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The initial value problem for the elliptic-hyperbolic Davey-Stewartson equation

By

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Abstract

We present the local and the global existence theorems for the elliptic-hyperbolic Davey-Stewartson equation which does not allows the classical energy estimates. To overcome this difficulty, we make use of the smoothing property of linear Schrödinger type equations which was obtained by S. Doi. Then under the smallness condition to L^2-norm of the initial data, we get the local solution. Moreover we show the global existence of small amplitude solutions by the *a priori* estimates for which the null gauge condition of Y. Tsutsumi plays an important role.

1. Introduction

We study the initial value problem for the elliptic-hyperbolic Davey-Stewartson equation of the form

$$\partial_t u - i(\partial_x^2 + \partial_y^2)u = f(u) \quad \text{in } (0,\infty) \times \mathbf{R}^2, \tag{1.1}$$

$$u(0,x,y) = u_0(x,y) \quad \text{in } \mathbf{R}^2, \tag{1.2}$$

where $u(t,x,y)$ is $\mathbf{C}$-valued, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, and the nonlinear term $f(u)$ is defined by

$$f(u) = \sum_{j=0}^{2} a_j f_j(u), \quad f_1(u)(x,y) = \int_y^{+\infty} \partial_x(|u(x,y')|^2)\,dy' u(x,y),$$

$$f_0(u) = |u|^2 u, \quad f_2(u)(x,y) = \int_x^{+\infty} \partial_y(|u(x',y)|^2)\,dx' u(x,y),$$

$a_0, a_1, a_2, \in \mathbf{C}$ are constants.

We use the following notations. $(\xi,\zeta) \in \mathbf{R}^2$ means the dual variable of $(x,y) \in \mathbf{R}^2$ under the Fourier transform. $\partial_\xi = \partial/\partial\xi$ and $\partial_\zeta = \partial/\partial\zeta$. J_x and J_y are defined by

$$J_x u = e^{ix^2/4(1+t)} 2i(1+t)\partial_x(e^{-ix^2/4(1+t)}u) = (x + (1+t)\partial_x)u,$$

$$J_y u = e^{iy^2/4(1+t)} 2i(1+t)\partial_y(e^{-iy^2/4(1+t)}u) = (y + (1+t)\partial_y)u.$$

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$$\langle D_x\rangle=(1-\partial_x^2)^{1/2}, \langle D_y\rangle=(1-\partial_y^2)^{1/2}, \langle D_x;D_y\rangle=(1-\partial_x^2-\partial_y^2)^{1/2}, \langle x\rangle=\sqrt{1+x^2},$$

$$\langle y\rangle=\sqrt{1+y^2}, \langle x;y\rangle=\sqrt{1+x^2+y^2}.$$

$$W^{m,p}=W^{m,p}(\mathbf{R}^2)$$
$$=\left\{u\in\mathscr{S}'(\mathbf{R}^2)\,\middle|\, \|u\|_{W^{m,p}}=\left(\iint_{\mathbf{R}^2}|\langle D_x;D_y\rangle^m u|^p dxdy\right)^{1/p}<+\infty\right\},$$

$L^p=W^{0,p}$, $H^m=W^{m,2}$ for $m\in\mathbf{R}$ and $1\leqslant p<\infty$.

$$W^{m,\infty}=W^{m,\infty}(\mathbf{R}^2)=\{u\in\mathscr{S}'(\mathbf{R}^2)|\ \|u\|_{W^{m,\infty}}=\text{ees.sup}\,|\langle D_x;D_y\rangle^m u|<+\infty\},$$

$L^\infty=W^{0,\infty}$, for $m\in\mathbf{R}$.

$$H^{m,n}=H^{m,n}(\mathbf{R}^2)=\{u\in\mathscr{S}'(\mathbf{R}^2)|\ \|\langle x;y\rangle^n\langle D_x;D_y\rangle^m u\|_{L^2}<+\infty\}$$

for $m,n\in\mathbf{R}$. $\|\cdot\|_m$ means H^m-norm. Especially $\|\cdot\|$ and $(\cdot,\cdot)$ mean L^2-norm and L^2-inner product respectively. $\mathscr{S}=\mathscr{S}(\mathbf{R}^2)$ and $\mathscr{S}'=\mathscr{S}'(\mathbf{R}^2)$ denote the Schwartz class and its topological dual respectively. Let Ω be an open subset of some Euclidean space. $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ are the dual pair of the class of test functions of $C_0^\infty(\Omega)$ and the class of distributions on Ω. $\mathscr{B}^0(\Omega)$ is the Banach space of all bounded linear continuous functions on Ω. $\mathscr{B}^\infty(\Omega)$ is the Fréchet space of all C^∞ functions on Ω whose derivatives of any order are all bounded. Let E and F be Fréchet spaces. $\mathscr{L}(E,F)$ denotes the set of all bounded linear operators of E to F. $\mathscr{L}(E)=\mathscr{L}(E,E)$. Let (X,Y) is a dual pair of locally convex spaces X and Y. $\langle y,x\rangle$ means the operation of $y\in Y$ on $x\in X$. $C([0,T];E)$ and $C_w([0,T];E)$ are the sets of all strongly and weakly continuous E-valued functions on $[0,T]$ respectively. $[s]$ means the largest integer less than or equal to $s\in\mathbf{R}$. $\mathbf{N}=\{1,2,3,\dots\}$ and $\mathbf{Z}_+=\mathbf{N}\cup\{0\}$. We denote the positive constants by the same letter C.

Originally the Davey-Stewartson systems are written as

$$\partial_t u-i(\delta\partial_x^2+\partial_y^2)u=i\gamma|u|^2u+ib(\partial_x\varphi)u, \tag{1.3}$$

$$(\partial_x^2+c\partial_y^2)\varphi=\partial_x(|u|^2), \tag{1.4}$$

where $\delta,\gamma=\pm1$, $b\in\mathbf{R}$ and $c\in\mathbf{R}\backslash\{0\}$. In [6] J.-M. Ghidaglia and J.-C. Saut classified (1.3)–(1.4) as elliptic-elliptic, hyperbolic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic according to the respective sign of $(\delta,c)=(+,+),(-,+),(+,-)$ and $(-,-)$, and studied the initial value problem for them.

For the cases of the elliptic-elliptic and the hyperbolic-elliptic (i.e. $c>0$), (1.3)–(1.4) becomes

$$\partial_t u-i(\delta\partial_x^2+\partial_y^2)u=i\gamma|u|^2u+ib(R_c(|u|^2))u, \tag{1.5}$$

where R_c is a singular integral operator whose symbol is $\xi^2/(\xi^2+c\zeta^2)$. Since R_c is a bounded linear operator of $L^p(\mathbf{R}^2;\mathbf{R})$ to $L^p(\mathbf{R}^2;\mathbf{R})$ for any $1<p<+\infty$, (1.5) is similar to

$$\partial_t u - i(\delta\partial_x^2 + \partial_y^2)u = i\gamma'|u|^2 u, \quad \gamma' \in \mathbf{R}\backslash\{0\}.$$

Then J.-M. Ghidaglia and J.-C. Saut ([6]) obtained the complete results on the local existence, the global existence and the blow-up of the initial value problem for (1.5) under the condition $\delta = +1$.

On the other hand, in the cases of the elliptic-hyperbolic and the hyperbolic-hyperbolic (i.e. $c < 0$), one assume the radiation condition

$$\varphi(t, x, y) \to 0 \quad \text{as } x + y, x - y \to +\infty \tag{1.6}$$

in order that the subsystem (1.4) is solvable. Here we put $c = -1$ for simplicity. With the transformation $x := (x + y)/2$, $y := (x - y)/2$, the system (1.3)–(1.4)–(1.6) becomes

$$\partial_t u - i(\partial_x^2 + \partial_y^2)u = \tilde{f}(u) \quad \text{if } \delta = 1, \tag{1.7}$$

or

$$\partial_t u + 2i\partial_x\partial_y u = \tilde{f}(u) \quad \text{if } \delta = -1,$$

where

$$\tilde{f}(u) = i\left(\gamma - \frac{b}{2}\right)f_0(u) + i\frac{b}{4}f_1(u) + i\frac{b}{4}f_2(u).$$

Thus we consider the nonlinear term $f(u)$ as in (1.1). Because $f(u)$ contains $\partial_x u$, $\partial_x \bar{u}$, $\partial_y u$ and $\partial_y \bar{u}$, the classical energy estimates are not available for the cases of the elliptic-hyperbolic and the hyperbolic-hyperbolic. F. Linares and G. Ponce ([11]) proved the local existence of small solutions to the initial value problems for the cases of the elliptic-hyperbolic and the hyperbolic-hyperbolic by the sharp smoothing estimates on $e^{it(\partial_x^2+\partial_y^2)}$ and $e^{-2it\partial_x\partial_y}$, which are basically due to C. E. Kenig, G. Ponce and L. Vega ([9]). Recently, using so-called abstract Cauchy-Kowalewski theorem, N. Hayashi and J.-C. Saut ([7]) have shown the local and the global existence of analytic solutions to the initial value problems for the cases of the elliptic-hyperbolic and the hyperbolic-hyperbolic.

The purpose of this paper is to show the global existence of small amplitude solutions to (1.1)–(1.2). The main results are the following.

Theorem 1.1 (Local existence). *Let m_1 be a sufficiently large integer. We put $a = \max(|a_1|, |a_2|)$. Then for any*

$$u_0 \in H^m (m \in \mathbf{N} \geqslant m_1) \textit{ satisfying } \|u_0\| = \|u_0\|_{L^2} < \frac{1}{2\sqrt{ae}}, \tag{1.8}$$

there exists a time $T = T(\|u_0\|_{m_1}) > 0$ such that the initial value problem (1.1)–(1.2) *possesses a unique solution*

$$u \in C_{\mathrm{w}}([0, T); H^m) \cap C([0, T); H^{m-1}).$$

Theorem 1.2 (Global existence). *Let m_2 be a sufficiently large integer greater then or equal to m_1. Then there exists a constant $\delta > 0$ such that for any*

$$u_0 \in \bigcap_{j=0}^{5} H^{m-j,j} \ (m \in \mathbf{N} \geqslant m_2 + 3) \ \textit{satisfying} \ \sum_{j=0}^{5} \|u_0\|_{H^{m_2-j,j}} \leqslant \delta,$$

the initial value problem (1.1)–(1.2) *possesses a unique solution*

$$u \in \bigcap_{j=0}^{5} (C_{\mathrm{w}}([0,\infty); H^{m-j,j}) \cap C([0,\infty); H^{m-1-j,j})).$$

Remark 1.1. Since our analysis is based on the symbolic calculus of pseudo-differential operators, it is troublesome to determine the minimum of m_1 and m_2.

Remark 1.2. In [14], using the inverse scattering technique, A. S. Fokas and L. Y. Sung proved the global existence for (1.3)–(1.4)–(1.6) with large initial data under the conditions $\delta = +1$, $c = -1$ and $2\gamma + b = 0$.

Now we explain the idea of the proofs. Theorem 1.1 follows from the energy inequality. Theorem 1.2 is proved by the *a priori* estimates which consist of the energy and the decay estimates.

For the energy estimates, we make use of S. Doi's method ([5]) for linear Schrödinger type equations

$$\partial_t u - i\Delta u + \sum_{j=1}^{N} b_j(t,x)\partial_j u + c(t,x)u = f(t,x) \quad (0,T) \times \mathbf{R}^N,$$

where $\partial_j = \partial/\partial x_j$ $(j = 1, \dots, N)$, $\nabla = (\partial_1, \dots, \partial_N)$, $\Delta = \nabla \cdot \nabla = \partial_1^2 + \cdots + \partial_N^2$ and $b_j(t,x), c(t,x) \in C^\infty([0,T]; \mathscr{B}^\infty(\mathbf{R}^N))$. Roughly speaking, under the appropriate condition on $\operatorname{Im} b_j(t,x)$, there exists a automorphic $u \mapsto Ku$ in $L^2(\mathbf{R}^N)$ such that $[K, -i\Delta]K^{-1}$ is elliptic which is stronger than $\sum_{j=1}^{N} \operatorname{Im} b_j(t,x)\partial_j$. Because one can choose $[K, -i\Delta]K^{-1}$ as sufficiently strong, $([K, -i\Delta]K^{-1} + \sum_{j=1}^{N} \operatorname{Im} b_j(t,x)\partial_j)^{1/2}$ gives the smoothing estimates of order $1/2$. We use this property to get the energy estimates for (1.1)–(1.2). We remark that S. Doi's method is also available to solve the (local) semilinear Schrödinger equations (see [1], [2], [3] and [4]).

We explain the outline of the decay estimates. We note that $f(u)$ satisfies the gauge invariance, that is for any $\theta \in \mathbf{R}$ and for any $u \in \mathscr{S}(\mathbf{R}^2)$, $f(e^{i\theta}u) = e^{i\theta}f(u)$. Then, J_x and J_y act well on $f(u)$ and we can make use of the Libnitz formula with respect to J_x and J_y, that is, for instance,

$$J_x^\alpha f_0(u) = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}(-1)^{\alpha_2} J_x^{\alpha_1} u \overline{J_x^{\alpha_2} u} J_x^{\alpha_3} u.$$

Moreover, using the Gagliardo-Nirenberg inequalities with J_x and J_y, we get the decay estimates. For example, it follows that the inequality

$$\|f_0(u)\| = \||e^{-i(x^2+y^2)/4(1+t)}u|^2 u\|$$

$$\leq \|e^{-i(x^2+y^2)/4(1+t)}u\|_{L^\infty}^2 \|u\|$$

$$C \leqslant \|u\|^2 \sum_{\alpha+\beta=2} \|\partial_x^\alpha \partial_y^\beta e^{-i(x^2+y^2)/4(1+t)}u\|$$

$$C(1+t)^{-2}\|u\|^2 \sum_{\alpha+\beta=2} \|J_x^\alpha J_y^\beta u\|.$$

$f_1(u)$ and $f_2(u)$ behave as if they were cubic terms in one space dimension because of their nonlocality. Then the expected decay rate of $f_1(u)$ and $f_2(u)$ is at most $O((1+t)^{-1})$ by the elementary nonlinear estimate. This is not enough to prove the existence of the global small amplitude solution. Then we need an extra time-decay. Fortunately we can use the null gauge condition of Y. Tsutsumi ([13])

$$\partial_x(|u|^2) = \frac{1}{2i(1+t)}(J_x u\bar{u} - u\overline{J_x u}) \tag{1.9}$$

and we can get the extra time-decay. (see also [8]).

Fortunately, for the elliptic-hyperbolic case, we can find the operator K such that the commutator of the principal part and it does not bring about the loss of time-decay. Then, we can extend the local existence theorem to the global one provided that the initial data is sufficiently small. On the other hand, for the hyperbolic-hyperbolic case, the commutator causes the loss of time-decay. Our method is not available to study the global existence for the hyperbolic-hyperbolic equation.

The organizations of this paper is as follows. §2 is devoted to obtain the smoothing effect of $e^{it(\partial_x^2+\partial_y^2)}$. §3 contains preliminary results. In §4 and §5 we prove Theorems 1.1 and 1.2 respectively.

2. Linear estimates

In this section, following S. Doi [5], we obtain the smoothing effect of $e^{it(\partial_x^2+\partial_y^2)}$. We use the symbolic calculus of pseudo-differential operators on $\mathbf{R}$ (see H. Kumano-go [10]). Let T be a positive time and let t belong to the interval $[0, T]$. We define the pseudo-differential operators $K(t) = k(t, x, D_x)$ and $H(t) = h(t, y, D_y)$ by

$$k(t,x,\xi) = \exp\left(-\int_0^x \phi(t,s)\,ds\,\xi\langle\xi\rangle^{-1}\right), \quad k'(t,x,\xi) = k(t,x,\xi)^{-1},$$

$$h(t,y,\zeta) = \exp\left(-\int_0^y \psi(t,s)\,ds\,\zeta\langle\zeta\rangle^{-1}\right), \quad h'(t,y,\zeta) = h(t,y,\zeta)^{-1},$$

$$\phi(t,s), \psi(t,s) \in C^1([0,T]; L^1(\mathbf{R})) \cap C([0,T]; \mathscr{B}^\infty(\mathbf{R})),$$

$$\phi(t,s), \psi(t,s) \geqslant 0 \quad \text{for } (t,s) \in [0,T] \times \mathbf{R}.$$

For the convenience, we put

$$B_K(t) = \sup_{(x,\xi)\in\mathbf{R}^2} \sum_{\alpha+\beta\leqslant l} (|\partial_x^\beta \partial_\xi^\alpha k(t,x,\xi)| + |\partial_x^\beta \partial_\xi^\alpha k'(t,x,\xi)|),$$

$$B_H(t) = \sup_{(y,\zeta)\in\mathbf{R}^2} \sum_{\alpha+\beta\leqslant l} (|\partial_y^\beta \partial_\zeta^\alpha h(t,y,\zeta)| + |\partial_y^\beta \partial_\zeta^\alpha h'(t,y,\zeta)|),$$

$$B_\phi^0(t) = \int_{-\infty}^{+\infty} \phi(t,s)\,ds, \quad B_\psi^0(t) = \int_{-\infty}^{+\infty} \psi(t,s)\,ds,$$

$$B_\phi^1(t) = \sup_{x\in\mathbf{R}} \left| \int_0^x \partial_t\phi(t,s)\,ds \right|, \quad B_\psi^1(t) = \sup_{y\in\mathbf{R}} \left| \int_0^y \partial_t\psi(t,s)\,ds \right|,$$

$$B_\phi^\infty(t) = \sup_{x\in\mathbf{R}} \sum_{\alpha\leqslant l} (|\partial_x^\alpha \phi(t,x)| + |\partial_t\partial_x^\alpha \phi(t,x)|),$$

$$B_\psi^\infty(t) = \sup_{y\in\mathbf{R}} \sum_{\alpha\leqslant l} (|\partial_y^\alpha \psi(t,y)| + |\partial_t\partial_y^\alpha \psi(t,y)|)$$

where $l \in \mathbf{N}$ is some large integer (see Remark 1.1). We note that $B_K(t)$ and $B_H(t)$ are greater than one. The property of the transformation $K(t)$ and $H(t)$ are the following.

Lemma 2.1. *$K(t)$ and $H(t)$ are automorphic on $L^2(\mathbf{R}^2)$. More precisely, for any $u \in L^2(\mathbf{R}^2)$ and any $t \in [0, T]$, we have*

$$\|u\| \leqslant CB_K(t)^2(1 + B_\phi^0(t))(1 + B_\phi^\infty(t))(\|K(t)u\| + \|u\|_{-1}), \tag{2.1}$$

$$(\|K(t)u\| + \|u\|_{-1}) \leqslant CB_K(t)\|u\|, \tag{2.2}$$

$$\|u\| \leqslant CB_H(t)^2(1 + B_\psi^0(t))(1 + B_\psi^\infty(t))(\|H(t)u\| + \|u\|_{-1}), \tag{2.3}$$

$$(\|H(t)u\| + \|u\|_{-1}) \leqslant CB_H(t)\|u\|. \tag{2.4}$$

Proof. The inequalities (2.2) and (2.4) follow from the L^2-boundedness theorem of pseudo-differential operators directly. On the other hand, we prove (2.1) by using the identity

$$K'(t)K(t) = 1 + R_0(t),$$

$$\begin{aligned}\sigma(R_0(t)))(x,\xi) &\equiv e^{ix\xi}R_0(t)(e^{-ix\xi}) \\ &= \frac{i}{2\pi}\left(\int_0^x \phi(t,s)\,ds\right)\xi\langle\xi\rangle^{-1} \\ &\quad \times \int_0^1 \iint_{\mathbf{R}\times\mathbf{R}} e^{-iz\eta}\langle\xi+\theta\eta\rangle^{-3} k'(t,x,\xi+\theta\eta) \\ &\quad \times \phi(t,x+z)k(t,x+z,\xi)\,dz\,d\eta\,d\theta.\end{aligned}$$

Similarly we can show (2.3).

Now we obtain the smoothing estimates.

Lemma 2.2. *We put* $f^{\varepsilon} = \partial_t u - (i+\varepsilon)(\partial_x^2 + \partial_y^2)u$, $(\varepsilon \in [0,1])$. *Then there exists a constant* $C_1 > 0$ *which is independent of* $\varepsilon \in [0,1]$, *such that*

$$\begin{aligned}\frac{d}{dt}\|K(t)u(t)\|^2 \leqslant\ & -4(\phi(t,x)\langle D_x\rangle^{1/2}K(t)u(t), \langle D_x\rangle^{1/2}K(t)u(t)) \\ & - 2\varepsilon(\|\partial_x K(t)u(t)\|^2 + \|\partial_y K(t)u(t)\|^2) \\ & + C_1(B_\phi^1(t) + B_\phi^\infty(t))B_{etc}(t)(\|K(t)u(t)\| + \|u(t)\|_{-1})^2 \\ & + 2\,Re(K(t)f^{\varepsilon}(t), K(t)u(t)),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\frac{d}{dt}\|H(t)u(t)\|^2 \leqslant\ & -4(\psi(t,y)\langle D_y\rangle^{1/2}H(t)u(t), \langle D_y\rangle^{1/2}H(t)u(t)) \\ & - 2\varepsilon(\|\partial_x H(t)u(t)\|^2 + \|\partial_y H(t)u(t)\|^2) \\ & + C_1(B_\psi^1(t) + B_\psi^\infty(t))B'_{etc}(t)(\|H(t)u(t)\| + \|u(t)\|_{-1})^2 \\ & + 2\,Re(H(t)f^{\varepsilon}(t), H(t)u(t)),\end{aligned} \tag{2.6}$$

for $u \in C([0,T]; H^2) \cap C^1([0,T]; L^2)$ *and* $t \in [0,T]$, *where*

$$B_{etc}(t) = B_K(t)^3(1 + B_\phi^\infty(t))^3(1 + B_\phi^0(t)),$$

$$B'_{etc}(t) = B_H(t)^3(1 + B_\psi^\infty(t))^3(1 + B_\psi^0(t)).$$

Proof. Here we show (2.5). Simple calculation yields

$$\begin{aligned}K(t)f^{\varepsilon} =\ & \partial_t(K(t)u) - (i+\varepsilon)(\partial_x^2 + \partial_y^2)(K(t)u) \\ & + 2(1 - i\varepsilon)\langle D_x\rangle^{1/2}\phi(t,x)\langle D_x\rangle^{1/2}(K(t)u) + R_1^{\varepsilon}(t)u.\end{aligned} \tag{2.7}$$

The third and the forth terms of the right hand side of (2.7) is the commutator $[K(t), \partial_t - (i+\varepsilon)(\partial_x^2 + \partial_y^2)]$. The former is the principal part of it which gives the smoothig estimate. The later is the lower order term and a L^2-bounded operator. More precisely we have

$$R_1^{\varepsilon}(t) = R_2(t) - (i+\varepsilon)R_3(t),$$

$$\sigma(R_2(t))(x,\xi) = -\partial_t k(t,x,\xi) = \int_0^x \partial_t\phi(t,s)ds\,\xi\langle\xi\rangle^{-1}k(t,x,\xi),$$

$$\begin{aligned}R_3(t) &= [K(t), \partial_x^2] - 2i\langle D_x\rangle^{1/2}\phi(t,x)\langle D_x\rangle^{1/2} \\ &= R_4(t) + R_5(t) + R_6(t)K(t),\end{aligned}$$

$$\sigma(R_4(t))(x,\xi) = (-2i\phi(t,x)\langle\xi\rangle^{-1} - \phi(t,x)^2\xi^2\langle\xi\rangle^{-2} + \partial_x\phi(t,x)\xi\langle\xi\rangle^{-1})k(t,x,\xi),$$

$$\sigma(R_5(t))(x,\xi) = -\frac{i}{2\pi}\xi\langle\xi\rangle^{-1}\int_0^1\iint_{\mathbf{R}\times\mathbf{R}} e^{-iz\eta}(\xi+\theta\eta)\langle\xi+\theta\eta\rangle^{-1}$$
$$\times\,\phi(t,x+z)k(t,x+z,\xi)\,dz\,d\eta\,d\theta,$$
$$\sigma(R_6(t))(x,\xi) = -\frac{1}{2\pi}\langle\xi\rangle^{1/2}\int_0^1\iint_{\mathbf{R}\times\mathbf{R}} e^{-iz\eta}(\xi+\theta\eta)\langle\xi+\theta\eta\rangle^{-3/2}$$
$$\times\,\partial_x\phi(t,x+z)\,dz\,d\eta\,d\theta.$$

We observe the relationship between the operator norm of $R_1^\varepsilon(t)$ and the coefficients of it in detail. Then, the L^2-boundedness theorem of pseudo-differential operators implies that there exists a constant $C_1' > 0$ which is independent of $\varepsilon \in [0,1]$, such that

$$\|R_1^\varepsilon(t)\|_{\mathscr{L}(L^2)} \leqslant C_1'(B_\phi^1(t) + B_\phi^\infty(t))(1 + B_\phi^\infty(t))B_K(t) \tag{2.8}$$

for $t \in [0,T]$ and $\varepsilon \in [0,1]$. We take the L^2-inner product of (2.7) and $K(t)u$. Using (2.1), (2.7) and (2.8), we obtain (2.5).

3. Preliminaries

This section is devoted to the estimates on the nonlinear term $f(u)$. Especially the Gagliardo-Nirenberg inequalities (see e.g., L. Nirenberg [12])

$$\|v\|_{L^\infty(\mathbf{R})} \leqslant C\|\partial_x v\|_{L^2(\mathbf{R})}^{1/2}\|v\|_{L^2(\mathbf{R})}^{1/2} \quad \text{for } v \in H^1(\mathbf{R}^2), \tag{3.1}$$

$$\|u\|_{L^\infty(\mathbf{R})} \leqslant C\sum_{\alpha+\beta=2}\|\partial_x^\alpha\partial_y^\beta u\|_{L^2(\mathbf{R}^2)}^{1/2}\|u\|_{L^2(\mathbf{R}^2)}^{1/2} \quad \text{for } u \in H^2(\mathbf{R}^2), \tag{3.2}$$

play important roles to get the decay estimates of solutions.

Let l be the same integer as in §2. We put

$$\phi(x) = M\int_{-\infty}^{+\infty}|u(x,y)|^2\,dy, \quad \psi(y) = M\int_{-\infty}^{+\infty}|u(x,y)|^2\,dx,$$

with some constant $M > 0$. Similarly we define $K = k(x,D_x)$, $K' = k'(x,D_x)$, $H = h(y,D_y)$ and $H' = h'(y,D_y)$ by

$$k(x,\xi) = \exp\Big(-\int_0^x \phi(x')\,dx'\xi\langle\xi\rangle^{-1}\Big), \quad k'(x,\xi) = k(x,\xi)^{-1},$$

$$h(y,\zeta) = \exp\Big(-\int_0^x \psi(y')\,dy'\zeta\langle\zeta\rangle^{-1}\Big), \quad h'(y,\zeta) = h(y,\zeta)^{-1}.$$

We put $R_0 = K'K - 1$ and we define B_K, B_H, B_ϕ^∞ and B_ψ^∞ in the same way as in §2. To resolve the loss of derivatives, we prepare

Lemma 3.1. *Let m be an integer $\geqslant l+1$. Then we have*

$$\left|\left(K\left\{\left(\int_y^{+\infty} u'\partial_x Pv\,dy'\right)u\right\}, Kw\right)\right|$$

$$\leqslant \frac{1}{2M}\left(1+\sup_{(x,\xi)\in\mathbf{R}^2}|k(x,\xi)p(x,\xi)|^2\right)(\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw)$$

$$+ CB_K^4\int_{-\infty}^{+\infty}\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|^2\,dy\|Kw\|^2 \tag{3.3}$$

for $u\in H^m$ *and* $w\in H^1$, *where*

$$u'=\bar{u},\ P=K',\ v=Kw \quad or \quad u'=u,\ P=K,\ v=\overline{Kw},$$

and

$$\left|\left(H\left\{\left(\int_x^{+\infty} u'\partial_y Qv\,dx'\right)u\right\}, Hw\right)\right|$$

$$\leqslant \frac{1}{2M}\left(1+\sup_{(y,\zeta)\in\mathbf{R}^2}|h(y,\zeta)q(y,\zeta)|^2\right)(\psi(x)\langle D_y\rangle^{1/2}Hw, \langle D_y\rangle^{1/2}Hw)$$

$$+ CB_H^4\int_{-\infty}^{+\infty}\sup_{y\in\mathbf{R}}|\langle D_y\rangle^l u(x,y)|^2\,dx\|Hw\|^2 \tag{3.4}$$

for $u\in H^m$ *and* $w\in H^1$, *where*

$$u'=\bar{u},\ Q=H',\ v=Hw \quad or \quad u'=u,\ Q=H,\ v=\overline{Hw},$$

Proof. We have only to show (3.3). Here we regard u and u' as coefficients. Simple calculation for the commutator gives

$$\left|\left(K\left\{\left(\int_y^{+\infty} u'\partial_x Pv\,dy'\right)u\right\}, Kw\right)\right|$$

$$\leqslant \left|\left(\int_y^{+\infty} r_8(y,y^1,x,D_x)v\,dy', Kw\right)\right|$$

$$+\left|\left(\int_y^{+\infty} u' r_9(x,D_x)\langle D_x\rangle^{1/2}v\,dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right|, \tag{3.5}$$

$$r_8(y,y',x,D_x) = [K, u(x,y)u'(x,y')]\partial_x P$$

$$+ [u(x,y)u'(x,y')K\partial_x P, \langle D_x\rangle^{1/2}]\langle D_x\rangle^{-1/2}$$

$$+ \langle D_x\rangle^{1/2}u(x,y)u'(u,y')(K\partial_x P\langle D_x\rangle^{-1/2} - r_9(x,D_x)\langle D_x\rangle^{1/2}),$$

$$r_9(x,\xi) = ik(x,\xi)p(x,\xi)\xi\langle\xi\rangle^{-1}.$$

The L^2-boundedness theorem of pseudo-differential operators on $\mathbf{R}$ yields

$$\|r_8(y,y',\cdot,\cdot)\|_{\mathscr{L}(L^2(\mathbf{R}))} \leqslant CB_K^2\left(\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|\right)\left(\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y')|\right).$$

Then we have

$$\begin{aligned}
&\left|\left(\int_y^{+\infty} r_8(y,y',x,D_x)v\,dy', Kw\right)\right| \\
&\quad\leqslant \iiint |r_8(y,y',x,D_x)v(x,y')|\,|Kw(x,y)|dxdydy' \\
&\quad\leqslant \iint\left(\int |r_8(y,y',x,D_x)v(x,y')|^2dx\right)^{1/2}\left(\int |Kw(x,y)|^2dx\right)^{1/2}dydy' \\
&\quad\leqslant CB_K^2\iint\left(\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|\right)\left(\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y')|\right) \\
&\qquad\times\left(\int |v(x,y')|^2dx\right)^{1/2}\left(\int |Kw(x,y)|^2dx\right)^{1/2}dydy' \\
&\quad= CB_K^2\left\{\int \sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|\left(\int |Kw(x,y)|^2\,dx\right)^{1/2}dy\right\}^2 \qquad (3.6)\\
&\quad\leqslant CB_K^2\left(\int \sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|^2\,dy\right)\|Kw\|^2. \qquad (3.7)
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\left|\left(\int_y^{+\infty} u' r_9(x,D_x)\langle D_x\rangle^{1/2}v\,dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right| \\
&\quad\leqslant \iiint |u(x,y')|\,|r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')|\,|u(x,y)|\,|\langle D_x\rangle^{1/2}Kw(x,y)|dxdydy' \\
&\quad\leqslant \int\left(\int |u(x,y')|^2\,dy'\right)^{1/2}\left(\int |r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')|^2dy'\right)^{1/2} \\
&\qquad\times\left(\int |u(x,y)|^2\,dy\right)^{1/2}\left(\int |\langle D_x\rangle^{1/2}Kw(x,y)|^2\,dy\right)^{1/2}dx \\
&\quad= \frac{1}{M}\int \phi(x)\left(\int |r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')|^2\,dy'\right)^{1/2} \\
&\qquad\times\left(\int |\langle D_x\rangle^{1/2}Kw(x,y)|^2\,dy\right)^{1/2}dx \\
&\quad\leqslant \frac{1}{M}\|\phi(x)^{1/2}r_9(x,D_x)\langle D_x\rangle^{1/2}v\|\,\|\phi(x)^{1/2}\langle D_x\rangle^{1/2}Kw\|
\end{aligned}$$

$$\leqslant \frac{1}{2M}\{\|\phi(x)^{1/2}r_9(x,D_x)\langle D_x\rangle^{1/2}v\|^2 + \|\phi(x)^{1/2}\langle D_x\rangle^{1/2}Kw\|^2\}$$

$$= \frac{1}{2M}\{\mathrm{Re}(r_9(x,D_x)^*\phi(x)r_9(x,D_x)\langle D_x\rangle^{1/2}v, \langle D_x\rangle^{1/2}v)$$

$$+ (\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw)\}.$$

We put

$$r_{10}(x,\xi) = |k(x,\xi)p(x,\xi)|^2\xi^2\langle\xi\rangle^{-2}.$$

We note that $\phi(x)r_{10}(x,\xi)$ is the symbol of the principal part of $r_9(x,D_x)^*\phi(x)r_9(x,D_x)$. Then we have

$$\left|\left(\int_y^{+\infty} u' r_9(x,D_x)\langle D_x\rangle^{1/2}v\,dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right|$$

$$\leqslant \frac{1}{2M}\{\mathrm{Re}(\phi(x)r_{10}(x,D_x)\langle D_x\rangle^{1/2}v, \langle D_x\rangle^{1/2}v)$$

$$+ (\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw)\} + CB_\phi^\infty B_K^4\|Kw\|^2.$$

We note the trivial inequality

$$\phi(x)\sup_{(x,\xi)\in\mathbf{R}^2}|k(x,\xi)p(x,\xi)|^2 \geqslant \phi(x)r_{10}(x,\xi).$$

Using the sharp Gårding inequality we get

$$\left|\left(\int_y^{+\infty} u' r_9(x,D_x)\langle D_x\rangle^{1/2}v\,dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right|$$

$$\leqslant \frac{1}{2M}\left\{1 + \sup_{(x,\xi)\in\mathbf{R}^2}|k(x,\xi)p(x,\xi)|^2\right\}(\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw)$$

$$+ CB_\phi^\infty B_K^4\|Kw\|^2$$

$$\leqslant \frac{1}{2M}\left\{1 + \sup_{(x,\xi)\in\mathbf{R}^2}|k(x,\xi)p(x,\xi)|^2\right\}(\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw)$$

$$+ C\int\sup_{x\in\mathbf{R}}|\langle D_x\rangle^l u(x,y)|^2\,dy B_K^4\|Kw\|^2. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.5), we obtain (3.3).

To prove Theorem 1.1, we prepare the following two lemmata.

Lemma 3.2. *Let m be an integer greater than or equal to 2. Then there exists a constant $C > 0$ depending only on m such that*

$$\|f_0(u)\|_m + \sum_{\substack{\alpha+\beta\leqslant m \\ \alpha\leqslant m-1}} \|\partial_x^\alpha \partial_y^\beta f_1(u)\| + \sum_{\substack{\alpha+\beta\leqslant m \\ \alpha\leqslant m-1}} \|\partial_x^\alpha \partial_y^\beta f_2(u)\| \leqslant C\|u\|^2_{[(m+1)/2]+1}\|u\|_m, \quad (3.9)$$

$$\sum_{j=0}^{2} \|f_j(u) - f_j(v)\|_{m-1} \leqslant C(\|u\|_m + \|v\|_m)^2 \|u - v\|_m \quad (3.10)$$

for $u, v \in H^m$, and

$$\left\|\partial_x^m f_1(u) - \left(\int_y^{+\infty} \bar{u}\partial_x^{m+1} u\, dy'\right) u - \left(\int_y^{+\infty} u\partial_x^{m+1} \bar{u}\, dy'\right) u\right\|$$
$$+ \left\|\partial_y^m f_2(u) - \left(\int_x^{+\infty} \bar{u}\partial_y^{m+1} u\, dx'\right) u - \left(\int_x^{+\infty} u\partial_y^{m+1} \bar{u}\, dx'\right) u\right\|$$
$$\leqslant C\|u\|^2_{[(m+1)/2]+1}\|u\|_m \quad (3.11)$$

for $u \in H^{m+1}$.

Lemma 3.3. *Let m be an integer greater than or equal to $l + 1$. Then we have*

$$\left|\left(K\left[\left\{\int_y^{+\infty} (\bar{u}\partial_x^{m+1} u + u\partial_x^{m+1}\bar{u})\, dy'\right\} u\right], K\partial_x^m u\right)\right|$$
$$\leqslant \frac{1}{2M}(3 + e^{4M\|u\|^2})(\phi(x)\langle D_x\rangle^{1/2} K\partial_x^m u, \langle D_x\rangle^{1/2} K\partial_x^m u)$$
$$+ CB_K^5\|u\|^2_{l+1}\|K\partial_x^m u\|(\|K\partial_x^m u\| + \|u\|_{m-1}), \quad (3.12)$$

$$\left|\left(H\left[\left\{\int_y^{+\infty} (\bar{u}\partial_y^{m+1} u + u\partial_y^{m+1}\bar{u})\, dy'\right\} u\right], H\partial_y^m u\right)\right|$$
$$\leqslant \frac{1}{2M}(3 + e^{4M\|u\|^2})(\psi(x)\langle D_y\rangle^{1/2} H\partial_y^m u, \langle D_y\rangle^{1/2} H\partial_y^m u)$$
$$+ CB_H^5\|u\|^2_{l+1}\|H\partial_y^m u\|(\|H\partial_y^m u\| + \|u\|_{m-1}) \quad (3.13)$$

for any $u \in H^{m+1}$.

Proof of Lemma 3.2. The Gagliardo-Nirenberg inequality (3.2) implies

$$\|f_0(u)\|_m \leqslant C\|u\|^2_{L^\infty}\|u\|_m. \quad (3.14)$$

Let α and β be non-negative integers satisfying $\alpha + \beta \leqslant m$ and $\alpha \leqslant m - 1$. We have

$$\partial_x^\alpha \partial_y^\beta f_1(u) = g_1^{\alpha\beta}(u) + g_2^{\alpha\beta}(u)$$

$$g_1^{\alpha\beta}(u) = -\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3 \leqslant \alpha}} \sum_{\beta_1+\beta_2+\beta_3=\beta-1} \frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!} \frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}$$

$$\times \partial_x^{\alpha_1}\partial_y^{\beta_1} u \partial_x^{\alpha_2}\partial_y^{\beta_2}\bar{u}\partial_x^{\alpha_3}\partial_y^{\beta_3} u$$

$$g_2^{\alpha\beta}(u) = -\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3 \leqslant \alpha}} \frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!} \left(\int_y^{+\infty} \partial_x^{\alpha_1} u \partial_x^{\alpha_2}\bar{u}\, dy' \right) \partial_x^{\alpha_3}\partial_y^\beta u.$$

In the same way as (3.14), we get

$$\|g_1^{\alpha\beta}(u)\| \leqslant C\|u\|_{L_\infty}^2 \|u\|_m. \tag{3.15}$$

Using the Schwarz inequality with respect to y', we have

$$\|g_2^{\alpha\beta}(u)\| \leqslant \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3 \leqslant \alpha}} \frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!}$$

$$\times \left\{ \int \left(\int |\partial_x^{\alpha_1} u|^2\, dy_1 \right) \left(\int |\partial_x^{\alpha_2} u|^2\, dy_2 \right) \left(\int |\partial_x^{\alpha_3}\partial_y^\beta u|^2\, dy_3 \right) dx \right\}^{1/2}$$

$$\leqslant C \sup_{x\in\mathbf{R}} \int |\langle D_x; D_y\rangle^{[(m+1)/2]} u|^2\, dy \|u\|_m$$

$$\leqslant C\|u\|_{[(m+1)/2]+1}^2 \|u\|_m. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\sum_{\substack{\alpha+\beta\leqslant m\\ \alpha\leqslant m-1}} \|\partial_x^\alpha \partial_y^\beta f_1(u)\| \leqslant C\|u\|_{[(m+1)/2]+1}^2 \|u\|_m. \tag{3.17}$$

In the same way, we have

$$\sum_{\substack{\alpha+\beta\leqslant m\\ \beta\leqslant m-1}} \|\partial_x^\alpha \partial_y^\beta f_2(u)\| \leqslant C\|u\|_{[(m+1)/2]+1}^2 \|u\|_m. \tag{3.18}$$

Summing up (3.14), (3.17) and (3.18), we obtain (3.9). Similarly we can get (3.10) and (3.11).

Proof of Lemma 3.3. Here we note the identities

$$\partial_x^{m+1} u = \partial_x K' K \partial_x^m u - \partial_x R_0 \partial_x^m u, \quad \partial_x^{m+1}\bar{u} = \partial_x K \overline{K\partial_x^m u} - \overline{\partial_x R_0 \partial_x^m u},$$

where $R_0 = K'K - 1 \in \mathscr{L}(H^{-3}, L^2)$. Using (3.3), we obtain (3.12). Similarly we get (3.13) by (3.4).

Now we obtain the estimates on the nonlinear term $f(u)$ in order to prove Theorem 1.2. Using the Gagliardo-Nirenberg inequality (3.2), we have

$$\|u\|_{L^\infty} \leqslant C(1+t)^{-1} \sum_{\alpha'+\beta'=2} \|J_x^{\alpha'} J_y^{\beta'} u\|^{1/2} \|u\|^{1/2} \tag{3.19}$$

for $u \in H^2 \cap H^{0,2}$ (see [2], [8] or [13] for instance). Let us introduce

$$X_{m,n}(t) = \sum_{\substack{\alpha+\beta \leqslant m \\ \alpha'+\beta'=n}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} u(t)\|, \quad m, n \in \mathbf{Z}_+$$

for $u \in C[0,T]; H^{m+n} \cap H^{m,n})$, where T is a positive time. Here we remark some commutation relations concerning J_x and J_y, those are

$$[\partial_t - i(\partial_x^2 + \partial_y^2), J_x] = [\partial_t - i(\partial_x^2 + \partial_y^2), J_y] = 0,$$

$$[J_x, J_y] = [\partial_x, J_y] = [\partial_y, J_x] = 0, \quad [\partial_x, J_x] = [\partial_y, J_y] = 1.$$

We show the following two lemmata which correspond to Lemmata 3.2 and 3.3 respectively.

Lemma 3.4. *Let m be an integer greater then or equal to 6. Then we have*

$$\begin{aligned}
&\sum_{\substack{\alpha+\beta+\alpha'+\beta' \leqslant m \\ \alpha'+\beta' \leqslant 5}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} f_0(u(t))\| \\
&\quad + \sum_{\substack{\alpha+\beta+\alpha'+\beta' \leqslant m \\ \alpha'+\beta' \leqslant 5 \\ \alpha+\alpha' \leqslant m-1 \\ \alpha' \leqslant 4}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} f_1(u(t))\| \\
&\quad + \sum_{\substack{\alpha+\beta+\alpha'+\beta' \leqslant m \\ \alpha'+\beta' \leqslant 5 \\ \beta+\beta' \leqslant m-1 \\ \beta' \leqslant 4}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} f_2(u(t))\| \\
&\quad \leqslant C(1+t)^{-2} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t),
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\sum_{\substack{\alpha+\alpha'=m \\ \alpha' \leqslant 5}} \|\partial_x^\alpha J_x^{\alpha'} f_1(u(t)) - \tilde{f}_1^{\alpha\alpha'}(u(t))\| \\
&\quad + \sum_{\substack{\beta+\beta'=m \\ \beta' \leqslant 5}} \|\partial_x^\beta J_x^{\beta'} f_2(u(t)) - \tilde{f}_2^{\beta\beta'}(u(t))\| \\
&\quad + \sum_{\alpha+\beta \leqslant m-6} \|\partial_x^\alpha \partial_y^\beta J_x^5 f_1(u(t))\| \\
&\quad + \sum_{\alpha+\beta \leqslant m-6} \|\partial_x^\alpha \partial_y^\beta J_y^5 f_2(u(t))\| \\
&\quad \leqslant C(1+t)^{-1} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t)
\end{aligned} \tag{3.21}$$

where

$$\tilde{f}_1^{\alpha\alpha'}(u) = \int_y^{+\infty} (\partial_x^{\alpha+1} J_x^{\alpha'} u\bar{u} + (-1)^{\alpha'} u\overline{\partial_x^{\alpha+1} J_x^{\alpha'} u})\, dy' u,$$

$$\tilde{f}_2^{\beta\beta'}(u) = \int_x^{+\infty} (\partial_y^{\beta+1} J_y^{\beta'} u\bar{u} + (-1)^{\beta'} u\overline{\partial_y^{\beta+1} J_y^{\beta'} u})\, dx' u.$$

Lemma 3.5. *Let m be an integer greater than or equal to* $\max(l+1, 6)$. *Then we have*

$$\begin{aligned}
&\sum_{\substack{\alpha+\alpha'=m\\ \alpha'\leqslant 5}} |(K(t)\tilde{f}_1^{\alpha\alpha'}(u(t)), K(t)\partial_x^{\alpha} J_x^{\alpha'} u(t))| \\
&\quad \leqslant \frac{1}{2M}(3 + e^{4M\|u(t)\|^2}) \\
&\qquad \times (\phi(t,x)\langle D_x\rangle^{1/2} K(t)(\partial_x^{\alpha} J_x^{\alpha'} u(t)), \langle D_x\rangle^{1/2} K(t)(\partial_x^{\alpha} J_x^{\alpha'} u(t)),) \\
&\qquad + CB_K(t)^4 (1+t)^{-1} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \|K(t)(\partial_x^{\alpha} J_x^{\alpha'} u(t))\|^2 \\
&\qquad + CB_K(t)^3 B_{\phi}^{\infty}(t)(1+t)^{-1} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \\
&\qquad \times \|K(t)(\partial_x^{\alpha} J_x^{\alpha'} u(t))\| \, \|\partial_x^{\alpha} J_x^{\alpha'} u(t)\|,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\sum_{\substack{\beta+\beta'=m\\ \beta'\leqslant 5}} |(H(t)\tilde{f}_2^{\beta\beta'}(u(t)), H(t)\partial_y^{\beta} J_y^{\beta'} u(t))| \\
&\quad \leqslant \frac{1}{2M}(3 + e^{4M\|u(t)\|^2}) \\
&\qquad \times (\psi(t,y)\langle D_y\rangle^{1/2} H(t)(\partial_y^{\beta} J_y^{\beta'} u(t)), \langle D_y\rangle^{1/2} H(t)(\partial_y^{\beta} J_y^{\beta'} u(t)),) \\
&\qquad + CB_H(t)^4 (1+t)^{-1} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \|H(t)(\partial_y^{\beta} J_y^{\beta'} u(t))\|^2 \\
&\qquad + CB_H(t)^3 B_{\psi}^{\infty}(t)(1+t)^{-1} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \\
&\qquad \times \|H(t)(\partial_y^{\beta} J_y^{\beta'} u(t))\| \, \|\partial_y^{\beta} J_y^{\beta'} u(t)\|
\end{aligned} \tag{3.23}$$

$$\textit{for } t \in [0,T], \quad u \in \bigcap_{j=0}^{5} C([0,T]; H^{m+1-j,j}).$$

Proof of Lemma 3.4. Elementary calculus gives

$$\begin{aligned}
&\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m\\ \alpha'+\beta'\leqslant 5}} \|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}f_0(u(t))\| \\
&\leqslant C(1+t)^{-2}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\sum_{j=0}^{5}X_{m-j,j}(t) \qquad (3.24)
\end{aligned}$$

(see e.g., [2] or [8]). Let α,β,α' and β' be non-negative integers satisfying $\alpha+\beta+\alpha'+\beta'\leqslant m, \alpha'+\beta'\leqslant 5, \alpha+\alpha'\leqslant m-1$ and $\alpha'\leqslant 4$. We decompose $\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}f_1(u)$ into two parts.

$$\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}f_1(u) = g_3^{\alpha\beta\alpha'\beta'}(u) + g_4^{\alpha\beta\alpha'\beta'}(u).$$

One is a local nonlinear term and another is non-local, those are

$$\begin{aligned}
g_3^{\alpha\beta\alpha'\beta'}(u) = &-\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\alpha'_1+\alpha'_2+\alpha'_3=\alpha'+1}\ \sum_{\beta_1+\beta_2+\beta_3=\beta}\ \sum_{\beta'_1+\beta'_2+\beta'_3=\beta'-1} \\
&\times\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha'_1+\alpha'_2)}{\alpha'_1!\alpha'_2!\alpha'_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}\frac{\beta'!}{\beta'_1!\beta'_2!\beta'_3!(\beta'-\beta'_3)}(-1)^{\alpha'_2+\beta'_2} \\
&\times\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha'_1}J_y^{\beta'_1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha'_2}J_y^{\beta'_2}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha'_3}J_y^{\beta'_3}u \\
&-\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\alpha'_1+\alpha'_2+\alpha'_3=\alpha'}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1} \\
&\times\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha'_1+\alpha'_2)}{\alpha'_1!\alpha'_2!\alpha'_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha'_2} \\
&\times\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha'_1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha'_2}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha'_3}J_y^{\beta'}u \\
&+\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\substack{\alpha'_1+\alpha'_2+\alpha'_3=\alpha'\\ \alpha'_1\geqslant 1}}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1} \\
&\times\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha'_1+\alpha'_2)}{\alpha'_1!\alpha'_2!\alpha'_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha'_2} \\
&\times\alpha'_1\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha'_1-1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha'_2}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha'_3}J_y^{\beta'}u \\
&+\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\substack{\alpha'_1+\alpha'_2+\alpha'_3=\alpha'\\ \alpha'_2\geqslant 1}}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1} \\
&\times\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha'_1+\alpha'_2)}{\alpha'_1!\alpha'_2!\alpha'_3!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha'_2} \\
&\times\alpha'_2\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha'_1-1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha'_2-1}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha'_3}J_y^{\beta'}u,
\end{aligned}$$

$$g_4^{\alpha\beta\alpha'\beta'}(u) = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \sum_{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'+1} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}$$

$$\times (-1)^{\alpha_2'} \frac{1}{2i(1+t)} \left(\int_y^{+\infty} \partial_x^{\alpha_1} J_x^{\alpha_1'} u \overline{\partial_x^{\alpha_2} J_x^{\alpha_2'} u}\, dy' \right) \partial_x^{\alpha_3} \partial_y^{\beta} J_x^{\alpha_3'} J_y^{\beta'} u.$$

Here we modified $g_4^{\alpha\beta\alpha'\beta'}(u)$ by using the null gauge condition (1.9). In the same way as (3.24), we get

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m\\ \alpha'+\beta'\leqslant 5\\ \alpha+\alpha'\leqslant m-1\\ \alpha'\leqslant 4}} \|g_3^{\alpha\beta\alpha'\beta'}(u(t))\|$$

$$\leqslant C(1+t)^{-2} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t). \tag{3.25}$$

Using the Gagliardo-Nirenberg inequality (3.1), we have

$$\int \sup_{x\in\mathbf{R}} |u(t,x,y)|^2\, dy \leqslant C(1+t)^{-1} \sum_{\alpha+\beta\leqslant 1} \|\partial_x^{\alpha} J_x^{\beta} u(t)\|^2. \tag{3.26}$$

Combining (3.26) and the same calculation as in (3.16), we get

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m\\ \alpha'+\beta'\leqslant 5\\ \alpha+\alpha'\leqslant m-1\\ \alpha'\leqslant 4}} \|g_4^{\alpha\beta\alpha'\beta'}(u(t))\|$$

$$\leqslant C(1+t)^{-2} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t) \tag{3.27}$$

Summing up (3.25) and (3.27), we obtain

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m\\ \alpha'+\beta'\leqslant 5\\ \alpha+\alpha'\leqslant m-1\\ \alpha'\leqslant 4}} \|\partial_x^{\alpha} \partial_y^{\beta} J_x^{\alpha'} J_y^{\beta'} f_1(u(t))\|$$

$$\leqslant C(1+t)^{-2} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t). \tag{3.28}$$

Similarly we can get

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m\\ \alpha'+\beta'\leqslant 5\\ \beta+\beta'\leqslant m-1\\ \beta'\leqslant 4}} \|\partial_x^{\alpha} \partial_y^{\beta} J_x^{\alpha'} J_y^{\beta'} f_2(u(t))\|$$

$$\leqslant C(1+t)^{-2} \left(\sum_{j=0}^{4} X_{m-j-1,j}(t) \right)^2 \sum_{j=0}^{5} X_{m-j,j}(t). \tag{3.29}$$

Combining (3.24), (3.28) and (3.29), we obtain (3.20). The proof of (3.21) is almost same as that of (3.20). Here we remark that we do not use the null gauge condition (1.9) to get (3.21).

Proof of Lemma 3.5. The proof of Lemma 3.5 is basically same as that of Lemma 3.3. Of course, to get the time decay, we make use of the same technique as in the proof of Lemma 3.4.

4. Proof of Theorem 1.1

We prove Theorem 1.1 by the parabolic regularization and the uniform estimates which follow from Lemma 2.2. First we consider

$$\partial_t u^\varepsilon - (i+\varepsilon)(\partial_x^2 + \partial_y^2)u^\varepsilon = f(u^\varepsilon) \quad \text{in } (0,\infty) \times \mathbf{R}^2, \tag{4.1}$$

$$u^\varepsilon(0,x,y) = u_0(x,y) \quad \text{in } \mathbf{R}^2, \tag{4.2}$$

where $\varepsilon \in (0,1]$. We remark that the initial data u_0 is independent of $\varepsilon \in (0,1]$. Since the elliptic term $-\varepsilon(\partial_x^2 + \partial_y^2)$ gains the regularity of order 1 and resolves the loss of derivatives, we obtain the local existence theorem for (4.1)–(4.2).

Lemma 4.1. *Let m be an integer greater than or equal to 2. For any $u_0 \in H^m$, there exists a time $T_\varepsilon = T(\varepsilon, \|u_0\|_2) > 0$ such that the initial value problem (4.1)–(4.2) possesses a unique solution $u^\varepsilon \in C([0,T_\varepsilon); H^m)$. Moreover the mapping $u_0 \mapsto u^\varepsilon$ is continuous between the above spaces.*

Proof. Let $\{U^\varepsilon(t)\}_{t\geqslant 0}$ be a semigroup generated by the linear part of (4.1). We consider the integral equation which is equivalent to (4.1)–(4.2)

$$u^\varepsilon(t) = U^\varepsilon(t)u_0 + \int_0^t U^\varepsilon(t-\tau)f(u^\varepsilon(\tau))\,d\tau. \tag{4.3}$$

Using the smoothing property $U^\varepsilon(t)$ and the nonlinear estimates (3.9) and (3.10), we can prove Lemma 4.1. Here we omit the rigorous proof.

Secondly we prove the existence of a solution to (1.1)–(1.2) by the uniform estimates on $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$. More precisely we show that there exists a time $T > 0$ which is independent $\varepsilon \in (0,1]$ such that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. Let $m_1 = l+3$ where l is the same integer as in §2, and let m be an integer greater then or equal to m_1. We put $M = ae$. We define

$$K^\varepsilon(t) = k^\varepsilon(t,x,D_x), \quad H^\varepsilon(t) = h^\varepsilon(t,y,D_y),$$

$$k^\varepsilon(t,x,\xi) = \exp\left(-\int_0^x \phi^\varepsilon(t,x')\,dx'\xi\langle\xi\rangle^{-1}\right),$$

$$h^\varepsilon(t,y,D_y) = \exp\left(-\int_0^y \psi^\varepsilon(t,y')\,dy'\zeta\langle\zeta\rangle^{-1}\right),$$

$$\phi^\varepsilon(t,x) = M\int |u^\varepsilon(t,x,y)|^2 dy, \quad \psi^\varepsilon(t,y) = M\int |u^\varepsilon(t,x,y)|^2\, dx,$$

$$N_m^\varepsilon(t) = \sum_{\substack{\alpha+\beta\leqslant m\\ \alpha,\beta\leqslant m-1}} \|\partial_x^\alpha \partial_y^\beta u^\varepsilon(t)\| + \|K^\varepsilon(t)\partial_x^m u^\varepsilon(t)\| + \|H^\varepsilon(t)\partial_y^m u^\varepsilon(t)\|.$$

Since the initial data u_0 is independent of $\varepsilon \in (0,1], N_m^\varepsilon(0)$ is also independent of $\varepsilon \in (0,1]$ and then we denote them by the same notation N_m. It follows from simple calculation that there exists an increasing function $A(\cdot)$ on $[0,+\infty)$ such that

$$\begin{aligned} &B_{K^\varepsilon}(t), B_{H^\varepsilon}(t), B^0_{\phi^\varepsilon}(t), B^0_{\psi^\varepsilon}(t), B^1_{\phi^\varepsilon}(t), B^1_{\psi^\varepsilon}(t), B^\infty_{\phi^\varepsilon}(t), B^\infty_{\psi^\varepsilon}(t)\\ &\leqslant A(\|u^\varepsilon(t)\|_{m_1-1}) \leqslant A(N^\varepsilon_{m_1}(t)). \end{aligned}$$

We put

$$T_\varepsilon^* = \left\{ 0 \leqslant T < \frac{1/2\sqrt{ae} - \|u_0\|}{8C_2(N_{m_1})^3} \,\middle|\, N^\varepsilon_{m_1}(t) < 2N_{m_1}, 0 \leqslant t < T \right\},$$

where C_2 is a positive constant appearing in the estimate $\|f(u)\| \leqslant C_2\|u\|_2^3$. Lemma 4.1 shows $T_\varepsilon^* > 0$. By the integral equation (4.3), we have

$$\begin{aligned} \|u^\varepsilon(t)\| &\leqslant \|u_0\| + \int_0^t \|f(u^\varepsilon(\tau))\|\, d\tau\\ &\leqslant \|u_0\| + C_2\int_0^t \|u^\varepsilon(\tau)\|_2^3\, d\tau\\ &\leqslant \|u_0\| + 8C_2(N_{m_1})^3 t < \frac{1}{2\sqrt{ae}} \quad \text{for } 0 \leqslant t < T_\varepsilon^*. \end{aligned}$$

Here we note that the local well-posedness justifies the validity of the following energy estimates. Let n be an integer which ranges in $m_1 \leqslant n \leqslant m$. Using (3.9), we get

$$\begin{aligned} \frac{d}{dt}\sum_{\substack{\alpha+\beta\leqslant n\\ \alpha,\beta\leqslant n-1}} \|\partial_x^\alpha\partial_y^\beta u^\varepsilon(t)\|^2 &\leqslant 2\sum_{\substack{\alpha+\beta\leqslant n\\ \alpha,\beta\leqslant n-1}} \|\partial_x^\alpha\partial_y^\beta f(u^\varepsilon(t))\|\,\|\partial_x^\alpha\partial_y^\beta u^\varepsilon(t))\|\\ &\leqslant C\|u^\varepsilon(t)\|^2_{n-1} N_n^\varepsilon(t)^2. \end{aligned} \tag{4.4}$$

The linear estimate (2.5) gives

$$\begin{aligned} \frac{d}{dt}\|K^\varepsilon\partial_x^n u^\varepsilon(t)\|^2 \leqslant &-4(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t))\\ &+ C\|u^\varepsilon(t)\|^2_{n-1}N_n^\varepsilon(t)^2 + 2\,\mathrm{Re}(K^\varepsilon(t)\partial_x^n f(u^\varepsilon(t)), K^\varepsilon(t)\partial_x^n u^\varepsilon(t)). \end{aligned} \tag{4.5}$$

Here we decompose $\partial_x^n f(u^\varepsilon(t))$ in the last term of the right hand side of (4.5) as

$$\partial_x^n f(u^\varepsilon(t)) = g_5^n(u^\varepsilon(t)) + g_6^n(u^\varepsilon(t)),$$

$$g_5^n(u^\varepsilon(t)) = a_0\partial_x^n f_0(u^\varepsilon(t)) + a_2\partial_x^n f_2(u^\varepsilon(t))$$

$$+ a_1\left\{\partial_x^n f_1(u^\varepsilon(t)) - \left(\int_y^{+\infty}(\partial_x^{n+1}u^\varepsilon(t)\bar{u}^\varepsilon(t) + u^\varepsilon(t)\partial_x^{n+1}\bar{u}^\varepsilon(t))\,dy'\right)u^\varepsilon(t)\right\},$$

$$g_6^n(u^\varepsilon(t)) = a_1\left\{\left(\int_y^{+\infty}(\partial_x^{n+1}u^\varepsilon(t)\bar{u}^\varepsilon(t) + u^\varepsilon(t)\partial_x^{n+1}\bar{u}^\varepsilon(t))\,dy'\right)u^\varepsilon(t)\right\}.$$

Using the nonlinear estimates (3.11) and (3.12), we have

$$\begin{aligned}
&2\,\mathrm{Re}(K^\varepsilon(t)\partial_x^n f(u^\varepsilon(t)), K^\varepsilon(t)\partial_x^n u^\varepsilon(t)) \\
&\quad \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\
&\qquad + \frac{a}{M}(3 + e^{4M\|u^\varepsilon\|})(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)) \qquad (4.6)\\
&\quad \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\
&\qquad + \left(\frac{3}{e} + 1\right)(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)) \qquad (4.7)
\end{aligned}$$

for $t \in [0, T_\varepsilon^*)$. Substituting (4.7) into (4.5), we get

$$\begin{aligned}
&\frac{d}{dt}\|K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\|^2 \\
&\quad \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\
&\qquad - 3(1 - 1/e)(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)) \\
&\quad \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2. \qquad (4.8)
\end{aligned}$$

Similarly, using (2.6), (3.11) and (3.13), we have

$$\frac{d}{dt}\|H^\varepsilon(t)\partial_y^n u^\varepsilon(t)\|^2 \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2. \qquad (4.9)$$

Summing up (4.4), (4.8) and (4.9), we obtain

$$\frac{d}{dt}N_n^\varepsilon(t) \leqslant C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t) \quad \text{for } t \in [0, T_\varepsilon^*).$$

The Gronwall inequality implies

$$N_n^\varepsilon(t) \leqslant N_n \exp\left(C\int_0^t \|u^\varepsilon(\tau)\|_{n-1}^2 d\tau\right) \quad \text{for } t \in [0, T_\varepsilon^*). \qquad (4.10)$$

In particular, if $n = m_1$ and $t = T_\varepsilon^*$, then the energy estimate (4.10) becomes $2 \leqslant$

$\exp(4CT_\varepsilon^*(N_{m_1})^2)$. By the definition of T_ε^*, we obtain

$$T_\varepsilon^* \geqslant T \equiv \min\left(\frac{\log 2}{4C(N_{m_1})^2}, \frac{1/2\sqrt{ae} - \|u_0\|}{8C_2(N_{m_1})^3}\right) > 0.$$

This means that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^{m_1})$. Using (4.10) successively, we show that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. Then the standard compactness arguments imply that there exist a subsequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in L^\infty(0,T;H^m)$ such that

$$u^\varepsilon \xrightarrow{\mathrm{W}^*} u \quad \text{in } L^\infty(0,T;H^m) \quad \text{as } \varepsilon \downarrow 0,$$

$$u^\varepsilon \longrightarrow u \quad \text{in } C([0,T];H^{m-\delta}_{\mathrm{loc}}),\ \delta > 0 \quad \text{as } \varepsilon \downarrow 0.$$

The second one of the above implies that u satisfies (1.2). Since u belongs to $L^\infty(0,T;H^m)$ and $C([0,T];H^{m-\delta}_{\mathrm{loc}})$, it is easy to see that u is also in $C_{\mathrm{W}}([0,T];H^m)$.

To see u is a solution to the equation (1.1), we have only to check

$$f_j(u^\varepsilon) \to f_j(u) \quad \text{in } \mathscr{D}'((0,T)\times\mathbf{R}^2) \quad \text{as } \varepsilon\downarrow 0, \quad j = 0,1,2. \tag{4.11}$$

It is easy to see the case of $j = 0$ in (4.11). Here we show the cases of $j = 1$. For this purpose, we introduce some classes of finite Radon measures as follows.

$$\mathscr{B}_0^0 \equiv C_0((0,T)\times\mathbf{R}^2) \text{ equipped with } \mathscr{B}^0 \text{ norm},$$

$$\tilde{\mathscr{B}}^0 \equiv \{v(t,x,y) \in \mathscr{B}^0((0,T)\times\mathbf{R}^2) \mid \exists R > 0, \exists w(t,x) \in C_0((0,T)\times(-R,+R)) \quad \text{s.t.}$$

$$\operatorname{supp}[v] \subset (0,T)\times(-R,+R)\times(-R,+\infty), \quad v(t,x,y) = w(t,x) \text{ for } y > R\}$$

equipped with $\mathscr{B}^0$ norm,

$\mathscr{M}$ and $\tilde{\mathscr{M}}$ denote the topological dual of $\mathscr{B}_0^0$ and $\tilde{\mathscr{B}}^0$ respectively. Clearly $\mathscr{B}_0^0$ and $\tilde{\mathscr{B}}^0$ are separable and not complete. $\mathscr{B}_0^0 \subset \tilde{\mathscr{B}}^0$ implies $\tilde{\mathscr{M}} \subset \mathscr{M}$. The properties of $\tilde{\mathscr{M}}$ are the following.

Lemma 4.2. *We assume that μ and ν belong to $\tilde{\mathscr{M}}$ and that $\mu = \nu$ in $\mathscr{D}'((0,T)\times\mathbf{R}^2)$. Then $\mu = \nu$ in $\tilde{\mathscr{M}}$.*

Proof. Lemma 4.2 is well known if we replace $\tilde{\mathscr{M}}$ by $L^1_{\mathrm{loc}}((0,T)\times\mathbf{R}^2)$. Since $\mathscr{D}$ is dense in $\mathscr{B}_0^0, \mu = \nu$ in $\mathscr{D}'$ implies $\mu = \nu$ in $\mathscr{M}$. It is enough to consider the case of $\mu, \nu \geqslant 0$ and to prove

$$\langle \mu - \nu, v\rangle = 0 \qquad \text{for any } v \in \tilde{\mathscr{B}}^0, v \geqslant 0. \tag{4.12}$$

Let $\{\gamma_n\}_{n\geqslant 1}$ be a sequence of functions belonging to $C_0(\mathbf{R})$ and satisfying $1 \geqslant \gamma_n \geqslant 0$ and

$$\gamma_n(y) = \begin{cases} 1 & (|y| \leqslant n) \\ 0 & (|y| \geqslant n+1). \end{cases}$$

Clearly, $\gamma_n(y)v(t,x,y)$ is in $\mathscr{B}_0^0$ and

$$0 \leqslant \gamma_1(y)v(t,x,y) \leqslant \gamma_2(y)v(t,x,y) \leqslant \cdots \leqslant \gamma_n(y)v(t,x,y) \leqslant \cdots \to v(t,x,y)$$

for any $v \in \mathscr{B}_0^0$ satisfying $v \geqslant 0$. Here we note that we can see μ and v as positive finite measures on Borel sets of $(0,T) \times \mathbf{R}^2$. Then $\mu = v$ in $\mathscr{M}$ means

$$\int_{(0,T)\times\mathbf{R}^2} \gamma_n(y)v(t,x,y)\,d\mu = \int_{(0,T)\times\mathbf{R}^2} \gamma_n(y)v(t,x,y)\,dv$$

for all $n \in \mathbf{N}$. Using the Beppo-Levi theorem, we obtain (4.12).

Now we return to the proof of (4.11). Clearly

$$|u^\varepsilon|^2 \to |u|^2 \quad \text{in } \mathscr{D}'((0,T)\times\mathbf{R}^2) \quad \text{as } \varepsilon \downarrow 0$$

and $|u|^2 \in \tilde{\mathscr{M}}$. Since $\{|u^\varepsilon|^2\}_{\varepsilon\in(0,1]}$ is bounded in $\tilde{\mathscr{M}}$ and $\tilde{\mathscr{B}}^0$ is a separable normed vector space, there exist a subsequence $\{|u^\varepsilon|^2\}_{\varepsilon\in(0,1]}$ and $\mu \in \tilde{\mathscr{M}}$ such that

$$|u^\varepsilon| \xrightarrow{\mathrm{w}^*} \mu \quad \text{in } \tilde{\mathscr{M}} \quad \text{as } \varepsilon \downarrow 0.$$

$\tilde{\mathscr{M}} \subset \mathscr{D}'$ yields $\mu = |u|^2$ in $\mathscr{D}'$. Then Lemma 4.2 implies

$$|u^\varepsilon| \xrightarrow{\mathrm{w}^*} |u|^2 \quad \text{in } \tilde{\mathscr{M}} \quad \text{as } \varepsilon \downarrow 0. \tag{4.13}$$

Let $\alpha(t,x,y)$ belong to $\mathscr{D}((0,T)\times\mathbf{R}^2)$. Using the Fubini theorem and the integration by parts with respect to x, We have

$$\langle f_1(u^\varepsilon) - f_1(u), \alpha\rangle = F_1^\varepsilon(\alpha) + F_2^\varepsilon(\alpha),$$

$$F_1^\varepsilon(\alpha) = \int_0^T \iint_{\mathbf{R}^2} \int_y^{+\infty} \partial_x|u^\varepsilon(t,x,y')|^2 dy'$$

$$\times (u^\varepsilon(t,x,y) - u(t,x,y))\alpha(t,x,y)dxdydt,$$

$$F_2^\varepsilon(\alpha) = \int_0^T \iint_{\mathbf{R}^2} (|u^\varepsilon(t,x,y')|^2 - |u(t,x,y')|^2)v(t,x,y';\alpha)dxdy'dt,$$

$$v(t,x,y';\alpha) = \partial_x \int_{-\infty}^{y'} u(t,x,y)\alpha(t,x,y)dy \in \tilde{\mathscr{B}}^0.$$

It is easy to see $F_1^\varepsilon(\alpha) \to 0$ as $\varepsilon \downarrow 0$. It follows that $F_2^\varepsilon(\alpha) \to 0$ as $\varepsilon \downarrow 0$ from (4.13). Then we have finished proving the case of $j = 1$ in (4.11). Similarly we can show the case of $j = 2$ in (4.11). This completes the proof of the existence of a solution to (1.1)–(1.2).

The uniqueness of the solution can be proved by the same energy method as above. More precisely, let $u, v \in L^\infty(0,T;H^m)$ be solutions to (1.1)–(1.2). We note the identity

$$\partial_x u\bar{u} + u\partial_x\bar{u} - \partial_x v\bar{v} - v\partial_x\bar{v} = \partial_x(u-v)\bar{u} + \partial_x v(\overline{u-v}) + u\partial_x(\overline{u-v}) + (u-v)\partial_x\bar{v}.$$

Then, we define the symbol of the transformations $\partial_x(u-v) \mapsto K(t)\partial_x(u-v)$ and $\partial_y(u-v) \mapsto H(t)\partial_y(u-v)$ by using u and we evaluate $\|u-v\|_1$. These procedure imply the uniqueness of the solution. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Finally we prove Theorem 1.2 by the *a priori* estimates. We take $m_2 \in \mathbf{N}$ as $m_2 = \max(l+4, 6) \geqslant m_1 + 1$. In view of Theorem 1.1, we have only to obtain the *a priori* estimate of $\|u(t)\|_{m_2-1}$ $(< 1/2\sqrt{ae})$. In the same way as §4, we define $K(t), H(t), B_K(t)$ and etc. We denote $Y(t)$ by

$$
\begin{aligned}
Y(t) &= \sum_{j=0}^{4} X_{m_2-1-j,j}(t) + (1+t)^{-1/2} X_{m_2-6,5}(t) \\
&\quad + \sum_{j=0}^{4} \sum_{\substack{\alpha+\beta=m_2-j \\ \alpha'+\beta'=j \\ \alpha+\alpha',\beta+\beta' \leqslant m_2-1}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} u(t)\| \\
&\quad + (1+t)^{-1/2} \left(\sum_{\substack{\alpha+\alpha'=m_2 \\ \alpha' \leqslant 5}} \|K(t)\partial_x^\alpha J_x^{\alpha'} u(t)\| + \sum_{\substack{\beta+\beta'=m_2 \\ \beta' \leqslant 5}} \|H(t)\partial_x^\beta J_x^{\beta'} u(t)\| \right).
\end{aligned}
$$

We suppose

$$\sup_{t \in [0,T)} Y(t) \leqslant R \quad \text{for some } T > 0,$$

where $R > 0$ is smaller that $1/2\sqrt{ae}$. Using the equation (1.1), We have

$$
\begin{aligned}
&\left| \int_0^x \partial_t \phi(t,x')\,dx' \right| \\
&\quad = M \left| \int_0^x \int_{-\infty}^{+\infty} \partial_t |u(t,x',y)|^2 dy dx' \right| \\
&\quad = M \left| \int_0^x \int_{-\infty}^{+\infty} (\partial_t u\bar{u} + u\partial_t \bar{u}) dy dx' \right| \\
&\quad = M \left| \int_0^x \int_{-\infty}^{+\infty} \{ i(\partial_x^2 + \partial_y^2) u\bar{u} - iu(\partial_x^2 + \partial_y^2)\bar{u} + (f(u)u + \overline{f(u)u}) \} dy dx' \right| \\
&\quad \leqslant 2M \sup_{x \in \mathbf{R}} \left| \int_{-\infty}^{+\infty} (\partial_x u\bar{u} - u\partial_x \bar{u}) dy \right| + 2M \|f(u(t))\| \, \|u(t)\| \\
&\quad \leqslant C(1+t)^{-1} \sum_{j=0}^{2} X_{2-j,j}(t)^2 (1 + \|u(t)\|^2) \\
&\quad \leqslant C(1+t)^{-1} Y(t)^2.
\end{aligned}
$$

Here we used the same technique as in (3.20). This means that the transformation $u(t) \mapsto K(t)u(t)$ does not bring about the loss of time-decay because of the structural nice property of $\phi(t,x)$. This was first pointed out by S. Katayama and Y. Tsutsumi ([8]). Similarly we have

$$B_K(t), B_H(t), B_\phi^0(t), B_\psi^0(t) \leqslant C,$$

$$B_\phi^1(t), B_\psi^1(t), B_\phi^\infty(t), B_\psi^\infty(t) \leqslant CR^2(1+t)^{-1}.$$

By the assumption, the nonlinear estimates (3.20), (3.21), (3.22) and (3.23) become

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta' \leqslant m_2 \\ \alpha',\beta' \leqslant 5 \\ \alpha+\alpha',\beta+\beta' \leqslant m_2-1 \\ \alpha',\beta' \leqslant 4}} \|\partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} f(u(t))\| \leqslant CR^3(1+t)^{-3/2}, \tag{5.1}$$

$$\sum_{\alpha+\beta \leqslant m_2-6} \{\|\partial_x^\alpha \partial_y^\beta J_x^5 f(u(t))\| + \|\partial_x^\alpha \partial_y^\beta J_y^5 f(u(t))\|\} \leqslant CR^3(1+t)^{-1/2}, \tag{5.2}$$

$$\sum_{\substack{\alpha+\alpha'=m_2 \\ \alpha' \leqslant 5}} \|(K(t)\partial_x^\alpha J_x^{\alpha'} f(u(t)), K(t)\partial_x^\alpha J_x^{\alpha'} u(t))\|$$

$$\leqslant CR^4 + \frac{1}{2}\left(1+\frac{3}{e}\right)$$

$$\times \sum_{\substack{\alpha+\alpha'=m_2 \\ \alpha' \leqslant 5}} (\phi(t,x)\langle D_x\rangle^{1/2} K(t)\partial_x^\alpha J_x^{\alpha'} u(t), \langle D_x\rangle^{1/2} K(t)\partial_x^\alpha J_x^{\alpha'} u(t)), \tag{5.3}$$

$$\sum_{\substack{\beta+\beta'=m_2 \\ \beta' \leqslant 5}} \|(H(t)\partial_y^\beta J_y^{\beta'} f(u(t)), H(t)\partial_y^\beta J_y^{\beta'} u(t))\| \tag{5.4}$$

$$\leqslant CR^4 + \frac{1}{2}\left(1+\frac{3}{e}\right)$$

$$\times \sum_{\substack{\beta+\beta'=m_2 \\ \beta' \leqslant 5}} (\psi(t,y)\langle D_y\rangle^{1/2} H(t)\partial_y^\beta J_y^{\beta'} u(t), \langle D_y\rangle^{1/2} H(t)\partial_y^\beta J_y^{\beta'} u(t)). \tag{5.5}$$

Using (2.5), (2.6), (5.3) and (5.5), we get

$$\frac{d}{dt}\left(\sum_{\substack{\alpha+\alpha'=m_2 \\ \alpha' \leqslant 5}} \|K(t)\partial_x^\alpha J_x^{\alpha'} u(t)\|^2 + \sum_{\substack{\beta+\beta'=m_2 \\ \beta' \leqslant 5}} \|H(t)\partial_y^\beta J_y^{\beta'} u(t)\|^2\right) \leqslant CR^4.$$

Integrating with respect to t, we have

$$(1+t)^{-1/2}\left(\sum_{\substack{\alpha+\alpha'=m_2\\ \alpha'\leqslant 5}}\|K(t)\partial_x^{\alpha}J_x^{\alpha'}u(t)\|+\sum_{\substack{\beta+\beta'=m_2\\ \beta'\leqslant 5}}\|H(t)\partial_y^{\beta}J_y^{\beta'}u(t)\|\right)\leqslant C(\delta+R^2). \quad (5.6)$$

On the other hand, by the energy estimate with (5.1) and (5.2), we get

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leqslant m_2\\ \alpha',\beta'\leqslant 5\\ \alpha+\alpha',\beta+\beta'\leqslant m_2-1\\ \alpha',\beta'\leqslant 4}}\|\partial_x^{\alpha}\partial_y^{\beta}J_x^{\alpha'}J_y^{\beta'}u(t)\|\leqslant C(\delta+R^3), \quad (5.7)$$

$$(1+t)^{-1/2}\sum_{\alpha+\beta\leqslant m_2-6}(\|\partial_x^{\alpha}\partial_y^{\beta}J_x^5u(t)\|+\|\partial_x^{\alpha}\partial_y^{\beta}J_y^5u(t)\|)\leqslant C(\delta+R^3). \quad (5.8)$$

Summing up (5.6), (5.7) and (5.8), we obtain

$$\sup_{t\in[0,T)} Y(t)\leqslant C(\delta+R^2).$$

Let R_1 be a positive small constant satisfying $CR_1^2\leqslant R_1/4$. We can take R as $R\leqslant R_1$. Then we have

$$\sup_{t\in[0,T)} Y(t)\leqslant \frac{R}{2}$$

provided that δ is sufficiently small. This completes the proof of Theorem 1.2.

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