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Meyer's inequality of the type $L(\log L)^{\alpha}$ on abstract Wiener spaces

By

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1. Introduction

Let (W, H, μ) be an abstract Wiener space, that is,

(i) W is a real separable Banach space,

(ii) H is a real separable Hilbert space which is continuously and densely imbedded in W,

(iii) μ is a Borel probability measure on W such that

$$\int_W \exp\left(\sqrt{-1}(f, w)\right) d\mu(w) = \exp\left(-\frac{1}{2}\|f\|_{H^*}^2\right)$$

for any $f \in W^*$. Here we identify H^* with H by the Riesz theorem so that $W^* \hookrightarrow H^* = H$. As usual, we denote by D H-derivative and by L be the Ornstein-Uhlenbeck operator, cf. [13].

As for the continuity of operators D and L with respect to L^p-norms on the Wiener space, the following inequality due to Meyer [8] is well-known: for $1 < p < \infty$,

$$\|D\phi\|_p \lesssim \|(I - L)^{1/2}\phi\|_p \lesssim \|D\phi\|_p + \|\phi\|_p. \tag{1.1}$$

Here $A(\phi) \lesssim B(\phi)$ means that there exists a constant $k > 0$ independent of ϕ such that $A(\phi) \le kB(\phi)$ holds for every ϕ. (1.1) implies in particular that the operator $D(I - L)^{-1/2}$ is continuous from $L^p(\mu)$ to $L^p(\mu : H)$ (L^p-space with values in $H^* = H$) if $1 < p < \infty$. However this continuity fails when $p = 1$ and the main purpose of this article is to study the continuity of the operator $D(I - L)^{-1/2}$ with respect to the $L(\log L)^{\alpha}$-topology.

Meyer's proof of (1.1) relies on the Littlewood-Paley inequality. Recently, a simplified proof was given by Pisier [9], or Feyel [3], by using the L^p-continuity of the Hilbert transform on the circle. L^1-continuity of the Hilbert transform no longer holds. However, it is continuous as an operator from L log L to L^1 (cf. [6]). Taking this fact into account, a natural question arises: Can we show that

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the operator $D(I-L)^{-1/2}$ is continuous from $L \log L$ to L^1, more generally, form $L(\log L)^{\alpha+1}(\mu)$ to $L(\log L)^{\alpha}(\mu; H^*)$ for $\alpha \geq 0$? We could not answer this question; however, we could obtain the following partial answer which is our main result in this article: $D(I-L)^{-1/2}$ is continuous from $L(\log L)^{\alpha+3/2}(\mu)$ to $L(\log L)^{\alpha}(\mu; H^*)$ for any $\alpha \geq 0$.

Our method in this article is as follows. We first show the Doob inequality for a right-continuous submartingale $X = (X_t)_{t\geq 0}$ in the frame of $L(\log L)^{\alpha}$-spaces as Theorem 3.1. This inequality implies that if $X_\infty \in L(\log L)^{\alpha+1}$ then $X_\infty^* := \sup_{t\geq 0}|X_t| \in L(\log L)^{\alpha}$. In Section 4, we study the Burkholder inequality for vector valued martingales. In Section 5, we apply the inequality of Doob and Burkholder thus obtained to prove the continuity of the Hilbert transform on the circle T from $L(\log L)^{\alpha+1}(T)$ to $L(\log L)^{\alpha}(T)$ (Theorem 5.1). In Section 8, we show that the projection J_1 of $L^2(\mu)$ onto the first order Wiener chaos is continuous from $L(\log L)^{1/2}(\mu)$ to $L^1(\mu)$ (Theorem 8.1). This result is obtained by applying the hypercontractivity of Ornstein-Uhlenbeck semigroup and an extrapolation theorem (Lemma 8.2). By the continuity of the Hilbert transform on the circle and the projection operator J_1 thus established, we prove in Section 9 our main result that the operator $D(I-L)^{-1/2}$ is continuous from $L(\log L)^{\alpha+3/2}(\mu)$ to $L(\log L)^{\alpha}(\mu; H^*)$ for any $\alpha \geq 0$. This theorem holds not only for real-valued functions but also for vector-valued functions. By this fact and also the fact that $\alpha \geq 0$ can be arbitrary, we can obtain an estimate of $E[\|D^n f\|_{H^{*\otimes n}}]$ as Corollary 9.3.

This article also includes a probabilistic proof of Stein's theorem and its generalization (Theorem 6.4) which asserts the equivalence of the following two statements for a nonnegative L^1-functions on the circle T:

(i) f is in $L(\log L)^{\alpha+1}(T)$.

(ii) Its Hilbert transform Hf is in $L(\log L)^{\alpha}(T)$.

Here we used the 'reversed' Doob inequality (Theorem 3.6) which claims that for a nonnegative continuous martingale $X = (X_t)$, $X_\infty^* \in L(\log L)^{\alpha}$ and $X_0 \in L(\log \mathrm{L})^{\alpha+1}$ (particularly X_0 is a constant), imply $X_\infty \in L(\log L)^{\alpha+1}$. Also we need a generalization of a characterization of $\mathcal{M}^1$-martingales (Theorem 6.1) which is a natural generalization of Janson's result in [5].

2. Orlicz space $L(\log L)^{\alpha}$

In this section we introduce definitions and some basic properties of the Orlicz spaces following Adams [1]. For $\alpha > 0$ we define functions on $[0, \infty)$ as

$$\phi_\alpha(x) = \log^{\alpha}(1+x),$$

$$A_\alpha(x) = \int_0^x \phi_\alpha(y)dy,$$

$$B_\alpha(x) = \int_0^x \phi_\alpha^{-1}(y)dy,$$

and for $\alpha = 0$, we define $\phi_0(x) = 1$ and $A_0(x) = x$. however, we do not define $B_0(x)$. From the above definition it is easy to see that $A_\alpha(x)$ is convex. Moreover it is easy to see that $A_\alpha(x)$ satisfies Δ_2-condition, that is, there exist a constant C_2 (which may depend on α) satisfying

$$A_\alpha(2x) \leq C_2 A_\alpha(x)$$

for any $x > 0$. Combining the two properties above, we obtain

$$A_\alpha(x + y) \leq \frac{C_2}{2}(A_\alpha(x) + A_\alpha(y))$$

for any $x, y > 0$. From the Δ_2-condition we obtain that, for any $r > 0$, there exists a constant C_r which satisfies,

$$A_\alpha(rx) \leq C_r A_\alpha(x) \tag{2.1}$$

for any $x > 0$.

It is also verified that there is a constant $c = c_\alpha$ which satisfies

$$\frac{1}{c_\alpha} x\phi_\alpha(x) \leq A_\alpha(x) \leq x\phi_\alpha(x). \tag{2.2}$$

Now we define the Orlicz Spaces $L(\log L)^\alpha$ and their norms.

Definition 2.1. Let (Ω, P) be a probability space and K be a separable Hilbert space. We define the Orlicz norms of a K-valued measurable function f by

$$\|f\|_{L(\log L)^\alpha(P;K)} = \inf\left\{r > 0; E\left(A_\alpha\left(\frac{\|f\|_K}{r}\right)\right) \leq 1\right\} \tag{2.3}$$

and the Orlicz spaces by

$$L(\log L)^\alpha(K) = \{f; f \text{ is measurable and } \|f\|_{L(\log L)^\alpha(P;K)} < \infty\}. \tag{2.4}$$

When $K = R$ we simple write space as $L(\log L)^\alpha$.

We see by (2.2) that $L(\log L)^\alpha(K)$ is equal to the space of measurable functions satisfying $E[\|f\|_K \log^\alpha(1 + \|f\|_K)] < \infty$. Note that for $\alpha = 0$, $L(\log L)^0(K) = L^1(K)$. It is known that $L(\log L)^\alpha(K)$ becomes a Banach space with this norm.

The following two relations between the norm and the integral of the type $E[A_\alpha(\|f\|_K)]$ are basic;

$$\|f\|_{L(\log L)^\alpha(K)} \leq 1 + E[A_\alpha(\|f\|_K)] \tag{2.5}$$

and if $\|f\|_{L(\log L)^\alpha(K)}$ is finite, then

$$E[A_\alpha(\|f\|_K)] \leq C_{\|f\|_{L(\log L)^\alpha(K)}} \tag{2.6}$$

where $C_{\|f\|_{L(\log L)^\alpha(K)}}$ is the constant in (2.1).

From (2.5) and (2.6), we can see that a linear oprator T is continuous from $L(\log L)^{\alpha}(K_1)$ to $L(\log L)^{\beta}(K_2)$ if there exists some constant C such that

$$E[A_{\beta}(\|Tf\|_{K_2})] \leq C(1 + E[A_{\alpha}(\|f\|_{K_1})]) \tag{2.7}$$

for any f in a dense subspace of $L(\log L)^{\alpha}(K)$.

Next we show several properties of $L(\log L)^{\alpha}$.

Proposition 2.2. *For t, $s > 0$, it holds that*

$$ts \leq A_{\alpha}(t) + B_{\alpha}(s),$$

in particular,

$$t\phi_{\alpha}(t) = A_{\alpha}(t) + B_{\alpha}(\phi_{\alpha}(t)).$$

And if we set

$$\|f\|_{B_{\alpha}} = \inf\left\{r > 0; E\left[B_{\alpha}\left(\frac{|f|}{r}\right)\right] \leq 1\right\},$$

then the following Hölder type inequality holds:

$$E[fg] \leq 2\|f\|_{L(\log L)^{\alpha}}\|g\|_{B_{\alpha}}.$$

Proof. See Section 8 of Adams [1] for a proof. The first inequality is known as Young's inequality.

Let f_n and f be functions in $L(\log L)^{\alpha}$. It is known that $\lim_{n\to\infty}\|f_n - f\|_{L(\log L)^{\alpha}} = 0$ if and only if $\lim_{n\to\infty} A_{\alpha}(|f_n - f|) = 0$. (For a proof, see Section 8 of Adams [1]). However, it does not seem to be obvious that the convergence in norm implies $\lim_{n\to\infty} E[A_{\alpha}(|f_n|)] = E[A_{\alpha}(|f|)]$. So we will give a proof.

Proposition 2.3. *Let f_n and f be functions in $L(\log L)^{\alpha}$ and assume that $\lim_{n\to\infty}\|f_n - f\|_{L(\log L)^{\alpha}} = 0$. Then $\lim_{n\to\infty} A_{\alpha}(|f_n|) = A_{\alpha}(|f|)$ in L^1.*

Proof. It is easy to see that

$$\|f\|_{B_{\alpha}} \leq 1 + E[B_{\alpha}(|f|)].$$

Applying the above inequality and Proposition 2.2, we have

$$\begin{aligned}
E[|A_{\alpha}(|f_n|) - A_{\alpha}(|f|)|] &= E\left[\left|(|f_n| - |f|)\frac{A_{\alpha}(|f_n|) - A_{\alpha}(|f|)}{|f_n| - |f|}\right|\right] \\
&\leq E[|f_n - f|\phi_{\alpha}(|f_n| + |f|)] \\
&\leq \|f_n - f\|_{L(\log L)^{\alpha}}\|\phi_{\alpha}(|f_n| + |f|)\|_{B_{\alpha}} \\
&\leq \|f_n - f\|_{L(\log L)^{\alpha}}E[1 + B_{\alpha}(\phi_{\alpha}(|f_n| + |f|))] \\
&\leq \|f_n - f\|_{L(\log L)^{\alpha}}E[1 + \phi_{\alpha}(|f_n| + |f|)\phi_{\alpha}^{-1}(\phi_{\alpha}(|f_n| + |f|))]
\end{aligned}$$

$$\leq \|f_n - f\|_{L(\log L)^\alpha} E[1 + (|f_n| + |f|)\phi_\alpha(|f_n| + |f|)]$$

$$\leq \|f_n - f\|_{L(\log L)^\alpha} E[1 + c_\alpha A_\alpha(|f_n| + |f|)]$$

$$\leq \|f_n - f\|_{L(\log L)^\alpha} E[1 + c_\alpha C_2\{A_\alpha(|f_n|) + A_\alpha(|f|)\}].$$

Since $\|f_n\|_{L(\log L)^\alpha}$ is bounded, $A_\alpha(|f_n|)$ is also bounded by (2.6). This completes the proof.

3. The Doob inequality

Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $(\mathscr{F}_t)$. In this section we assume that all the processes are defined on $(\Omega, \mathscr{F}, P)$ and $(\mathscr{F}_t)$-adapted. We show the Doob inequality in the following form:

Theorem 3.1. *Let $(X_t)_{0 \leq t \leq \infty}$ be a non-negative right continuous submartingale and $X_t^* = \sup_{s \leq t} X_s$. Then there is a constant K_α dependeng on α but not on X such that*

$$E[A_\alpha(X_\infty^*)] \leq K_\alpha(1 + E[A_{\alpha+1}(X_\infty)]).$$

Remark 3.2. This theorem holds withuot assuming that $X_\infty = \lim_{t\to\infty} X_t$. But for the rest of this paper we will always assume that $X_\infty = \lim_{t\to\infty} X_t$ for the process X.

Before giving a proof we need the following properties of $A_\alpha(x)$:

Proposition 3.3. *For t, $s > 0$, it holds that*

$$(1+x)\log^{\alpha+1}(1+x) = A_{\alpha+1}(x) + (\alpha+1)A_\alpha(x), \tag{3.1}$$

$$B_{\alpha+1}(\log^{\alpha+1}(1+x)) \leq (\alpha+1)A_\alpha(x). \tag{3.2}$$

Proof. (3.1) is a direct consequence of an integration by parts. (3.2) is obtained by using (2.2) and (3.1) as

$$\begin{aligned} B_{\alpha+1}(\log^{\alpha+1}(1+x)) &= x\log^{\alpha+1}(1+x) - A_{\alpha+1}(x) \\ &= A_{\alpha+1}(x) + (\alpha+1)A_\alpha(x) - \log^{\alpha+1}(1+x) - A_{\alpha+1}(x) \\ &\leq (\alpha+1)A_\alpha(x). \end{aligned}$$

This competes the proof.

To prove the theorem, it is enough to show the following discrete case.

Lemma 3.4. *Let $X_1, X_2, \ldots, X_N$ be a non-negative submartigale and $X^* = \sup_{k\leq N} X_k$. Then there is a constant K_α independent of N and (X_k) such that*

$$E[A_\alpha(X^*)] \leq K_\alpha(1 + E[A_{\alpha+1}(X_N)]).$$

Proof. In the proof, we use the following notation: We write $f \lesssim g$ if there exist a constant $c > 0$ such that $f \leq cg$ (the constant c may depend only on α).

By a well-known inequality for submartingales we have

$$P[X^* \geq x+1] \leq \frac{1}{x+1} E[X_N I_{\{X^* \geq x+1\}}].$$

On the other hand,

$$\begin{aligned} E[A_\alpha((X^*-1)_+)] &= E\left[\int_0^{(X^*-1)_+} \log^\alpha(1+x)dx\right] \\ &= \int_0^\infty P[X^* \geq x+1] \log^\alpha(1+x)dx \\ &\leq \int_0^\infty \frac{1}{x+1} E[X_N I_{\{X^* \geq x+1\}}] \log^\alpha(1+x)dx \\ &= E\left[X_N \int_0^{(X^*-1)_+} \frac{\log^\alpha(1+x)}{1+x} dx\right] \\ &= E\left[X_N \frac{1}{\alpha+1} \log^{\alpha+1}(1+(X^*-1)_+)\right]. \end{aligned}$$

Choose $\varepsilon > 0$ so that $\varepsilon < \dfrac{1}{\alpha+1}$. Then the right hand side (R.H.S.) equals to

$$\begin{aligned} &\varepsilon E\left[\frac{X_N}{\varepsilon(\alpha+1)} \log^{\alpha+1}(1+(X^*-1)_+)\right] \\ &\quad \leq \varepsilon E\left[A_{\alpha+1}\left(\frac{X_N}{\varepsilon(\alpha+1)}\right)\right] + \varepsilon E[B_{\alpha+1}(\log^{\alpha+1}(1+(X^*-1)_+)] \\ &\quad \leq \varepsilon E\left[A_{\alpha+1}\left(\frac{X_N}{\varepsilon(\alpha+1)}\right)\right] + \varepsilon(\alpha+1) E[A_\alpha((X^*-1)_+)]. \end{aligned}$$

Here we used the Young inequality and the previous lemma. From this and the Δ_2-condition of A_α, we have finally

$$\begin{aligned} E[A_\alpha((X^*-1)_+)] &\leq \frac{\varepsilon}{1-\varepsilon(\alpha+1)} E\left[A_{\alpha+1}\left(\frac{X_N}{\varepsilon(\alpha+1)}\right)\right] \\ &\lesssim E[A_{\alpha+1}(X_N)]. \end{aligned}$$

Noting that $X^* - (X^*-1)_+ \leq 1$, we can complete the proof as

$$\begin{aligned} E[A_\alpha((X^*))] &\leq E[A_\alpha((X^*-1)_+ + (X^* - (X^*-1)_+))] \\ &\lesssim E[A_{\alpha+1}(X_N)] + 1. \end{aligned}$$

Theorem 3.1 implies that, for a right-continuous martingale (X_t), $X_\infty \in$

$L(\log L)^{\alpha+1}$ implies $X_\infty^* \in L(\log L)^\alpha$. For a non-negative continuous martingale, the converse also holds.

Before giving a proof, we will show a lemma (in Section 6.2 of Durrett [2]).

Lemma 3.5. *Let $(X_t)_{0 \le t \le \infty}$ be a non-negative, continuous martingale such that $X_0 = c > 0$. Then for any $\lambda > c$*

$$\lambda P(X_\infty^* > \lambda) \ge E[X_\infty I_{(X_\infty > \lambda)}].$$

Proof. Set $T = \inf\{s; X_s > \lambda\} \wedge t$. Then T is a bounded stopping time. Since X_t has continuous paths and $X_0 = c < \lambda$, we have $X_T = \lambda$ on $\{T < t\} = \{X_t^* > \lambda\}$. So we have

$$\begin{aligned}\lambda P(X_t^* > \lambda) &= E[X_T; X_t^* > \lambda] \\ &= E[X_t; X_t^* > \lambda] \\ &= E[X_t; T < t] \\ &= E[X_t; T_\lambda < t],\end{aligned}$$

where $T_\lambda = \inf\{s; X_s > \lambda\}$. Remember that a non-negative, continuous supermartingale has a limit X_∞. Taking the limit of the above inequality as $t \to \infty$, we have by Fatou's lemma

$$\begin{aligned}\lambda P(X_\infty^* > \lambda) &\ge E[X_\infty I_{(T_\lambda < \infty)}] \\ &\ge E[X_\infty I_{(X_\infty > \lambda)}].\end{aligned}$$

This completes the proof.

Theorem 3.6. *Let (X_t) be a non-negative continuous martingale. If $X_\infty^* \in L(\log L)^\alpha$ and $X_0 \in L(\log L)^{\alpha+1}$ (particularly if X_0 is a constant), then $X_\infty \in L(\log L)^{\alpha+1}$.*

Proof. We first consider the case X_0 is constant, i.e., $X_0 = c > 0$.

$$\begin{aligned}E[A_\alpha(X_\infty^*)] &= E\left[\int_0^\infty I_{(X_\infty^* > \lambda)} \log^\alpha(1+\lambda)d\lambda\right] \\ &= \int_0^c \log^\alpha(1+\lambda)d\lambda + E\left[\int_c^\infty I_{(X_\infty^* > \lambda)} \log^\alpha(1+\lambda)d\lambda\right] \\ &\ge A_\alpha(c) + \int_c^\infty \frac{1}{\lambda} E[X_\infty I_{(X_\infty > \lambda)}] \log^\alpha(1+\lambda)d\lambda] \\ &\ge A_\alpha(c) + E\left[X_\infty \int_c^\infty I_{(X_\infty > \lambda)} \frac{\log^\alpha(1+\lambda)}{1+\lambda} d\lambda\right] \\ &= A_\alpha(c) + \frac{1}{\alpha+1} E[X_\infty\{\log^{\alpha+1}(1+X_\infty) \vee \log^{\alpha+1}(1+c) - \log^{\alpha+1}(1+c)\}\end{aligned}$$

$$\geq A_\alpha(c) + \frac{1}{\alpha+1} E[X_\infty\{\log^{\alpha+1}(1+X_\infty) - \log^{\alpha+1}(1+c)\}]$$

$$\geq \frac{1}{\alpha+1} E[A_{\alpha+1}(X_\infty)] + A_\alpha(c) - \frac{c}{\alpha+1}\log^{\alpha+1}(1+c).$$

Next let us consider the case X_0 is random. Note that we may assume that the probability space is a standard measurable space. Then for the regular conditional probability $P^\omega(\cdot)$ given $\mathscr{F}_0$, we can apply the above estimate to obtain

$$E^\omega[A_\alpha(X_\infty^*)] \geq \frac{1}{\alpha+1} E^\omega[A_\alpha(X_\infty)] + A_\alpha(X_0) - \frac{X_0}{\alpha+1}\log^{\alpha+1}(1+X_0).$$

Integrating both sides we finally have

$$E[A_\alpha(X_\infty^*)] \geq \frac{1}{\alpha+1} E[A_\alpha(X_\infty)] + E\left[A_\alpha(X_0) - \frac{X_0}{\alpha+1}\log^{\alpha+1}(1+X_0)\right].$$

Note that $A_\alpha(X_0) - \frac{X_0}{\alpha+1}\log^{\alpha+1}(1+X_0)$ is integrable provided that $X_0 \in L(\log L)^{\alpha+1}$.

4. The Burkholder inequality

In this section we will prove the Burkholder inequality for vector valued martingales in a similar way as Lenglart, Lepingle and Pratteli [7] and Revuz and Yor [10].

Definition 4.1. Let η be a positive real function defined on $(0, a]$ such that $\lim_{x\to 0}\eta(x) = 0$ and let $\beta > 1$. An ordered pair (X, Y) of positive random variables is said to satisfy the "good λ inequality" $I(\eta, \beta)$ if

$$P[X \geq \beta\lambda;\, Y < \delta\lambda] \leq \eta(\delta) P[X \geq \lambda]$$

for every $\lambda > 0$ and $\delta \in (0, a]$. We will write $(X, Y) \in I(\eta, \beta)$.

In the following, F will be a moderate function, that is, an increasing, continuous function vanishing at 0 and satisfying the Δ_2-condition.

Lemma 4.2. *There is a constant c depending only on η, β, F such that, if $(X, Y) \in I(\eta, \beta)$, then*

$$E[F(X)] \leq cE[F(Y)].$$

Proof. See Lemma 4.9 in chapter IV of Revuz and Yor [10].

Lemma 4.3. *Let $A(t)$ and $B(t)$ be continuous adapted increasing process satisfying $A(0) = B(0) = 0$. If*

$$E[(A(T) - A(S))^p] \leq CE[B(T)^p I_{(S<T)}]$$

for some positive real numbers p, C and every stopping times S, T such that $S \leq T$, then $(A(\infty), B(\infty)) \in I(\eta, \beta)$ for every $\beta > 1$ and $\eta(\delta) = C(\beta-1)^{-p}\delta^p$.

Proof. See exercise 4.25 in chapter IV of Revuz and Yor [10].

Let $\vec{M}(t)$ be an $\mathbf{R}^d$-valued martingale, that is, $\vec{M}(t) = (M_1(t), M_2(t), \ldots, M_d(t))$ and each M_i is a continuous martingale. $\|\vec{M}(t)\|_{\mathbf{R}^d} = \|\vec{M}(t)\| = \sqrt{M_1(t)^2 + \cdots + M_d(t)^2}$. Set $\langle\vec{M}\rangle(t) = \langle M_1\rangle(t) + \cdots + \langle M_d\rangle(t)$ so that $\|\vec{M}(t)\|^2 - \langle\vec{M}\rangle(t)$ is a martingale.

Theorem 4.4. *Let $\vec{M}(t)$ be a continuous $\mathbf{R}^d$-valued martingale starting at 0 and F be a moderate function. Set $\|\vec{M}\|^*(t) = \sup_{s\le t}\|\vec{M}(s)\|$. Then there are constants c_F and C_F (independent of the dimension d) such that*

$$E[F(\|\vec{M}\|^*(t))] \le c_F E[F(\langle\vec{M}\rangle(t)^{1/2})],$$

$$E[F(\langle\vec{M}\rangle(t)^{1/2})] \le C_F E[F(\|\vec{M}\|^*(t))].$$

Proof. Note that the following argument does not depend on the dimension. Without loss of generality we may assume that M_i are equi-integrable martingales. Take two stopping times T, S such that $S \le T$ and fix them. We set

$$\vec{N}(t) = \{\vec{M}((S+t)\wedge T) - \vec{M}(S)\}I_{(S<T)}.$$

Then

$$\begin{aligned}
\langle\vec{N}\rangle(\infty) &= \sum_{i=1}^{d}\langle N_i\rangle(\infty) \\
&= \sum_{i=1}^{d}\{\langle M_i\rangle(T) - \langle M_i\rangle(S)\}I_{(S<T)} \\
&\le \sum_{i=1}^{d}\langle M_i\rangle(T)I_{(S<T)} \\
&= \langle\vec{M}\rangle(T)I_{(S<T)}.
\end{aligned} \tag{4.1}$$

One the other hand, by the triangular inequality, we obtain

$$\|\vec{M}\|^*(T) - \|\vec{M}\|^*(S) \le \|\vec{N}\|^*(\infty) \le 2\|\vec{M}\|^*(T)I_{(S<T)}. \tag{4.2}$$

First we show by (4.1) and (4.2) that $(\langle\vec{M}\rangle(t), \|\vec{M}\|^*(t)^2)$ satisfies the assumption of Lemma 4.3:

$$\begin{aligned}
E[\langle\vec{M}\rangle(T) - \langle\vec{M}\rangle(S)] &= E[\{\langle\vec{M}\rangle(T) - \langle\vec{M}\rangle(S)\}I_{(S<T)}] \\
&= E[\langle\vec{N}\rangle(\infty)] \\
&= \sum_{i=1}^{d} E[\langle N_i\rangle(\infty)]
\end{aligned}$$

$$= \sum_{i=1}^{d} E[N_i(\infty)^2]$$

$$\leq E[\|\vec{N}\|^*(\infty)^2]$$

$$\leq 4E[\|\vec{M}\|^*(T)^2 I_{(S<T)}].$$

Next we show again by (4.1) and (4.2) that $(\|\vec{M}\|^*(t), \langle\vec{M}\rangle(t)^{1/2})$ also satisfies the assumption of Lemma 4.3:

$$E[\{\|\vec{M}\|^*(T) - \|\vec{M}\|^*(S)\}^2] = E[\|\vec{N}\|^*(\infty)^2]$$

$$\leq \sum_{i=1}^{d} E[N_i^*(\infty)^2]$$

$$\leq 4\sum_{i=1}^{d} E[\langle N_i\rangle(\infty)]$$

$$= 4\sum_{i=1}^{d} E[\{\langle M\rangle_i(T) - \langle M_i\rangle(S)\} I_{(S<T)}]$$

$$\leq 4E[\langle\vec{M}\rangle(T) I_{(S<T)}].$$

This completes the proof.

5. Continuity of the Hilbert transform

In this section we will prove the continuity of the Hilbert transform. In the sequel, we use the following notations:

(i) $D = \{z \in \mathbf{C}; |z| < 1\}$ is the unit disk.

(ii) $T = \{z \in \mathbf{C}; |z| = 1\}$ is the circle: We identify T with a closed interval $[-\pi, \pi]$.

(iii) $d\theta$ is the normalized Lesbegue measure on T.

The Hilbert transform H is a composition $H = RQP : L^2(T) \to T^2(T)$ of three mappings P, Q and R described below.

(1) $P : L^2(T) \to H^2(D)$ is the Poisson integral, i.e., for $f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta} \in L^2(T)$, $Pf(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}_n r^{|n|} e^{in\theta}$. Another expression of Pf is;

$$Pf(r, \theta) = (P_r * f)(\theta),$$

where $P_r(\theta) = \dfrac{1 - r^2}{1 - 2r\cos\theta + r^2}$ is the Poisson kernel. Note that Pf is harmonic in D.

(2) $Q : H^2(D) \to H^2(D)$ is the map taking conjugate harmonic function vanishing at the origin $z = 0$. If $u \in H^2(D)$ is written in the form $u(r, \theta) =$

$\sum_{n=-\infty}^{\infty} \hat{u}_n r^{|n|} e^{in\theta}$, then $Qu(r,\theta) = -i\sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\hat{u}_n r^{|n|} e^{in\theta}$, where

$$\mathrm{sgn}(n) = \begin{cases} +1 & (n > 0), \\ -1 & (n < 0), \\ 0 & (n = 0). \end{cases}$$

(3) $R : H^2(D) \to L^2(T)$ is the map taking boundary value. All the function spaces we considered above are real valued.

Let K be a separable Hilbert space. We generalize Hilbert transform H to the spaces of K-valued functions as an operator acting only on function part, that is, if a K-valued function $\vec{f}$ is written in the following form

$$\vec{f} = \sum_{i=1}^{m} f_i(\theta)e_i,$$

where $e_i \in K$ and $f_i(\theta)$ are real valued functions on T, then $H\vec{f}$ is defined by

$$H\vec{f} = \sum_{i=1}^{m} (Hf_i)(\theta)e_i.$$

In this section we will prove the following result.

Theorem 5.1. *There is a constant C_α satisfying*

$$\int_{-\pi}^{\pi} A_\alpha(\|H\vec{f}(\theta)\|_K)d\theta \leq C_\alpha\left(1 + \int_{-\pi}^{\pi} A_{\alpha+1}(\|\vec{f}(\theta)\|_K)d\theta\right)$$

for any $\vec{f} \in L(\log L)^{\alpha+1}(T; K)$. From this we see that $H : L(\log L)^{\alpha+1}(T: K) \to L(\log L)^{\alpha}(T; K)$ is continuous.

Proof. It is sufficient to show the theorem for $\vec{f} = \sum_{i=1}^{m} f_i(\theta)e_i$, where $f_i \in C^\infty(T)$ and e_i are orthonormal and $m = 1, 2, \ldots$. We may assume further that $\int_{-\pi}^{\pi} \vec{f} d\theta = 0$ in K since H vanishes on constant functions.

Let $B_t = B_t^1 + iB_t^2$ be a complex Brownian motion starting at 0. For r $(0 < r < 1)$ we set a stopping time $\tau_r = \inf\{t; |B_t| = r\}$. By the rotational invariance of a Brownian motion, we have for $0 < r < 1$

$$\begin{aligned} \int_{-\pi}^{\pi} A_\alpha(\|QP\vec{f}(re^{i\theta})\|)d\theta &= E[A_\alpha(\|QP\vec{f}(B_{\tau_r})\|)] \\ &\leq E[A_\alpha(\|QP\vec{f}(B^{\tau_r})\|^*(\infty))], \end{aligned} \tag{5.1}$$

where $B^{\tau_r}(t) = B(\tau_r \wedge t)$. Using the Burkholder inequality,

$$\text{R.H.S. of (5.1)} \lesssim E[A_\alpha(\langle QP\vec{f}(B^{\tau_r})\rangle^{1/2}(\theta))]. \tag{5.2}$$

Since $P\vec{f}(B_t^{\tau_r}) + iQP\vec{f}(B_t^{\tau_r})$ is a conformal martingale, we have

$$\begin{aligned} E[A_\alpha(\langle QP\vec{f}(B^{\tau_r})\rangle^{1/2}(\infty))] &= E[A_\alpha(\langle P\vec{f}(B^{\tau r})\rangle^{1/2}(\infty)))] \\ &\lesssim E[A_\alpha(\|P\vec{f}(B^{\tau r})\|^*(\infty))] \\ &\lesssim 1 + E[A_{\alpha+1}(\|P\vec{f}(B(\tau_r))\|)] \\ &= 1 + \int_{-\pi}^{\pi} A_{\alpha+1}(\|P\vec{f}(re^{i\theta})\|)d\theta. \end{aligned}$$

Here we used the Burkholder inequality and the Doob inequality.

Because f_i's are taken from $C^\infty(T)$ we do not need to worry about troubles which may occur in limiting procedure. So we can finish the proof by applying the dominated convergence theorem to the both sides as $r \uparrow 1$.

There is another expression of H as a singular integral: H is the formal convolution with $\cot\frac{\theta}{2}$. (Because of the singularity of $\cot\frac{\theta}{2}$ at $\theta = 0$ we used the word "formal").

Proposition 5.2.

$$Hf(\beta) = \text{p.v.}\int_T \cot\frac{\theta}{2} f(\beta - \theta)d\theta, \tag{5.3}$$

where p.v. *denotes the principal value.*

Proof. It is well-known. So we omit the proof.

6. A generalization of Stein's theorem

In this section we first generalize Janson's characterization of $\mathscr{M}^1$ martingales and then prove a generalized Stein's theorem. Here we only consider the case of a probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)$ which is generated by a d-dimensional Brownian motion. We introduce the following notation.

$$\mathscr{M}(\log\mathscr{M})^\alpha = \Big\{X = (X_t); X \text{ is a } (\mathscr{F}_t)\text{-continuous martingale}$$
$$\text{such that } X_0 = \text{const. and } E\Big[A_\alpha\Big(\sup_{0\le t\le\infty}|X_t|\Big)\Big] < \infty\Big\},$$

$$\mathscr{K}(\log\mathscr{K})^\alpha = \Big\{X = (X_t); X \text{ is a } (\mathscr{F}_t)\text{-continuous matingale}$$
$$\text{such that } X_0 = \text{const. and } \sup_{0\le t\le\infty} E[A_\alpha(|X_t|)] < \infty\Big\}.$$

Next we define the martingale transform. Since our filtration is generated by a d-dimensional Brownian motion, every adapted local martingale X_t is written in the form

$$X_t = X_0 + \int_0^t H_s \cdot dB_s$$

for some locally square-integrable predictable $\mathbf{R}^d$-valued process H. We can define the transform of X by any $d \times d$ matrix Q as

$$(Q * X)_t = \int_0^t QH_s \cdot dB_s.$$

By definition it is obvious that $\mathscr{H}(\log \mathscr{H})^\alpha \supset \mathscr{M}(\log \mathscr{M})^\alpha$. The next theorem characterizes $\mathscr{M}(\log \mathscr{M})^\alpha$.

Theorem 6.1. *The following two conditions are equivalent.*

(i) $X \in \mathscr{M}(\log \mathscr{M})^\alpha$.

(ii) *There exist matrices $Q_1, \dots, Q_m$ $(m \geq 1)$ which do not have a common eigenvector in $\mathbf{R}^d$ and which satisfy $Q_i * X \in \mathscr{H}(\log \mathscr{H})^\alpha$ for any $i = 0, 1, \dots, m$, where $Q_0 * X = X$.*

Proof. First we prove that (i) implies (ii). The bracket of $Q * X$ is computed as follows

$$\begin{aligned}\langle Q * X \rangle_t &= \int_0^t |QH_s|^2 ds \\ &\leq \|Q\|_{op}^2 \int_0^t |H_s|^2 ds \\ &= \|Q\|_{op}^2 \langle X \rangle_t,\end{aligned}$$

where $\|Q\|_{op}$ is the operator norm of Q. By the Burkholder inequality we conclude that $Q * X \in \mathscr{M}(\log \mathscr{M})^\alpha \subset \mathscr{H}(\log \mathscr{H})^\alpha$.

Next we prove that (ii) implies (i). We need the following lemma in Section 6.7 of Durrett [2].

Lemma 6.2. *Assume the condition* (ii) *of Theorem* 6.1. *Then there is a $p_0 < 1$ (that depends only on the matrices Q_i's) such that if $F_t = (1 + \sum_{i=0}^m (Q_i * X)_t^2)^{1/2}$, then F_t^p is a local submartingale for all $p > p_0$.*

Remark 6.3. In Durrett [2], it is assumed that $X_0 = 0$. But the proof in [2] is also valid in our case that $X_0 = \text{const}$.

We also need the following inequalities. There is a constant K_α such that for any $x, y \geq 0$,

$$(x + y)^\alpha \leq K_\alpha (x^\alpha + y^\alpha). \tag{6.1}$$

Indeed, setting $K_\alpha = 1$ for $0 \leq \alpha \leq 1$ and $K_\alpha = 2^{\alpha-1}$ for $\alpha \geq 1$, we see by differentiation that (6.1) holds. Let $p < 1$. There is a constant M_p such that for any $x \geq 0$,

$$0 \leq \log(1+x^p) - p\log(1+x) \leq M_p. \tag{6.2}$$

Indeed, setting $M_p = (1-p)\log 2$, we see by differenciation and by (6.1) that (6.2) holds. It is also verified that

$$|X_t| \leq F_t \leq 1 + \sum_{i=0}^{m} |(Q_i * X)_t|. \tag{6.3}$$

Choose such a $p \in (p_0, 1)$ as in Lemma 6.2. Then by (2.2) (6.2) and (6.3), we have

$$\begin{aligned}
p^\alpha c_{p\alpha}^{-1/p} E\left[\sup_{0\leq t\leq\infty} A_\alpha(|X_t|)\right] &\leq p^\alpha c_{p\alpha}^{-1/p} E\left[\sup_{0\leq t\leq\infty} |X_t|\log^\alpha(1+|X_t|)\right] \\
&\leq c_{p\alpha}^{-1/p} E\left[\sup_{0\leq t\leq\infty} |X_t|\log^\alpha(1+|X_t|^p)\right] \\
&= c_{p\alpha}^{-1/p} E\left[\sup_{0\leq t\leq\infty} \{|X_t|^p\log^{p\alpha}(1+|X_t|^p)\}^{1/p}\right] \\
&\leq E\left[\sup_{0\leq t\leq\infty} \{A_{p\alpha}(|X_t|^p)\}^{1/p}\right] \\
&\leq E\left[\sup_{0\leq t\leq\infty} \{A_{p\alpha}(F_t^p)\}^{1/p}\right].
\end{aligned} \tag{6.4}$$

By Lemma 6.2, $A_{p\alpha}(F_t^p)$ is a local submartingale (in fact, it is a submartingale). For $1/p > 1$, we can apply the Doob inequality for the right hand side of (6.4).

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq\infty} \{A_{p\alpha}(F_t^p)\}^{1/p}\right] \\
&\quad\leq \frac{1/p}{1/p-1} \sup_{0\leq t\leq\infty} E[\{A_{p\alpha}(F_t^p)\}^{1/p}] \\
&\quad\leq \frac{1}{1-p} \sup_{0\leq t\leq\infty} E[\{F_t^p\log^{p\alpha}(1+F_t^p)\}^{1/p}] \\
&\quad= \frac{1}{1-p} \sup_{0\leq t\leq\infty} E[F_t\log^\alpha(1+F_t^p)] \\
&\quad\leq \frac{1}{1-p} \sup_{0\leq t\leq\infty} E[F_t\{p\log(1+F_t)+M_p\}^\alpha] \\
&\quad\leq \frac{K_\alpha}{1-p} \sup_{0\leq t\leq\infty} E[F_t\{p^\alpha\log^\alpha(1+F_t)+M_p^\alpha\}]
\end{aligned}$$

$$\le \frac{K_\alpha p^\alpha}{1-p} \sup_{0\le t\le\infty} E[F_t \log^\alpha(1+F_t)] + \frac{K_\alpha M_p^\alpha}{1-p} \sup_{0\le t\le\infty} E[F_t]$$

$$\le \frac{K_\alpha p^\alpha c_\alpha}{1-p} \sup_{0\le t\le\infty} E[A_\alpha(F_t)] + \frac{K_\alpha M_p^\alpha}{1-p} \sup_{0\le t\le\infty} E[F_t]$$

$$\le \frac{K_\alpha p^\alpha c_\alpha}{1-p} \cdot \frac{C_{m+2}}{m+2} \left\{ A_\alpha(1) + \sum_{i=0}^{m} \sup_{0\le t\le\infty} E[A_\alpha(|(Q_i * X)_t|)] \right\}$$

$$+ \frac{K_\alpha M_p^\alpha}{1-p} \left\{ 1 + \sum_{i=0}^{m} \sup_{0\le t\le\infty} E[|(Q_i * X)_t|] \right\}. \tag{6.5}$$

Here we used (2.1) (2.2) (6.1) (6.2) and (6.3). By the assumption of the theorem the right hand side of (6.5) is finite. This proves the theorem.

Theorem 6.4. *Suppose f is in $L(\log L)^\alpha(T)$ and $f \ge 0$. Under this condition the following two conditions are equivalent.*

(i) *f is in $L(\log L)^{\alpha+1}(T)$.*

(ii) *Its Hilbert transform Hf is in $L(\log L)^\alpha(T)$.*

Proof. The implication (i) $\Rightarrow$ (ii) is immediate from the continuity of Hilbert transform. We prove the implication (ii) $\Rightarrow$ (i). It is easy to see that, if we set $u(z) = Pf(r,\theta) = (P_r * f)(\theta)$, then $u(z) \ge 0$ and $u(0) = \int_T f(\theta)d\theta$. We also set $v(z) = QPf(r,\theta) = PHf(r,\theta)$. Let $B_t = B_t^1 + iB_t^2$ be a complex Brownian motion starting at 0. Define a stopping time τ_r by $\tau_r = \inf\{t; |B_t| = r\}$ and set $B_t^{\tau_r} = B_{t\wedge\tau_r}$. Then by the Ito formula we have

$$u(B_t^{\tau_r}) = \int_T f(\theta)d\theta + \int_0^{t\wedge\tau_r} \nabla u(B_s)\cdot dB_s. \tag{6.6}$$

and by Ito's formula and the Cauchy-Riemann equation we have

$$\begin{aligned} v(B_t^{\tau_r}) &= \int_0^{t\wedge\tau_r} \nabla v(B_s)\cdot dB_s \\ &= \int_0^{t\wedge\tau_r} (Q_1 \nabla u)(B_s)\cdot dB_s \\ &= (Q_1 * u(B^{\tau_r}))_t, \end{aligned}$$

where we set

$$Q_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that Q_1 is a rotation and does not have an eigenvector.

Now we can apply Theorem 6.1:

$$
\begin{aligned}
&E[A_\alpha(u(B^{\tau_r})^*)] \\
&\quad \le k_1 + k_2\left\{\sup_t E[A_\alpha(u(B_t^{\tau_r}))] + \sup_t E[A_\alpha(v(B_t^{\tau_r}))]\right\} \\
&\qquad + k_3\left\{\sup_t E[|u(B_t^{\tau_r})|] + E[|v(B_t^{\tau_r})|]\right\} \\
&\quad = k_1 + k_2\left\{E[A_\alpha(u(B_{\tau_r}))] + \sup_t E[A_\alpha(v(B_{\tau_r}))]\right. \\
&\qquad \left. + k_3\{E[|u(B_{\tau_r})|] + E[|v(B_{\tau_r})|]\right\} \\
&\quad = k_1 + k_2\left\{\int_T A_\alpha(u(re^{i\theta}))d\theta + \int_T A_\alpha(v(re^{i\theta}))d\theta\right\} \\
&\qquad \times k_3\left\{\int_T |u(re^{i\theta})|d\theta + \int_T |v(re^{i\theta})|d\theta\right\} \\
&\quad \le k_1 + k_2\left\{\int_T A_\alpha(f(\theta))d\theta + \int_T A_\alpha(Hf(\theta))d\theta\right\} \\
&\qquad + k_3\left\{\int_T |f(\theta)|d\theta + \int_T |Hf(\theta)|d\theta\right\},
\end{aligned} \tag{6.7}
$$

where k_i's are constants corresponding to those in the proof of Theorem 6.1, i.e.,

$$
\begin{aligned}
k_2 &= p^{-\alpha}c_{p\alpha}^{1/p}\frac{K_\alpha p^\alpha c_\alpha}{1-p}\cdot\frac{C_{m+2}}{m+2} \\
k_3 &= p^{-\alpha}c_{p\alpha}^{1/p}\frac{K_\alpha M_p^\alpha}{1-p} \\
k_1 &= k_2 A_\alpha(1) = k_3.
\end{aligned}
$$

The last inequality in (6.7) is verified by Lemma 6.5 which will be given later.

From (6.7) combined with Theorem 3.6, we obtain

$$
\begin{aligned}
&\frac{1}{\alpha+1}\int_T A_{\alpha+1}(Pf(r,\theta))d\theta \\
&\quad = \frac{1}{\alpha+1}E[A_{\alpha+1}(u(B_{\tau_r}))] \\
&\quad \le E[A_\alpha(u(B_{\tau_r})^*)] + G(\|f\|_1) \\
&\quad \le k_1 + k_2\left\{\int_T A_\alpha(f(\theta))d\theta + \int_T A_\alpha(Hf(\theta))d\theta\right\} \\
&\qquad + k_3\left\{\int_T |f(\theta)|d\theta + \int_T |Hf(\theta)|d\theta\right\} + G(\|f\|_1),
\end{aligned} \tag{6.8}
$$

where $G(c) = -A_\alpha(c) + \dfrac{c}{\alpha+1}\log^{\alpha+1}(1+c)$. Since $f \in L^1(T)$, $\lim_{r\uparrow 1}\|P_r * f - f\|_{L^1} = 0$. Hence there is a sequence $\{r_n\}$ such that $r_n \uparrow 1$ as $n \to \infty$ and $P_{r_n} * f(\theta) \to f(\theta)$ for $a.a.\theta(d\theta)$ as $n \to \infty$. By Fatou's lemma the proof is completed.

Lemma 6.5. *For any f in $L(\log L)^\alpha$, $\int_T A_\alpha(P_r * f(\theta))d\theta$ is increasing and converges to $\int_T A_\alpha(f(\theta))d\theta$* as $r \uparrow 1$.

Proof. Since $P_r * f(\theta)$ is harmonic in the open disk, $\int_T A_\alpha(P_r * f(\theta))d\theta$ is increasing in r. We will prove the convergence. For a continuous function f on T, it is well-known that $\lim_{r\uparrow 1}\|P_r * f - f\|_{\mathbf{C}(T)} = 0$. By Proposition 2.3 we have that the lemma holds for $f \in \mathbf{C}(T)$ and

$$\int_T A_\alpha(P_r * f(\theta))d\theta \le \int_T A_\alpha(f(\theta))d\theta.$$

Replacing f in the above inequality by f/λ, we have

$$\|P_r * f\|_{L(\log L)^\alpha} \le \|f\|_{L(\log L)^\alpha}.$$

This shows that the convolution operator with P_r is a contraction for any r. By the usual argument for strong convergence of operators, we see that, for any f in $L(\log L)^\alpha$, $\lim_{r\uparrow 1}\|P_r * f - f\|_{L(\log L)^\alpha} = 0$. By Proposition 2.3 we have that the lemma holds for any $f \in L(\log L)^\alpha$. This completes the proof.

7. Computation of $D_h(I - L)^{-1/2}$

In this section we compute $D_h(I-L)^{-1/2}\phi$ (firstly for a real-valued function ϕ). Let (W, H, μ) be an abstract Wiener space. Let D be the H-derivative L be the Ornstein-Uhlenbeck operator and P_t be the Ornstein-Uhlenbeck semigroup. P_t can be expressed as $P_t\phi(x) = \int_W \phi(e^{-t}x + (1-e^{-2t})^{1/2}y)\mu(dy)$. We define the space S of rapidly decreasing cylinder functions on W by

$$S = \{\phi; \phi(w) = F([h_1](w), \dots, [h_n](w)) \text{ for some } n, h_1, \dots, h_n \in H, \text{ and } F \in S(\mathbf{R}^d)\}$$

where $S(\mathbf{R}^d)$ is the Schwartz space of rapidly decreasing C^∞ functions.

We define a transform R_θ on $W \times W$ by

$$R_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

In the following we always set $\cos\theta = e^{-t}$.

First we compute the derivative of $P_t\phi$ for $\phi \in S$ in the direction of $h \in H$.

$$\begin{aligned} D_hP_t\phi(x) &= \frac{d}{ds}P_t\phi(x+sh)\Big|_{s=0} \\ &= \frac{d}{ds}\int \phi(e^{-t}(x+sh) + (1-e^{-2t})^{1/2}y)\mu(dy)\Big|_{s=0} \end{aligned}$$

$$= \frac{d}{ds}\int \phi\left(e^{-t}x + (1-e^{-2t})^{1/2}\left(y + \frac{se^{-t}}{(1-e^{-2t})^{1/2}}h\right)\right)\mu(dy)\Big|_{s=0}$$

$$= \frac{d}{ds}\int \phi(e^{-t}x + (1-e^{-2t})^{1/2}y)\mathscr{E}\left\{\frac{se^{-t}}{(1-e^{-2t})^{1/2}}h\right\}\mu(dy)\Big|_{s=0}$$

$$= \frac{e^{-t}}{(1-e^{-2t})^{1/2}}\int \phi(e^{-t}x + (1-e^{-2t})^{1/2}y)[h](y)\mu(dy),$$

where $\mathscr{E}(h) = \exp([h](y) - \frac{1}{2}\|h\|_H^2)$ for $h \in H$.

Next we compute $D_h(I-L)^{-1/2}\phi$. For this we represent $(I-L)^{-1/2}\phi$ as follows:

$$(I-L)^{-1/2}\phi(x) = \Gamma(1/2)^{-1}\int_0^\infty t^{-1/2}e^{-t}P_t\phi(x)dt.$$

Because D_h is closable, it is sufficient to compute the following limit.

$$\lim_{\varepsilon\to 0} D_h\int_\varepsilon^\infty t^{-1/2}e^{-t}P_t\phi(x)dt.$$

Now

$$D_h\int_\varepsilon^\infty t^{-1/2}e^{-t}P_t\phi(x)dt \tag{7.1}$$

$$= \int_\varepsilon^\infty t^{-1/2}\frac{e^{-2t}}{(1-e^{-2t})^{1/2}}\int_W \phi(e^{-t}x + (1-e^{-2t})^{1/2}y)[h](y)\mu(dy)dt$$

$$= \int_\rho^{\pi/2}(-\log\cos\theta)^{-1/2}\frac{\cos^2\theta}{\sin\theta}\int_W(\phi\otimes 1)(R_\theta(x,y))[h](y)\mu(dy)\tan\theta d\theta$$

$$= \int_\rho^{\pi/2}(-\log\cos\theta)^{-1/2}\cos\theta\int_W(\phi\otimes 1)(R_\theta(x,y))[h](y)\mu(dy)d\theta, \tag{7.2}$$

where $\rho = -\log\cos\varepsilon$ and $(f\otimes g)(x,y) = f(x)g(y)$.

Setting the R.H.S. of (7.2) by I and making the transform $y\to -y$, we have

$$I = -\int_\rho^{\pi/2}(-\log\cos\theta)^{-1/2}\cos\theta\int_W(\phi\otimes 1)(R_{-\theta}(x,y))[h](y)\mu(dy)d\theta.$$

On the other hand, making the transform $\theta\to -\theta$, we see

$$I = \int_{-\pi/2}^{-\rho}(-\log\cos\theta)^{-1/2}\cos\theta\int_W(\phi\otimes 1)(R_{-\theta}(x,y))[h](y)\mu(dy)d\theta.$$

By averaging these, the R.H.S. of (7.2) equals to

$$\frac{1}{2}\left(\int_{-\pi/2}^{-\rho} - \int_{\rho}^{\pi/2}\right)(-\log\cos\theta)^{-1/2}\cos\theta\int_W (\phi\otimes 1)(R_{-\theta}(x,y))[h](y)\mu(dy)d\theta$$

$$= \int_W [h](y)\left[\left(\int_{-\pi}^{-\rho} + \int_{\rho}^{\pi}\right)K(\theta)(\phi\otimes 1)(R_{-\theta}(x,y))d\theta\right]\mu(dy)$$

where $K(\theta)$ is defined as follows

$$K(\theta) = \begin{cases} -\frac{1}{2}(-\log\cos\theta)^{-1/2}\cos\theta & \left(0 \le \theta \le \frac{\pi}{2}\right), \\ \frac{1}{2}(-\log\cos\theta)^{-1/2}\cos\theta & \left(-\frac{\pi}{2} \le \theta \le 0\right), \\ 0 & (\text{otherwise}). \end{cases}$$

Note that as $\theta \downarrow 0$, or equivalently $t \downarrow 0$,

$$K(\theta) = -\frac{e^{-t}}{2\sqrt{t}} = -\frac{1}{2\sqrt{t}} + \text{a bounded function},$$

$$\cot\frac{\theta}{2} = \sqrt{\frac{1+e^{-t}}{1-e^{-t}}} = \frac{\sqrt{2}}{\sqrt{t}} + \text{a bounded function}.$$

So we have

$$K(\theta) = -\frac{1}{2\sqrt{2}}\cot\frac{\theta}{2} + \text{a bounded function}.$$

Let $\rho \downarrow 0$. Then we have

$$\lim_{\rho\downarrow 0}\int_W [h](y)\left[\left(\int_{-\pi}^{-\rho} + \int_{\rho}^{\pi}\right)K(\theta)(\phi\otimes 1)(R_{-\theta}(x,y))d\theta\right]\mu(dy)$$

$$= \int_W [h](y)\left[\lim_{\rho\downarrow 0}\left(\int_{-\pi}^{-\rho} + \int_{\rho}^{\pi}\right)K(\theta)(\phi\otimes 1)(R_{-\theta}(x,y))d\theta\right]\mu(dy)$$

$$= \int_W [h](y)\left[\text{p.v.}\int_{-\pi}^{\pi} K(\theta)(\phi\otimes 1)(R_{-\theta}(x,y))d\theta\right]\mu(dy).$$

Here the first equality (a change of $\lim_\rho$ and $\int_W$) is guaranteed because we took ϕ from S.

For convienience we set

$$UF(x,y) = \text{p.v.}\int_{-\pi}^{\pi} K(\theta)F(R_{-\theta}(x,y))d\theta$$

for $F : W\times W \to \mathbf{R}$. Note that if (x,y) is substituted by $R_\beta(x,y)$ it is the convolution of $K(\theta)$ and $F(R_\theta(x,y))$.

From the argument in this section, we conclude that

$$\Gamma(1/2)D_h(I-L)^{-1/2}\phi(x) = \int_W [h](y)U(\phi\otimes 1)(x,y)\mu(dy). \tag{7.3}$$

Note that $D_h(I-L)^{-1/2}$ can also be generalized to act on K-valued functions as an operator acting only on function part. If a K-valued function ϕ is written in the following form

$$\phi(x) = \sum_{i=1}^{m} \phi_i(x) e_i,$$

where $e_i \in K$, then

$$\Gamma(1/2) D_h(I-L)^{-1/2}\phi(x) = \sum_{i=1}^{m} \int_W [h](y) U(\phi_i \otimes 1)(x,y)\mu(dy)\, e_i. \tag{7.4}$$

8. Continuity of J_1

In this section we will show the continuity of the projection J_1 of $L^2(\mu)$ onto the first order Wiener chaos. Unfortunately J_1 is not continuous from $L^1(\mu)$ to $L^1(\mu)$. For example consider one dimensional standard normal distribution and functions $\exp(nx)$, $(n \in \mathbf{N})$. In fact $\|e^{nx}\|_{L^1} = e^{n^2/2}$ and

$$\|J_1(e^{nx})\|_{L^1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{nx} e^{-x^2/2} dx \|x\|_{L^1} = n e^{n^2/2} \|x\|_{L^1}.$$

Using an extrapolation theorem we will show, however, in the following theorem that $J_1 : L(\log L)^{1/2}(\mu) \to L^1(\mu)$ is continuous.

Theorem 8.1. *Let J_1 be the projection of $L^2(\mu; K)$ onto the first Wiener chaos and $F \in L^2(K)$. Then there is a constant $\hat{C}$ satisfying*

$$E[\|J_1 F\|_K] \le \hat{C}(1 + A_{1/2}(\|F\|_K)). \tag{8.1}$$

Proof. We will prove that the semigroup $\vec{P}_t$ is hypercontractive where

$$\vec{P}_t F = \sum_{i=1}^{m} P_t F_i(x) e_i,$$

for any F of the form $F = \sum_{i=1}^{m} F_i(x) e_i$ and $F_i \in L^2(\mu)$, $e_i \in K$.

Since the Ornstein-Uhlenbeck semigroup is written as $P_t\phi(x) = \int_W \phi(e^{-t}x + (1-e^{-2t})^{1/2} y)\mu(dy)$, it is easily verified by interchanging the order of the norm and the integral that

$$\|\vec{P}_t F(x)\|_K \le P_t \|F\|_K(x).$$

Since the hypercontractivity of P_t in the scalar-valued case is well-known, the above inequality shows the hypercontractivity of $\vec{P}_t$, that is,

$$\|\vec{P}_t F\|_q \le \|F\|_p,$$

where $p - 1 = e^{-2t}(q-1)$.

And the hypercontractivity of $\vec{P}_t$ implies, for $1 < p \le 2$,

$$\|J_1 F\|_p \le \frac{1}{\sqrt{p-1}} \|F\|_p. \tag{8.2}$$

The proof for scalar-valued case of inequality (8.2) in Ikeda and Watanaba [4] is also applicable to this vector-valued case.

We will finish the proof by te following lemma on the extrapolation, which is a generalization of the result in the scalar-valued case by Yano [14]:

Lemma 8.2. *Let T be a transformation which transforms every integrable K-valued function to a measurable K-valued function, both being defined on a probability space (Ω, P), such that*

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu(x) \quad \textit{implies} \quad \|Tf\|_K \le \sum_{\nu=0}^{\infty} \|Tf_\nu\|_K, \tag{8.3}$$

$$\|T(-f)\|_K = \|Tf\|_K, \tag{8.4}$$

and the inequality

$$\{E[\|Tf\|_K^p]\}^{1/p} \le C_p \{E[\|f\|_K^p]\}^{1/p} \tag{8.5}$$

holds with the constant C_p satisfying the inequality

$$C_p \le \frac{C}{(p-1)^k} \tag{8.6}$$

for any p, $1 < p \le 2$, for some $k > 0$ and for a constant C. Then we have

$$E[\|Tf\|_K] \le a_k E[A_k(\|f\|_K)] + b_k, \tag{8.7}$$

where a_k and b_k are constants depending only on k.

Proof. The following proof is a slight modification of that in Yano [14].

Fix a $k > 0$. It is sufficient to show (8.7) for f satisfying $\|f(x)\| \ge 1$ for almost all x, since any f can be expressed as

$$\begin{aligned} f(x) &= f(x) I_{\{(f(x), v)_K \ge 0\}} + f(x) I_{\{(f(x), v)_K < 0\}} \\ &= (f(x) I_{\{(f(x), v)_K \ge 0\}} + v) - (-f(x) I_{\{(f(x), v)_K < 0\}} + v) \\ &= f^+(x) - f^-(x), \end{aligned}$$

where $v \in K$ is a unit vector in K. Note that $\|f^+(x)\|_K \ge |(f^+(x), v)_K| \ge 1$. By the same reason we see that $\|f^-(x)\|_K \ge 1$. Set $\Omega_\nu = \{2^\nu \le \|f(x)\|_K < 2^{\nu+1}\}$. Then $\Omega = \sum_{\nu=0}^{\infty} \Omega_\nu$. And set $f_\nu(x) = f(x) I_{\Omega_\nu}(x)$ and $g_\nu(x) = 2^{-\nu} f_\nu(x)$. Then $f(x) = \sum_{\nu=0}^{\infty} f_\nu(x) = \sum_{\nu=0}^{\infty} 2^\nu g_\nu(x)$ and $1 \le \|g_\nu(x)\|_K < 2$. By (8.3), (8.5) and (8.6) we have

$$\|Tf\|_K = \sum_{\nu=0}^{\infty} 2^\nu \|Tg_\nu(x)\|_K.$$

Integrating both sides, we obtain

$$
\begin{aligned}
E[\|Tf\|_K] &\le \sum_{v=0}^{\infty} 2^v E[\|Tg_v(x)\|_K] \\
&= \sum_{v=0}^{\infty} 2^v \{E[\|Tg_v(x)\|_K^{p_v}]\}^{1/p_v} \\
&\le C_{p_v} \sum_{v=0}^{\infty} 2^v \{E[\|g_v(x)\|_K^{p_v}]\}^{1/p_v} \\
&\le \frac{4C}{(p_v - 1)^k} \sum_{v=0}^{\infty} 2^v \{E[\|g_v(x)\|_K]\}^{1/p_v}.
\end{aligned}
$$

Here we used the fact that $1 \le \|g_v(x)\|_K < 2$ and $1 \le p_v < 2$.

We substitute $p_v = 1 + 1/v$. Then we have

$$
\begin{aligned}
E[\|Tf\|_K] &\lesssim \sum_{v=0}^{\infty} 2^v v^k \{E[\|g_v(x)\|_K]\}^{v/(1+v)} \\
&\le \sum_{v=0}^{\infty} 2^v v^k (4E[\|g_v(x)\|_K] + 4^{-v}) \\
&\le 4 \sum_{v=0}^{\infty} 2^v v^k E[\|g_v(x)\|_K] + b_k \\
&\lesssim \sum_{v=0}^{\infty} E[\|f_v\|_K \log^k(\|f_v\|_K)] + 1 \\
&\lesssim E[A_k(\|f_v\|_K)] + 1.
\end{aligned}
$$

Here in the second inequality, we used Lemma 8.3 below.

Lemma 8.3. *Let $x \ge 0$ and $v \ge 1$. Then*

$$
x^{v/(v+1)} \le 4x + 4^{-v}.
$$

Proof. Let $F(x) = 4x^{(v+1)/v}$. Then $f(x) = F'(x) = 4\frac{v+1}{v}x^{1/v}$. The inverse function g of f is calculated as

$$
g(x) = \left\{\frac{v}{4(v+1)}x\right\}^v.
$$

Set

$$
G(x) = \int_0^x g(y)dy = \left\{\frac{v}{4(v+1)}\right\}^v \frac{x^{v+1}}{v+1}.
$$

Then by Young's inequality, we have

$$\begin{aligned} x^{\nu/(\nu+1)} &= x^{\nu/(\nu+1)} \cdot 1 \\ &\le F(x^{\nu/(\nu+1)}) + G(1) \\ &= 4x + \left\{\frac{\nu}{4(\nu+1)}\right\}^{\nu} \frac{1}{\nu+1} \\ &\le 4x + 4^{-\nu} \end{aligned}$$

and this completes the proof.

9. Continuity of $D(I-L)^{-1/2}$

First let us compute H^*-norm of $D(I-L)^{-1/2}\phi(x)$. Generally for $G \in L^2(\mu)$, the linear functional

$$\left(h \to \int_W [h](w)G(w)\mu(dw)\right) \in H^*$$

corresponds to an element $g \in H$ through the canonical isomorphism $H^* \cong H$, where

$$g = \sum_{i=1}^{\infty}\left(\int_W [h_i](w)G(w)\mu(dw)\right)h_i \in H$$

($\{h_i\}$ is an O.N.B. of H). Then we see that

$$\delta g(w) = J_1 G(w) \tag{9.1}$$

where δ is the divergence operator. Hence we have

$$E[|J_1G(w)|] = E[|\delta g(w)|] = \|g\|_H E\left[\left|\delta\left(\frac{g}{\|g\|_H}\right)(w)\right|\right]$$

and noting that $\delta\left(\frac{g}{\|g\|_H}\right)$ has the standard normal distribution, we obtain

$$\|g\|_H = \sqrt{\frac{\pi}{2}}E[|J_1G(w)|].$$

This relation holds even when $\|g\|_H = 0$.

Next we consider the case of K-valued functions

$$\left(h \to \sum_{i=1}^{m}\int_W [h](w)G_i(w)\mu(dw)e_i\right) \in H^* \otimes K,$$

where e_i are orthonormal in K. This element corresponds to an element $g = \sum_{i=1}^m g_i \otimes e_i \in H \otimes K$ through the canonical isomorphism $H^* \cong H$. Then we see that (9.1) holds in the sense of K-valued functions. And by Proposition 9.1 below we see that

$$E[\|J_1G(w)\|_K] = \|g\|_{H\otimes K}E\left[\left\|\delta\left(\frac{g}{\|g\|_{H\otimes K}}\right)(w)\right\|_K\right] \geq \sqrt{\frac{2}{\pi}}\|g\|_{H\otimes K}. \tag{9.2}$$

Proposition 9.1. *Let $g \in H \otimes K$ satisfy $\|g\|_{H\otimes K} = 1$. Then*

$$E[\|\delta g\|_K] \geq \sqrt{\frac{2}{\pi}}. \tag{9.3}$$

Proof. Since the mapping $\delta : H \otimes K \to L^2(K) \to L^1(K)$ is continuous, it is sufficient to prove (9.3) for elements of the form

$$g = \sum_{i=1}^{m} g_i \otimes e_i,$$

where e_i are orthonormal in K, $g_i \in H$ and $\|g\|_{H\otimes K}^2 = \sum_{i=1}^m \|g_i\|_H^2 = 1$. Suppose such an element g be given. Then we may identify $\mathscr{L}\langle e_1, e_2, \ldots, e_m\rangle \cong \mathbf{R}^m$ and

$$\delta g = \sum_{i=1}^{m} (\delta g_i)(x) e_i$$

induces a Gaussian measure on $\mathbf{R}^m$ whose covariance matrix is $V = (\langle g_i, g_j\rangle_H)_{i,j=1}^m$. Let $t_1, t_2, \ldots, t_m$ be the eigenvalues of V. Then there exists an orthogonal matrix U which diagonalizes V:

$$U^{-1}VU = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_m \end{pmatrix}.$$

Note that

$$\sum_{i=1}^{m} t_i = 1 \quad \text{and that } t_i \geq 0. \tag{9.4}$$

We first consider non-degenerate cases, that is, all t_i's are positive. The left hand side of (9.3) is equal to

$$\begin{aligned}
&(2\pi)^{-m/2}\frac{1}{\sqrt{\det V}}\int_{\mathbf{R}^m} |x| \exp\left(-\frac{1}{2}x^T Vx\right)dx \\
&= (2\pi)^{-m/2}\frac{1}{\sqrt{\det V}}\int_{\mathbf{R}^m} |Ux| \exp\left(-\frac{1}{2}x^T U^{-1}VUx\right)dx \\
&= (2\pi)^{-m/2}\int_{\mathbf{R}^m}\left(\sum_{i=1}^{m} t_i x_i^2\right)^{1/2} \exp\left(-\frac{1}{2}|x|^2\right)dx.
\end{aligned} \tag{9.5}$$

The last expression in (9.5) includes degenerate cases, too.

It is sufficient to show that, for each m, the minimum under the condition (9.4) of the right hand side of (9.5) is larger than or equal to $\sqrt{\frac{2}{\pi}}$. We will prove this claim by induction for m.

For $m = 1$, it is already proved.

For $m = 2$, set, for $0 \leq t \leq 1$,

$$I(t) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \sqrt{tx_1^2 + (1-t)x_2^2}\exp\left(-\frac{1}{2}|x|^2\right)dx,$$

and

$$I_r(t) = \frac{1}{2\pi}\int_{\mathbf{R}^2\setminus B_r} \sqrt{tx_1^2 + (1-t)x_2^2}\exp\left(-\frac{1}{2}|x|^2\right)dx,$$

where B_r is the ball of radius r centered at 0. It is easy to see that, for any t, $I_r(t) \to I(t)$ as $r \to 0$. By the diferentiation under the integral sign, we can calculate the first and the second derivatives of $I_r(t)$:

$$\frac{dI_r}{dt}(t) = \frac{1}{4\pi}\int_{\mathbf{R}^2\setminus B_r} \frac{x_1^2 - x_2^2}{\sqrt{tx_1^2 + (1-t)x_2^2}}\exp\left(-\frac{1}{2}|x|^2\right)dx, \tag{9.6}$$

and

$$\frac{d^2I_r}{dt^2}(t) = -\frac{1}{8\pi}\int_{\mathbf{R}^2\setminus B_r} \frac{(x_1^2 - x_2^2)^2}{(tx_1^2 + (1-t)x_2^2)^{3/2}}\exp\left(-\frac{1}{2}|x|^2\right)dx. \tag{9.7}$$

From (9.6) and (9.7) we can easily see that $\frac{dI_r}{dt}\left(\frac{1}{2}\right) = 0$ and $\frac{d^2I_r}{dt^2}(t) \leq 0$. From this $I_r(t)$ increases on $[0, 1/2]$, and decreases on $[1/2, 1]$. And so does $I(t) = \lim_{r\to 0} I_r(t)$. Hence the minimum is $I(0) = I(1) = \sqrt{\frac{2}{\pi}}$.

For $m \geq 3$, by a similar proof we can see that the minimum can not be attained in the interior (= the points of non-degenerate cases), but attained on the boundary (= the points of degenerate cases). But in the degenerate cases, the minimum problem is reduced to the one for a lower dimension.

By the above inequality (9.2) (taking $G_i(\cdot) = U(\phi_i \otimes 1)(x, \cdot)$) and the results (7.4) and (8.1) of previous sections,

$$\begin{aligned}\|D(I-L)^{-1/2}\phi(x)\|_{H^*\otimes K} &\leq \text{const.} \times E^y[\|J_1^y U(\phi\otimes 1)(x,y)\|_K] \\ &\leq kE^y[1 + A_{1/2}(\|U(\phi\otimes 1)(x,y)\|_K)],\end{aligned}$$

where E^y and J_1^y are operations for the variable y, and k is a positive constant.

Then we will show the main result of this article.

Theorem 9.2. *$D(I-L)^{-1/2}$ is continuous from $L(\log L)^{\alpha+3/2}(\mu; K)$ to $L(\log L)^{\alpha}(\mu; H^*\otimes K)$ for any $\alpha \geq 0$.*

Proof. We have

$$\begin{aligned} E^x[A_\alpha(\|D(I-L)^{-1/2}\phi(x)\|_{H^*\otimes K})] &\le E^x[A_\alpha(kE^y[1+A_{1/2}(\|U(\phi\otimes 1)(x,y)\|_K)])] \\ &\lesssim 1+E^x[A_\alpha(E^y[A_{1/2}(\|U(\phi\otimes 1)(x,y)\|_K)])] \\ &\le 1+E^xE^y[A_\alpha(A_{1/2}(\|U(\phi\otimes 1)(x,y)\|_K)])]. \quad (9.8) \end{aligned}$$

Here we used Δ_2 condition for the second inequality and the convexity for the third. It is easy to show that

$$\lim_{x\to\infty}\frac{A_\alpha(A_{1/2}(x))}{A_{\alpha+1/2}(x)}=1.$$

From this we see that $A_\alpha(A_{1/2}(x))\lesssim 1+A_{\alpha+1/2}(x)$.

So from (9.8) we have

$$\begin{aligned} \text{R.H.S. of (9.8)} &\lesssim 1+E^xE^y[A_{\alpha+1/2}(\|U(\phi\otimes 1)(x,y)\|_K)] \\ &= 1+E^xE^y\left[\int_{-\pi}^{\pi}A_{\alpha+1/2}(\|U(\phi\otimes 1)(R_\beta(x,y)\|_K)d\beta\right]. \quad (9.9) \end{aligned}$$

Here we used the fact that the product measure $\mu\times\mu$ is invariant under the rotation R_β. By the continuity of the Hilbert transform and of a convolution operator with a bounded function, we have

$$\begin{aligned} \text{R.H.S. of (9.9)} &\lesssim 1+E^xE^y\left[\int_{-\pi}^{\pi}A_{\alpha+3/2}(\|(\phi\otimes 1)(R_\beta(x,y)\|_K)d\beta\right] \\ &= 1+E^xE^y[A_{\alpha+3/2}(\|(\phi\otimes 1)(x,y\|_K)] \\ &= 1+E^x[A_{\alpha+3/2}(\|\phi(x)\|_K)]. \end{aligned}$$

This completes the proof.

By this theorem we can estimate L^1-norm of $D^n f$.

Corollary 9.3. *Let $f\in S$ be a real valued function. Then*

$$\|D^n f\|_{L^1(\mu;H^{*\otimes n})}\lesssim\|(I-L)^{n/2}f\|_{L(\log L)^{3n/2}L(\mu)} \quad (9.10)$$

for $n=1,2,\ldots$.

Proof. For $n=1$ it is almost trivial. For $n=2$ we use a slightly modified version of theorem 9.2, that is, the continuity of $D(2I-L)^{-1/2}$. Then we have

$$\begin{aligned} \|D^2 f\|_{L^1(\mu;H^*\otimes H^*)} &\lesssim \|(2I-L)^{1/2}Df\|_{L(\log L)^{3/2}(\mu;H^*)} \\ &= \|D(I-L)^{1/2}f\|_{L(\log L)^{3/2}(\mu;H^*)} \\ &\lesssim \|(I-L)f\|_{L(\log L)^3(\mu)}. \end{aligned}$$

We complete the proof by induction.

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