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Level structure over $\widehat{E(n)}$ and stable splitting by Steinberg idempotent

By

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0. Introduction

Let $\widehat{E(n)}$ be the I_n-adic complete Johnson-Wilson spectrum. Its coefficient ring is given by

$$\widehat{E(n)}_* = \mathcal{O}[[u_1, \cdots, u_{n-1}]][u, u^{-1}]$$

where $\mathcal{O}$ is the Witt ring of the finite field $\mathbf{F}_{p^n}$. The generators u_i have degree 0 and u has degree 2. In [6] we construct a spectrum $F_n(p)$ whose $\widehat{E(n)}$-cohomology is rationally isomorphic to the extension of $\widehat{E(n)}_*$ obtained by adjoining all roots of the p-series $[p](x)$ for the associated formal group law. The Galois group of this extension is isomorphic to the general linear group $GL_n(\mathbf{F}_p)$ and acts on the spectrum $F_n(p)$. In this note we consider the stable splitting of $F_n(p)$ localized at p by the Steinberg idempotent in the group ring $\mathbf{Z}_{(p)}[GL_n(\mathbf{F}_p)]$.

Let $Sp^n S^0$ be the n-fold symmetric product of the sphere spectrum localized at p. We denote by $D(n)$ the cofiber of the diagonal map $Sp^{p^{n-1}}S^0 \to Sp^{p^n}S^0$. Let e_n be the Steinberg idempotent. Our main theorem is the following.

Theorem 0.1 (Theorem 2.6). *There is a homotopy equivalence*

$$e_n F_n(p) \simeq \Sigma^{-n} D(n).$$

From the construction of $F_n(p)$, we see that $F_n(p)$ contains the S-dual of the Tits building in bottom cells. Therefore the first nontrivial mod p cohomology group of $F_n(p)$ is the Steinberg representation. The image of the action of the Steenrod operations on the bottom cells is contained in the stable wedge summands of $F_n(p)$ corresponding to the Steinberg representation.

The Galois group $GL_n(\mathbf{F}_p)$ acts also on the diagram $\bar{\boldsymbol{D}}_1$ of [6]. We show that the splitted diagram by Steinberg idempotent degenerates and reduces to a part of the Kuhn's exact sequence which was used to solve the Whitehead conjecture [2] [3] [5]. Our main theorem follows from this fact. From the theorem, we see that the mod p cohomology of $e_n F_n(p)$ has the basis corresponding to the admissible Steenrod operations of length $\leq n$.

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The paper is organized as follows. In §1 we calculate the multiplicity of the Steinberg representation in the induced representation from a parabolic subgroup. In §2 we study the relation between the Steinberg representation in the diagram $\bar{\boldsymbol{D}}_1$ of [6] and the Kuhn's exact sequence, and prove the main theorem.

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1. Induced representaion and Steinberg idempotent

Let P_k be a maximal parabolic subgroup of $GL_n(\mathbf{F}_p)$ which is the stabilizer subgroup of the k-dimensional subspace spanned by the first k-basis vectors:

$$P_k=\left\{\begin{pmatrix} A & B \\ O & C \end{pmatrix} \in GL_n(\mathbf{F}_p) \,\middle|\, A \in GL_k(\mathbf{F}_p),\ C \in GL_{n-k}(\mathbf{F}_p)\right\}.$$

Then P_k is isomorphic to the semi-direct product

$$P_k \cong (GL_k(\mathbf{F}_p) \times GL_{n-k}(\mathbf{F}_p)) \ltimes Q$$

where Q is the subgroup of P_k such that A and C are identity matrixes. Let V be a right $GL_k(\mathbf{F}_p)$ module over $\mathbf{F}_p$. We regard V as a representation of P_k by using the homomorphism

$$P_k \to GL_k(\mathbf{F}_p) \times GL_{n-k}(\mathbf{F}_p) \to GL_k(\mathbf{F}_p).$$

We denote by V^{GL_n} the induced representation of V from P_k to $GL_n(\mathbf{F}_p)$. In this section we study the multiplicity of the Steinberg representation in V^{GL_n}.

Let Σ_n be the subgroup of $GL_n(\mathbf{F}_p)$ which consists of the permutation matrixes. Then $\Sigma_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma)\cdot\sigma$ in the group ring $\mathbf{Z}_{(p)}[GL_n(\mathbf{F}_p)]$ is denoted by $\tilde{\Sigma}_n$. Let B_n be a Borel subgroup which consists of upper triangular matrixes. We denote $\Sigma_{b\in B_n} b$ by $\bar{B}_n$. Then the Steinberg idempotent in $\mathbf{Z}_{(p)}[GL_n(\mathbf{F}_p)]$ is defined as

$$e_n=\tilde{\Sigma}_n\bar{B}_n/[GL_n(\mathbf{F}_p): U_n]$$

where U_n is a unipotent subgroup which consists of the upper triangular matrixes with all diagonal entries equal to 1. For a $GL_n(\mathbf{F}_p)$ module M over $\mathbf{F}_p$, the dimension of the vector space Me_n is equal to the multiplicity of the Steinberg representation in M.

Lemma 1.1. (i) *For* $1 \le k < n-1$, $V^{GL_n}e_n=0$.
(ii) *For* $k=n-1$, $\dim V^{GL_n}e_n = \dim Ve_{n-1}$.

Proof. We note that the Steinberg representation is a modular representation of defect zero. Therefore it is sufficient to prove the corresponding statement of the ordinary representation. Let St_k be the Steinberg character of GL_k. Then the restriction of St_n to P_k is the induced character of $St_k \times St_{n-k}$ from $GL_k \times GL_{n-k}$ to P_k (cf. [1]). Let λ be a character of a finite group G and μ a character of a subgroup H. Then we denote by λ_H the restriction of λ to H and by μ^G the induced character from H to G. For two class functions λ_1, λ_2 of G, there is the scalar product $(\lambda_1,\lambda_2)_G$. Let χ be a character of GL_k and $\bar{\chi}$ the pull-back to P_k. Then we have

$$\begin{aligned}(\bar{\chi}^{GL_n}, St_n)_{GL_n} &= (\bar{\chi}, (St_n)_{P_k})_{P_k}\\ &= (\bar{\chi}, (St_k \times St_{n-k})^{P_k})_{P_k}\\ &= (\chi, St_k)_{GL_k}\cdot(1_{GL_{n-k}}, St_{n-k})_{GL_{n-k}}\end{aligned}$$

where $1_{GL_{n-k}}$ is the trivial character of GL_{n-k}. This completes the proof.

For $k=n-1$, there is a homomorphism $\bar{\varphi}: V \to V^{GL_n}$ defined by $\bar{\varphi}(v) = v\otimes 1_n$ where 1_n is the identity matrix. Let $i: Ve_{n-1} \hookrightarrow V$ be the inclusion and $\pi: V^{GL_n} \to V^{GL_n}e_n$ the projection. We consider the composition $\varphi = \pi\circ\bar{\varphi}\circ i: Ve_{n-1} \to V^{GL_n}e_n$.

Lemma 1.2. *The homomorphism φ is an isomorphism as vector spaces.*

Proof. It is sufficient to prove that φ is injective by Lemmma 1.1. Let

$$\tau = (1,2,\cdots,n) = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

be the cyclic permutation. For $1\le i\le n$, we denote τ^i by τ_i. Since the intersection of the group generated by τ and P_{n-1} is $\{1_n\}$, we take $\tau_1,\cdots,\tau_n,\tau_{n+1},\cdots,\tau_s$ as a complete set of representatives of the left coset decomposition $P_{n-1}\backslash GL_n(\mathbf{F}_p)$. Then any element of V^{GL_n} is uniquely written as $\Sigma_{i=1}^s v_i\otimes\tau_i$, $(v_i\in V)$. Let $\bar{\phi}: V^{GL_n}\to V$ be a homomorphism defined by $\bar{\phi}(\Sigma v_i\otimes\tau_i) = v_1$. Then we denote by ϕ the composition $\pi'\circ\bar{\phi}\circ i'$ where $i': V^{GL_n}e_n \to V^{GL_n}$ is the inclusion and $\pi': V\to Ve_{n-1}$ the projection. We show that $\phi\circ\varphi = id$ up to sign. For $v\in V$, we have $(v\otimes 1_n)\tilde{\Sigma}_n = \Sigma_\sigma \mathrm{sgn}(\sigma)\cdot v\otimes\sigma = \Sigma_{i=1}^n \mathrm{sgn}(\tau_i)\cdot v\tilde{\Sigma}_{n-1}\otimes\tau_i$. Let

$$b = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & & b' & \end{pmatrix} \in B_n,\ b'\in B_{n-1}.$$

Then $\tau_i b\tau_1^{-1}\in P$ $(1\le i\le n)$ if and only if $i=1$ and $b_{12}=\cdots=b_{1n}=0$. In this case we have

$$\tau_1 b\tau_1^{-1} = \begin{pmatrix} b' & 0 \\ 0 & b_{11}\end{pmatrix}.$$

Therefore

$$(v\otimes 1_n)e_n = \pm ve_{n-1}\otimes\tau_1 + \sum_{i\neq 1} v_i\otimes\tau_i.$$

This shows that $\phi\circ\varphi = id$ up to sign.

2. Stable splitting of $F_n(p)$

In this section we study the Steinberg representation in the diagram $\bar{D}_1$ of [6] and the Kuhn's exact sequence by using the results of §1. Then we prove the main theorem. In this section we discuss in the category of p-local spectra.

Let X be a naive $GL_k(\mathbf{F}_p)$ spectrum. We can regard X as a naive P_k spectrum as usual. We let $(\ )_+$ denote the suspension spectrum of the pointed space with disjoint base point. Then $GL_n(\mathbf{F}_p)_+ \wedge_{P_k} X$ is a naive $GL_n(\mathbf{F}_p)$ spectrum. By Lemma 1.1, we obtain the following lemma.

Lemma 2.1. *If $k<n-1$, then the spectrum $e_n(GL_n(\mathbf{F}_p)_+ \wedge_{P_k} X)$ is contractible.*

For $k=n-1$, there is a projection $\Phi: GL_n(\mathbf{F}_p)_+ \wedge_{P_{n-1}} X \to P_{n-1+} \wedge_{P_{n-1}} X = X$. Then we obtain the following lemma from Lemma 1.2.

Lemma 2.2. *The map Φ induces a homotopy equivalence*

$$e_n(GL_n(\mathbf{F}_p)_+ \wedge_{P_{n-1}} X) \simeq e_{n-1} X.$$

Let $V=\mathbf{F}_p^n$ be the n-dimensional vector space over $\mathbf{F}_p$. We take EV as a contractible free V space on which $GL_n(\mathbf{F}_p)$ acts. For a subspace W of V the quotient space EV/W is the classifying space BW. For the subset T of $\{0,1,\dots,n-1\}$, we denote by W_T the set of all flags of type T. That is, W_T is the set of all expanding sequences of subspaces of $\mathbf{F}_p^n$ whose i-th subspace has dimension t_i if $T=\{t_1<\cdots<t_k\}$:

$$W_T=\{\{W_1 \subset W_2 \subset \cdots \subset W_k \subset V\} | \dim W_i = t_i\}.$$

For $w=\{W_1 \subset \cdots \subset W_k \subset V\} \in W_T$, we let Bw be the classifying space BW_1 and BW_T the disjoint union $\coprod_{w \in W_T} Bw$. In particular, $BW_\emptyset = BV$. We note that $GL_n(\mathbf{F}_p)$ acts on BW_T. Let $W^{(k)}$ be the k-dimensional subspace spanned by the first k-basis vectors, and $T_k=\{k,k+1,\cdots,n-1\}$ for $0 \le k \le n$. We denote by $w^{(k)}$ the distinguished flag of type T_k as follows:

$$w^{(k)}=\{W^{(k)} \subset W^{(k+1)} \subset \cdots \subset W^{(n-1)} \subset V\}.$$

Then there is a projection $\Phi_k: BW_{T_k} \to Bw^{(k)}$. By induction, we obtain the following theorem from Lemma 2.1 and 2.2.

Theorem 2.3. *If $T=T_k$ for some $0 \le k \le n$, then $\Phi_k: BW_{T_k} \to Bw^{(k)}$ induces a homotopy equivalence $e_n BW_{T_k +} \simeq e_n Bw^{(k)}_+$. Otherwise $e_n BW_{T+}$ is contractible.*

Let $\mathscr{E}=\{\epsilon=(\epsilon_1,\cdots,\epsilon_n) \in \mathbf{Z}^n \mid 0 \le \epsilon_i \le 2\}$ and $\bar{\mathscr{E}}=\{\epsilon \in \mathscr{E} \mid \epsilon_i = 0 \text{ or } 1\}$. We define the order in $\mathscr{E}$ as

$$\epsilon=(\epsilon_1, \cdots, \epsilon_n) \leq \epsilon'=(\epsilon'_1, \cdots, \epsilon'_n) \Leftrightarrow \epsilon_i \leq \epsilon'_i \text{ for all } i.$$

Then we regard $\mathscr{E}$ as a small category and $\bar{\mathscr{E}}$ a subcategory. Let $\bar{\boldsymbol{D}}'$ be the functor from $\bar{\mathscr{E}}$ to the category of topological spaces such that $\bar{\boldsymbol{D}}'(\epsilon)=BW_T$ where $i-1 \in T$ if and only if $\epsilon_i=0$, and $\bar{\boldsymbol{D}}'(\epsilon) \to \bar{\boldsymbol{D}}'(\epsilon')$ is a canonical covering for $\epsilon<\epsilon'$. Then $GL_n(\mathbf{F}_p)$ acts on $\bar{\boldsymbol{D}}'(\epsilon)$ for all ϵ and the maps $\bar{\boldsymbol{D}}'(\epsilon) \to \bar{\boldsymbol{D}}'(\epsilon')$ are equivariant. Corresponding to the equivariant transfer $\bar{\boldsymbol{D}}'(\epsilon')_+ \to \bar{\boldsymbol{D}}'(\epsilon)_+$ for $\epsilon<\epsilon'$, we obtain the homotopy commutative diagram in the category of $GL_n(\mathbf{F}_p)$ spectra. By the result of [6], we lift the homotopy commutative diagram to the strictly commutative diagram. Then we extend the diagram to the contravariant functor $\tilde{\boldsymbol{D}}$ from $\mathscr{E}$ to the category of $GL_n(\mathbf{F}_p)$ spectra so that

$$\begin{aligned}\tilde{\boldsymbol{D}}(\epsilon_1, \cdots, \epsilon_{i-1}, 2, \epsilon_{i+1}, \cdots, \epsilon_n) &\to \tilde{\boldsymbol{D}}(\epsilon_1, \cdots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \cdots, \epsilon_n) \\ &\to \tilde{\boldsymbol{D}}(\epsilon_1, \cdots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \cdots, \epsilon_n)\end{aligned}$$

are fiber sequences. We take the restriction of this diagram to the category of naive $GL_n(\mathbf{F}_p)$ spectra as the diagram $\boldsymbol{D}_1$ of [6]. Let $F_n(p)=\boldsymbol{D}_1(2, \cdots, 2)$. We note that for a naive G spectrum X there is a G map $X \to i^*i_*X$ which induces a nonequivariant equivalence, where i is the inclusion from $\mathbf{R}^\infty$ to a complete G universe. Then X and i^*i_*X induce an equivalent G representation in any cohomology theory. In particular, the stable splitting of X and i^*i_*X by an idempotent in the group ring of G are homotopy equivalent. We consider the stable splitting of the diagram $\boldsymbol{D}_1$ by the Steinberg idempotent.

We recall the Tits building for a group with BN-pair. The Tits building is a simplicial complex whose vertexes are maximal parabolic subgroups. Its simplexes are the sets of the maximal parabolic subgroups whose intersection is a parabolic subgroup. In the case $GL_n(\mathbf{F}_p)$ the simplexes are identified with the flags $\{0<W_1< \cdots <W_k<V\}$ in V. Let K be the Tits building for $GL_n(\mathbf{F}_p)$ and K_+ the suspension spectrum. We denote by $\tilde{K}$ the cofiber of the obvious map $K_+ \to S^0$. From the construction, we see that $\boldsymbol{D}_1(0, 2, \cdots, 2)$ is the S-dual of $\tilde{K}$. We note that the map $\boldsymbol{D}_1(1, 0, \cdots, 0) \to \boldsymbol{D}_1(0, \cdots, 0)$ induces trivial homomorphism in mod p cohomology. From these we obtain the following.

Lemma 2.4. *If $i<-n$, then $H^i(F_n(p); \mathbf{Z}/p)=0$. The $GL_n(\mathbf{F}_p)$ module $H^{-n}(F_n(p); \mathbf{Z}/p)$ is the Steinberg representation.*

From the lemma, we see that $H^{-n}(e_nF_n(p); \mathbf{Z}/p) \cong \mathbf{Z}/p$. We denote by v_n the generator for the map $e_nF_n(p) \to H\mathbf{Z}/p$.

We recall the Kuhn's exact sequence [2] [3]. Let $L(k)=\Sigma^{-k}Sp^{p^k}S^0/Sp^{p^{k-1}}S^0$ and $\partial_k: L(k+1) \to L(k)$ be a connecting map. Kuhn has shown that the sequence

$$\cdots \to \Sigma L(k+1) \xrightarrow{\partial_k} \Sigma L(k) \to \cdots \to \Sigma L(2) \xrightarrow{\partial_1} \Sigma L(1) \xrightarrow{\partial_0} \Sigma L(0) \xrightarrow{\eta} \Sigma H\mathbf{Z}_{(p)}$$

is exact where η is the inclusion of the bottom cell. Let $D(k)$ be the cofiber of the diagonal map $Sp^{p^{k-1}}S^0 \to Sp^{p^k}S^0$ and $M(k)=\Sigma^{-k}D(k)/D(k-1)$. There is a homotopy equivalence $M(k)\simeq L(k)\vee L(k-1)$ [4]. The mod p Kuhn's exact sequence is

$$\cdots \to M(k+1) \xrightarrow{\delta_k} M(k) \to \cdots \to M(1) \xrightarrow{\delta_0} M(0) \xrightarrow{\eta} H\mathbf{Z}/p$$

where δ_k is the connecting map. On the other hand, there is a homotopy equivalence $M(k)\simeq e_k B(\mathbf{Z}/p)^k_+$ [4]. Let $tr: B(\mathbf{Z}/p)_+ \to S^0$ be the transfer. The sequence

$$\cdots \to M(k+1) \xrightarrow{d_k} M(k) \to \cdots \to M(1) \xrightarrow{d_0} M(0) \xrightarrow{\eta} H\mathbf{Z}/p$$

is equivalent to the mod p Kuhn's exact sequence, where d_k is the composition

$$M(k+1) \to B(\mathbf{Z}/p)^{k+1}_+ \xrightarrow{tr\wedge 1} B(\mathbf{Z}/p)^k_+ \to M(k).$$

By Lemma 2.2, we see that $e_n\boldsymbol{D}_1(1^k,0^{n-k})\simeq M(k)$ for $0\le k\le n$. Unless ϵ is $(1^k,0^{n-k})$ for some $0\le k\le n$, then $e_n\boldsymbol{D}_1(\epsilon)$ is contractible by Lemma 2.1. Furthermore, the map $e_n\boldsymbol{D}_1(1^{k+1},0^{n-k-1})\to e_n\boldsymbol{D}_1(1^k,0^{n-k})$ is equivalent to d_k. So we obtain the following proposition.

Proposition 2.5. *The sequence*

$$e_n\boldsymbol{D}_1(1,\cdots,1)\to e_n\boldsymbol{D}_1(1,\cdots,1,0)\to\cdots\to e_n\boldsymbol{D}_1(0,\cdots,0)$$

is equivalent to a part of the Kuhn's exact sequence

$$M(n)\to M(n-1)\to\cdots\to M(0).$$

Let $u_n: D(n)\to H\mathbf{Z}/p$ be a generator for $H^0(D(n);\mathbf{Z}/p)$. The mod p Steenrod algebra $\mathscr{A}_p$ acts on $H^*(D(n);\mathbf{Z}/p)$. For a finite sequence $I=(\epsilon_0, r_1, \epsilon_1, r_2, \cdots)$, $r_j\ge 0$, $\epsilon_j=0,1$, we let $\theta^I=\beta^{\epsilon_0}\mathscr{P}^{r_1}\beta^{\epsilon_1}\mathscr{P}^{r_2}\cdots$ where β is the Bockstein operation and $\mathscr{P}^r$ is the Steenrod reduced power. (For $p=2$, and $\beta=Sq^1$ and $\mathscr{P}^i=Sq^{2i}$.) We say that I is admissible if $r_i\ge pr_{i+1}+\epsilon_i$ for all i. Then the admissible θ^I are a basis for $\mathscr{A}_p$. The length $l(I)$ is defined by $l(I)=n$ if $r_i=0$ for $i>n$ and $\epsilon_i=0$ for $i\ge n$. Then $H^*(D(n);\mathbf{Z}/p)$ has a basis $\{\theta^I(u_n)\mid I\text{: admissible, } l(I)\le n\}$ [4].

Theorem 2.6 (Theorem 0.1). *There is a homotopy equivalence*

$$e_nF_n(p)\simeq\Sigma^{-n}D(n).$$

In particular, $H^*(e_nF(p);\mathbf{Z}/p)$ *has a basis* $\{\theta^I(v_n)\mid I\text{: admissible, } l(I)\le n\}$.

Proof. Let

$$\begin{tikzcd}[column sep=small] & E_1 \arrow[dr] & & E_0 \arrow[dr] & \\ \cdots M(2) \arrow[ur] & & M(1) \arrow[ur] & & M(0) \to H\mathbf{Z}/p \end{tikzcd}$$

be the Kuhn's exact sequence where $E_k \to M(k) \to E_{k-1}$ are exact. We obtain the following sequence

$$H\mathbf{Z}/p \to \Sigma E_0 \to \Sigma^2 E_1 \to \cdots.$$

Then $D(k) \xrightarrow{u_k} H\mathbf{Z}/p \to \Sigma^{k+1} E_k$ are cofiber sequences. We let $D(k)' = \Sigma^k e_n \boldsymbol{D}_1(2^k, 0^{n-k})$. Then there is a sequence

$$\begin{tikzcd}[column sep=small] \cdots \Sigma^{-2} D(2') \arrow[dr] & & \Sigma^{-1} D(1)' \arrow[dr] & \\ & M(2) \arrow[ur] & & M(1) \to M(0) \end{tikzcd}$$

where $\Sigma^{-k} D(k)' \to M(k) \to \Sigma^{-k+1} D(k-1)'$ are cofiber sequences. There are maps $v_k : D(k)' \to H\mathbf{Z}/p$ which are generators for $H^*(D(k)'; \mathbf{Z}/p)$. We let $\Sigma^{k+1} E_k'$ be the cofiber of v_k.

We prove that $D(k)' \simeq D(k)$ and the map $v_k : D(k)' \to H\mathbf{Z}/p$ is equivalent to u_k by induction on k. The case $k=1$ is trivial. Then we assume that the case $k-1$ is true. There is a homotopy commutative diagram

$$\begin{tikzcd} \Sigma^{-1} M(k) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & M(k) \arrow[d] \\ \Sigma^{-k} D(k-1) \arrow[r, "u_{k-1}"] \arrow[d] & \Sigma^{-k} H\mathbf{Z}/p \arrow[r] \arrow[d, equal] & E_{k-1} \arrow[d] \\ \Sigma^{-k} D(k)' \arrow[r, "v_k"] & \Sigma^{-k} H\mathbf{Z}/p \arrow[r] & \Sigma E_k' \end{tikzcd}$$

where all vertical and horizontal sequences are cofiber sequences. By the above diagram, we obtain a homotopy commutative triangle

$$\begin{tikzcd} & M(k) \arrow[dl] \arrow[dr] & \\ E_{k-1} \arrow[rr] & & \Sigma^{-k+1} D(k-1) \end{tikzcd}$$

Then the uniqueness of the map $M(k) \to E_{k-1}$ which factors through $M(k) \to M(k-1)$, we see that $E_k' \simeq E_k$ and the map $H\mathbf{Z}/p \to \Sigma^{k+1} E_k'$ is equivalent to the map $H\mathbf{Z}/p \to \Sigma^{k+1} E_k$. Hence $D(k)' \simeq D(k)$ and $v_k : D(k)' \to H\mathbf{Z}/p$ is equivalent to u_k. This completes the proof.

Remark. The strictly commutative diagram $\bar{\boldsymbol{D}}'$ over $\bar{\mathscr{E}}$ extends to the strictly commutative diagram $\boldsymbol{D}'$ over $\mathscr{E}$ such that

$$\begin{aligned} \boldsymbol{D}'(\epsilon_1, \cdots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \cdots, \epsilon_n) &\to \boldsymbol{D}'(\epsilon_1, \cdots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \cdots, \epsilon_n) \\ &\to \boldsymbol{D}'(\epsilon_1, \cdots, \epsilon_{i-1}, 2, \epsilon_{i+1}, \cdots, \epsilon_n) \end{aligned}$$

are cofiber sequences. Let $C_n(p) = \boldsymbol{D}'(2, \cdots, 2)$. Then the stable splitting of $C_n(p)$ by the Steinberg idempotent is homotopy equivalent to $L(n)$ which is an indecomposable wedge summand of $B(\mathbf{Z}/p)^n_+$.

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