} \quad d^{k'}(P_k g, \mathrm{id}) \leq 1/k < 1,$$

where $k' = k$ or $= r$ according as $r = \infty$ or $r < \infty$. Therefore we can apply Lemma 7.1 to $\phi = P_k g$ and $D = M_1$, and see that $P_k g \in \mathrm{Diff}(\overline{M}_0) = G_0$. Then by

Lemma 8.2 we get

$$\int_{\overline{M}_0} \rho((P_k g)(p))dp > \beta_k.$$

This means that the 1st term in (8.4) of $F_k(g)$ is positive, and so $F_k(g) > 0$.

9.2. Proof of Theorem 6.1. Choose non-empty open sets O_k in such a way that $O_k \subset M_k \backslash \overline{M}_{k-1}$ for $k \geq 2$. Fix $\gamma > 1$, and for $k \geq 2$, put

$$U_k := \left\{ g \in \Omega; F_k(g) > \gamma \cdot \int_{O_k} \rho(g(p), \mathrm{id}(p))\, dp \right\}.$$

Since $G_n = \mathrm{Diff}(\overline{M}_n) = \{g \in G; \mathrm{supp}(g) \subset \overline{M}_n\}$, we see that, if $n < k$, then $g = \mathrm{id}$ on O_k. Then, by Lemma 9.1, $U_k \cap G_n = \Omega \cap G_n$, and this is τ_n-open in G_n. In particular, $G_0 = \Omega \cap G_0 \subset U_k$. Put $U := \bigcap_{k=2}^{\infty} U_k \subset \Omega$.

Lemma 9.2. *The subset U is τ_{ind}-open in G.*

Proof. For any $n \geq 2$, the intersection $U \cap G_n$ is τ_n-open in G_n, because

$$U \cap G_n = \bigcap\nolimits_{k=2}^{n} (U_k \cap G_n) \cap (\Omega \cap G_n).$$

Now we come to the final stage of the proof of Theorem 6.1, and it is enough for us to prove the following lemma.

Lemma 9.3. *There does not exist any τ_{ind}-neighbourhood V of $e \in G$ such that $V^2 \subset U$.*

Proof. Suppose the contrary and let V be such that $V^2 \subset U$. Since $V \cap G_0$ is τ_0-open and $W_{k,0}$'s form a basis of τ_0-neighbourhoods of $e \in G_0$, there exists a $W_{k,0}$ such that $V \cap G_0 \supset W_{k,0}$. Put $V_k = V \cap \mathrm{Diff}_0(O_k)$. Then

$$W_{k,0} V_k \subset V^2 \subset U \subset U_k \subset \Omega.$$

Hence, for any $g \in W_{k,0}$, $h \in V_k$,

$$F_k(g \circ h) > \gamma \cdot \int_{O_k} \rho((g \circ h)(p), \mathrm{id}(p))\, dp.$$

Note that $\mathrm{supp}(g) \subset \overline{M}_0$, $\mathrm{supp}(h) \subset M_k \backslash M_{k-1}$, and that

$$g \circ h = g \text{ on } \overline{M}_0, \quad g \circ h = h \text{ on } O_k, \quad g \circ h = \mathrm{id} \text{ anywhere else.}$$

Then we have

$$\left| \int_{\overline{M}_0} \rho((P_k g)(p))\, dp - \beta_k \right| > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Further, since $g \in W_{k,0} = W_k \cap G_0$, we have $P_k g = g$, and the above inequality turns out to be

$$\int_{\overline{M}_0} \rho(g(p))\, dp - \beta_k > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Taking the infimum over $g \in W_{k,0}$, we get 0 on the left hand side and so

$$0 = \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Hence $h = \mathrm{id}$. This means that $V \cap \mathrm{Diff}_0(O_k) = \{\mathrm{id}\}$. A contradiction. Q.E.D.

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