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On existence of local solutions for differential equations on Wiener space

By

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Abstract

We consider a differential equation on the Wiener space. We obtain some inequalities for capacity under the transformations of the Wiener space induced by the solutions of differential equation and show the existence of local solutions by using the inequalities for capacity.

1. Introduction

Let (X, H, μ) be an abstract Wiener space and A a vector field on X which is a mapping from X into H smooth in the sense of Malliavin (cf. [5]). We consider the following differential equation on X:

$$(1.1) \qquad \begin{cases} \dfrac{d}{dt}(U_t(x)) = A(U_t(x)), \\ U_0(x) = x. \end{cases}$$

In [8], the second author constructed, under the condition of Theorem 2.1, a solution $U_t(x)$ which satisfies (1.1) quasi everywhere (q.e.), i.e., for all x except in a set of (r, p)-capacity 0, for all $r \geq 0$ and $p > 1$. Here the capacities are associated with the Ornstein-Uhlenbeck operator on the Wiener space (cf. [5], [6]). By the way of its construction, we see that this $U_t(x)$ is a quasi continuous modification of a solution, in the sense of almost everywhere, of Cruzeiro [1].

In the previous paper [9], Yun obtained further refinements of this solution. He chose a quasi continuous modification $\tilde{A}(x)$ of $A(x)$ defined everywhere on X and constructed $U_t(x), t \in \mathbf{R}, x \in X$, such that $[t \mapsto U_t(x)] \in W_\infty^\infty(X; C([-T, T] \to X))$ (cf. [6]), $X \ni x \mapsto [t \mapsto U_t(x)] \in C([-T, T] \to X)$ is quasi continuous for every $T > 0$ and satisfies

$$(1.1)' \qquad U_t(x) = x + \int_0^t \tilde{A}(U_s(x))ds \qquad \text{for quasi every (q.e.) } x \in X,$$

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for all $t \in \mathbf{R}$. Furthermore, he chose $U_t(x)$ so that the mapping $x \to U_t(x)$ preserves the class of slim sets for every t and the flow property

$$U_t \circ U_s(x) = U_{t+s}(x), \tag{1.2}$$

holds q.e. for every t and s.

Cruzeiro proved the almost everywhere existence of local solutions of (1.1) under some integrability condition ([3], Theorem 3.2). In the present paper, we extend the almost everywhere existence to the quasi everywhere existence. In other words, under the conditions described in Theorem 2.6, we show that for all $\varepsilon > 0$ and $M > 0$, there exist $Z_{\varepsilon, M} \subset X$ and $U_t(x) \equiv V_t(x) + x$ satisfying (1.1) for all $x \in Z_{\varepsilon, M}$ and $|t| < M$ such that $C_{r,p}(Z^c_{\varepsilon, M}) < \varepsilon$ for all $r \geq 0$ and $q > p > 1$, where $C_{r,p}$ is (r, p)-capacity on X.

2. Inequality for capacities and existence of local solutions

Let (X, H, μ) be an abstract Wiener space introduced by Gross. Let E be a real separable Hilbert space. We set

$$W^p_r(X; E) := (1 - L)^{-r/2}(L^p(X, \mu; E))$$

for the Ornstein-Uhlenbeck operator L (cf. [8]). Then $W^p_r(X; E)$ becomes a Banach space and we can define the Sobolev space $W^p_r(X; E)$ with the differentiability index r and the integrability index p with a norm

$$\|f\|_{r,p} := \|u\|_{L^p} \qquad \text{for } f = (1 - L)^{-r/2}u, \quad u \in L^p(X, \mu; E).$$

We denote the space $\bigcap_r W^p_r(X; E)$ by $W^p_\infty(X; E)$ for $p \in [1, \infty)$ and $W^\infty_\infty(X; E) = \bigcap_p W^p_\infty(X; E)$. We can define the gradient operator $\nabla : W^p_{r+1}(X; E) \to W^p_r(X; E \otimes H)$ and its dual, the divergence operator, $\delta : W^p_{r+1}(X; E \otimes H) \to W^p_r(E)$ as usual (cf. [1]).

Next let us recall the notion of (r, p)-capacity. The (r, p)-capacity $C_{r,p}$ is defined as follows: for an open set $O \subset X$,

$$C_{r,p}(O) = \inf\{\|f\|^p_{r,p} : f \in W^p_r(X; \mathbf{R}),\ f \geq 1 \text{ a.e. on } O\}$$

and for an arbitrary set $B \subset X$,

$$C_{r,p}(B) = \inf\{C_{r,p}(G) : G \text{ is open and } G \supset B\}.$$

We say that a property holds quasi-everywhere (q.e. in abbreviation) if it holds except on a set of capacity 0 for all r, p. We note that

$$C_{r,p}(\{\|u\| > l\}) \leq \frac{1}{l^p}\|u\|^p_{W^p_r}. \tag{2.1}$$

Here u is taken to be quasi-continuous. For the quasi-sure extension, we need to consider the differential equation to be satisfied by $LV_t(x)$ where $V_t(x) \equiv U_t(x) - x$ and L is the Ornstein-Uhlenbeck operator. But since

$$\begin{cases} (dLV_t/dt)(x) = L(A(V_t(x)+x)), \\ LV_0(x) = 0, \end{cases}$$

and

$$\begin{aligned} L(A(V_t(x)+x)) &= \sum_i \frac{\partial^2}{\partial x_i^2}(A(V_t(x)+x)) - \sum_i \frac{\partial}{\partial x_i}(A(V_t(x)+x)) \cdot x_i \\ &= LA(V_t(x)+x) + \nabla A(V_t(x)+x) \cdot LV_t(x) \\ &\quad + \sum_i \nabla_{jk}^2 A(V_t(x)+x) \cdot \partial_i V_t^j(x) \cdot \partial_i V_t^k(x) \\ &\quad + 2\sum_i \partial_i \nabla_j A(V_t(x)+x) \cdot \partial_i V_t^j(x), \end{aligned}$$

we should consider the following system of differential equations to be satisfied by $[V_t(x), \nabla V_t(x), LV_t(x)]$:

$$\frac{d}{dt} V_t(x) = A(V_t(x)+x), \tag{2.2}$$

$$\frac{d}{dt} \nabla V_t(x) = \nabla A(V_t(x)+x) \cdot \nabla V_t(x) + \nabla A(V_t(x)+x), \tag{2.3}$$

$$\begin{aligned} \frac{d}{dt} LV_t(x) &= LA(V_t(x)+x) + \nabla A(V_t(x)+x) \cdot LV_t(x) \\ &\quad + \sum_i \nabla_{jk}^2 A(V_t(x)+x) \cdot \partial_i V_t^j(x) \cdot \partial_i V_t^k(x) \\ &\quad + 2\sum_i \partial_i \nabla_j A(V_t(x)+x) \cdot \partial_i V_t^j(x), \end{aligned} \tag{2.4}$$

$$V_0(x) = \nabla V_0(x) = LV_0(x) = 0.$$

More generally, in addition to (2.2) $\sim$ (2.4), we consider the following system of differential equations to be satisfied by

$$[L^m \nabla^n V_t(x) : m = 0, 1, \ldots, N, \ \ n = 0, 1, \ldots, 2N, \ \ 2m+n \leq 2N] :$$

$$\begin{gathered} \frac{d}{dt} L^m \nabla^n V_t = \nabla A \cdot L^m \nabla^n V_t + E^{m,n}(L^i \nabla^j A, L^l \nabla^r V_t : \quad i = 0, 1, \ldots, m, \\ j = 0, 1, \ldots, n, \qquad l = 0, 1, \ldots, m-1, \\ r = 0, 1, \ldots, n, \qquad 2i + j \leq 2m + n, \\ 2l + r \leq 2(m-1) + n), \\ L^m \nabla^n V_0(x) = 0, \qquad m = 0, 1, \ldots, k, \qquad n = 0, 1, \ldots, 2k, \\ 2m + n \leq 2k, \qquad k = 2, 3, \ldots, N, \end{gathered}$$

where $E^{m,n}$ is some polynomial which can be obtained successively (cf. [8]).

Theorem 2.1 (Yun [8], Theorem 5.5). *If A is a vector field on X satisfying the following conditions*

(i) $A \in W_\infty^\infty(X;H)$ *and* $\forall\lambda > 0$, $\int_X \exp(\lambda\|A(x)\|)d\mu(x) < +\infty$,

(ii) $\forall\lambda > 0$, $\forall n = 1,2,\dots,\int_X \exp(\lambda\|\nabla^n A(x)\|)d\mu(x) < +\infty$,

(iii) $\forall\lambda > 0, \int_X \exp(\lambda|\delta A(x)|)d\mu(x) < +\infty$,

then $V_t(x)$ exists for all $t \in \mathbf{R}$, q.e.x and satisfy the following differential equation

$$\begin{cases} (dV_t/dt)(x) = A(V_t(x) + x), \\ V_0(x) = 0. \end{cases} \tag{2.5}$$

Furthermore, it can be shown (cf. [9]) that we can modify the solution $U_t(x)$ so that it is defined for every $t \in \mathbf{R}$ and $x \in X$, satisfies $(1.1)'$ for q.e. $x \in X$ for all $t \in \mathbf{R}$ and also has the quasi sure flow property, i.e., satisfies (1.2) for q.e. $x \in X$, for all $t, s \in \mathbf{R}$ (cf. [9]).

For the proof of Theorem 2.1, we first consider (1.1) in a finite dimensional case. We show that for any $k \in \mathbf{N}$, $(L^k V_t)$ exists for all $t \in \mathbf{R}, \mu$-a.e.x and thereby (V_t) admits a quasi-continuous modification as a $C([0,T] \to X)$-valued function for any $T > 0$. In finite dimensional case, one point has a positive (r,p)-capacity for sufficiently large r and p. Therefore we can show that a solution to (1.1) exists for every initial value $x \in X$. To deal with (LV_t), for example, we have to consider the above system of differential equations (2.2)$\sim$(2.4). To proceed to the infinite dimensional case, we adopt a finite dimensional approximation. To be precise, we take a sequence $\{A^{(n)}\}$ converging to A such that $A^{(n)}$ depends only on finite number of coordinates and takes values in a finite dimensional subspace of H. Denoting a solution for $A^{(n)}$ by $(V_t^{(n)})$, we show that $(V_t^{(n)})$ converges quasi-everywhere to the limit $V_t(x)$ and $U_t(x) = V_t(x) + x$ satisfies (1.1).

For the existence of local solutions, consider another Sobolev space on $[-M,M] \times X$ as follows:
For fixed $M > 0$ and a Hilbert space E, the Sobolev space W_r^p on $[-M,M] \times X$ is defined by

$$W_r^p([-M,M] \times X;E) := (1 - \Delta - L)^{-r/2}(L^p([-M,M] \times X, dt/2M \otimes d\mu; E)),$$

where $\Delta = \frac{\partial^2}{\partial t^2}$ is Laplacian on $[-M,M]$ and

$$\begin{aligned} &L^p([-M,M] \times X, dt/2M \otimes d\mu; E) \\ &\quad := \left\{ u : [-M,M] \times X \to E; \int_{-M}^{M} \int_X \|u(t,x)\|_E^p d\mu \frac{dt}{2M} < +\infty \right\}. \end{aligned}$$

The Sobolev space $W_r^p([-M,M] \times X;E)$ becomes a Banach space with a norm

$$\|f\|_{r,p} := \|u\|_{L^p} \qquad \text{for } f = (1 - \Delta - L)^{-r/2}u,$$
$$u \in L^p([-M,M] \times X, dt/2M \otimes d\mu; E).$$

The (r,p)-capacity $\bar{C}_{r,p}$ on $[-M,M]\times X$ is defined as follows. For $[0,\infty]$-valued lower semicontinuous (l.s.c. in abbreviation) function h, define $\bar{C}_{r,p}(h)$ by

$$\bar{C}_{r,p}(h) := \inf\{\|g\|_{r,p}^p; g\in W_r^p([-M,M]\times X;E),\ \ g\geq h, dt/2M\otimes\mu\text{-a.e.}\}$$

and for an arbitrary $[-\infty,\infty]$-valued function f (not assumed to be $dt/2M\otimes\mu$-measurable),

$$\bar{C}_{r,p}(f) := \inf\{\bar{C}_{r,p}(h); h \text{ is l.s.c. and } h(t,x)\geq|f(t,x)|,\ \forall(t,x)\in[-M,M]\times X\}.$$

For a set G, we define $\bar{C}_{r,p}(G)=\bar{C}_{r,p}(1_G)$, where 1_G denotes the indicator function of G.

Since the above norm $\|f\|_{r,p}$ is equivalent to the norm

$$\left(\int_{-M}^{M}\int_X\sum_{0\leq i+j\leq r}\left\|\left(\frac{\partial}{\partial t}\right)^i\nabla^j u\right\|_E^p d\mu\frac{dt}{2M}\right)^{1/p}$$

by Meyer's equivalence, we use only the latter in this paper.

Proposition 2.2. *If A satisfies the hypothesis of Theorem 2.1, then for every $\varepsilon>0$, there exists $F\subset[-M,M]\times X$ with $\bar{C}_{r,p}(F^c)<\varepsilon$ for every r, p and $V_t^{(n)}(x)\to V_t(x)$ uniformly on F. Here $V_t^{(n)}$ are defined as above.*

Proof. Since

$$\begin{aligned}&\left|\int_0^t A^{(n)}(V_s^{(n)}(x)+x)-A^{(m)}(V_s^{(m)}(x)+x)ds\right|^p\\ &\quad\leq\left(\int_0^t|A^{(n)}(V_s^{(n)}(x)+x)-A^{(m)}(V_s^{(m)}(x)+x)|ds\right)^p\\ &\quad\leq t^p\cdot\int_0^t|A^{(n)}(V_s^{(n)}(x)+x)-A^{(m)}(V_s^{(m)}(x)+x)|^p\frac{ds}{t},\end{aligned}$$

we have

$$\begin{aligned}&\|V^{(n)}-V^{(m)}\|_{L^p(dt/2M\otimes d\mu)}\\ &\leq t\cdot\left(\int_{-M}^{M}\int_X\int_0^t|A^{(n)}(V_s^{(n)}(x)+x)-A^{(m)}(V_s^{(n)}(x)+x)|^p\frac{ds}{t}d\mu\frac{dt}{2M}\right)^{1/p}\\ &\quad+t\cdot\left(\int_{-M}^{M}\int_X\int_0^t|A^{(n)}(V_s^{(m)}(x)+x)-A^{(m)}(V_s^{(m)}(x)+x)|^p\frac{ds}{t}d\mu\frac{dt}{2M}\right)^{1/p}\\ &\quad\to 0 \text{ as } n,m\to\infty \text{ uniformly}\end{aligned}$$

(see also the proof of Theorem 5.5 (cf. [8])).

And by the facts

$$\frac{\partial V_t^{(n)}}{\partial t} = A^{(n)}(V_t^{(n)}(x) + x) \qquad \text{and}$$

$$\nabla V_t^{(n)}(x) = \int_0^t (\nabla A^{(n)}(V_s^{(n)}(x) + x) \cdot \nabla V_s^{(n)}(x) + \nabla A^{(n)}(V_s^{(n)}(x) + x))ds,$$

we have

$$\begin{aligned}
&\left(\int_X \left\| \frac{\partial V_t^{(n)}}{\partial t} - \frac{\partial V_t^{(m)}}{\partial t} \right\|^p d\mu \right)^{1/p} \\
&\qquad = (E[|A^{(n)}(V_t^{(n)}(x) + x) - A^{(m)}(V_t^{(m)}(x) + x)|^p])^{1/p} \\
&\qquad \leq (E[|A^{(n)}(V_t^{(n)}(x) + x) - A^{(m)}(V_t^{(n)}(x) + x)|^p])^{1/p} \\
&\qquad\quad + (E[|A^{(m)}(V_t^{(n)}(x) + x) - A^{(m)}(V_t^{(m)}(x) + x)|^p])^{1/p} \\
&\qquad \to 0 \qquad \text{as } n, m \to \infty \text{ uniformly}
\end{aligned}$$

and

$$\begin{aligned}
&\int_X \|\nabla V_t^{(n)} - \nabla V_t^{(m)}\|^p d\mu \\
&\qquad \leq t^p \cdot \int_X \int_0^t |\nabla A^{(n)}(V_s^{(n)}(x) + x) \cdot \nabla V_s^{(n)}(x) + \nabla A^{(n)}(V_s^{(n)}(x) + x) \\
&\qquad\quad - \nabla A^{(m)}(V_s^{(m)}(x) + x) \cdot \nabla V_s^{(m)}(x) - \nabla A^{(m)}(V_s^{(m)}(x) + x)|^p \frac{ds}{t} d\mu \\
&\qquad \to 0 \quad \text{as } n, m \to \infty \text{ uniformly.}
\end{aligned}$$

Thus we have

$$\|V^{(n)} - V^{(m)}\|_{W_1^p} \to 0 \qquad \text{as } n, m \to \infty.$$

Similarly, we can prove that

$$\|V^{(n)} - V^{(m)}\|_{W_r^p} \to 0 \qquad \text{as } n, m \to \infty \qquad \text{for all } r, p.$$

Adopting a finite dimensional approximation used in the proof of Theorem 5.5 (cf. [8]), we can prove the existence of F.

Lemma 2.3. *For any $1 < p < p_1$ and $r \geq 0$, if $g \in W_r^{p_1}(X; \mathbf{R})$, then the family $\{g \circ U_t^{(n)}\}$ forms a bounded set in $W_r^p([-M, M] \times X; \mathbf{R})$. To be more precise, there exists a constant C_1 such that $\|g \circ U_t^{(n)}\|_{r,p} \leq C_1 \cdot \|g\|_{r,p_1}$ for all n and $g \in W_r^{p_1}(X; \mathbf{R})$. Here $U_t^{(n)}(x) = V_t^{(n)}(x) + x$.*

Proof. We prove only the case $r = 1$. The general case can be proved similarly. Since $g \in W_r^{p_1}(X;\ \mathbf{R})$, $g(V_t^{(n)}(x) + x) \in L^p(dt/2M \otimes d\mu)$. Note that

$$\left(\int_{-M}^{M}\int_X\left(\left\|\frac{\partial(g\circ V^{(n)})}{\partial t}\right\| + \|\nabla(g\circ V_t^{(n)})\|\right)^p d\mu\frac{dt}{2M}\right)^{1/p}$$

$$\leq \left(\int_{-M}^{M}\int_X\left\|\frac{\partial(g\circ V^{(n)})}{\partial t}\right\|^p d\mu\frac{dt}{2M}\right)^{1/p} + \left(\int_{-M}^{M}\int_X\|\nabla(g\circ V_t^{(n)})\|^p d\mu\frac{dt}{2M}\right)^{1/p}.$$

If we put $k_t^{(n)}(x) = (d(U_t^{(n)})_*\mu/d\mu)(x)$, then for all $p > 1$, $\sup_n\{\|k_t^{(n)}\|_p + \|LV_t^{(n)}\|\} < \infty$, and $\sup_n\|\nabla V_t^{(n)}\|_p < \infty$ (cf. [8]). Since

$$\left\|\frac{\partial V^{(n)}}{\partial t}\right\|_p = \|A^{(n)}(V_t^{(n)}(x) + x)\|_p \leq \|k_t^{(n)}\|_{2p}\cdot\|A^{(n)}\|_{2p},$$

$$\|\nabla V_t^{(n)}(x)\|_p = \left\|\int_0^t \nabla A^{(n)}(V_s^{(n)}(x) + x)\cdot\nabla V_s^{(n)}(x)ds + \nabla A^{(n)}(V_s^{(n)}(x) + x)\right\|_p,$$

we have

$$\begin{aligned}\left\|\frac{\partial g(V_t^{(n)}(x) + x)}{\partial t}\right\|_p &= \|(\nabla g)(V_t^{(n)}(x) + x)\cdot A^{(n)}(V_t^{(n)}(x) + x)\|_p\\ &\leq \|(\nabla g)(V_t^{(n)}(x) + x)\|_{2p}\cdot\|A^{(n)}(V_t^{(n)}(x) + x)\|_{2p}\\ &\leq \|\nabla g\|_{4p}\cdot\|A^{(n)}\|_{4p}\cdot 2\|k_t^{(n)}\|_{4p}\end{aligned}$$

and

$$\begin{aligned}&\|\nabla(g(V_t^{(n)}(x) + x))\|_p\\ &\quad\leq\left\|(\nabla g)(V_t^{(n)}(x) + x)\nabla V_s^{(n)}(x) + \nabla A^{(n)}(V_s^{(n)}(x) + x)ds\right\|_p\\ &\quad\leq\|\nabla g\|_{4p}\cdot\|k_t^{(n)}\|_{4p}\left\|\int_0^t\nabla A^{(n)}(V_s^{(n)}(x) + x)\cdot\nabla V_s^{(n)}(x)ds\right\|_{2p}\\ &\quad\leq\|\nabla g\|_{4p}\cdot\|k_t^{(n)}\|_{4p}t\cdot\left(\int_0^t\|k_s^{(n)}\|_{8p}^{2p}\cdot\|\nabla A^{(n)}\|_{8p}^{2p}\cdot\|\nabla V_s^{(n)}\|_{4p}^{2p}\frac{ds}{t}\right)^{1/2p},\end{aligned}$$

we can find a constant C_1 such that $\|g\circ U_t^{(n)}\|_{r,p} \leq C_1\cdot\|g\|_{r,p_1}$.

We define $U : [-M, M]\times X \ni (t, x) \mapsto U_t(x) \in X$.

Lemma 2.4. *For any* $1 < p < p_1$ *and* $r \geq 0$, *if* $g \in W_r^{p_1}(X;\mathbf{R})$, *then* $g\circ U \in W_r^p([-M, M]\times X;\mathbf{R})$; *more precisely, there exists a constant* C *such that* $\|g\circ U\|_{r,p} \leq C\cdot\|g\|_{r,p_1}$.

Proof. By Lemma 2.3, $\|g \circ U^{(n)}\|_{r,p} \leq C_1 \cdot \|g\|_{r,p_1}$ for some constant C_1. Thus the proof is complete if we prove that $g \circ U^{(n)}$ converges to $g \circ U$ in $W_r^p([-M, M] \times X; \mathbf{R})$. Note that

$$\|g \circ U^{(n)} - g \circ U\|_{W_1^p}$$

$$= \left(\int_{-M}^{M} \int_X \left(\left\| \frac{\partial(g(V_t^{(n)}(x) + x))}{\partial t} - \frac{\partial(g(V_t(x) + x))}{\partial t} \right\|\right.\right.$$

$$\left.\left. + \|\nabla(g(V_t^{(n)}(x) + x)) - \nabla(g(V_t(x) + x))\|\right)^p d\mu \frac{dt}{2M}\right)^{1/p}$$

$$\leq \left(\int_{-M}^{M} \int_X \left\| \frac{\partial(g(V_t^{(n)}(x) + x))}{\partial t} - \frac{\partial(g(V_t(x) + x))}{\partial t} \right\|^p d\mu \frac{dt}{2M}\right)^{1/p}$$

$$+ \left(\int_{-M}^{M} \int_X \|\nabla(g(V_t^{(n)}(x) + x)) - \nabla(g(V_t(x) + x))\|^p d\mu \frac{dt}{2M}\right)^{1/p}.$$

We can complete the proof since the last two integrals are calculated by

$$\left\| \frac{\partial(g(V_t^{(n)}(x) + x))}{\partial t} - \frac{\partial(g(V_t(x) + x))}{\partial t} \right\|_p$$

$$\leq \|(\nabla g)(V_t^{(n)}(x) + x) \cdot (A_n(V_t^{(n)}(x) + x) - A(V_t(x) + x))\|_p$$

$$+ \|A(V_t(x) + x)((\nabla g)(V_t^{(n)}(x) + x) - (\nabla g)(V_t(x) + x))\|_p$$

$$\leq \|k_t^{(n)}\|_{4p} \cdot \|\nabla g\|_{4p} \cdot \|A_n(V_t^{(n)}(x) + x) - A(V_t(x) + x)\|_{2p}$$

$$+ \|A(V_t(x) + x)\|_{2p} \cdot \|(\nabla g)(V_t^{(n)}(x) + x) - (\nabla g)(V_t(x) + x)\|_{2p}$$

and

$$\|\nabla(g(V_t^{(n)}(x) + x)) - \nabla(g(V_t(x) + x))\|_p$$

$$\leq \|(\nabla g)(V_t^{(n)}(x) + x) \cdot \nabla V_t^{(n)}(x) - (\nabla g)(V_t^{(n)}(x) + x) \cdot \nabla V_t(x)\|_p$$

$$+ \|(\nabla g)(V_t^{(n)}(x) + x) \cdot \nabla V_t(x) - (\nabla g)(V_t(x) + x) \cdot \nabla V_t(x)\|_p$$

$$\leq \|k_t^{(n)}\|_{4p} \cdot \|\nabla g\|_{4p} \cdot \|\nabla V_t^{(n)}(x) - \nabla V_t(x)\|_{2p}$$

$$+ \|\nabla V_t(x)\|_{2p} \cdot \|(\nabla g)(V_t^{(n)}(x) + x) - (\nabla g)(V_t(x) + x)\|_{2p}.$$

Theorem 2.5. *For $1 < p < p_1$ and $r \geq 0$, there exists a constant $C > 0$ such that*

$$\bar{C}_{r,p}(U^{-1}(B)) \leq C \cdot (C_{r,p_1}(B))^{p/p_1}, \quad \forall B \subset X.$$

Proof. Let O be an open set in X. Then by Lemma 2.4,

$$\begin{aligned}
&\bar{C}_{r,p}(U^{-1}(O)) \\
&\quad = \inf\{\|f\|_{r,p}^p : f \in W_r^p([-M,M] \times X; \mathbf{R}),\ f \geq 1 \text{ a.e. on } U^{-1}(O)\} \\
&\quad = \inf\{\|f\|_{r,p}^p : f \in W_r^p([-M,M] \times X; \mathbf{R}),\ f \circ U^{-1} \geq 1 \text{ a.e. on } O\} \\
&\quad = \inf\{\|g \circ U\|_{r,p}^p : g \circ U \in W_r^p([-M,M] \times X; \mathbf{R}),\ g \geq 1 \text{ a.e. on } O\} \\
&\quad \leq C \cdot \inf\{\|g\|_{r,p_1}^p : g \in W_r^{p_1}(X; \mathbf{R}), g \geq 1 \text{ a.e. on } O\} \\
&\quad = C \cdot (C_{r,p_1}(O))^{p/p_1}.
\end{aligned}$$

For an arbitrary set $B \subset X$, we take an open set $G \supset B$. Then

$$\bar{C}_{r,p}(U^{-1}(B)) \leq \bar{C}_{r,p}(U^{-1}(G)) \leq C \cdot (C_{r,p_1}(G))^{p/p_1}.$$

Taking the infimum for all open sets $G \supset B$, we have

$$\bar{C}_{r,p}(U^{-1}(B)) \leq C \cdot (C_{r,p_1}(B))^{p/p_1}.$$

Theorem 2.6. *Suppose that a vector field $A \in W_\infty^q(X; H)$ satisfies*

$$\sum_{n \geq 0} \int_X (|\nabla^n A(x)|^q_{(H^*)^{\otimes n} \otimes H} + |\nabla^n \delta A(x)|^q_{(H^*)^{\otimes n}}) \lambda^n d\mu < +\infty$$

for some $q > 2$ and $\lambda > 0$. Then for all $\varepsilon > 0$ and $M > 0$, we can find $Z_{\varepsilon,M} \subset X$ with $C_{r,p}(Z^c_{\varepsilon,M}) < \varepsilon$ for all $r \geq 0$ and $1 < p < q$ such that there exists $V_t(x)$ for all $x \in Z_{\varepsilon,M}$ and $|t| < M$ which satisfies (2.5).

Proof. Let $f_A(x) = \sum_{n \geq 0} (\|\nabla^n A(x)\|^2_{(H^*)^{\otimes n} \otimes H} + \|\nabla^n \delta A(x)\|^2_{(H^*)^{\otimes n}}) \lambda^n$. Then $f_A \in W_r^{q/2}$ for all r (cf. [3]). Let $\psi : \mathbf{R} \to \mathbf{R}$ be $|\psi| < 1$ and C^∞ with compact support such that

$$\psi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| > 2. \end{cases}$$

Define $\hat{A}_l(x) = A(x) \cdot \psi_l(f_A(x))$ where $\psi_l(t) = \psi(t/l)$. Then $\hat{A}_l(x)$ satisfies the hypothesis of Theorem 2.1. In fact, since

$$\begin{aligned}
\nabla \hat{A}_l(x) &= \nabla A(x) \cdot \psi_l(f_A(x)) + A(x) \cdot \psi_l'(f_A(x)) \cdot \nabla(f_A(x)), \\
\nabla(f_A(x)) &= \sum_{n \geq 0} 2(\nabla^n A, \nabla^n A)\lambda^n + \sum_{n \geq 0} 2\nabla^n \delta A \cdot \nabla^{n+1} \delta A \cdot \lambda^n,
\end{aligned}$$

$$\begin{aligned}\sum_{n\geq 0}(\nabla^n A, \nabla^n A)\lambda^n &\leq \sum_{n\geq 0}\|\nabla^n A\|\lambda^n \cdot \|\nabla^{n+1}A\|\lambda^n \\ &\leq \lambda^{-1/2}\cdot\left(\sum_{n\geq 0}(\nabla^n A)^2\lambda^n\right)^{1/2}\cdot\left(\sum_{n\geq 0}(\nabla^{n+1}A)^2\lambda^{n+1}\right)^{1/2} \\ &= \lambda^{-1/2}\cdot(f_A(x))\end{aligned}$$

and

$$\delta\hat{A}_l = \delta(A\cdot\psi_l) = \delta A\cdot\psi_l - \langle A, \nabla f_A\rangle\cdot\psi_l'(f_A(x)),$$

we have

$$\|\nabla\hat{A}_l\| \leq C\cdot\|\hat{A}_l\| \leq C\cdot l \qquad \text{and} \qquad \|\delta\hat{A}_l\| \leq C\cdot l.$$

We can have a similar proof to obtain estimate $\|\nabla^n\hat{A}_l\| \leq C_n\cdot l$ for all n. Thus the solution $V_t^{\hat{A}_l}(x) \equiv V_t(x)$ of (2.5) with respect to the vector field $\hat{A}_l$ exists q.e. x.

By definition of $\hat{A}_l$, the solutions $\hat{V}_t$ with respect to the vector field A exist for all $|t| < M$ on $Z_l \equiv \{x\in X : \forall t\in[-M, M], |f_A(\hat{V}_t(x)+x)| \leq l\}$. Let $\Omega_l = \{x\in X : |f_A(x)| \leq l\}$ and $\Gamma_l = \{(t,x)\in[-M,M]\times X : |f_A(\hat{V}_t(x)+x)| > l\}$. Then $\hat{U}^{-1}(\Omega_l^c) = \Gamma_l$ and $C_{r,p}(Z_l^c) \leq C_2\cdot\bar{C}_{r+1,p}(\Gamma_l)$ for some constant C_2 (cf. [5]). Thus by Theorem 2.5 and (2.1),

$$\begin{aligned}C_{r,p}(Z_l^c) &\leq C_2\cdot\bar{C}_{r+1,p}(\Gamma_l) \\ &\leq C_1\cdot(C_{r+1,p_1}(\Omega_l^c))^{p/p_1} \\ &\leq C/l^{p_1}\cdot(|f_A|_{W_{r+1}^{p_1}})^{p/p_1}\end{aligned}$$

for some constant C. Taking l sufficiently large, we can complete the proof.

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