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On maximality of two-sheeted unlimited covering surfaces of the unit disc

By

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1. Introduction

Let R be a Riemann surface. If there exists a conformal mapping $\imath$ of R onto a subregion of a Riemann surface $\tilde{R}$, then we call $\tilde{R}$, or more precisely the pair $(\tilde{R}, \imath)$, an extension of R. We often identify $\imath(R)$ with R and consider R as a subregion of $\tilde{R}$. According to this definition R itself is an extension of R. It is called a proper extension if $\tilde{R} \backslash \imath(R) \neq \varnothing$. A Riemann surface is called maximal if it has no proper extensions. An extension $\tilde{R}$ of R is called a maximal extension if $\tilde{R}$ is a maximal Riemann surface. In connection with the classification theory of Riemann surfaces we know that if R has a small ideal boundary then R is maximal. For example if R with no planar ends belongs to the class O_{HD}, O_{KD}, or O_{γ}, then R is maximal; see [SO, X.5C].

By a neighborhood of the ideal boundary of R we mean the exterior of a compact set of R. We call a connected component V of a neighborhood of the ideal boundary an end if it is not relatively compact.

Sakai [Sa3] has obtained a new characterization of non-maximal Riemann surfaces.

Theorem A ([Sa3] Theorem 4.1). *Let R be a Riemann surface. Then R is not maximal if and only if one of the following conditions holds for R.*

(a) *R has a planar end.*

(b) *R has a border.*

(c) *R has a disc with crowded ideal boundary.*

See the next section for the definition of a disc with crowded ideal boundary.

It is natural to ask whether there exists a maximal Riemann surface which does not belong to the class O_X-type above referred or not. Sakai has proved in Proposition 6.1 of [Sa3] that if a Riemann surface R has no planar ends and belongs to the class $\mathscr{S}^1_{KD}$ then R is maximal. The class $\mathscr{S}^1_{KD}$ is defined in [Sa1]. He also showed in Example 2 of [Sa3] that there exists a two-sheeted unlimited covering surface of the unit disc which belongs to the class $\mathscr{S}^1_{KD}$.

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Obviously it does not belong to the class O_X-type. Then our final goal is to know where the class of all maximal Riemann surfaces has place in the classification theory of Riemann surfaces. In [J] we have obtained sufficient conditions for a Riemann surface to be maximal.

Theorem B ([J] Theorems 2 and 3). *Let R be a Riemann surface of infinite genus having no planar ends.*

(1) *If R satisfies the condition $\Gamma_{h0}(R) \cap \Gamma^*_{h0}(R) \neq \{0\}$, then R is maximal.*

(2) *Suppose that there exists a harmonic function u on a neighborhood V of the ideal boundary of R such that u is non-constant in each component of V and has Γ_{he}- and Γ_{hm}-behaviors simultaneously. Then R is maximal.*

See the next section for the definitions of $\Gamma_{h0}(R)$ and Γ_{he}- and Γ_{hm}-behaviors.

In this paper we are mainly concerned with a two-sheeted unlimited covering surface R of the unit disc U with the projection mapping π. The pair (R, π) is also called a covering surface. As is known from Sakai's characterization of non-maximal Riemann surfaces, we need some informations about the neighborhood of the ideal boundary. In order to obtain them we consider the Kuramochi compactification R^* of R. We call $R^*\backslash R$ the Kuramochi ideal boundary of R and denote it by Δ_R. We shall show in Proposition 2 that the projection π is continuously extended to $R \cup \Delta_R$ and in Theorem 4 determine the set $I^\theta = \pi^{-1}(e^{i\theta})$ of the Kuramochi boundary points over $e^{i\theta} \in \partial U$. We obtain a sufficient condition for R to be maximal in terms of I^θ.

Theorem 1. *If I^θ consists of one minimal point for every $e^{i\theta} \in \partial U$, then R is a maximal Riemann surface.*

Later we shall give a theorem, that is Theorem 5, which includes Theorem 1. We also prove in Theorem 6 that the converse of Theorem 1 is not true.

2. Preliminaries

We summarize here the definitions concerning Riemann surfaces and covering surfaces.

A continuous mapping of the open interval $(0, 1)$ into a Riemann surface R is an open arc. We say that an open arc starts from the ideal boundary if $\bigcap_{0<\tau<1} \overline{f((0, \tau])}$ is an empty set and terminates at the ideal boundary if $\bigcap_{0<\tau<1} \overline{f([\tau, 1))} = \varnothing$.

We say that a plane point set E which is compact and totally disconnected belongs to the class N_D or is an N_D-set if $\mathbf{C}\backslash E$ belongs to the class O_{AD} (cf. [SO, p. 255]). Let D be a simply connected subregion of R. Suppose that its relative boundary ∂D consists of a countable number of analytic simple open arcs $\{\gamma_j\}$ such that each γ_j starts from the ideal boundary and terminates at the ideal boundary, $\gamma_j \cap \gamma_k = \varnothing$ if $j \neq k$, and $\{\gamma_j\}$ does not accumulate in R. Then a Riemann mapping ϕ of D onto the unit disc U is continuously extended over ∂D and $\phi(\partial D)$ is a relatively open subset of ∂U. We denote by I the complement of $\phi(\partial D)$ with respect to ∂U. We call D a disc with crowded ideal boundary if I is totally disconnected and is not an N_D-set.

Let R and S be Riemann surfaces. We say that R is an unlimited covering surface of S if there is an analytic mapping π of R onto S such that for any curve $\gamma = \gamma(t)$, $t \in [0, 1]$ on S and any point $P_0 \in \pi^{-1}(\gamma(0))$ there is a curve $\tilde{\gamma} = \tilde{\gamma}(t)$, $t \in [0, 1]$ on R such that $\tilde{\gamma}(0) = P_0$ and $\pi(\tilde{\gamma}(t)) = \gamma(t)$. We call the mapping π the projection mapping. The pair (R, π) is also called a covering surface. We know that if R is an unlimited covering surface of S, then for every point $q \in S$, $\pi^{-1}(q)$ contains the same number of points provided a branch point of order $n-1$ is counted n points; see [Sp, Theorem 4.2]. The number n is called the number of sheets. We say that a covering surface of S is ramified if it has branch points.

We recall some definitions of first order differentials on R. A differential $\omega = a(x, y)\,dx + b(x, y)dy$ is called real if all local coefficients $a(x, y)$ and $b(x, y)$ are real-valued functions and called of C^∞ class of $a(x, y)$ and $b(x, y)$ are so. We say that ω is square integrable if local coefficients are measurable and

$$\int_R (a^2 + b^2)dxdy = \int_R \omega \wedge \omega^*$$

is finite, where $\omega^* = -b(x, y)dx + a(x, y)dy$ is the conjugate differential of ω. The positive square root of this integral is denoted by $\|\omega\|_R$, and we call it the norm of ω. Let $\Gamma(R)$ be the space of all real square integrable differentials on R. We know that $\Gamma(R)$ is a Hilbert space with the inner product

$$(\omega_1, \omega_2) = (\omega_1, \omega_2)_R = \int_R \omega_1 \wedge \omega_2^*.$$

Set

$$\Gamma_{e0}^\infty(R) = \{df; f \in C_0^\infty(R)\} \quad \text{and} \quad \Gamma_{e0}(R) = \overline{\Gamma_{e0}^\infty(R)},$$

where $C_0^\infty(R)$ is a class of infinitely differentiable functions with compact support on R. We denote by $\Gamma_h(R)$ the subspace of $\Gamma(R)$ which consists of harmonic differentials.

We introduce important subspaces of $\Gamma_h(R)$. Let $\Gamma_{he}(R)$ (resp. $\Gamma_{hse}(R)$) be the subspace of $\Gamma_h(R)$ whose elements ω are exact (resp. semiexact) on R, that is,

$$\int_\gamma \omega = 0 \quad \text{for every (resp. every dividing) 1-cycle } \gamma \text{ on } R.$$

We often use notation $\Gamma, \Gamma_h, \Gamma_{he}, \cdots$ instead of $\Gamma(R), \Gamma_h(R), \Gamma_{he}(R), \cdots$. Given a closed subspace Γ_y of Γ_h, the orthogonal complement of Γ_y in Γ_h is denoted by $\Gamma_y^\perp$. Set $\Gamma_y^* = \{\omega^*; \omega \in \Gamma_y\}$. Since $(\omega_1, \omega_2) = (\omega_1^*, \omega_2^*)$ holds, we have $(\Gamma_y^*)^\perp = (\Gamma_y^\perp)^*$. Then we shall write it simply $\Gamma_y^{*\perp}$. We need the subspaces of harmonic measures Γ_{hm} and Γ_{h0}; see [AS, V.15C, 10B and 14C] for definition. By [AS, V.15D] and [AS, V.10C] we have $\Gamma_{hm} = \Gamma_{hse}^{*\perp}$ and $\Gamma_{h0} = \Gamma_{he}^{*\perp}$. By definition it follows that $\Gamma_h \supset \Gamma_{hse} \supset \Gamma_{he}$ and $\Gamma_{he} \supset \Gamma_{hm}$. We have $\Gamma_{hse} \supset \Gamma_{h0} \supset \Gamma_{hm}$ because they are orthogonal complements of Γ^*_{hm}, Γ^*_{he}, and Γ^*_{hse}, respectively. See also [AS, V.15E]. We summarize the inclusion relations here:

$$\begin{array}{ccccc} \Gamma_h & \supset & \Gamma_{hse} & \supset & \Gamma_{he} \\ & & \cup & & \cup \\ & & \Gamma_{h0} & \supset & \Gamma_{hm}. \end{array}$$

If the differential dh of a function h of the class C^1 is square integrable, then we call the integral $\int_R (h_x^2 + h_y^2) dxdy = \|dh\|_R^2$ the Dirichlet integral of h and say that h has finite Dirichlet integral. Let $HD(R)$ be the class of real-valued harmonic functions on R with finite Dirichlet integral and $KD(R)$ be the subclass of $HD(R)$ whose elements u have the property

$$\int_\gamma du^* = 0 \quad \text{for every dividing 1-cycle } \gamma \text{ on } R.$$

Let $AD(R)$ be the class of analytic functions on R with finite Dirichlet integral. We denote by $\Re AD(R)$ the class of real-valued harmonic functions u such that there is a single-valued conjugate harmonic function u^* of u and $u + iu^*$ belongs to $AD(R)$. By the Cauchy-Riemann equation we have $du^* = -u_y dx + u_x dy = (u^*)_x dx + (u^*)_y dy = d(u^*)$. It is easily seen that $u \in \Re AD(R)$ if and only if $u \in HD(R)$ and

$$\int_\gamma du^* = 0 \quad \text{for every 1-cycle } \gamma \text{ on } R.$$

The relations

$$\{du; u \in HD(R)\} = \Gamma_{he}(R)$$

$$\{du; u \in KD(R)\} = \Gamma_{he}(R) \cap \Gamma^*_{hse}(R)$$

$$\{du; u \in \Re AD(R)\} = \Gamma_{he}(R) \cap \Gamma^*_{he}(R)$$

hold. We say that a Riemann surface R belongs to the class O_{HD}, O_{KD} or O_{AD} if and only if $HD(R)$, $KD(R)$ or $\Re AD(R)$ consists of only constant functions, respectively.

Let ω be a real differential defined in a neighborhood of the ideal boundary of R and Γ_χ be any closed subspace of Γ_{he}. Then ω is said to have Γ_χ-behavior if the following representation holds in some neighborhood of the ideal boundary of R:

$$\begin{cases} \omega = \omega_1 + df_0, \\ \omega^* = \omega_2 + df_1, \end{cases}$$

where $\omega_1 \in \Gamma_\chi$, $\omega_2 \in \Gamma_\chi^{*\perp}$, and f_0 and f_1 are C^∞-functions on R such that df_0 and df_1 belong to Γ_{e0}. We say that a function u has Γ_χ-behavior if du does.

3. Results

Let S be a Riemann surface and R be a two-sheeted unlimited covering surface of S. We obtain a sufficient condition for R to be maximal as follows.

Theorem 2. *Let S be a Riemann surface and R be a two-sheeted unlimited covering surface of S with the projection mapping π. If R has no planar ends and $\pi^{-1}(Q)$ consists of one point for quasi every $Q \in \Delta_S$, then R is maximal.*

The next theorem is a generalization of Theorem 2.

Theorem 3. *Let S be a Riemann surface and R be a two-sheeted unlimited covering surface of S with the projection mapping π. If R is of positive genus and $\pi^{-1}(Q)$ consists of one point for quasi every $Q \in \Delta_S$, then there are a maximal extension $(\tilde{R}, \iota)$ of R and an extension $(\tilde{S}, \iota')$ of S such that $\tilde{R}$ is a two-sheeted unlimited covering surface of $\tilde{S}$ with the projection mapping $\tilde{\pi}$ which satisfies $\iota' \circ \pi = \tilde{\pi} \circ \iota$ on R.*

We know that the Kuramochi boundary of the unit disc U is homeomorphic to $\partial U = \{|z| = 1\}$ and every Kuramochi boundary point is minimal. We shall show the following theorem.

Theorem 4. *Let (R, π) be a two-sheeted unlimited covering surface of the unit disc U. Then for $e^{i\theta} \in \partial U$ the fiber $I^\theta = \pi^{-1}(e^{i\theta})$ is one of the following sets.*

(a) *I^θ consists of two minimal points.*

(b) *I^θ is homeomorphic to $I = [0, 1]$, and two minimal points correspond to* 0 *and* 1.

(c) *I^θ consists of one minimal point.*

We say that R has (W)-property if $\Gamma_{he}(R) \cap \Gamma^*_{hse}(R) \subset \Gamma^*_{he}(R)$ holds. We shall show the next theorem. We note that the assertion of Theorem 1 is (a) $\Rightarrow$ (h) in this theorem.

Theorem 5. *For a two-sheeted unlimited covering surface R of the unit disc U which has infinitely many branch points we have the relation*

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i) \textit{ and } (i) \not\Rightarrow (h)$$

among the following conditions (a)–(i):

(a) *I^θ consists of one minimal point for every $e^{i\theta} \in \partial U$.*

(b) *I^θ consists of one minimal point for quasi every $e^{i\theta} \in \partial U$.*

(c) *$\Gamma_{he}(R) \cap \Gamma^1_h(R) = \{0\}$.*

(d) *$HD(R) = HD(U)$. That is, for every $u \in HD(R)$ there is $\underline{u} \in HD(U)$ such that $u = \underline{u} \circ \pi$.*

(e) *$HD(R) = \Re AD(R)$ holds on R.*

(f) *R has (W)-property.*

(g) *$\Gamma_{h0}(R) \cap \Gamma^*_{h0}(R) \neq \{0\}$.*

(h) *R is a maximal Riemann surface.*

(i) *$\Re AD(R) = HD(U)$.*

Remark. It is not known whether or not (g) $\Rightarrow$ (f) or (h) $\Rightarrow$ (g).

Finally we shall show that the converse of Theorem 1 is not true.

Theorem 6. *There exists a two-sheeted unlimited covering surface* (R, π) *of* U *such that* R *is maximal and* I^θ *is homeomorphic to* $[0,1]$ *for every* $e^{i\theta} \in \partial U$.

Theorems 2 and 3 are proved in Section 6 and Theorem 4 is proved in Section 7. The proof of Theorem 5 is in Section 8 and the proof of Theorem 6 is in Section 9.

4. Kuramochi boundary

We shall recall the definition of the Kuramochi (ideal) boundary of R and some properties of it; see [CC] and [O].

We fix a closed parametric disc K_0 on R and a point $p_0 \in R \backslash K_0$. Let $\{R_n\}_{n \geq 1}$ be an exhaustion of R, that is, R_n is a regular subregion of R with $\overline{R_n} \subset R_{n+1}$, $R \backslash R_n$ contains no compact component and $R = \bigcup R_n$. We assume that $R_1 \supset K_0$. Then there is n_0 such that R_{n_0} contains p_0. For any $n \geq n_0$ there exists a function $\mathrm{N}_n(p, p_0)$ on $R_n \backslash K_0$ which satisfies

i) $\mathrm{N}_n(p, p_0)$ has a singularity $-\log|z|$ at p_0, where z is a local parameter about p_0,

ii) $\mathrm{N}_n(p, p_0)$ is harmonic in $(R_n \backslash K_0) \backslash \{p_0\}$,

iii) $\mathrm{N}_n(p, p_0) = 0$ if $p \in \partial K_0$,

and

iv) the (inner) normal derivative $(\partial \mathrm{N}_n(\cdot, p_0))/\partial \nu$ vanishes on ∂R_n.

The sequence $\{\mathrm{N}_n(p, p_0)\}_{n \geq n_0}$ converges uniformly on every compact subset of $((R \backslash K_0) \backslash \{p_0\}) \cup \partial K_0$. Denote the limit function by $\mathrm{N}_{p_0}(p) = \mathrm{N}(p, p_0)$. We know that $\|d\mathrm{N}_{p_0} - d\mathrm{N}_n(\cdot, p_0)\|_{R_n \backslash K_0}$ tends to zero as $n \to \infty$ (cf. [O, Theorem 4]). Since $\mathrm{N}_n(p, p_0) = \mathrm{N}_n(p_0, p)$ holds, we have $\mathrm{N}(p, p_0) = \mathrm{N}(p_0, p)$.

We say that a sequence $\{p_m\}_{m \geq 1}$ converges to the ideal boundary of R if $\{p_m\}$ does not accumulate in R. A sequence $\{p_m\}$ converging to the ideal boundary of R is called a fundamental sequence if $\{\mathrm{N}_{p_m}\}$ converges uniformly on every compact subset of $R \backslash K_0$. Among the family of fundamental sequences we define an equivalence relation: two fundamental sequences $\{p_m\}$ and $\{p'_m\}$ are equivalent if the limit functions coincide. We call an equivalence class a Kuramochi boundary point. To every Kuramochi boundary point p_0 which is an equivalence class of $\{p_m\}$ there corresponds a unique function $\mathrm{N}_{p_0}(p) = \lim_{m \to \infty} \mathrm{N}_{p_m}(p)$. We call the set of all Kuramochi boundary points the Kuramochi ideal boundary or simply Kuramochi boundary of R and denote it by Δ_R. Set $R^* = R \cup \Delta_R$. This R^* is called the Kuramochi compactification of R. We define a distance $d(p, p')$ on $R^* \backslash K_0$ by

$$d(p, p') = \sup_{P \in K_1} \left| \frac{\mathrm{N}(P, p)}{1 + \mathrm{N}(P, p)} - \frac{\mathrm{N}(P, p')}{1 + \mathrm{N}(P, p')} \right|,$$

where K_1 is a closed parametric disc in $R \backslash K_0$. We call it the Kuramochi distance on $R^* \backslash K_0$.

We know that this compactification does not depend on the choice of K_0. That is, let K_0' be another closed parametric disc in R and Δ_R' be the Kuramochi boundary constructed on $R \backslash K_0'$. Then there is a homeomorphism φ of $R \cup \Delta_R$ onto $R \cup \Delta_R'$ such that $\varphi|_R$ is an identity mapping (cf. [O, Theorem 12]).

For any compact set $K \subset R \backslash K_0$ and any compact set $K' \subset R^* \backslash K_0$ disjointed from K, $\mathrm{N}(p, p_0)$ is continuous on $K \times K'$ by Harnack's inequality (cf. [O, p. 278]). Obviously $\mathrm{N}_{p_0}(p) = \mathrm{N}(p, p_0)$, as a function of p, is positive on $R \backslash K_0$ and equal to zero on ∂K_0. If p_0 is a point in Δ_R, then $\mathrm{N}_{p_0}(p)$ is harmonic in $R \backslash K_0$. If p_0 is a point in $R \backslash K_0$, then $\mathrm{N}_{p_0}(p)$ is harmonic in $(R \backslash K_0) \backslash \{p_0\}$ and has a singularity $-\log|z|$ at p_0. Moreover in this case if we define values of N_{p_0} at $p \in \Delta_R$ by $\mathrm{N}_p(p_0)$, then N_{p_0} has a continuous extension over Δ_R.

We can define the value of N_{p_0} at $p \in \Delta_R$ for $p_0 \in \Delta_R$ so that N_{p_0} is lower semicontinuous in $R^* \backslash K_0$.

Now we obtain a function $\mathrm{N}_{p_0}(p) = \mathrm{N}(p, p_0)$ on $(R^* \backslash K_0) \times (R^* \backslash K_0)$ and we call it the Kuramochi kernel function. It is known that the Kuramochi kernel function has the following properties:

i) $\mathrm{N}(p, p_0)$ is lower semi-continuous on $(R^* \backslash K_0) \times (R^* \backslash K_0)$.

ii) If $p_0 \in R \backslash K_0$, then N_{p_0} is continuous on $(R^* \backslash K_0) \backslash \{p_0\}$.

iii) If $p_0 \in \Delta_R$, then N_{p_0} is continuous on $R \backslash K_0$ and lower semi-continuous on $R^* \backslash K_0$.

iv) $\mathrm{N}(p, p_0) = \mathrm{N}(p_0, p)$.

See [CC, Satz 17.1 and p.178] for the Kuramochi kernel functions. In [CC] the Kuramochi kernel function is denoted by $\tilde{g}$.

Denote the set of all minimal boundary points by Δ_R^1. Set $\Delta_R^0 = \Delta_R \backslash \Delta_R^1$. We know that Δ_R^1 is a G_δ set and Δ_R^0 is an F_σ set; see [CC, p.134].

We take $\mathrm{N}(p, p')$ as a kernel of potential. For any positive measure μ in $R^* \backslash K_0$ we can define the potential $\mathrm{N}\mu(p) = \int \mathrm{N}(p, p') d\mu(p')$ if it is not equal to $+\infty$. A positive measure μ on $R^* \backslash K_0$ is said to be canonical if $\mu(\Delta_R^0) = 0$. We know that any potential $\mathrm{N}\mu(p)$ has a canonical representation, that is, there exists uniquely a canonical measure $\tilde{\mu}$ such that $\mathrm{N}\mu = \mathrm{N}\tilde{\mu}$; see Satz 16.2 in [CC] or Corollary of Theorem 24 and Theorem 27 in [O].

We call a subregion G of R admissible if its relative boundary ∂G consists of a finite number of analytic Jordan curves and its closure $\bar{G} = G \cup \partial G$ is disjoint from K_0. For example if $\Omega \supset K_0$ is a regular subregion of R then each component of $R \backslash \bar{\Omega}$ is admissible. For $f \in C_0^\infty(R)$ and an admissible subregion G let us denote by $\mathscr{D}_G^f$ the family of all Dirichlet finite functions of C^1 class on G with boundary values f on ∂G. Then there exists uniquely $f^{\partial G} \in \mathscr{D}_G^f$ which minimizes the Dirichlet integral in $\mathscr{D}_G^f$.

The following facts for an admissible subregion G of R are useful (see [O, Theorem 5]):

i) If $p_0 \in G$, then $\mathrm{N}_{p_0} \geq (\mathrm{N}_{p_0})^{\partial G}$ in G.

ii) If $p_0 \notin G \cup \partial G$, then $\mathrm{N}_{p_0} = (\mathrm{N}_{p_0})^{\partial G}$ in G.

Denote by $\mathcal{N}(R)$ the class of continuous functions f in R for which there exists a regular subregion $\Omega \supset K_0$ such that $f(p) = f^{\partial V}(p)$ in each component V of $R\backslash\bar{\Omega}$. If a regular subregion Ω' contains $\bar{\Omega}$, then $f = f^{\partial V'}$ holds in each component V' of $R\backslash\overline{\Omega'}$ and f is harmonic in some neighborhood of $R\backslash\overline{\Omega'}$. Thus considering Ω' instead of Ω we may assume from the beginning that f is harmonic in some neighborhood of $R\backslash\Omega$. We know that every $f \in \mathcal{N}(R)$ has a continuous extension on Δ_R. See [CC, p.167 and p.170].

We shall use boundary behaviors of the Green function and the Kuramochi kernel function. Denote the Green function on $R\backslash K_0$ with a pole at p_0 by $g_{p_0}(p) = g(p, p_0)$. We know that N_{p_0} and g_{p_0} have finite Dirichlet integral over some neighborhood of the ideal boundary. We show the next lemma.

Lemma 1. *Suppose that* $p_0 \in R\backslash K_0$. *Then differentials* $dN^*_{p_0}$ *and* dg_{p_0} *admit the following representations in some neighborhood of the ideal boundary:*

$$dN^*_{p_0} = \omega_{h0} + df_0 \quad \textit{and} \quad dg_{p_0} = df_1,$$

where $\omega_{h0} \in \Gamma_{h0}(R)$, $f_0, f_1 \in C^\infty(R)$, *and* $df_0, df_1 \in \Gamma_{e0}(R)$.

Proof. Set $V_0 = \{p; \mathrm{N}_{p_0}(p) \geq M\}$. Then V_0 is a closed parametric disc centered at p_0 for sufficiently large $M > 0$. Let Ω be a relatively compact subregion of R such that $\Omega \supset V_0 \cup K_0$ and $\partial\Omega$ consists of a finite number of analytic curves. We show that $d\mathrm{N}^*_{p_0}|_{R\backslash\Omega}$ can be extended to a closed differential σ of C^∞ class on R by using the same arguments as in [Y, Lemma 1]. Let $\{\mathrm{N}_n(p, p_0)\}$ be the sequence which is defined on the top of this section. Since $\{\mathrm{N}_n(p, p_0)\}$ converges uniformly to N_{p_0} on some neighborhoods of ∂K_0 and ∂V_0, we have

$$\int_{\partial K_0 \cup \partial V_0} d\mathrm{N}^*_{p_0} = \lim_{n\to\infty} \int_{\partial K_0 \cup \partial V_0} d\mathrm{N}_n(\cdot, p_0)^* = 0.$$

Take a quadrilateral subregion W of $\Omega\backslash(V_0 \cup K_0)$ such that one pair of opposite sides consists of subarcs of ∂K_0 and ∂V_0, and that the other pair of opposite sides consists of arcs in $\Omega\backslash(V_0 \cup K_0)$. Let $\tilde{W}$ be the interior of $W \cup V_0 \cup K_0$. Then $\tilde{W}$ is a simply connected region and

$$\int_{\partial\tilde{W}} d\mathrm{N}^*_{p_0} = \int_{\partial W} d\mathrm{N}^*_{p_0} + \int_{\partial K_0 \cup \partial V_0} d\mathrm{N}^*_{p_0} = 0.$$

We can choose u of C^∞ class in a neighborhood of $\partial\tilde{W}$ so that $du = d\mathrm{N}^*_{p_0}$ and extend u over $\tilde{W}$ so that $u \in C^\infty(\tilde{W})$. Then define a closed differential σ of C^∞ class as follows:

$$\sigma = \begin{cases} d\mathrm{N}^*_{p_0} & \text{on } R\backslash\tilde{W} \\ du & \text{on } \tilde{W}. \end{cases}$$

In this proof we often use well-known orthogonal decompositions $\Gamma_c(R) = \Gamma_h(R) + \Gamma_{e0}(R)$ and $\Gamma(R) = \Gamma_c(R) + \Gamma^*_{e0}(R)$, where $\Gamma_c(R)$ is the class of square integrable closed differentials; see [AS, V.10A]. Then we have $\sigma = \omega + df_0$, where ω and df_0 belong to $\Gamma_h(R)$ and $\Gamma_{e0}(R)$, respectively. It is easily seen that f_0 is of $C^\infty(R)$. We show that this ω of σ is ω_{h0} which we want. It suffices to show that ω belongs to $\Gamma_{h0}(R) = \Gamma_{he}^{*\perp}(R)$. For any $dv \in \Gamma_{he}$

$$(\omega, dv^*)_R = (\sigma, dv^*)_R = (\sigma, dv^*)_{R\setminus\Omega} + (\sigma, dv^*)_\Omega = (d\mathrm{N}_{p_0}^*, dv^*)_{R\setminus\Omega} + \int_{\partial\Omega} v\sigma$$

holds. Since $\|d\mathrm{N}_{p_0} - d\mathrm{N}_n(\cdot, p_0)\|_{R_n\setminus K_0}$ tends to zero as $n \to \infty$,

$$\begin{aligned}(d\mathrm{N}_{p_0}^*, dv^*)_{R\setminus\Omega} &= \lim_{n\to\infty} (d\mathrm{N}_n(\cdot, p_0)^*, dv^*)_{R_n\setminus\Omega} \\ &= \lim_{n\to\infty} \int_{-\partial\Omega} v d\mathrm{N}_n(\cdot, p_0)^* = -\int_{\partial\Omega} v d\mathrm{N}_{p_0}^* = -\int_{\partial\Omega} v\sigma.\end{aligned}$$

Hence $(\omega, dv^*)_R = 0$. We deduce that ω is an element of Γ_{h0}.

For the Green function we set $U_0 = \{p; g_{p_0}(p) \geq M\}$. For sufficiently large $M > 0$, U_0 is a closed parametric disc centered at p_0. Let $g_n(p, p_0)$ be the Green function on $R_n\setminus K_0$ with a pole at p_0. We know that a sequence $\{g_n(p, p_0)\}$ converges to $g_{p_0}(p)$ uniformly on every compact subset of $((R\setminus K_0)\setminus\{p_0\}) \cup \partial K_0$ and $\|dg_{p_0} - dg_n(\cdot, p_0)\|_{R_n\setminus K_0} \to 0$ as $n \to \infty$. Since $g_{p_0} = 0$ on analytic boundary ∂K_0, g_{p_0} is extended to be harmonic in some neighborhood of ∂K_0. Then there is a function $\tilde{f}$ on R such that $\tilde{f} \in C^\infty(R\setminus\{p_0\})$ and $\tilde{f} = g_{p_0}$ in $R\setminus K_0$. Let ρ be a function of $C^\infty(R)$ such that $\rho = 1$ in $R\setminus U_0$ and $\rho = 0$ in some neighborhood of p_0. Set $f_1 = \rho\tilde{f}$. Then f_1 belongs to $C^\infty(R)$ and is equal to g_{p_0} in the neighborhood of the ideal boundary. In order to prove $df_1 \in \Gamma_{e0}(R)$ it suffices to show that $(df_1, \tau)_R = 0$ holds for every $\tau \in \Gamma_h(R)$. Note that

$$\begin{aligned}(df_1, \tau)_R &= (df_1, \tau)_{R\setminus(K_0\cup U_0)} + (df_1, \tau)_{U_0} + (df_1, \tau)_{K_0} \\ &= \lim_{n\to\infty} (dg_n(\cdot, p_0), \tau)_{R_n\setminus(K_0\cup U_0)} + \int_{\partial U_0} f_1\tau^* + \int_{\partial K_0} f_1\tau^* \\ &= -\lim_{n\to\infty} \int_{\partial U_0} g_n(\cdot, p_0)\tau^* + \int_{\partial U_0} M\tau^* \\ &= -\int_{\partial U_0} g_{p_0}\tau^* = -\int_{\partial U_0} M\tau^* = 0.\end{aligned}$$

We have a conclusion.

Remark. Let K_0 and K_0' be mutually disjoint closed parametric discs of R. We can construct the Kuramochi kernel functions and the Kuramochi compactification with respect to $R\setminus(K_0 \cup K_0')$ in the same way as above. All statements in this section are true if we use $K_0 \cup K_0'$ instead of K_0. We shall show that Lemma 1 is true even if we choose $K_0 \cup K_0'$ instead of K_0.

Let N_{p_0} and $\tilde{\mathrm{N}}_{p_0}$ be the Kuramochi kernel functions on $R\setminus K_0$ and $R\setminus (K_0 \cup K_0')$, respectively. By Lemma 1 we have

$$d\mathrm{N}_{p_0}^* = \omega_{h0} + df, \quad \omega_{h0} \in \Gamma_{h0}(R), \quad f \in C^\infty(R), \quad \text{and} \quad df \in \Gamma_{e0}(R)$$

in the neighborhood of the ideal boundary. Since $\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0}$ is harmonic in $R \backslash (K_0 \cup K_0')$ and satisfies $\int_{\partial K_0 \cup \partial K_0'} d(\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0})^* = 0$, we can choose a closed differential σ of C^∞ class on R such that $\sigma = d(\mathrm{N}_{p_0} - \tilde{N}_{p_0})^*$ holds in the neighborhood of the ideal boundary by the same way as in the proof of Lemma 1. Hence it suffices to show that $\sigma \in \Gamma_{h0}(R) + \Gamma_{e0}(R)$. For any $dv \in \Gamma_{he}$ we have $(\sigma, dv^*)_R = 0$ by the same argument as in the proof of Lemma 1. Therefore $\sigma \in \Gamma_{h0}(R) + \Gamma_{e0}(R)$ and $d\tilde{\mathrm{N}}_{p_0}^* = d\mathrm{N}_{p_0}^* - \sigma$ admits a representation $\tilde{\omega}_{h0} + d\tilde{f}$ with $\tilde{\omega}_{h0} \in \Gamma_{h0}(R)$, $\tilde{f} \in C^\infty(R)$, and $d\tilde{f} \in \Gamma_{e0}(R)$ in the neighborhood of the ideal boundary.

We shall remind the definition of the Kuramochi capacity. See [CC, p. 185]. We denote by $C(F)$ the Kuramochi capacity of a subset F of $R^* \backslash K_0$. If F is a compact subset of $R^* \backslash K_0$, then

$$C(F) = \sup\{\mu(F); \mu \text{ is a positive canonical measure and } \mathrm{N}\mu \leq 1 \text{ on } F\}.$$

If D is an open set in $R^* \backslash K_0$, then

$$C(D) = \sup\{C(F); F \text{ is a compact set with } F \subset D\}.$$

For a set $A \subset R^* \backslash K_0$ the Kuramochi (outer) capacity is defined by

$$C(A) = \inf\{C(D); D \text{ is an open set including } A\}.$$

We say that a set E is (full) polar if the Kuramochi capacity of E is equal to 0. We know that compact subsets of Δ_R^0 are polar; see p.185 of [CC]. Since Δ_R^0 is an F_σ set, from subadditivity of capacity it follows that Δ_R^0 is polar. See also [CC, pp. 186–189]. We say that a statement is true for quasi every $Q \in A$ or quasi everywhere on A if the subset of A for which the statement is false has vanishing capacity.

We consider the Kuramochi boundary of a Riemann surface $R' = R \backslash K_0$.

Proposition 1. *There is a homeomorphism $\imath$ of $(R \backslash K_0) \cup \Delta_R \cup \partial K_0$ onto $R' \cup \Delta_{R'}$ such that $\imath$ is the identity mapping in $R \backslash K_0$ and for a subset A of Δ_R A is polar with respect to R if and only if $\imath(A)$ is polar with respect to R'.*

Proof. (cf. [O, Theorem 12]) It is easily seen that there is a homeomorphism $\imath$ of $(R \backslash K_0) \cup \Delta_R \cup \partial K_0$ onto $R' \cup \Delta_{R'}$ such that $\imath$ is the identity mapping in $R \backslash K_0$.

We prove the remaining assertion. We choose a closed parametric disc $\tilde{K}_0$ on R'. Let $\tilde{\mathrm{N}}_{p_0}$ be the Kuramochi kernel function of $(R' \cup \Delta_{R'}) \backslash \tilde{K}_0$ for $p_0 \in (R' \cup \Delta_{R'}) \backslash \tilde{K}_0$.

Let K_1 and $\tilde{K}_1$ be closed parametric discs in R such that $K_1 \cap \tilde{K}_1 = \emptyset$, $K_1 \backslash \partial K_1 \supset K_0$, and $\tilde{K}_1 \backslash \partial \tilde{K}_1 \supset \tilde{K}_0$. Fix regular subregion Ω of R such that $(K_1 \cup \tilde{K}_1) \subset \Omega$. Since $\mathrm{N}(p, q)$ is continuous and positive on $(R^* \backslash \Omega) \times (\partial K_1 \cup \partial \tilde{K}_1)$, we have

$$0 < m = \min_{(R^* \backslash \Omega) \times (\partial K_1 \cup \partial \tilde{K}_1)} \mathrm{N}(p, q) < \max_{(R^* \backslash \Omega) \times (\partial K_1 \cup \partial \tilde{K}_1)} \mathrm{N}(p, q) = M < \infty.$$

For the same reason

$$0 < \tilde{m} = \min_{(R^*\backslash\Omega)\times(\partial K_1\cup\partial\tilde{K}_1)} \tilde{\mathrm{N}}(p,q) < \max_{(R^*\backslash\Omega)\times(\partial K_1\cup\partial\tilde{K}_1)} \tilde{\mathrm{N}}(p,q) = \tilde{M} < \infty$$

holds. Set $a = \max(\tilde{M}/m, 1)$ and $b = \max(M/\tilde{m}, 1)$. If $p_0 \in R\backslash\Omega$, then

$$\begin{aligned}(a\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0})^{\partial G} &= (a-1)\mathrm{N}_{p_0}^{\partial G} + (\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0})^{\partial G}\\ &\le (a-1)\mathrm{N}_{p_0} + \mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0}\\ &= a\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0}\end{aligned}$$

in G, where $G = R\backslash(K_1 \cup \tilde{K}_1)$ is an admissible subregion of R. From the inequality

$$\inf_G (a\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0})^{\partial G} = \min_{\partial K_1\cup\partial\tilde{K}_1} (a\mathrm{N}_{p_0} - \tilde{\mathrm{N}}_{p_0}) \ge am - \tilde{M} \ge 0,$$

it follows that $a\mathrm{N}_{p_0} \ge \tilde{\mathrm{N}}_{p_0}$ in G. For the same reason $b\tilde{\mathrm{N}}_{p_0} \ge \mathrm{N}_{p_0}$ in G holds for $p_0 \in R\backslash\Omega$. Denote by $C(A)$ and $\tilde{C}(A)$ the Kuramochi capacities of $A \subset R^*\backslash\Omega$ with respect to N_{p_0} and $\tilde{\mathrm{N}}_{p_0}$, respectively. If F is a compact subset of $R\backslash\Omega$ and a positive measure μ on F satisfies $\mathrm{N}\mu \le 1$ on F, then $(1/a)\tilde{\mathrm{N}}\mu \le 1$ holds. Then $a\tilde{C}(F) \ge C(F)$ follows. For the same reason $bC(F) \ge \tilde{C}(F)$ holds. If D is an open set in $R^*\backslash\Omega$, then by Folgesatz 17.6 of [CC] $C(D) = C(D\cap R)$ and $\tilde{C}(D) = \tilde{C}(D\cap R)$ hold and hence we have $\frac{1}{a}C(D) \le \tilde{C}(D) \le bC(D)$. Therefore $\frac{1}{a}C(A) \le \tilde{C}(A) \le bC(A)$ holds for every subset A of $R^*\backslash\Omega$. In particular $A \subset R^*\backslash\Omega$ is polar with respect to N if and only if it is polar with respect to $\tilde{\mathrm{N}}$.

5. Kuramochi boundary of two-sheeted unlimited covering surfaces

In this section let S be a Riemann surface and R be a two-sheeted unlimited covering surface of S with the projection mapping π. Let j be a sheet interchange of R, that is, j is a conformal automorphism of R which satisfies $j \circ j =$ the identity and $\pi = \pi \circ j$. We fix a closed parametric disc $\underline{K}_0$ on S. When (R, π) is ramified, we may choose $\underline{K}_0$ such that it contains just one point of the projection of branch points. Then $K_0 = \pi^{-1}(\underline{K}_0)$ is a simply connected subregion with analytic boundary. If (R, π) does not have branch points, then $K_0 = \pi^{-1}(\underline{K}_0)$ consists of mutually disjoint two simply connected subregions with analytic boundary. As is seen in Remark given after Lemma 1 of Section 4, we can construct the Kuramochi kernel functions $\mathrm{N}(p, p_0)$ and the Kuramochi compactification R^* with respect to $R\backslash K_0$ in the second case, too. Denote the Kuramochi boundary of S by Δ_S and the Kuramochi compactification of S by S^*. We shall use notation p, p_0 as points of R^* and q, q_0 as points of S^*. Let $\underline{\mathrm{N}}(q, q_0)$ be the Kuramochi kernel functions for $S\backslash\underline{K}_0$.

We shall show the next proposition.

Proposition 2 (cf. [JMS] Proposition 2.1). (1) *Suppose that* $N, \underline{N}, j$ *and* π *be as mentiond above. Then*

$$N(p, p_0) = N(j(p), j(p_0)) \tag{1-1}$$

and

$$\underline{N}(\pi(p), \pi(p_0)) = N(p, p_0) + N(p, j(p_0)) = N(p, p_0) + N(j(p), p_0) \tag{1-2}$$

hold on $(R \backslash K_0) \times (R \backslash K_0)$.

(2) j *and* π *can be extended continuously over* Δ_R. *Moreover* $j \circ j$ *is the identity and* $\pi = \pi \circ j$ *holds for extended* j *and* π.

(3) (1-1) *and* (1-2) *hold on* $(R^* \backslash K_0) \times (R^* \backslash K_0)$.

Proof. (1) Let $\{S_n\}_{n \geq 1}$ be an exhaustion of S with $S_1 \supset \underline{K}_0$. Set $R_n = \pi^{-1}(S_n)$. Since there is n_0 such that R_{n_0} is connected for all $n \geq n_0$, we may assume $n_0 = 1$. Then $\{R_n\}$ is an exhaustion of R. It is easily seen that $\mathrm{N}_n(p, p_0) = \mathrm{N}_n(j(p), j(p_0))$ holds for N_n defined in Section 4, and hence (1-1) follows.

Let $\underline{\mathrm{N}}_n(q, q_0)$ be the Kuramochi kernel function of $S_n \backslash \underline{K}_0$. We can easily show that a function $\underline{\mathrm{N}}_n(\pi(p), q_0)$ is equal to $\mathrm{N}_n(p, p_0) + \mathrm{N}_n(p, j(p_0))$, where $q_0 = \pi(p_0)$. We obtain (1-2) as n tends to infinity.

(2) For a point $p_0 \in \Delta_R$ and any fundamental sequence $\{p_m\}$ defining p_0 we have

$$\lim_{m \to \infty} \mathrm{N}(p, j(p_m)) = \lim_{m \to \infty} \mathrm{N}(j(p), p_m) = \mathrm{N}(j(p), p_0)$$

and

$$\lim_{m \to \infty} \underline{\mathrm{N}}(\pi(p), \pi(p_m)) = \lim_{m \to \infty} \{\mathrm{N}(p, p_m) + \mathrm{N}(j(p), p_m)\} = \mathrm{N}(p, p_0) + \mathrm{N}(j(p), p_0)$$

in $R \backslash K_0$ by (1). Then each $\{j(p_m)\}$ (resp. $\{\pi(p_m)\}$) is also a fundamental sequence on R (resp. S) and defines a Kuramochi boundary point in Δ_R (resp. Δ_S). We note that this boundary point $j(p_0)$ (resp. $\pi(p_0)$) is determined independently of the choice of $\{p_m\}$ defining p_0. With this definition we can extend the mapping j and π over Δ_R. For extended j and π it is easily seen that (1-1) and (1-2) hold on $(R \backslash K_0) \times (R^* \backslash K_0)$. It follows that $j \circ j$ is the identity and $\pi = \pi \circ j$ holds. It is easily checked that j and π are continuous on Δ_R with respect to the Kuramochi distance on $R \backslash K_0$ and $S \backslash \underline{K}_0$.

(3) On account of the symmetry of the Kuramochi kernel function on $(R^* \backslash K_0) \times (R^* \backslash K_0)$ it suffices to show that (1-1) and (1-2) hold on $\Delta_R \times \Delta_R$. Fix $p_0 \in \Delta_R$. Denote by $\mathscr{D}_{R \backslash \bar{R}_n}^{\mathrm{N}_{p_0}}$ the family of all Dirichlet finite functions of C^1 class on $R \backslash \bar{R}_n$ with boundary value N_{p_0} on ∂R_n. Then there is a unique function $F_{p_0}^n$ (resp. $F_{j(p_0)}^n$) which minimizes the Dirichlet integral in $\mathscr{D}_{R \backslash \bar{R}_n}^{\mathrm{N}_{p_0}}$ (resp. $\mathscr{D}_{R \backslash \bar{R}_n}^{\mathrm{N}_{j(p_0)}}$). Since

$$F_{p_0}^n(p) = \mathrm{N}_{p_0}(p) = \mathrm{N}_{j(p_0)}(j(p_0)) = (F_{j(p_0)}^n \circ j)(p)$$

holds on ∂R_n, we conclude that $F_{p_0}^n = F_{j(p_0)}^n \circ j$ on Δ_R. We know that $F_{p_0}^n$ is continuous on $R^*\backslash R_n$ and the value $\mathrm{N}_{p_0}(p)$ for $p \in \Delta_R$ is defined by $\lim_{n\to\infty} F_{p_0}^n(p)$. Therefore we obtain (1-1) on $\Delta_R \times \Delta_R$ as $n \to \infty$.

As for (1-2) by a similar argument as above we have

$$\underline{F}^n_{\pi(p_0)}(\pi(p)) = F^n_{p_0}(p) + F^n_{j(p_0)}(p)$$

on $R\backslash R_n$, where $\underline{F}^n_{\pi(p_0)}(\pi(p))$ is the unique function which minimizes the Dirichlet integral in $\mathscr{D}^{\underline{\mathrm{N}}_{\pi(p_0)}}_{S\backslash\bar{S}_n}$.

Since each side of this equation is continuous in $R^*\backslash R_n$, the equality holds also on Δ_R. Then (1-2) on $\Delta_R \times \Delta_R$ follows as $n \to \infty$.

The following lemma about the relation between polar sets in $R^*\backslash K_0$ and polar sets in $S^*\backslash \underline{K}_0$ is shown in [JMS, Lemma 2.3].

Lemma 2. *Let E be a subset of $S^*\backslash\underline{K}_0$. Then E is polar if and only if $\pi^{-1}(E)$ is a polar subset of $R^*\backslash K_0$.*

We have the next proposition about the relation between the sets Δ_R^1 and Δ_S^1 of minimal points. For the proof see Theorem 1 of [JMS].

Proposition 3. *Let S be a Riemann surface and (R, π) be a two-sheeted unlimited covering surface of S. Then we have $\pi(\Delta_R^1) = \Delta_S^1$. Moreover the fiber $\pi^{-1}(Q)$ contains at most two minimal points for every $Q \in \Delta_S^1$.*

6. Proof of Theorems 2 and 3

We denote by $HD(\overline{R\backslash K_0})$ the class of harmonic functions in some neighborhood of $\overline{R\backslash K_0} = (R\backslash K_0) \cup \partial K_0$ which have finite Dirichlet integral over $R\backslash K_0$. Let $g_{p_0}(p) = g(p, p_0)$ be the Green function on $R\backslash K_0$ with a pole at p_0. Set

$$H_{p_0}(p) = \{\mathrm{N}_{p_0}(p) - \mathrm{N}_{j(p_0)}(p)\} - \{g_{p_0}(p) - g_{j(p_0)}(p)\}$$

for $p_0 \in R\backslash K_0$. Then we have the next lemma, which will be used to prove Theorem 2.

Lemma 3. *If $h \in HD(\overline{R\backslash K_0})$, then*

$$(dh, dH_{p_0})_{R\backslash K_0} = 2\pi\{h(p_0) - h(j(p_0))\} - \int_{\partial K_0} h(p)\{dN^*_{p_0}(p) - dN^*_{j(p_0)}(p)\}.$$

Proof. Note that $H_{p_0} \in HD(\overline{R\backslash K_0})$. Set $U_0(M) = \{p; g_{p_0}(p) \geq M\}$. If M is sufficiently large, then $U_0(M)$ is a closed disc. Then for every $h \in HD(\overline{R\backslash K_0})$ we have

$$(dh, dg_{p_0})_{R\backslash(K_0 \cup U_0(M))} = 0$$

and

$$(dh, d\mathrm{N}_{p_0})_{R\backslash(K_0\cup U_0(M))} = -\int_{\partial K_0 \cup \partial U_0(M)} h(p) d\mathrm{N}^*_{p_0}(p)$$

by the same argument as in Lemma 1. Note that

$$(dh, d\mathrm{N}_{p_0} - dg_{p_0})_{R\backslash K_0} = \lim_{M\to\infty}(dh, d\mathrm{N}_{p_0} - dg_{p_0})_{R\backslash(K_0\cup U_0(M))}$$

$$= -\int_{\partial K_0} h(p)d\mathrm{N}^*_{p_0}(p) - \lim_{M\to\infty}\int_{\partial U_0(M)} h(p)d\mathrm{N}^*_{p_0}(p).$$

It is easily seen that

$$\lim_{M\to\infty}\int_{\partial U_0(M)} h(p)d\mathrm{N}^*_{p_0}(p) = -2\pi h(p_0).$$

Hence we obtain the required conclusion.

For differentials on R the pull back induced by j is denoted by $j^{\#}$. Every $\omega \in \Gamma_h(R)$ has a representation $\omega = 2^{-1}(\omega + j^{\#}(\omega)) + 2^{-1}(\omega - j^{\#}(\omega))$. Put $\omega_0 = 2^{-1}(\omega + j^{\#}(\omega))$ and $\omega_1 = 2^{-1}(\omega - j^{\#}(\omega))$. Then we have $\omega_0 = j^{\#}(\omega_0)$ and $\omega_1 = -j^{\#}(\omega_1)$. Set $\Gamma_h^0(R) = \{\omega \in \Gamma_h(R); \omega = j^{\#}(\omega)\}$ and $\Gamma_h^1(R) = \{\omega \in \Gamma_h(R); \omega = -j^{\#}(\omega)\}$. From the equation

$$(\omega, \sigma)_R = (j^{\#}(\omega), j^{\#}(\sigma))_R$$

it follows that $\Gamma_h^0(R) \perp \Gamma_h^1(R)$. Hence we have the orthogonal decomposition

$$\Gamma_h(R) = \Gamma_h^0(R) + \Gamma_h^1(R).$$

We say that a function f on R^* is quasicontinuous if for any $\varepsilon > 0$ there exists an open set G_ε such that the capacity of G_ε is less than ε and f is continuous as a function on $R^*\backslash G_\varepsilon$. We know that every $f \in CD(R)$ has a quasicontinuous extension over Δ_R and by this extension $f \in CD(R)$ with $df \in \Gamma_{e0}(R)$ is equal to some constant quasi everywhere on Δ_R. See [CC, Satz 17.9 and Satz 17.10]. Now we show the following proposition before proving Theorem 2.

Proposition 4. *Let S be a Riemann surface and (R, π) be a two-sheeted unlimited covering surface of S. Then $\Gamma_{he}(R) \cap \Gamma_h^1(R) = \{0\}$ if and only if $\pi^{-1}(Q)$ consists of only one point for quasi every $Q \in \Delta_S$.*

Proof. When Δ_S is polar or equivalently $S \in O_G$ (cf. [CC, p.189]), by Lemma 2 Δ_R is polar and $R \in O_G \subset O_{HD}$. Hence the conclusion is true.

We assume that $S \notin O_G$. We shall show

Claim 1: *$\Gamma_{he}(R) \cap \Gamma_h^1(R) = \{0\}$ if and only if $H_{p_0} = 0$ for every $p_0 \in R\backslash K_0$* and

Claim 2: *$H_{p_0} = 0$ for every $p_0 \in R\backslash K_0$ if and only if $\pi^{-1}(Q)$ consists of one point for quasi every $Q \in \Delta_S$.*

Claim 1. Let u be a function of $HD(R)$ such that du belongs to $\Gamma_{he}(R) \cap \Gamma_h^1(R)$. Note that $u \circ j \in HD(R)$ and

$$d(u \circ j) = j^{\#}(du) = -du.$$

Thus $d(u+u\circ j)=0$ and $u+u\circ j$ is constant c in R. Hence $u-c/2=-(u-c/2)\circ j$ holds. We shall replace $u-c/2$ by u. Then u satisfies $u=-u\circ j$. Let ρ be a function of $C^{\infty}(R)$ such that

$$\begin{cases} \rho=0 & \text{in a neighborhood of } K_0 \\ \rho=1 & \text{in a neighborhood of the ideal boundary} \\ 0\le\rho\le 1 & \text{otherwise.} \end{cases}$$

Then $\rho u\in C^{\infty}(R)$ has finite Dirichlet integral. We consider $R_0=R\backslash K_0$ as a Riemann surface. By Proposition 1 the Kuramochi compactification of R_0 is homeomorphic to $R_0\cup\Delta_R\cup\partial K_0$. Let $(\rho u)|_{R_0}=v_h+v_0$ be the Royden decomposition of $(\rho u)|_{R_0}$ in R_0, where $v_h\in HD(R_0)$ and v_0 is a Dirichlet potential in R_0 (cf. [CC, Satz 7.6]). By Satz 7.5 of [CC] we know $dv_0\in\Gamma_{e0}(R_0)$. Since $v_0=(\rho u)|_{R_0}-v_h$ is harmonic in a neighborhood of ∂K_0 in R_0, v_0 is continuously extended to be constant 0 on ∂K_0 by Lemma 5 of [J] and Satz 17.10 of [CC]. Thus v_h is also continuously extended to be constant 0 on ∂K_0. On the other hand $v_h=u$ quasi everywhere on Δ_R by Satz 17.10 of [CC] and Proposition 1. From the uniqueness of the Royden decomposition $(\rho u)|_{R_0}\circ j=v_h\circ j+u_0\circ j$ is the Royden decomposition of $(\rho u)|_{R_0}\circ j$ in R_0. In the neighborhood of the ideal boundary of R, $v_h+v_h\circ j$ is equal to $(u+u\circ j)-(v_0+v_0\circ j)=-v_0-v_0\circ j$. Thus $v_h+v_h\circ j=0$ quasi everywhere on Δ_R. Therefore $v_h+v_h\circ j$ is a harmonic function and a Dirichlet potential in R_0 and hence $v_h+v_h\circ j=0$ in R_0. Then v_h belongs to $HD(\overline{R\backslash K_0})$ and satisfies $v_h=0$ on ∂K_0, and $v_h(p)=-v_h(j(p))$. Hence

$$(dv_h,dH_{p_0})_{R\backslash K_0}=4\pi v_h(p_0).$$

If $H_{p_0}=0$ for every $p_0\in R\backslash K_0$, then $v_h=0$ in R_0 and $(\rho u)|_{R_0}$ is equal to v_0 in R_0. Hence $u=0$ quasi everywhere on Δ_R and we have $u\equiv 0$. Since u is arbitrary, $\Gamma_{he}(R)\cap\Gamma_h^1(R)=\{0\}$.

Next assume $\Gamma_{he}(R)\cap\Gamma_h^1(R)=\{0\}$. We extend H_{p_0} continuously over K_0 by putting $H_{p_0}=0$. Then H_{p_0} is a Dirichlet function on R, and a harmonic part u of the Royden decomposition of H_{p_0} belongs to $HD(R)$ and satisfies $u=H_{p_0}$ quasi everywhere on Δ_R (cf. [CC, Satz 7.6]). It is easily seen that du belongs to $\Gamma_{he}(R)\cap\Gamma_h^1(R)$. Hence we have $u\equiv 0$. It follows that $H_{p_0}=0$ quasi everywhere on Δ_R. Therefore H_{p_0} is a Dirichlet potential in R_0 by Satz 17.10 of [CC]. Hence we conclude that $H_{p_0}\equiv 0$.

Claim 2. For the Green function g_{p_0} we set $U_0=\{g_{p_0}(p)\ge M\}$. For sufficiently large $M>0$, U_0 is compact. Since $\min(g_{p_0},M)$ has finite Dirichlet integral, it is a Dirichlet potential by definition; see [CC, p.79]. Hence we have $g_{p_0}(P)=0$ for quasi every $P\in\Delta_R$ by Satz 17.10 of [CC].

Suppose that $H_{p_0}\equiv 0$ for $p_0\in R\backslash K_0$. Then $\mathrm{N}_{p_0}(P)=\mathrm{N}_{j(p_0)}(P)$ holds for quasi every $P\in\Delta_R$. We can choose a countable set $\{p_n\}_{n\ge 1}$ which is dense in $R\backslash K_0$. Set $I_n=\{P\in\Delta_R;\mathrm{N}_{p_n}(P)=\mathrm{N}_{j(p_n)}(P)\}$ and $I=\bigcap_{n=1}^{\infty}I_n$. Since $\Delta_R\backslash I_n$ is polar, $\Delta_R\backslash I=\bigcup_{n=1}^{\infty}(\Delta_R\backslash I_n)$ is also polar by subadditivity of capacity; see [CC, p.188]. For every $P \in I$, $\mathrm{N}_{p_n}(P) = \mathrm{N}_{p_n}(j(P))$ holds by 3) in Proposition 2. Hence $\mathrm{N}_P + \mathrm{N}_{j(P)}$ on $R \backslash K_0$ or equivalently $P = j(P)$. Consequently by 2) in Proposition 2 we have $\mathrm{N}_P(p) = 2^{-1}\underline{\mathrm{N}}_{\pi(P)}(\pi(p))$ on $R \backslash K_0$.

Let $Q \in \Delta_S$. We show that if $\pi^{-1}(Q)$ contains a minimal point P which satisfies $P = j(P)$ then $\pi^{-1}(Q) = \{P\}$. If there is another point $P' \in \pi^{-1}(Q)$, then $\mathrm{N}_{P'}(p) + \mathrm{N}_{j(P')}(p) = \underline{\mathrm{N}}_Q(\pi(p)) = 2\mathrm{N}_P(p)$ holds. Since $\mathrm{N}_P(p)$ is a minimal function, there exists some $t > 0$ such that $\mathrm{N}_{P'}(p) = t\mathrm{N}_P(p)$. This means that P' is also minimal. But this is a contradiction. Hence we have $\pi^{-1}(Q) = \{P\}$.

Set $E = \{Q \in \Delta_S; \pi^{-1}(Q)$ contains at least two points$\}$. If a minimal point P belongs to $\pi^{-1}(E)$, then $P \neq j(P)$. By the above observation P is not an element in I. Hence we have $\pi^{-1}(E) \subset (\Delta_R \backslash I) \cup \Delta_R^0$. Therefore $\pi^{-1}(E)$ is polar and by Lemma 2, E is also polar.

Conversely suppose that $\pi^{-1}(Q)$ consists of one point for quasi every $Q \in \Delta_S$. By Lemma 2, $\mathrm{N}_P = \mathrm{N}_{j(P)}$ holds in $R \backslash K_0$ for quasi every $P \in \Delta_R$. It follows that $\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)} = 0$ quasi everywhere on Δ_R for every $p_0 \in R \backslash K_0$. Hence we have $H_{p_0} \equiv 0$ for every $p_0 \in R \backslash K_0$. This completes the proof.

Proof of Theorems 2 and 3. If $S \in O_G$, then $R \in O_G \subset O_{HD}$. Since $\Gamma_h(R) = \Gamma_{h0}(R) + \Gamma_{he}^{*\perp}(R)$ and $\Gamma_{he}(R) = \{0\}$, $\Gamma_{h0}(R) = \Gamma_h(R)$ holds. Thus $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ holds. When $S \notin O_G$, by Claim 2 in the proof of Proposition 4 we have shown that if $\pi^{-1}(Q)$ consists of one point for quasi every $Q \in \Delta_S$ then $\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)}$ is equal to $g_{p_0} - g_{j(p_0)}$ for every $p_0 \in R \backslash K_0$. In view of Lemma 1 and Remark after Lemma 1 we have the following representation of $d(\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)})$ and $d(\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)})^*$ in some neighborhood of the ideal boundary:

$$d(\mathrm{N}_{p_0} - N_{j(p_0)})^* = \omega_{h0} + df_0 \quad \text{and} \quad d(\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)}) = df_1,$$

where $\omega_{h0} \in \Gamma_{h0}$, $f_0, f_1 \in C^\infty$, and $df_0, df_1 \in \Gamma_{e0}$. This shows that $\mathrm{N}_{p_0} - \mathrm{N}_{j(p_0)}$ has Γ_{he}- and Γ_{hm}-behaviors. In each case, by Theorem B, if R has no planar ends then R is maximal. Thus Theorem 2 is proved.

Suppose that R has planar ends. By Theorem 3′ of [J] there is a maximal extension $(\tilde{R}, \iota)$ of R such that $\tilde{R} \backslash \iota(R)$ is a closed N_D-set. The mapping $\iota \circ j \circ \iota^{-1}$ is a conformal mapping of $\iota(R)$ onto $\iota(R)$. Since $\tilde{R} \backslash \iota(R)$ is a closed N_D-set, $\iota \circ j \circ \iota^{-1}$ is extended to be a conformal automorphism of $\tilde{R}$ by Lemma 4 of [Re]. We denote the extended one by $\tilde{j}$. It is obvious that $\tilde{j} \circ \tilde{j}$ is an identity mapping of $\tilde{R}$. Hence we obtain a Riemann surface $\tilde{S} = \tilde{R}/\tilde{j}$ and $\tilde{R}$ is a two-sheeted unlimited covering surface of $\tilde{S}$ with the natural projection mapping $\tilde{\pi}$ which satisfies $\tilde{\pi} = \tilde{\pi} \circ \tilde{j}$ (cf. [FK, III. 7.8]).

If $q \in S$ is a projection of a branch point of (R, π), then a branch point $\pi^{-1}(q)$ satisfies $j(\pi^{-1}(q)) = \pi^{-1}(q)$. Hence $\iota(\pi^{-1}(q))$ satisfies $\tilde{j}(\iota(\pi^{-1}(q))) = \iota(\pi^{-1}(q))$. This means that $\iota(\pi^{-1}(q))$ is a branch point of $(\tilde{R}, \tilde{\pi})$ and the point $(\tilde{\pi} \circ \iota \circ \pi^{-1})(q)$ is well-defined.

If $q \in S$ is not a projection of a branch point of (R, π), then $\pi^{-1}(q)$ consists of two points p_0 and p_1. Since $j(p_0) = p_1$, $\tilde{j}(\iota(p_0)) = \iota(p_1)$ holds. Then we have $\tilde{\pi}(\iota(p_0)) = \tilde{\pi}(\iota(p_1))$. Hence the image $(\tilde{\pi} \circ \iota \circ \pi^{-1})(q)$ is well-defined. Since π and

$\tilde{\pi}$ are conformal mappings in some neighborhood of a non-branch point, $\iota' = \tilde{\pi} \circ \iota \circ \pi^{-1}$ is a conformal mapping of S into $\tilde{S}$ and satisfies $\iota' \circ \pi = \tilde{\pi} \circ \iota$.

7. Kuramochi boundary of two-sheeted unlimited covering surfaces of the unit disc

In this section we treat a two-sheeted unlimited covering surface R of the unit disc U with the projection mapping π. Since U is simply connected, if an unlimited covering surface of U is not ramified then it is conformally equivalent to the unit disc. Thus we may assume that R is ramified and has branch points.

First we shall show that (R, π) is uniquely determined by the set $\{z_\nu\}$ of the projection of branch points.

Proposition 5. *Let (R, π) and (R', π') be two-sheeted unlimited covering surfaces of the unit disc $U = \{|z| < 1\}$. Let j (resp. j') be the sheet interchange of (R, π) (resp. (R', π')) and $\{z_\nu\}$ (resp. $\{w_\nu\}$) be the projection of branch points of (R, π) (resp. (R', π')).*

Then the following conditions are equivalent:

(a) *There is a Möbius transformation $T(z)$ such that $T(U) = U$ and $T(\{z_\nu\}) = \{w_\nu\}$.*

(b) *There is a conformal map ψ of R onto R' such that $\psi \circ j = j' \circ \psi$.*

Proof. (b) $\Rightarrow$ (a): In this proof we shall use notation $U_z = \{|z| < 1\}$ and $U_w = \{|w| < 1\}$. By assumption $(\pi' \circ \psi)(\pi^{-1}(z))$ consists of one point for every $z \in U_z$. Set $w = T(z) = (\pi' \circ \psi)(\pi^{-1}(z))$ in U_z. It is easily seen that $T(z)$ is a bijection of U_z to U_w and $T(\{z_\nu\}) = \{w_\nu\}$. Since π (resp. π') is a locally conformal mapping of $R \backslash \{\pi^{-1}(z_\nu)\}$ (resp. $R' \backslash \{(\pi')^{-1}(w_\nu)\}$) onto $U_z \backslash \{z_\nu\}$ (resp. $U_w \backslash \{w_\nu\}$), $T(z)$ is a conformal mapping of $U_z \backslash \{z_\nu\}$ to $U_w \backslash \{w_\nu\}$. Because isolated points $\{z_\nu\}$ are removable for an analytic function $T(z)$ in $U_z \backslash \{z_\nu\}$, $T(z)$ is a conformal mapping of U_z onto U_w with $T(\{z_\nu\}) = \{w_\nu\}$.

(a) $\Rightarrow$ (b): Since $(R', T^{-1} \circ \pi')$ is also a two-sheeted unlimited covering surface of the unit disc with projection $\{z_\nu\}$ of branch points and the sheet interchange of $(R', T^{-1} \circ \pi')$ is equal to j', the identity map of R' satisfies condition (b). Then it is enough to prove in the case when $\{z_\nu\} = \{w_\nu\}$.

By Weierstrass' Theorem there exists an analytic function $f(z)$ in U_z such that $f(z)$ has a single zero at every z_ν and $f(z) \neq 0$ if $z \neq z_\nu$; see [Ru, p. 326, 15.11 Theorem]. We know that a Riemann surface (R_f, π_f) of an analytic configuration of $\sqrt{f(z)}$ is a two-sheeted unlimited covering surface of the unit disc with projection $\{z_\nu\}$ of branch points; see for example [Sp, Chapter 3].

If $\sqrt{f} \circ \pi$ defines a single valued analytic function on (R, π), then we can easily construct a conformal mapping ψ_f of R onto R_f such that $\psi_f \circ j = j_f \circ \psi_f$, where j_f is the sheet interchange of (R_f, π_f).

Now it suffices to show that $\sqrt{f} \circ \pi$ is a single valued analytic function on (R, π). Fix a reference point $p_0 \in R$ which is not a branch point of π. We may assume that $0 \notin \{z_\nu\}$ and $\pi(p_0) = 0$. We can choose mutually disjoint closed discs U_ν in U which is centered at z_ν and does not contain 0. Let l_ν be a finite union of segments in $U\setminus\bigcup_\nu \overline{U_\nu}$ which does not intersect itself and starts from 0 and terminates at a point of ∂U_ν. Set $c_\nu = \partial U_\nu$ and $\gamma_\nu = l_\nu^{-1} c_\nu l_\nu$. We can show that a closed curve $\gamma \in U\setminus\{z_\nu\}$ which issues from 0 is homotopic to some $\gamma_{\nu_1}^{\alpha_1}\cdots\gamma_{\nu_k}^{\alpha_k}$, where $\alpha_1,\dots,\alpha_k \in \mathbf{Z}\setminus\{0\}$, $\nu_1,\dots,\nu_k \in \mathbf{N}$ and $\nu_j \neq \nu_{j\pm1}$. Set $\tilde{R} = R\setminus\{\text{branch points}\}$. Let σ be an arbitrary closed curve in $\tilde{R}$ which starts from p_0. It can be shown that if $\pi(\sigma)$ is homotopic to $\gamma_{\nu_1}^{\alpha_1}\cdots\gamma_{\nu_k}^{\alpha_k}$, then $\sum_{j=1}^k \alpha_j \equiv 0 \pmod 2$ holds. Note that

$$\begin{aligned}
2^{-1}\int_{\pi(\sigma)} d\arg f(z) &= 2^{-1}\int_{\gamma_{\nu_1}^{\alpha_1}\cdots\gamma_{\nu_k}^{\alpha_k}} d\arg f(z)\\
&= 2^{-1}\sum_{j=1}^{k}\int_{\gamma_{\nu_j}^{\alpha_j}} d\arg f(z)\\
&= 2^{-1}\sum_{j=1}^{k}\int_{\gamma_{\nu_j}^{\alpha_j}} d\arg(z - z_{\nu_j})\\
&= 2^{-1}\sum_{j=1}^{k} 2\pi\alpha_j = \pi\sum_{j=1}^{k}\alpha_j.
\end{aligned}$$

Then $\sqrt{f(z)} = \exp(2^{-1}(\log|f(z)| + i\arg f(z)))$ defines the same function element about $z = 0$ if it is continued analytically along $\pi(\sigma)$. Hence $\sqrt{f}\circ\pi$ defines a single valued analytic function on (R,π).

Remark. The uniqueness of covering surfaces with the same branch points does not hold generally. We consider a Riemann surface R_1 of $\sqrt{(z-\sqrt{2})(z+\sqrt{2})}$ and a Riemann surface R_2 of $\sqrt{z(z-\sqrt{2})(z+\sqrt{2})}$. Then R_1 and R_2 are two-sheeted unlimited covering surfaces of $\mathbf{C}\cup\{\infty\}$ with the natural projection mappings π_1 and π_2, respectively. Then the inverse image $R_1' = \pi_1^{-1}(A)$ and $R_2' = \pi_2^{-1}(A)$ of the annulus $A = \{1 < |z| < 2\}$ are two-sheeted unlimited covering surfaces of A with projection of branch points $\pm\sqrt{2} \in A$. But they are not conformally equivalent because R_1' has four boundary components and R_2' has two boundary components. It is easily seen that if $n \geq 3$ there are many n-sheeted unlimited covering surfaces of the unit disc with the same projection of branch points $\{z_\nu\}$.

Now we prove Theorem 4.

Proof of Theorem 4. By Proposition 3 we know that $I^\theta = \pi^{-1}(e^{i\theta})$ contains one or two minimal points.

Let $\{z_\nu\}_{\nu\geq1}$ be the projection of branch points. Set $\Omega_r = \{z \in \overline{U};\ |z - e^{i\theta}| \leq r\}$. Suppose that $e^{i\theta}$ is not an accumulation point of $\{z_\nu\}$. Then there exists $r_0 > 0$ such that $\Omega_{r_0} \cap \{z_\nu\} = \emptyset$. The inverse image $\pi^{-1}(\Omega_{r_0})$ consists of just two components and each of them determines a border of R. Then I^θ consists of two minimal points. This is the case (a).

Assume that $e^{i\theta}$ is an accumulation point of $\{z_\nu\}$. There exists a subsequence

$\{z_{\nu_k}\}$ such that $\lim_{k\to\infty} z_{\nu_k} = e^{i\theta}$. Note that $\{z_{\nu_k}\}$ is a fundamental sequence on U. By Proposition 2, I^θ is closed. Since $e^{i\theta}$ is a limit point of $\{z_{\nu_k}\}$, every Ω_r contains some z_{n_k}. Hence $\pi^{-1}(\Omega_r)$ is connected. Therefore I^θ, which is equal to $\bigcap_{n=1}^{\infty}\pi^{-1}(\Omega_{1/n})$, is connected. Since $\mathrm{N}(p,\pi^{-1}(z_{\nu_k})) = 2^{-1}\underline{\mathrm{N}}(\pi(p), z_{\nu_k})$ and $\{z_{\nu_k}\}$ is a fundamental sequence on U, $\{\pi^{-1}(z_{\nu_k})\}$ is also a fundamental sequence converging to some ideal boundary point $P^\theta_{1/2}\in I^\theta$ which satisfies $\mathrm{N}(p,P^\theta_{1/2}) = 2^{-1}\underline{\mathrm{N}}(\pi(p),e^{i\theta})$. Let P^θ be an arbitrary point in I^θ. By Proposition 3 we have $j(P^\theta)\in I^\theta$ and $2^{-1}\{\mathrm{N}(p,P^\theta)+\mathrm{N}(p,j(P^\theta))\} = 2^{-1}\underline{\mathrm{N}}(\pi(p),e^{i\theta}) = \mathrm{N}(p,P^\theta_{1/2})$. Hence $P^\theta = j(P^\theta)$ if and only if $P^\theta = P^\theta_{1/2}$. Let P^θ_1 be a minimal point in I^θ. One of the two cases occurs: i) P^θ_1 coincides with $j(P^\theta_1)$ or ii) P^θ_1 differs from $P^\theta_0 = j(P^\theta_1)$.

In case i) assume that there exists another point P^θ in I^θ. Since P^θ_1 coincides with $P^\theta_{1/2}$, the equation $\mathrm{N}(p,P^\theta)+\mathrm{N}(p,j(P^\theta)) = 2\mathrm{N}(p,P^\theta_1)$ holds. But this contradicts the fact that $\mathrm{N}(p,P^\theta_1)$ is minimal. Hence I^θ consists of one minimal point $P^\theta_1 (= P^\theta_{1/2})$. This is the case (c).

In case ii) I^θ contains just two minimal points P^θ_1 and P^θ_0. By (1-2) in Proposition 2 the equation

$$\mathrm{N}(p,P^\theta)+\mathrm{N}(p,j(P^\theta)) = \mathrm{N}(p,P^\theta_0)+\mathrm{N}(p,P^\theta_1)$$

holds for any point P^θ in I^θ. Then $\mathrm{N}(p,P^\theta)$ has the canonical representation $t\mathrm{N}(p,P^\theta_0)+s\mathrm{N}(p,P^\theta_1)$ with some $s,t\in[0,1]$. By Proposition 2

$$\mathrm{N}(p,j(P^\theta)) = s\mathrm{N}(p,P^\theta_0)+t\mathrm{N}(p,P^\theta_1)$$

holds. It follows that $t+s=1$ and $\mathrm{N}(p,P^\theta) = t\mathrm{N}(p,P^\theta_0)+(1-t)\mathrm{N}(p,P^\theta_1)$. This correspondence defines a mapping ψ of I^θ to $[0,1]$ by $\psi(P^\theta)=t$. By the uniqueness of canonical representation injectivity of ψ follows. Suppose that $\mathrm{N}(p,P^\theta)$ (resp. $\mathrm{N}(p,\hat{P}^\theta)$) has a representation $t\mathrm{N}(p,P^\theta_0)+(1-t)\mathrm{N}(p,P^\theta_1)$ (resp. $\hat{t}\mathrm{N}(p,P^\theta_0)+(1-\hat{t})\mathrm{N}(p,P^\theta_1)$). Then

$$\begin{aligned}
&d(P^\theta,\hat{P}^\theta)\\
&\quad= \sup_{p\in K_1}\left|\frac{\mathrm{N}(p,P^\theta)}{1+\mathrm{N}(p,P^\theta)} - \frac{\mathrm{N}(p,\hat{P}^\theta)}{1+\mathrm{N}(p,\hat{P}^\theta)}\right|\\
&\quad= \sup_{p\in K_1}\left|\frac{(t-\hat{t})\{\mathrm{N}(p,P^\theta_0)-\mathrm{N}(p,P^\theta_1)\}}{\{1+t\mathrm{N}(p,P^\theta_0)+(1-t)\mathrm{N}(p,P^\theta_1)\}\{1+\hat{t}\mathrm{N}(p,P^\theta_0)+(1-\hat{t})\mathrm{N}(p,P^\theta_1)\}}\right|\\
&\quad\geq |t-\hat{t}|\frac{\sup_{p\in K_1}|\mathrm{N}(p,P^\theta_0)-\mathrm{N}(p,P^\theta_1)|}{\sup_{p\in K_1}\{1+\mathrm{N}(p,P^\theta_0)+\mathrm{N}(p,P^\theta_1)\}^2}.
\end{aligned}$$

Since

$$\frac{\sup_{p\in K_1}|\mathrm{N}(p,P^\theta_0)-\mathrm{N}(p,P^\theta_1)|}{\sup_{p\in K_1}\{1+\mathrm{N}(p,P^\theta_0)+\mathrm{N}(p,P^\theta_1)\}^2}$$

is positive finite, the continuity of ψ follows. Since I^θ is connected, the mapping is surjective. Therefore the inverse mapping is also continuous. Hence I^θ is homeomorphic to $[0, 1]$. This is the case (b).

We shall obtain a sufficient condition that I^θ consists of one minimal point for every $e^{i\theta} \in \partial U$. We denote the distance between z_ν and $\{z_\mu\}_{\mu \neq \nu} \cup \partial U$ by d_ν. Set $B(z_\nu, r) = \{|z - z_\nu| < r\}$. It is easily seen that $B(z_\nu, d_\nu/2) \cap B(z_\mu, d_\mu/2) = \emptyset$ if $\nu \neq \mu$. The next result is essentially due to Example 1.5 of [Sa1].

Proposition 6. *Suppose that there is a positive number k, $1/2 < k < 1$, such that for any positive integer ν_0 an open set $\bigcup_{\nu \geq \nu_0} B(z_\nu, kd_\nu)$ contains a smooth Jordan curve γ_{ν_0} which separates $z = 0$ from ∂U. Then I^θ consists of one minimal point for every $e^{i\theta} \in \partial U$.*

To prove this proposition we use the following fact (cf. [AS, p.147]).

Lemma 4. *If $u(z)$ is a harmonic function in U with finite Dirichlet integral and vanishes at $z = 0$, then $|u(z)| \leq \dfrac{\rho}{\sqrt{\pi}(1 - \rho)} \|du\|_U$ holds in $|z| \leq \rho$, $0 < \rho < 1$.*

Proof of Proposition 6. Let u be an arbitrary function in $HD(R)$. If $z \in U$ and $z \neq z_\nu$, then $\pi^{-1}(z)$ consists of just two points p_1, p_2. Set $\underline{u}(z) = |u(p_1) - u(p_2)|$ if $z \neq z_\nu$ and $\underline{u}(z_\nu) = 0$. Then $\underline{u}(z)$ is a non-negative subharmonic function in U. Since $\{z_\nu\}$ has no accumulation point in U, for any $\varepsilon > 0$ there is a positive integer ν_0 such that $\bigcup_{\nu \geq \nu_0} B(z_\nu, d_\nu) \subset U \backslash \{|z| \leq 1 - \varepsilon\}$. Denote by D_{ν_0} a Jordan domain bounded by γ_{ν_0}. Then D_{ν_0} contains $\{|z| \leq 1 - \varepsilon\}$. By the maximum principle for subharmonic functions $\underline{u}(z) \leq \max_{z \in \gamma_{\nu_0}} \underline{u}(z)$ holds in D_{ν_0} and also in $\{|z| \leq 1 - \varepsilon\}$. A function $\varphi(p) = \sqrt{\pi(p) - z_\nu}/\sqrt{d_\nu}$ becomes a single-valued analytic function in a simply connected subregion $\pi^{-1}(B(z_\nu, d_\nu))$ of R and φ maps $\pi^{-1}(B(z_\nu, d_\nu))$ conformally onto a unit disc $\{|w| < 1\}$ with $\varphi(\pi^{-1}(z_\nu)) = 0$. It is easily seen that $\varphi(p_1) = -\varphi(p_2)$ for $p_1, p_2 \in \pi^{-1}(z)$ and $\varphi(\pi^{-1}(B(z_\nu, kd_\nu))) = \{|w| < \sqrt{k}\}$. The function $\tilde{u}_\nu(w) = u|_{\pi^{-1}(B(z_\nu, d_\nu))} \circ \varphi^{-1}(w) - u(\pi^{-1}(z_\nu))$ satisfies the condition of Lemma 4. Hence

$$|\tilde{u}_\nu(w)| \leq \frac{\sqrt{k}}{\sqrt{\pi}(1 - \sqrt{k})} \|du\|_{\pi^{-1}(B(z_\nu, d_\nu))}$$

holds in $|w| < \sqrt{k}$. If $z \in B(z_\nu, kd_\nu)$, then

$$\begin{aligned} \underline{u}(z) &= |u(p_1) - u(p_2)| = |\tilde{u}_\nu(\varphi(p_1)) - \tilde{u}_\nu(\varphi(p_2))| \\ &= |\tilde{u}_\nu(\varphi(p_1)) - \tilde{u}_\nu(-\varphi(p_1))| \leq \frac{2\sqrt{k}}{\sqrt{\pi}(1 - \sqrt{k})} \|du\|_{\pi^{-1}(B(z_\nu, d_\nu))} \end{aligned}$$

This implies that

$$\underline{u}(z) \leq \frac{2\sqrt{k}}{\sqrt{\pi}(1 - \sqrt{k})} \|du\|_{\pi^{-1}(U \backslash \{|z| \leq 1 - \varepsilon\})}$$

holds in $\{|z| \le 1-\varepsilon\}$. Since ε is arbitrary, we have $\underline{u}(z) \equiv 0$. Therefore $u(p_1) = u(p_2)$ holds if $\pi(p_1) = \pi(p_2)$. By Claim 1 of the proof of Proposition 4, $H_{p_0} = 0$ holds for every $p_0 \in R \backslash K_0$. Since the Green function $\underline{g}_{q_0}(q)$ on $U \backslash \underline{K}_0$ tends to zero as $q \to \partial U$, the Green function g_{p_0} on $R \backslash K_0$ is equal to 0 on Δ_R. Hence $\mathrm{N}(P, p_0) = \mathrm{N}(P, j(p_0))$ holds for every $P \in \Delta_R$. By Proposition 2 $P = j(P)$ holds. By Theorem 4 I^θ consists of one minimal point for every $e^{i\theta} \in \partial U$.

8. Proof of Theorem 5

Proof of Theorem 5. Clearly (a) implies (b). By Proposition 4, (b) $\Leftrightarrow$ (c) follows. We have shown in [J, Lemma 3] that (f) implies (g). Since R has infinitely many branch points, R does not have a planar end. By Theorem 2 in [J] we have (g) $\Rightarrow$ (h).

(c) $\Leftrightarrow$ (d): If $u \in HD(R)$, then $du = 2^{-1}(du + j^{\#}(du)) + 2^{-1}(du - j^{\#}(du))$ and $2^{-1}(du + j^{\#}(du)) \in \Gamma_h^0(R)$ and $2^{-1}(du - j^{\#}(du)) \in \Gamma_h^1(R)$, as observed in Section 6. Since $j^{\#}(du) = d(u \circ j) \in \Gamma_{he}(R)$, $2^{-1}(du - j^{\#}(du)) \in \Gamma_{he}(R) \cap \Gamma_h^1(R)$. Therefore (c) implies $du = d(u \circ j)$ and hence $u - u \circ j$ is a constant function. Because u and $u \circ j$ take the same value at each branch point, we have $u = u \circ j$. Then $\underline{u}(z) = 2^{-1}(u(p) + (u \circ j)(p))$, $p \in \pi^{-1}(z)$, belongs to $HD(U)$ and satisfies $u = \underline{u} \circ \pi$. Thus (c) $\Rightarrow$ (d) is shown.

Conversely if for every $u \in HD(R)$ there is $\underline{u} \in HD(U)$ such that $u = \underline{u} \circ \pi$, then $u = u \circ j$ holds. Hence $du - j^{\#}(du) = 0$ and $\Gamma_{he}(R) \cap \Gamma_h^1(R) = \{0\}$.

(d) $\Rightarrow$ (e): For every $u \in HD(R)$ there is $\underline{u} \in HD(U)$ such that $u = \underline{u} \circ \pi$. Since $HD(U) = \Re AD(U)$, there is a single-valued conjugate harmonic function $\underline{u}^*$ of $\underline{u}$. Then $\underline{u}^* \circ \pi$ is a single-valued conjugate harmonic function of u. Hence $u \in \Re AD(R)$ and $HD(R) = \Re AD(R)$ holds.

(e) $\Leftrightarrow$ (f): From the relations

$$\{du; u \in KD(R)\} = \Gamma_{he}(R) \cap \Gamma_{hse}^*(R)$$

$$\{du; u \in \Re AD(R)\} = \Gamma_{he}(R) \cap \Gamma_{he}^*(R),$$

in Section 2 it follows that R has (W)-property if and only if $KD(R) = \Re AD(R)$ holds. If K is a compact subset of R, then $\pi(K)$ is a compact subset of U, and hence $\pi(K)$ is contained in $\{|z| < r_0\}$ for some r_0, $0 < r_0 < 1$. Since R has infinitely many branch points, there is a branch point of p_0 of (R, π) such that $r_0 < |\pi(p_0)| < 1$. Then $\pi^{-1}(\{r_0 < |z| < 1\})$ is connected. Thus a neighborhood of the ideal boundary $R \backslash K$ contains a connected neighborhood of the ideal boundary $\pi^{-1}(\{r_0 < |z| < 1\})$. Hence R has only one ideal boundary component. If γ is a dividing curve on R, then some connected component of $R \backslash \gamma$ is a neighborhood of the ideal boundary. Hence γ is homologous 0. Therefore $\Gamma_{hse}(R) = \Gamma_h(R)$ holds. Hence $KD(R) = HD(R)$ holds. Therefore we have (e) $\Leftrightarrow$ (f).

(h) $\Rightarrow$ (i): We shall prove a contraposition. Suppose that there is $u \in$

$\Re AD(R)\backslash HD(U)$. If necessary by considering $u - u \circ j$ on R we may assume that $u(p) = -u(j(p))$ holds on R. Set $D = \{p \in R; u(p) > 0\}$. Since $u(p) = -u(j(p))$, D does not contain any branch points and $D \cap j(D) = \emptyset$. Let D_0 be a connected component of D. Then D_0 is conformally equivalent to $\pi(D_0)$ by the mapping $\pi|_{D_0}$. Hence D_0 is planar and the relative boundary ∂D_0 consists of piecewise analytic curves, which are part of level curves $\{p \in R; u(p) = 0\}$. Since du^* is exact on D_0, D_0 is simply connected. We can find a conformal mapping ϕ of D_0 to the upper half plane H. Set $I_\phi = \partial H \backslash \phi(\partial D_0)$. We can see that $u \circ \phi^{-1} \in HD(H)$ and $u \circ \phi^{-1} = 0$ on $\phi(\partial D_0)$. For $z \in H_-$, the lower half plane, define $u \circ \phi^{-1}(z) = -u \circ \phi^{-1}(\bar{z})$. Then the extended $u \circ \phi^{-1}$ belongs to $HD(\mathbf{C}\backslash I_\phi)$. We can easily show that $d(u \circ \phi^{-1})^*$ is exact in $\mathbf{C}\backslash I_\phi$. Then there exists a non-constant Dirichlet finite analytic function $u \circ \phi^{-1} + i(u \circ \phi^{-1})^*$ on $\mathbf{C}\backslash I_\phi$. Hence I_ϕ is not N_D-set and R has a disc with crowded ideal boundary. This implies that R is not maximal by Theorem A (c).

(e) $\Rightarrow$ (d): We have shown (e) $\Rightarrow$ (i). Immediately (e) $\Rightarrow$ (d) follows.

(c) $\Rightarrow$ (a): Let $g_{p_0}(p)$ be the Green function on $R\backslash K_0$ with a pole at p_0 and $\underline{g}_{z_0}(z)$ the Green function on $U\backslash \underline{K}_0$ with a pole at z_0. It is seen that $\lim_{z \to \partial U} \underline{g}_{z_0}(z) = 0$ and $g_{p_0}(p) + g_{j(p_0)}(p) = \underline{g}_{\pi(p_0)}(\pi(p))$. Hence g_{p_0} is extended to be continuous on $(R\backslash K_0) \cup \Delta_R$ by putting $g_{p_0} = 0$. In the proof of Proposition 4 if we use the condition $g_{p_0} = 0$ on Δ_R instead of the condition $g_{p_0} = 0$ quasi everywhere on Δ_R, then we conclude that (c) implies (a).

(i) $\not\Rightarrow$ (h): We shall present a counterexample. Set $z_n = 1 - n^{-1}$, $n \in \mathbf{N}$. Let (R, π) be a two-sheeted unlimited covering surface of the unit disc which has a branch point over each of $\{z_n\}$. For every bounded and Dirichlet finite analytic function f on R we define $\tilde{f}(z) = (f(p) - f(j(p)))^2$ where $z = \pi(p)$. Then $\tilde{f}(z)$ is also a bounded and Dirichlet finite analytic function on U and $\tilde{f}$ vanishes at every z_n. Since $\sum_{n=1}^{\infty}(1 - |z_n|) = \sum_{n=1}^{\infty} n^{-1} = \infty$, we have $\tilde{f} \equiv 0$. Therefore $f(p) = f(j(p))$ holds on R. Since the class of bounded and Dirichlet finite analytic functions is dense in $AD(R)$ with respect to the Dirichlet norm, the equation $f(p) = f(j(p))$ is valid for every $f \in AD(R)$; see [Sa2, Corollary 2.6]. Therefore (i) $\Re AD(R) = HD(U)$ holds on R.

On the other hand R is not maximal because R has a border; see Theorem A (b). This completes the proof.

9. Proof of Theorem 6

It is easily checked that Theorem 6 follows from the next two propositions.

Proposition 7. *Let (R, π) be a two-sheeted unlimited covering surface of the unit disc with projection $\{z_\nu\}_{\nu \ge 1}$ of branch points. Suppose that every $e^{i\theta} \in \partial U$ is an accumulation point of $\{z_\nu\}$. Then there is a sequence $\{\kappa_\nu\}_{\nu \ge 1}$, $0 < \kappa_\nu \le 1$, such that if $z'_\nu \in B(z_\nu, \kappa_\nu d_\nu)\backslash\{z_\nu\}$ then the two-sheeted unlimited covering surface (R_1, π_1), the projection of whose branch points is $\{z_\nu\} \cup \{z'_\nu\}$, satisfies that $\pi_1^{-1}(e^{i\theta})$ is homeomorphic to the closed interval $[0, 1]$ for every $e^{i\theta} \in \partial U$.*

Proposition 8. *Let (R, π) be a maximal two-sheeted unlimited covering surface of the unit disc with projection $\{z_\nu\}_{\nu \geq 1}$ of branch points. Then there is a sequence $\{\kappa_\nu\}_{\nu \geq 1}$, $0 < \kappa_\nu \leq 1$, such that if $z'_\nu \in B(z_\nu, \kappa_\nu d_\nu) \backslash \{z_\nu\}$, then the two-sheeted unlimited covering surface (R_2, π_2), the projection of whose branch points is $\{z_\nu\} \cup \{z'_\nu\}$, is also maximal.*

First we show Proposition 7.

Proof of Proposition 7. Set $U_1 = U \backslash \overline{B(z_1, d_1/8)}$. Let $u_\nu, \nu \geq 2$, be the solution of the Neumann-Dirichlet problem on $U_1 \backslash \overline{B(z_\nu, \kappa_\nu d_\nu)}$ for some κ_ν, $0 < \kappa_\nu < 1/8$, with boundary values $u_\nu = 0$ on $|z - z_1| = d_1/8$, $u_\nu = 1$ on $|z - z_\nu| = \kappa_\nu d_\nu$, and $\partial u_\nu / \partial n = 0$ on ∂U. We choose κ_ν to satisfy the condition $|u_\nu(z)| < 1/2^{\nu+1}$ in $U_1 \backslash \overline{B(z_\nu, d_\nu/4)}$ and $\|du_\nu\|_{U_1 \backslash \overline{B(z_\nu, \kappa_\nu d_\nu)}} < 1/2^\nu$. Set $u_\nu = 1$ on $B(z_\nu, \kappa_\nu d_\nu)$ and $u_\nu = 0$ on $B(z_1, d_1/8)$. Then u_ν becomes a continuous Dirichlet function in U and $\sum_{\nu \geq 2} u_\nu(z)$ converges uniformly on any compact set of U. Set $f(z) = \sum_{\nu \geq 2} u_\nu(z)$, $\kappa_1 = 1/8$, and $F = \overline{\bigcup_{\nu \geq 1} B(z_\nu, \kappa_\nu d_\nu)}$. It is easily seen that $f(z)$ is continuous in U, harmonic in $U \backslash F$, $0 \leq f(z) \leq 1/4$ in $U_1 \backslash \bigcup_{\nu \geq 2} \overline{B(z_\nu, d_\nu/4)}$, $f(z) \geq 1$ on F, and has finite Dirichlet integral over U. We show that $f(z)$ minimizes Dirichlet integral in the class $\tilde{\mathscr{D}}_F^f = \{h; h \in CD(U),\ h = f \text{ on } F\}$. By Satz 15.1 of [CC] there exists uniquely a function $f^F \in \tilde{\mathscr{D}}_F^f$ which minimizes Dirichlet integral in $\tilde{\mathscr{D}}_F^f$. We know that u_ν minimizes Dirichlet integral in $\tilde{\mathscr{D}}_{F_\nu}^{u_\nu} = \{h; h \in CD(U),\ h = 0 \text{ on } \overline{B(z_1, \kappa_1 d_1)},\ h = 1 \text{ on } \overline{B(z_\nu, \kappa_\nu d_\nu)}\}$, where $F_\nu = \overline{B(z_1, \kappa_1 d_1)} \cup \overline{B(z_\nu, \kappa_\nu d_\nu)}$. Since $f - f^F = 0$ on F_ν, we have $(du_\nu, df - df^F)_U = 0$ by Satz 15.1 a) of [CC]. It follows that

$$(df, df - df^F)_U = \sum_{\nu \geq 2} (du_\nu, df - df^F)_U = 0$$

and $\|df\|_U^2 = (df, df^F)_U$. Hence $0 \leq \|df - df^F\|_U^2 = \|df^F\|_U^2 - \|df\|_U^2$ and $\|df\|_U \leq \|df^F\|_U$. From the uniqueness of f^F we have $f = f^F$. Choose $z'_\nu \neq z_\nu$ in $B(z_\nu, \kappa_\nu d_\nu)$, $\nu \geq 1$. Let (R_1, π_1) be a two-sheeted unlimited covering surface of U with projection $\{z_\nu\} \cup \{z'_\nu\}$ of branch points. Since $\pi_1^{-1}(B(z_1, \kappa_1 d_1))$ contains just two branch points, $\partial \pi_1^{-1}(B(z_1, \kappa_1 d_1))$ consists of two analytic Jordan curves C_0, C_1. Choose $\varphi \in C_0^\infty(R_1)$ such that $\varphi = 0$ on C_0 and $\varphi = 1$ on C_1. Then $\varphi^{C_0 \cup C_1}$ belongs to $\mathscr{N}(R_1)$; recall that $\mathscr{N}(R_1)$ is, as defined before Lemma 1, the class of continuous functions f in R_1 for which there exists a regular subregion $\Omega \supset K_0$ such that $f(p) = f^{\partial V}(p)$ in each component V of $R_1 \backslash \overline{\Omega}$. Let j_1 be a sheet interchange of (R_1, π_1). From the relation $\varphi^{C_0 \cup C_1} \circ j_1 = (\varphi \circ j_1)^{C_0 \cup C_1}$ it follows that $\varphi^{C_0 \cup C_1} \circ j_1 + \varphi^{C_0 \cup C_1} \equiv 1$ on R_1. A line segment l_ν with endpoints z_ν and z'_ν lies in $B(z_\nu, \kappa_\nu d_\nu)$ for $\nu \geq 2$. We denote by G_0 one of the two components of $(R_1 \backslash \pi_1^{-1}\overline{(B(z_1, \kappa_1 d_1))}) \backslash \bigcup_{\nu \geq 2} \pi_1^{-1}(l_\nu)$ which has a border C_0. The projection mapping π_1 maps G_0 conformally onto $(U \backslash \overline{B(z_1, \kappa_1 d_1)}) \backslash \bigcup_{\nu \geq 2} l_\nu$ and $\varphi_0 = \varphi^{C_0 \cup C_1} \circ \pi_1^{-1}$ is a harmonic function with finite Dirichlet integral on it. Put $\varphi_0 = 0$ on $\overline{B(z_1, \kappa_1 d_1)}$. Then φ_0 is a continuous Dirichlet function on $U \backslash \bigcup_{\nu \geq 2} l_\nu$ and satisfies $\varphi_0 = \varphi_0^{F'}$, where $F' = F \backslash \bigcup_{\nu \geq 2} l_\nu$ is a closed set in $U \backslash \bigcup_{\nu \geq 2} l_\nu$. Since

$\varphi_0 \leq 1 \leq f$ holds on F', it follows that $\varphi_0^{F'} \leq f^{F'} = f^F$ in $U \backslash \bigcup_{\nu \geq 2} l_\nu$. Hence $\varphi_0 \leq 1/4$ in $U_1 \backslash \bigcup_{\nu \geq 2} \overline{B(z_\nu, d_\nu/4)}$. For any $e^{i\theta} \in \partial U$ there exists a sequence $\{\zeta_k\} \subset U_1 \backslash \bigcup_{\nu \geq 2} \overline{B(z_\nu, d_\nu/4)}$ converging to $e^{i\theta}$. The sequence $\{\pi_1^{-1}(\zeta_k)\} \subset R_1$ contains a subsequence $\{\pi_1^{-1}(\zeta_{k_l})\}$ which converges to some point P_0 in $\pi_1^{-1}(e^{i\theta})$. Since $\varphi^{C_0 \cup C_1}$ is a continuous function on the Kuramochi compactification of R_1, we have

$$\varphi^{C_0 \cup C_1}(P_0) = \lim_{l \to \infty} \varphi^{C_0 \cup C_1}(\pi_1^{-1}(\zeta_{k_l})) = \lim_{l \to \infty} \varphi_0(\pi_1^{-1}(\zeta_{k_l})) \leq \frac{1}{4}.$$

On the other hand $\{j_1 \circ \pi_1^{-1}(\zeta_{k_l})\}$ converges to $j_1(P_0)$ in $\pi_1^{-1}(e^{i\theta})$ and

$$\varphi^{C_0 \cup C_1}(j_1(P_0)) = \lim_{l \to \infty} \varphi^{C_0 \cup C_1}(j_1 \circ \pi_1^{-1}(\zeta_{k_l})) = 1 - \lim_{l \to \infty} \varphi_0(\pi_1^{-1}(\zeta_{k_l})) \geq \frac{3}{4}.$$

Therefore $\pi_1^{-1}(e^{i\theta})$ contains two different points P_0 and $j_1(P_0)$. By Theorem 4 we have a conclusion.

In order to prove Proposition 8 we prepare three lemmas.

Let γ be a closed Jordan curve on $\mathbf{C}$ and z_1 and z_2 be two distinct points on γ. Then $\gamma \backslash \{z_1, z_2\}$ consists of two open Jordan arcs γ_1 and γ_2. We say that γ is of bounded turning if there exists a constant $C > 0$ such that

$$\min(diam\, \gamma_1, diam\, \gamma_2) \leq C|z_1 - z_2|$$

holds for any pair (z_1, z_2) of γ, where $diam\, \gamma_j$ is the diameter of γ_j. In [LV, §8] the following has been shown: if γ is of bounded turning and ϕ_γ is a Riemann mapping of the Jordan domain D_γ bounded by γ onto the unit disc U then there is a quasiconformal mapping $\tilde{\phi}_\gamma$ of $\mathbf{C}$ onto itself such that $\tilde{\phi}_\gamma = \phi_\gamma$ in D_γ.

Lemma 5. *Let R and R' be Riemann surfaces. Suppose that there is a quasiconformal mapping f of R onto R'. If R has a planar end, a border, or a disc with crowded ideal boundary, then R' has a planar end, a border, or a disc with crowded ideal boundary, respectively.*

Proof. It is clear that if R has a planar end or a border then R' does.

If R has a disc D with crowded ideal boundary, then $D' = f(D)$ is a simply connected subregion of R'. Let ϕ and ϕ' be Riemann mappings of D and D' onto the unit disc $U_z = \{|z| < 1\}$ and $U_w = \{|w| < 1\}$, respectively. Then $F = \phi' \circ f \circ \phi^{-1}$ is a quasiconformal mapping of U_z onto U_w. By [LV, §8] F can be extended to a quasiconformal mapping of $\mathbf{C}$ onto itself. By definition $I = \partial U_z \backslash \phi(\partial D)$ does not belong to the class N_D. Since whether a compact set in $\mathbf{C}$ belongs to the class N_D or not is invariant under quasiconformal mappings of $\mathbf{C}$, $I' = F(I)$ does not belong to the class N_D. Note that

$$\begin{aligned} I' = F(I) &= F(\partial U_z \backslash \phi(\partial D)) \\ &= F(\partial U_z) \backslash F(\phi(\partial D)) = \partial U_w \backslash \phi'(f(\partial D)) \\ &= \partial U_w \backslash \phi'(\partial f(D)) = \partial U_w \backslash \phi'(\partial D'). \end{aligned}$$

Since $\phi'(\partial D')$ is a relatively open set in ∂U_w, it consists of at most countably infinite open intervals $\{J_n\}$, where $J_n = \{e^{i\theta}; a_n < \theta < b_n\}$ for some $0 \le a_n < b_n < 2\pi$. We may assume $b_n - a_n < \pi$. Let J_n' be a line segment with endpoints e^{ia_n} and e^{ib_n}. Then J_n' lies in U_w except two endpoints. Set $\gamma = I' \cup (\bigcup_n J_n')$. It can be shown that for any $0 \le \theta < 2\pi$ there exists uniquely a point $\gamma(\theta) \in \gamma$ such that $\arg \gamma(\theta) = \theta$. By this parametrization γ becomes a Jordan curve. Denote by U_γ a simply connected region bounded by γ. We shall show that $D_\gamma = \phi'^{-1}(U_\gamma)$ is a disc with crowded ideal boundary on R'. The relative boundary $\partial D_\gamma = \phi'^{-1}(\bigcup_n J_n')$ consists of analytic arcs. Let ϕ_γ be a Riemann mapping of U_γ. Then $\phi_\gamma \circ \phi'$ maps D_γ conformally onto the unit disc U. Let $I_\gamma = \partial U \backslash (\phi_\gamma \circ \phi')(\partial D_\gamma) = \partial U \backslash \phi_\gamma(\bigcup_n J_n')$. Since γ is of bounded turning, there is a quasiconformal mapping $\tilde{\phi}_\gamma$ of $\mathbf{C}$ onto itself such that $\tilde{\phi}_\gamma = \phi_\gamma$ in D_γ. Note that

$$I_\gamma = \tilde{\phi}_\gamma(\gamma) \backslash \tilde{\phi}_\gamma\left(\bigcup_n J_n'\right) = \tilde{\phi}_\gamma(I').$$

Since an N_D-set is preserved by quasiconformal mappings of $\mathbf{C}$, I_γ does not belong to the class N_D. Therefore D_γ is a disc with crowded ideal boundary.

Lemma 6. *Let A be the annulus $\{e^{-\rho} < |z| < e^{\rho}\}$, $\rho > 0$. For any points z_0, $z_1 \in A$, $|z_0| = e^{-\rho/3}$, $|z_1| = e^{\rho/3}$, there is a K-quasiconformal mapping of A onto A with $f(z_0) = -1$ and $f(z_1) = 1$ which is an identity map in $\{e^{-\rho} < |z| < e^{-5\rho/6}\} \cup \{e^{5\rho/6} < |z| < e^{\rho}\}$, where constant K depends only on ρ.*

Proof. Map A by the conformal mapping $w = \log z$ onto $G = \{w = u + iv; -\rho < u < \rho, -\pi \le v < \pi\}$. Let $w_0 = -\rho/3 + iv_0 = \log z_0$ and $w_1 = \rho/3 + iv_1 = \log z_1$. We construct a quasiconformal mapping of G. Let $\psi_0(t)$ be a C^∞ function on $\mathbf{R}$ such that $\psi_0(t) = 0$ if $|t| \ge 1$, $0 < \psi_0(t) \le 1$ if $|t| < 1$ and $\psi_0(0) = 1$. Define

$$\Phi_0(u,v) = \begin{cases} (u,v) & \text{if } |u| \ge \dfrac{2\rho}{3} \\ \left(u, v + (\pi - |v_0|)\psi_0\left(\dfrac{3}{\rho}\left(u + \dfrac{\rho}{3}\right)\right)\operatorname{sgn}(v_0)\right) \bmod 2\pi & \text{if } \dfrac{-2\rho}{3} < u < 0 \\ \left(u, v - v_1\psi_0\left(\dfrac{3}{\rho}\left(u - \dfrac{\rho}{3}\right)\right)\right) \bmod 2\pi & \text{if } 0 \le u < \dfrac{2\rho}{3}, \end{cases}$$

where $\operatorname{sgn}(v_0)$ is the signature of v_0. Then Φ_0 is quasiconformal in G, and it satisfies $\Phi_0(-\rho/3, v_0) = (-\rho/3, -\pi)$ and $\Phi_0(\rho/3, v_1) = (\rho/3, 0)$. By elementary calculation maximal dilatation of Φ_0 is less than $\dfrac{1+\mu_0(\rho)}{1-\mu_0(\rho)}$, where $\mu_0 = \dfrac{\nu_0(\rho)}{\sqrt{1+\nu_0(\rho)^2}}$ with $\nu_0(\rho) = \dfrac{3\pi}{2\rho}\max_{t\in\mathbf{R}}|\psi_0'(t)|$.

Next we construct a quasiconformal mapping Φ_1 of G with $\Phi_1(-\rho/3, -\pi) = (0, -\pi)$ and $\Phi_1(\rho/3, 0) = (0, 0)$. Let

$$\psi_1(x) = \begin{cases} \left(\frac{3}{2}\right)^{-4/3} x^{4/3} & \text{if } 0 \le x \le \frac{3}{2} \\ 2 - \left(\frac{3}{2}\right)^{-4/3} (3-x)^{4/3} & \text{if } \frac{3}{2} \le x \le 3 \end{cases}$$

and

$$h(t) = \begin{cases} 0 & \text{if } |t| \ge \frac{5}{6}\rho \\ -\frac{\rho}{6}\psi_1\left(\frac{6}{\rho}t + 5\right) & \text{if } -\frac{5}{6}\rho \le t \le -\frac{\rho}{3} \\ -\frac{\rho}{3} & \text{if } |t| \le \frac{\rho}{3} \\ \frac{\rho}{6}\psi_1\left(\frac{6}{\rho}t - 2\right) - \frac{\rho}{3} & \text{if } \frac{\rho}{3} \le t \le \frac{5}{6}\rho. \end{cases}$$

Then $h(t)$ is a C^1 function on $\mathbf{R}$ and satisfies

$$|h'(t)| \le \frac{8}{9} < 1.$$

Put $\Phi_1(u, v) = (u + (\cos v)h(u), v)$. Then Φ_1 is a quasiconformal mapping of G onto itself with $\Phi_1(-\rho/3, -\pi) = (0, -\pi)$ and $\Phi_1(\rho/3, 0) = (0, 0)$. The maximal dilatation of Φ_1 is less than $\frac{1 + \mu_1(\rho)}{1 - \mu_1(\rho)}$, where $\mu_1(\rho) = \sqrt{\frac{(8/9)^2 + \rho^2}{1 + \rho^2}} < 1$. Then the quasiconformal mapping $f(z) = \exp \circ \Phi_1 \circ \Phi_0 \circ \log z$ is what we want.

Lemma 7. *Let Γ_ν be the family of locally rectifiable curves γ in $U \backslash \overline{B(z_\nu, \kappa_\nu d_\nu)}$ which start from some points of $\partial B(z_\nu, \kappa_\nu d_\nu)$ and tend toward ∂U and Γ_ν^* be the family of locally rectifiable curves γ^* in $R \backslash \pi^{-1}\overline{(B(z_\nu, \kappa_\nu d_\nu))}$ which issue from some points of $\partial \pi^{-1}(B(z_\nu, \kappa_\nu d_\nu))$ and tend to the ideal boundary of R. Then $\lambda(\Gamma_\nu) = 2\lambda(\Gamma_\nu^*)$ holds.*

Proof. If $\rho = \rho(z)|dz|$ is admissible for Γ_ν, then $\int_{\pi(\gamma^*)} \rho |dz| \ge 1$ because $\pi(\gamma^*) \in \Gamma_\nu$. Hence the pull-back $\pi^\#(\rho)$ of ρ by π is admissible for Γ_ν^*. Note that $\iint_R (\pi^\#(\rho))^2 dxdy = 2\iint_U \rho^2 dxdy$. Therefore we have $\lambda(\Gamma_\nu) \le 2\lambda(\Gamma_\nu^*)$.

If ρ is admissible for Γ_ν^*, then the pull-back $j^\#(\rho)$ of ρ by the sheet interchange j is admissible for Γ_ν^*. Hence $\tilde{\rho} = 2^{-1}(\rho + j^\#(\rho))$ is also admissible for Γ_ν^*. Since $\tilde{\rho}$ satisfies $\tilde{\rho} = j^\#(\tilde{\rho})$, there exists a linear density ρ' on U which satisfies $\tilde{\rho} = \pi^\#(\rho')$. For any $\gamma \in \Gamma_\nu$ there is a lift γ' of γ, which belongs to Γ_ν^*. Then ρ' is admissible for Γ_ν. Note that

$$\iint_U \rho'^2 dxdy = \frac{1}{2}\iint_R \tilde{\rho}^2 dxdy$$

and

$$\iint_R \rho^2 dxdy - \iint_R \tilde{\rho}^2 dxdy$$

$$= \frac{1}{2}\iint_R (\rho^2 + j^\#(\rho)^2)dxdy - \frac{1}{4}\iint_R (\rho^2 + j^\#(\rho)^2 + 2\rho j^\#(\rho))dxdy$$

$$= \frac{1}{4}\iint_R (\rho - j^\#(\rho))^2 dxdy \geq 0.$$

Hence $2\iint_U \rho'^2 dxdy \leq \iint_R \rho^2 dxdy$. It follows that $\lambda(\Gamma_\nu) \geq 2\lambda(\Gamma_\nu^*)$. This completes the proof.

We call the extremal length $\lambda(\Gamma_\nu)$ the extremal distance between $\partial B(z_\nu, \kappa_\nu d_\nu)$ and ∂U.

Proof of Proposition 8. We choose $0 < \kappa_\nu < 1/6$ such that the extremal distance between $\partial B(z_\nu, \kappa_\nu d_\nu)$ and ∂U is greater than 1. Take $z'_\nu \in B(z_\nu, \kappa_\nu d_\nu)\setminus \{z_\nu\}$. Let (R_2, π_2) be the two-sheeted unlimited covering surface of the unit disc with projection $\{z_\nu\} \cup \{z'_\nu\}$ of branch points.

Let E_ν be the ellipse with foci z_ν and z'_ν such that the length of major axis is $5|z_\nu - z'_\nu|$. By G_ν we denote the Jordan domain bounded by E_ν. Then $\overline{G_\nu} \subset B(z_\nu, d_\nu/2)$. By the mapping $z = \psi_\nu(\zeta) = \dfrac{z'_\nu - z_\nu}{4}\left(\zeta + \dfrac{1}{\zeta}\right) + \dfrac{z'_\nu + z_\nu}{2}$ the annulus $\{1 < |\zeta| < 5 + \sqrt{24}\}$ is conformally mapped on $G_\nu \setminus l_\nu$ with $\psi_\nu(-1) = z_\nu$ and $\psi_\nu(1) = z'_\nu$, where l_ν is the line segment with endpoints z_ν and z'_ν. Let $\rho = \log(5 + \sqrt{24})$. By the reflection principle $\tilde{\psi}_\nu(\zeta) = (\pi_2^{-1} \circ \psi_\nu)(\zeta)$ is extended over $\{e^{-\rho} < |\zeta| \leq 1\}$ such that the extended $\tilde{\psi}_\nu$ maps $A = \{e^{-\rho} < |\zeta| < e^\rho\}$ conformally onto $\pi_2^{-1}(G_\nu)$ with $\tilde{\psi}_\nu(-1) = \pi_2^{-1}(z_\nu)$ and $\tilde{\psi}_\nu(1) = \pi_2^{-1}(z'_\nu)$.

Suppose that R_2 is not maximal. Then R_2 has a disc D_2 with crowded ideal boundary. Since D_2 is simply connected, $\pi_2^{-1}(G_\nu) \cap D_2$ contains neither C_ν^- nor C_ν^+, where $C_\nu^\pm = \tilde{\psi}_\nu(\{|\zeta| = e^{\pm\rho/3}\})$ are analytic closed Jordan curves. Then there exist $\zeta_\nu^\pm$, $|\zeta_\nu^\pm| = e^{\pm\rho/3}$, such that $\tilde{\psi}_\nu(\zeta_\nu^\pm) \notin D_2$. By Lemma 6 there is a K-quasiconformal mapping f_ν of A onto itself such that $f_\nu(\zeta_\nu^\pm) = \pm 1$, f_ν is an identity map in $\{e^{-\rho} < |z| < e^{-(5\rho)/6}\} \cup \{e^{(5\rho)/6} < |z| < e^\rho\}$, and that the constant K depends only on ρ. Define a mapping F on R_2 by

$$F(p) = \begin{cases} \tilde{\psi}_\nu \circ f_\nu \circ \tilde{\psi}_\nu^{-1}(p) & \text{if } p \in \pi_2^{-1}(G_\nu) \\ p & \text{if } p \in R_2 \setminus \bigcup_\nu \pi_2^{-1}(G_\nu). \end{cases}$$

Then F is a K-quasiconformal mapping of R_2 onto itself such that all the branch points $\{\pi_2^{-1}(z_\nu)\} \cup \{\pi_2^{-1}(z'_\nu)\}$ lie in $R_2 \setminus F(D_2)$. By the same argument as in the proof of Lemma 5 there is a disc D'_2 with crowded ideal boundary in $F(D_2)$, which does not contain any branch points of (R_2, π_2). By a Riemann mapping ϕ_2, D'_2 is conformally mapped onto U. Then ϕ_2 can be continuously extended over $\partial D'_2$ and $I_2 = \partial U \setminus \phi_2(\partial D'_2)$ does not belong to the class N_D. By some Möbius transformation T the unit disc U is mapped onto the upper half plane H such that $I_2^* = T(I_2)$ is contained in the closed interval $[-1, 1]$ of the real axis. Then I_2^* does not belong to the class N_D. There exists uniquely the vertical slit mapping $P_1(z)$ of $\mathbf{C}\backslash I_2^*$ such that P_1 minimizes $\Re a[S]$ in $\mathscr{V}$, where $\mathscr{V}$ is the family of univalent functions $S(z)$ on $\mathbf{C}\backslash I_2^*$ with the following expansion around ∞:

$$S(z) = z + \frac{a[S]}{z} + \cdots.$$

We know that $P_1(z)$ satisfies 1) $P_1(\bar{z}) = \overline{P_1(z)}$, 2) each connected component of $E^* = \mathbf{C}\backslash P_1(\mathbf{C}\backslash I_2^*)$ is a point on the real axis $\mathbf{R}$ or a vertical slit symmetric with respect to $\mathbf{R}$, 3) $P_1(H) = H\backslash E^*$, and 4) $P_1(\mathbf{R}\backslash I_2^*) \subset \mathbf{R}$. Since I_2^* does not belong to the class N_D and E^* is compact, there is a vertical slit $L_0 = \{x_0 + iy; |y| \leq y_0\}$ with maximal length among vertical slits in E^*. The open ball $B_0 = \{|z - z_0| < y_0\}$, $z_0 = x_0 + iy_0$, is contained in H. Let $B_n = \{|z - z_0| < 2^{-n} y_0\}$.

We know that $\pi_2^{-1}(l_\nu) \cap D_2'$ consists of a finite number of analytic arcs, $\{\gamma_k^{(\nu)}\}$, each of which has two end points on $\partial D_2'$. Set $\tilde{\phi}_2 = P_1 \circ T \circ \phi_2$. Then $\tilde{\phi}_2(\gamma_k^{(\nu)})$ is an analytic arc in $H\backslash E^*$ with two end points in $\mathbf{R}\backslash E^*$.

We show that if n is greater than $(4\pi)/\log 2$ then $B_n \cap \tilde{\phi}_2(D_2' \cap \bigcup_\nu \pi_2^{-1}(l_\nu)) = \emptyset$. See Chapter 1-D of [Ah] for the properties of extremal length. Suppose that there is an analytic arc $\gamma_k^{(\nu)}$ such that $B_n \cap \tilde{\phi}_2(\gamma_k^{(\nu)}) \neq \emptyset$. Denote by Λ_n the family of circles C_r centered at z_0 with radius r, $2^{-n} y_0 < r < y_0$, and by Λ_ν^* the family of rectifiable curves in $R_2\backslash \pi_2^{-1}(l_\nu)$ each of which issues from some point on $\pi_2^{-1}(l_\nu)$ and tends to the ideal boundary of R_2. By the property of extremal length and Lemma 7 the extremal length $\lambda(\Lambda_\nu^*)$ is greater than the half of the extremal distance between $\partial B(z_\nu, \kappa_\nu d_\nu)$ and ∂U, which is greater than one. Hence we have $\lambda(\Lambda_\nu^*) \geq 1/2$.

Every $C_r \in \Lambda_n$ contains a subarc C_r' which connects $\tilde{\phi}_2(\gamma_k^{(\nu)})$ and a vertical slit in E^*. Since $\tilde{\phi}_2^{-1}(C_r') \in \Lambda_\nu^*$, we have $\lambda(\Lambda_n) \geq \lambda(\Lambda_\nu^*) \geq 1/2$. On the other hand we know that $\lambda(\Lambda_n) = \dfrac{2\pi}{n \log 2} < \dfrac{1}{2}$. This is a contradiction. Hence $B_n \cap \tilde{\phi}_2(D_2' \cap \bigcup_\nu \pi^{-1}(l_\nu)) = \emptyset$ if $n > \dfrac{4\pi}{\log 2}$. It is clear that $\tilde{D}_2 = \tilde{\phi}_2^{-1}(B_n\backslash E^*)$ is a simply connected subregion of R_2 whose relative boundary consists of analytic arcs. Moreover $\tilde{D}_2$ is contained in $R_2\backslash \bigcup_\nu \pi^{-1}(l_\nu)$. It is easily seen that $R_2\backslash \bigcup_\nu \pi^{-1}(l_\nu)$ consists of two components $R_2^{(0)}$ and $R_2^{(1)}$ and π_2 maps each one of $R_2^{(0)}$ and $R_2^{(1)}$ conformally onto $U\backslash \bigcup_\nu l_\nu$. Since $\tilde{D}_2$ is connected, $\tilde{D}_2$ is contained in either $R_2^{(0)}$ or $R_2^{(1)}$. Therefore $\pi_2|_{\tilde{D}_2}$ is a conformal mapping of $\tilde{D}_2$ into $U\backslash \bigcup_\nu l_\nu$. Since $\pi_2(\tilde{D}_2)$ is also simply connected in $U\backslash\{z_\nu\}$, π^{-1} defines a conformal mapping of $\pi_2(\tilde{D}_2)$ into R. Thus we know that $D = \pi^{-1}(\pi_2(\tilde{D}_2))$ is a simply connected subregion of R with analytic relative boundary. Moreover by the mapping $\tilde{\Phi} = \tilde{\phi}_2 \circ \pi_2^{-1} \circ \pi$, D is mapped onto $B_n\backslash E^*$ with $\tilde{\Phi}(\partial D) = \partial B_n\backslash E^*$. It follows that D is a disc with crowded ideal boundary. This contradicts the maximality of R. Hence R_2 is maximal. This completes the proof.

Problem. Is the assertion of Proposition 8 true without the condition $z'_\nu \in B(z_\nu, \kappa_\nu d_\nu)$?

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