J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 577-587

Topological realization of level structures of the formal group law over $\widehat{E(n)}$

By

Takeshi TORII

0. Introduction

Let $\widehat{E(n)}$ be the I_n-adic complete Johnson-Wilson spectrum. For simplicity, we assume that $\widehat{E(n)}$ is 2-periodic. Then the coefficient ring is given by

$$\widehat{E(n)}_* = \mathcal{O}[[u_1,\ldots,u_{n-1}]][u,u^{-1}]$$

where $\mathcal{O}$ is the Witt ring of the finite field $\mathbf{F}_{p^n}$, the degree of u_i is 0 and the degree of u is 2. In case $n=1$, $\widehat{E(1)}$ is essentially the p-adic K-theory. There is a famous result of Atiyah on the K-theory of the classifying space of finite group [1]. In [5, 6], Hopkins, Kuhn and Ravenel gave a generalization of the Atiyah's theorem to $\widehat{E(n)}$-cohomology. For a finite group G, the $\widehat{E(n)}$-cohomology of the classifying space of G is, at least tensored with $\mathbf{Q}$, described in terms of the abelian subgroups in G. Let $\mathscr{A}(G)$ be the small category whose objects are abelian subgroups of G, with morphisms generated by inclusions and conjugations. Then there is an isomorphism

$$\widehat{E(n)}^*(BG)\otimes\mathbf{Z}[|G|^{-1}] \xrightarrow[A\in\mathscr{A}(G)]{\cong} \widehat{E(n)}^*(BA)\otimes\mathbf{Z}[|G|^{-1}].$$

Therefore it is fundamental to study the $\widehat{E(n)}$-cohomology of the classifying space of abelian groups and it intimately connects with the theory of formal group law.

In the following we concentrate in the degree 0 part. Then the formal group law associated to $\widehat{E(n)}$ induces the formal group law F over $\widehat{E(n)}_0$. The formal group law F is the universal deformation of the p-typical formal group law $\bar{F}$ over $\mathbf{F}_{p^n}$ whose p-series satisfies $[p]^{\bar{F}}(x)=x^{p^n}$. There is a notion of level structures in the classical theory of elliptic curves and modular forms. In [2] Drinfel'd defined the level structure of a formal module and prove that the set of deformations with the level structure of the formal group law $\bar{F}$ is represented by a complete regular local ring which is finite and flat over $\widehat{E(n)}_0$.

Let $\widehat{K(n)}$ be the p-adic Morava K-theory whose coefficient ring is given by

Supported by JSPS Research Fellowships for Young Scientists
Received March 15, 1999

$$\widehat{K(n)}_* = \mathcal{O}[u, u^{-1}].$$

The formal group law $\tilde{F}$ over $\mathcal{O}$ associated to $\widehat{K(n)}$ is a deformation of a formal $\mathcal{O}$-module $\bar{F}$. The representing ring of the set of all level r structures on the deformation $\tilde{F}$ over $\mathcal{O}$ is a degree $p^{nr}-p^{n(r-1)}$ totally ramified abelian extension obtained by a Lubin-Tate formal group law $\tilde{F}$. By using the classifying spaces of cyclic groups and the transfer maps between them, we constructed a sequence of spectra whose $\widehat{K(n)}$-cohomology is isomorphic to the tower of totally ramified abelian extensions obtained by the Lubin-Tate theory [12]. In this note we construct a spectrum whose $\widehat{E(n)}$-cohomology is rationally isomorphic to the representing ring of the level structure by using the classifying spaces of the subgroups of an abelian p-group and the transfer maps between them.

The paper is organized as follows. In §1 we recall the definition of the level structure of a deformation of the formal group law. In §2 we discuss the realization problem of the transfer map to the category of spectra. In §3 we construct a strictly commutative diagram consisting of the suspension spectra of the classifying spaces and transfer maps between them. In §4 we study the $\widehat{E(n)}$-cohomology tensored with $\mathbf{Q}$ of the above diagram and prove the main theorem (Theorem 4.4).

I would like to thank Professor Goro Nishida for suggestion of this work and many helpful conversations.

1. Level structures of deformation

Let p be a prime number. There is a universal p-typical formal group law F_{BP} over $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$. Let $\mathbf{F}_{p^n}$ be the finite field with p^n elements. There is a ring homomorphism

$$\theta: BP_* \to \mathbf{F}_{p^n}$$

which is defined by $\theta(v_i)=0$ for $i \neq n$, $\theta(v_n)=1$. Then the p-series of the induced formal group law $\bar{F}=\theta_* F_{BP}$ is given by

$$[p]^{\bar{F}}(x) = x^{p^n}.$$

In this section we recall the level structure of a deformation of $\bar{F}$.

Let $(R, \mathfrak{m})$ be a complete Noetherian local ring whose residue field is an $\mathbf{F}_{p^n}$-algebra. Then we say that the formal group law G over R is a deformation of $\bar{F}$ if $G \equiv \bar{F}$ mod $\mathfrak{m}$. Two lifts G and G' over R are said to be $*$-isomorphic if there is an isomorphism $g(x)$ between G and G' such that $g(x) \equiv x$ mod $\mathfrak{m}$. Then there is a one-to-one correspondence between the $*$-isomorphism classes of deformations over R and the continuous ring homomorphisms from $\mathcal{O}[[u_1, \ldots, u_{n-1}]]$ to R where $\mathcal{O}$ is the Witt ring of $\mathbf{F}_{p^n}$ [10, 8]. Let $\mathcal{R} = \mathcal{O}[[u_1, \ldots, u_{n-1}]]$. There is a ring homomorphism

$$\varphi:\mathbf{Z}_{(p)}[v_1,\ldots,v_n\ldots]\to\mathscr{R}$$

which is determined by $\varphi(v_i)=u_i$ for $i<n$, $\varphi(v_n)=1$, $\varphi(v_i)=0$ for $i>n$. Then we obtain the formal group law $F=\varphi_* F_{BP}$ over $\mathscr{R}$. The formal group law F over $\mathscr{R}$ is a universal deformation of $\bar{F}$.

For a finite abelian p group A, we define the level A structure of a deformation [2, 9, 11]. Let G be a deformation over R. Then $\mathfrak{m}$ is an abelian group with respect to the formal group sum $+_G$. Let $A(1)$ be the kernel of the p-times map:

$$A(1)=\mathrm{kernel}(p:A\to A).$$

A level A structure on G is a module homomorphism

$$\phi:A\to\mathfrak{m}$$

such that $[p]^G(x)$ is divisible by

$$\prod_{\alpha\in A(1)}(x-\phi(\alpha)).$$

A structure of level r on G means a level $(\mathbf{Z}/p^r)^n$ structure.

Let $\mathscr{C}$ be a category whose objects are complete Noetherian local rings with $\mathbf{F}_{p^n}$-algebras as residue fields. There is a functor Level(A) from $\mathscr{C}$ to the category of sets such that Level(A)(R) ($R\in\mathscr{C}$) is the set of all isomorphism classes of pairs (G,ϕ) where G is a deformation over R and ϕ a level A structure on G. We denote by rank(A) the p-rank of a finite abelian p-group A.

Theorem 1.1 ([2, 9, 11]). *If* rank$(A)>n$, *then* level$(A)=\emptyset$. *If* rank $(A)\leq n$, *then the functor* Level(A) *is represented by a regular local ring* $D(A)$. *Let* $A=\mathbf{Z}/p^{r_1}\times\cdots\times\mathbf{Z}/p^{r_s}$ *and* e_i *be the ith generator* $(1\leq i\leq s\leq n)$. *Let* α_i *be the images of* e_i *under the universal deformation of level* A. *Then* $D(A)=\mathscr{R}[\alpha_1,\ldots,\alpha_s]$ *and* $\alpha_1,\ldots,\alpha_s$, $u_s,\ldots,u_{n-1}$ *form a regular parameter system.*

We denote Level$((\mathbf{Z}/p^r)^n)$ by Level$_r$ and the representing ring $D((\mathbf{Z}/p^r)^n)$ by D_r. In the following we assume that rank$(A)\leq n$. We denote by A^* the character group of A:

$$A^*=\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z}).$$

Then $\widehat{E(n)}^0(BA)$ represents the functor from complete Noetherian local rings to sets such that $\mathrm{Hom}^c(\widehat{E(n)}^0(BA),R)$ is the set of all isomorphism classes of pairs (G,ϕ) where G is a deformation over R and ϕ is a module homomorphism from A^* to $\mathfrak{m}$. We denote by Hom(A^*) the functor represented by $\widehat{E(n)}^0(BA)$.

Let B be a subgroup of A and ϕ a level B^* structure on a deformation G. By following the dual map $A^*\to B^*$ with ϕ, we obtain a module homomorphism

$$A^* \to B^* \xrightarrow{\phi} \mathfrak{m}.$$

Hence we obtain a natural transformation $\mathrm{Level}(B^*) \to \mathrm{Hom}(A^*)$ and a homomorphism of representing rings $\widehat{E(n)}^0(BA) \to D(B^*)$.

Theorem 1.2 ([7]). *The ring homomorphisms* $\widehat{E(n)}^0(BA) \to D(B^*)$ *induce the following isomorphism*:

$$\widehat{E(n)}^0(BA)\otimes\mathbf{Q} \xrightarrow{\cong} \prod_{B\leq A} D(B^*)\otimes\mathbf{Q}.$$

2. Realization of transfer

In this section we study the realization problem of the diagram consisting of transfer maps, to the category of spectra.

Let $\mathbf{D}$ be a small category. We call a functor from $\mathbf{D}$ to the homotopy category of topological spaces as a homotopy commutative $\mathbf{D}$-diagram. Let $\mathbf{T}$ be the category of topological spaces and $h\mathbf{T}$ its homotopy category. There is a projection functor $\pi:\mathbf{T} \to h\mathbf{T}$. We say that a homotopy commutative $\mathbf{D}$-diagram $\bar{Y}$ has a realization if there is a functor Y from $\mathbf{D}$ to $\mathbf{T}$ such that $\pi\circ Y$ is naturally equivalent to $\bar{Y}$.

We recall a free category and the standard resolution [3]. A category is called free with S as the set of generators if every non-identity morphism is uniquely written as a finite composition of morphisms in S. For a small category $\mathbf{D}$, the free category $F\mathbf{D}$ on $\mathbf{D}$ is a free category with the same objects in $\mathbf{D}$ and with all non-identity morphisms as generators. The standard resolution $F_*\mathbf{D}$ of $\mathbf{D}$ is a simplicial category which in dimension k consists of the category $F_k\mathbf{D} = F^{k+1}\mathbf{D}$. There is an obvious functor $p:F_*\mathbf{D} \to \boldsymbol{D}$.

A map X from a simplicial category $\mathbf{E}$ to $\mathbf{T}$ consists of topological spaces $X(E)$ for all $E\in\mathbf{E}$ and continuous maps

$$X(E_1,E_2): |\mathbf{E}(E_1,E_2)| \times X(E_1) \to X(E_2)$$

for all $E_1,E_2\in\mathbf{E}$ subject to the obvious associativity and identity conditions where $|\mathbf{E}(E_1,E_2)|$ is the geometric realization of the simplicial set $\mathbf{E}(E_1,E_2)$.

For a homotopy commutative $\mathbf{D}$-diagram $\bar{Y}$, an ∞-homotopy commutative $\mathbf{D}$-diagram over $\bar{Y}$ is a map $Y_\infty:F_*\mathbf{D} \to \boldsymbol{T}$ such that the following diagram is commutative in the obvious sense:

$$\begin{array}{ccc} F_*\mathbf{D} & \xrightarrow{p} & \mathbf{D} \\ {\scriptstyle Y_\infty}\downarrow & & \downarrow{\scriptstyle \bar{Y}} \\ \mathbf{T} & \xrightarrow[\pi]{} & h\mathbf{T}. \end{array}$$

Then we have the following theorem.

Theorem 2.1 ([4]). *A homotopy commutative* **D**-*diagram has a realization if and only if there exists an* ∞-*homotopy commutative* **D**-*diagram over* $\bar{Y}$.

Let **P** be a partially ordered set with finite cardinality. We regard **P** as a small category. Let F' be a functor from **P** to **T** such that $F'(a)$ are finite complexes and the maps $F'(a) \to F'(a')$ are finite coverings. Let **S** be the category of spectra and h**S** its homotopy category. By corresponding the finite coverings $F'(a) \to F'(a')$ to the transfer maps $\Sigma^\infty F'(a)_+ \to \Sigma^\infty F'(a')_+$, we obtain a functor F from $\mathbf{P}^{op}$ to h**S**. As in the case of **T**, we say that a functor $\bar{Y}$ from a small category **D** to h**S** has a realization if there is a functor Y from **D** to **S** such that $\pi \circ Y$ is naturally equivalent to $\bar{Y}$ where $\pi: \mathbf{S} \to h\mathbf{S}$ is the projection. We consider the realization problem of F.

Let $a,b \in \mathbf{P}$ such that $a<b$. We denote by $P'(a,b)$ the set of all finite sequences $\mathbf{x}=(x_1,\ldots,x_n)$ in **P** such that $a<x_1<\cdots<x_n<b$. We define the ordering in $P(a,b)$ so that $\mathbf{x} \geq \mathbf{y}$ if $\mathbf{x}$ is a refinement of $\mathbf{y}$. Let $P(a,b)$ be the union of $P'(a,b)$ and the distinguished element $\emptyset$

$$P(a,b)=P'(a,b) \cup \{\emptyset\}.$$

We regard $\emptyset$ as the length 0 sequence and the minimum element. Then the simplicial set determined by the ordered set $P(a,b)$ is isomorphic to $F_*\mathbf{P}(a,b)$.

For $a \in \mathbf{P}$, we say that a map $i(a): F'(a) \to \mathbf{R}^{n(a)}$ is admissible if $F'(a,b) \times i(a): F'(a) \to F'(b) \times \mathbf{R}^{n(a)}$ is an embedding for all $b>a$ where $n(a)$ is a natural number. For example, an embedding $F'(a) \subsetneq \mathbf{R}^{n(a)}$ is admissible. We fix admissible maps $i(a)$. Let $\Sigma F'(a)=F'(a)_+ \wedge \bigwedge_{p \in \mathbf{P}} S^{n(b)}$. For an element $\mathbf{x}=(x_1,\ldots,x_k)$ of $P(a,a')$ and $b \in \mathbf{P}$, we define a relation $b \preceq \mathbf{x}$ if $x_i \leq b \leq x_{i+1}$ for some $0 \leq i \leq k$ where $x_0=a$ and $x_{k+1}=a'$. Let $\mathbf{x}_0<\cdots<\mathbf{x}_n$ be a non-singular n-simplex and $\Delta^n=\{(t_1,\cdots,t_n) \in \mathbf{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$ the standard n-simplex. For any $t \in \Delta^n$, we consider an embedding over the covering $F'(a,a')$:

$$i_t: F'(a) \to F'(a') \times \prod_{a \leq b < a'} \mathbf{R}^{n(b)}$$

where $p_b \circ i_t$ is $F'(a) \to F'(b) \xrightarrow{i(b)} \mathbf{R}^{n(b)} \xrightarrow{\cdot t_j} \mathbf{R}^{n(b)}$ if $b \preceq \mathbf{x}_{j-1}$ and $b \not\preceq \mathbf{x}_j$ $(t_{-1}=0,\ t_{n+1}=1)$. Then we obtain a homotopy of embedding $F'(a) \times \Delta^n \to F'(a') \times \prod_{a \leq b < a'} \mathbf{R}^{n(b)} \times \Delta^n$. By the Pontrjagin-Thom construction, there is a homotopy of transfer $\Sigma F'(a') \wedge \Delta^n_+ \to \Sigma F'(a) \wedge \Delta^n_+$. We identify the n-simplex $\mathbf{x}_0<\cdots<\mathbf{x}_n$ with Δ^n. Then it is easy to verify the above maps for all non-singular simplexes for $P(a,a')$ define a map

$$\Sigma F'(a') \wedge |P(a,a')|_+ \to \Sigma F'(a).$$

Hence we obtain the following theorem.

Theorem 2.2. *The functor* F *has a realization.*

Let G be a compact Lie group and $\{H\}$ be the finite set of the closed subgroups of G such that the index $[G:H]$ is finite. Then $\{H\}$ is a partially ordered set by means of the inclusions. There is a contractible free G-space EG and the quotient space EG/H is a classifying space BH. Let F'_G be the functor from $\{H\}$ to $\mathbf{T}$ with $F'_G(H)=EG/H$ as objects and the natural maps $F'_G(H)\to F_G(K)$ for $H\subset K$ as morphisms. By corresponding the finite covering $F'_G(H)\to F'_G(K)$ to the transfer map $\Sigma^\infty F'_G(K)_+\to\Sigma^\infty F'_G(H)_+$, we obtain a functor F_G from $\{H\}^{op}$ to $h\mathbf{S}$. By Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *The functor F_G has a realization.*

Let F'_0, F'_1 be two functors from $\mathbf{P}$ to $\mathbf{T}$ as above and F_0, F_1 the corresponding functors from $\mathbf{P}^{op}$ to $h\mathbf{S}$. We assume that there are maps $\alpha(a)$: $F'_0(a)\to F'_1(a)$ such that the following is a pull-back diagram for all $a<b$:

$$\begin{array}{ccc} F'_0(a) & \xrightarrow{\alpha(a)} & F'_1(a) \\ \downarrow & & \downarrow \\ F'_0(b) & \xrightarrow{\alpha(b)} & F'_1(b). \end{array}$$

Then we compare realizations of F_0 and F_1. The following lemma is easy.

Lemma 2.4. *Let $i(a):F'_1(a)\to\mathbf{R}^{n(a)}$ be an admissible map. Then $i(a)\circ\alpha(a):F'_0(a)\to\mathbf{R}^{n(a)}$ is also admissible.*

Let I be the ordered set $\{0,1\}$. Then we have a functor F from $\mathbf{P}^{op}\times I$ to $h\mathbf{S}$ such that $F(-\times i)=F(-)$ for $i=0,1$ and the morphism $F(a\times 0)\to F(a\times 1)$ is $\Sigma^\infty\alpha(a)_+$. For non-singular simplex $\mathbf{x}_0<\cdots<\mathbf{x}_n$ in $P(a,a')$, there is a commutative diagram

$$\begin{array}{ccc} F'_0(a)\times\Delta^n & \to & F'_0(a')\times\Pi_{a\le b<a'}\mathbf{R}^{n(b)} \\ {\scriptstyle\alpha(a)\times id}\downarrow & & \downarrow{\scriptstyle\alpha(a')\times id} \\ F'_1(a)\times\Delta^n & \to & F'_1(a')\times\Pi_{a\le b<a'}\mathbf{R}^{n(b)} \end{array}$$

where the top horizontal arrow is an embedding by means of $i(b)$ for $a\le b<a'$ and the bottom one is by means of $\alpha(a)\circ i(b)$. Then we easily obtain the following theorem from Theorem 2.1.

Theorem 2.5. *There are realizations G_0, G_1 of F_0, F_1 respectively and the natural transformation from G_0 to G_1 as functors from $\mathbf{P}^{op}$ to $\mathbf{S}$.*

Remark. We can easily generalize the discussion of this section to the equivariant situation.

3. Construction of the spectrum $F(A)$

In this section we construct a strictly commutative diagram, which is indexed by the flags in the vector space $\mathbf{F}_p^s$, consisting of the suspension spectra of the classifying spaces of the subgroups of A and the transfer maps between them by using the result of §2.

Let A be a finite abelian p-group of rank s ($s \leq n$). There is a contractible free A space EA. For the subgroup B of A, the quotient space EA/B is a classifying space of B. Let $\mathscr{W}$ be the set of all flags in $\mathbf{F}_p^s$. That is, $\mathscr{W}$ is a set of all expanding sequences of the subspaces of $\mathbf{F}_p^s$: $\mathscr{W} = \{W = \{W_1 \subset \cdots \subset W_k \subset \mathbf{F}_p^s\} | 0 \leq \dim W_1 < \cdots < \dim W_k < s,\ 0 \leq k \leq s\}$. There is a projection $\pi : A \to \mathbf{F}_p^s$. For $W \in \mathscr{W}$, we let BW be the classifying space $B\pi^{-1}(W_1)$. In particular, we have $B\{\mathbf{F}_p^n\} = BA$. If W' is a refinement of $W \in \mathscr{W}$, then there is an obvious finite covering map $BW' \to BW$. So we obtain a strictly commutative diagram, which is indexed by the flags in $\mathbf{F}_p^s$,consisting of the classifying spaces of the subgroups of A and the finite covering maps between them.

We define the ordering in $\mathscr{W}$ as follows:

$$W' \leq W \Leftrightarrow W' \text{ is a refinement of } W.$$

Then we regard $\mathscr{W}$ as a small category. From the above construction, there is a functor $\boldsymbol{B}$ from $\mathscr{W}$ to the category of spaces so that $\boldsymbol{B}W = BW$ and $\boldsymbol{B}W' \to \boldsymbol{B}W$ is an obvious finite covering for $W' < W$. By Corollary 2.3, we obtain the following lemma.

Lemma 3.1. *There is a functor* $\boldsymbol{E}$ *from* $\mathscr{W}$ *to the category of spectra such that* $\boldsymbol{E}W \simeq \Sigma^\infty BW_+$ *for* $W \in \mathscr{W}$ *and the morphism* $\boldsymbol{E}W \to \boldsymbol{E}W'$ *is a transfer map for* $W' < W$.

Let $\mathscr{E} = \{\epsilon = (\epsilon_1, \cdots, \epsilon_s) \in \boldsymbol{Z}^s | 0 \leq \epsilon_i \leq 2\}$ and $\bar{\mathscr{E}} = \{\epsilon \in \mathscr{E} | \epsilon_i = 0 \text{ or } 1\}$. We define the ordering in ϵ as follows:

$$\epsilon = (\epsilon_1, \cdots, \epsilon_s) \leq \epsilon' = (\epsilon'_1, \cdots, \epsilon'_s) \Leftrightarrow \epsilon_i \leq \epsilon'_i \text{ for all } i.$$

Then we regard $\mathscr{E}$ as a small category and $\bar{\mathscr{E}}$ a subcategory. For $\epsilon \in \bar{\mathscr{E}}$, we let K_ϵ be the subset of $\{0, 1, \cdots, s-1\}$ where $i-1 \in K_\epsilon$ if and only if $\epsilon_i = 0$. We denote the subset of $\mathscr{W}$ consisting of the flags of type K_ϵ by $\mathscr{W}_\epsilon$:

$$\mathscr{W}_\epsilon = \{W = \{W_1 \subset \cdots \subset W_k \subset \mathbf{F}_p^s\} | \dim W_j = i_j\}$$

if $K_\epsilon = \{i_1 < \cdots < i_k\}$. Let $\bar{\boldsymbol{D}}(\epsilon) = \Pi_{W \in \mathscr{W}_\epsilon} \boldsymbol{E}W$. Then the functor $\boldsymbol{E}$ induces a functor $\bar{\boldsymbol{D}}$ from $\bar{\mathscr{E}}$ to the category of spectra. Since $\bar{\boldsymbol{D}}$ is a strictly commutative diagram over $\bar{\mathscr{E}}$, we extend the diagram to a functor $\boldsymbol{D}$ from $\mathscr{E}$ to the category of spectra so that

$$\begin{aligned} \boldsymbol{D}(\epsilon_1, \cdots, \epsilon_{i-1}, 2, \epsilon_{i+1}, \cdots, \epsilon_s) &\to \boldsymbol{D}(\epsilon_1, \cdots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \cdots, \epsilon_s) \\ &\to \boldsymbol{D}(\epsilon_1, \cdots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \cdots, \epsilon_s) \end{aligned}$$

are fibre sequences. We define the spectrum $F(A)$ as $\boldsymbol{D}(2,\cdots,2)$. In particular, we denote $F((\boldsymbol{Z}/p^r)^n)$ by $F_n(p^r)$.

We denote the functors $\boldsymbol{B}$, $\boldsymbol{D}$, $\bar{\boldsymbol{D}}$ and $\boldsymbol{E}$ constructed above based on $(\boldsymbol{Z}/p^r)^n$ by $\boldsymbol{B}_r$, $\boldsymbol{D}_r$, $\bar{\boldsymbol{D}}_r$ and $\boldsymbol{E}_r$. We consider the relation between $\boldsymbol{D}_r$ and $\boldsymbol{D}_{r-1}$. Let $\pi_r:(\boldsymbol{Z}/p^r)^n \to (\boldsymbol{Z}/p^{r-1})^n$ be the projection. There is a π_r-equivariant map $E(\boldsymbol{Z}/p^r)^n \to E(\boldsymbol{Z}/p^{r-1})^n$. This map induces a natural transformation from $\boldsymbol{B}_r$ to $\boldsymbol{B}_{r-1}$ such that the following is a pull-back diagram for all $W' \leq W$:

$$\begin{array}{ccc} \boldsymbol{B}_r W' & \to & \boldsymbol{B}_{r-1} W' \\ \downarrow & & \downarrow \\ \boldsymbol{B}_r W & \to & \boldsymbol{B}_{r-1} W. \end{array}$$

By Theorem 2.5, we obtain functors $\boldsymbol{E}_r$ and $\boldsymbol{E}_{r-1}$ from $\mathscr{W}$ to the category of spectra, and a natural transformation from $\boldsymbol{E}_r$ to $\boldsymbol{E}_{r-1}$. Therefore we obtain the following theorem.

Theorem 3.2. *There are functors $\boldsymbol{D}_r$ from $\mathscr{E}$ to the category of spectra such that $\boldsymbol{D}_r(\epsilon) \simeq \Pi_{W \in \mathscr{W}_\epsilon} \boldsymbol{E}_r W$ and the morphism $\boldsymbol{D}_r(\epsilon) \to \boldsymbol{D}_r(\epsilon')$ is a transfer map for $\epsilon' < \epsilon, \epsilon, \epsilon' \in \bar{\mathscr{E}}$, and*

$$\begin{aligned} \boldsymbol{D}_r(\epsilon_1,\cdots,\epsilon_{i-1},2,\epsilon_{i+1},\cdots,\epsilon_n) &\to \boldsymbol{D}_r(\epsilon_1,\cdots,\epsilon_{i-1},1,\epsilon_{i+1},\cdots,\epsilon_n) \\ &\to \boldsymbol{D}_r(\epsilon_1,\cdots,\epsilon_{i-1},0,\epsilon_{i+1},\cdots,\epsilon_n) \end{aligned}$$

are fibre sequences. Furthermore, we have a natural transformation η_{r-1} from $\boldsymbol{D}_r$ to $\boldsymbol{D}_{r-1}$:

$$\boldsymbol{D}_1 \xleftarrow{\eta_1} \boldsymbol{D}_2 \xleftarrow{\eta_2} \boldsymbol{D}_3 \leftarrow \cdots.$$

4. $\widehat{E(n)}$-cohomology of $F(A)$ over $\mathbf{Q}$

In this section we study the $\widehat{E(n)}$-cohomology of $\boldsymbol{D}(\epsilon)$ tensored with $\mathbf{Q}$ and prove the main theorem (Theorem 4.4).

First we study the transfer maps under the decomposition of Theorem 1.2. Let A' and B be subgroups of A.

Lemma 4.1. *If $B \leq A'$, then the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{E(n)}^0(BA) & \to & D(B^*) \\ {\scriptstyle tr_*}\uparrow & & \uparrow{\scriptstyle \cdot|A/A'|} \\ \widehat{E(n)}^0(BA') & \to & D(B^*) \end{array}$$

where the left vertical arrow is a transfer map and the right vertical arrow is the index $|A/A'|$ times map.

Proof. There is a factorization

$$\widehat{E(n)}^0(BA) \to \widehat{E(n)}^0(BB) \to D(B^*).$$

Then the lemma follows from the double coset formula.

Lemma 4.2. *If* $B \not\leq A'$ *and* $A' \neq A$, *then the homomorphism*

$$\widehat{E(n)}^0(BA') \xrightarrow{tr*} \widehat{E(n)}^0(BA) \to D(B^*)$$

is a zero map.

Proof. From the following commutative diagram, we see that it is sufficient to prove the case $B=A$:

$$\begin{array}{ccccc} \widehat{E(n)}^0(BA) & \to & \widehat{E(n)}^0(BB) & \to & D(B^*) \\ \uparrow{\scriptstyle tr*} & & \uparrow{\scriptstyle tr*} & & \\ \widehat{E(n)}^0(BA') & \xrightarrow{\cdot|A/(A'+B)|} & \widehat{E(n)}^0(B(A'\cap B)). & & \end{array}$$

Let $A=\boldsymbol{Z}/p^{r_1}\times\cdots\times\boldsymbol{Z}/p^{r_s}$, $r_1\leq\cdots\leq r_s$. Let $C=(\boldsymbol{Z}/p^{r_s})^s$ and $p:C\to A$ be a canonical surjection. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \widehat{E(n)}^0(BA') & \xrightarrow{tr*} & \widehat{E(n)}^0(BA) & \to & D(A^*) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{E(n)}^0(Bp^{-1}(A')) & \xrightarrow{tr*} & \widehat{E(n)}^0(BC) & \to & D(C^*). \end{array}$$

where the right vertical arrow is injective. Then it is easy to prove that the composition of the bottom horizontal arrows is zero.

Let $\epsilon'=(\epsilon_{l+2},\cdots,\epsilon_s)$, $\epsilon_i=0$ or 1 $(l+2\leq i\leq s)$, and $\epsilon=(1^l,0,\epsilon')$. The following proposition follows from the induction and Lemma 4.1, 4.2.

Proposition 4.3. *The sequence*

$$0\to\widehat{E(n)}^0(\boldsymbol{D}(\epsilon_1))\otimes\boldsymbol{Q}\to\widehat{E(n)}^0(\boldsymbol{D}(\epsilon_2))\otimes\boldsymbol{Q}\to\widehat{E(n)}^0(\boldsymbol{D}(\epsilon_3))\otimes\boldsymbol{Q}\to 0$$

is exact where $\epsilon_1=(2^{k-1},0,1^{l-k},0,\epsilon')$, $\epsilon_2=(2^{k-1},1^{l-k+1},0,\epsilon')$ *and* $\epsilon_3=(2^k,1^{l-k+1},0,\epsilon')$. *There is an isomorphism*

$$\widehat{E(n)}^0(\boldsymbol{D}(2^k,1^{l-k},0,\epsilon'))\otimes\boldsymbol{Q}\cong\prod_{W\in\mathscr{W}_\epsilon}\ \prod_{\substack{\pi B\leq W_1\\ \dim\pi B\geq k}} D(B^*)\otimes\boldsymbol{Q}.$$

In particular, we obtain the following theorem from the case $k=l=s$.

Theorem 4.4. $\widehat{E(n)}^0(F(A))\otimes \boldsymbol{Q}\cong D(A^*)\otimes \boldsymbol{Q}$.

We recall that $F_n(p^r)=\boldsymbol{D}_r(2^n)$. From Theorem 3.2, there is a sequence of spectra:

$$F_n(p)\xleftarrow{\eta_1} F_n(p^2)\xleftarrow{\eta_2} F_n(p^3)\xleftarrow{\eta_3}\cdots.$$

Corollary 4.5. *The $\widehat{E(n)}$-cohomology of the above sequence tensored with* $\mathbf{Q}$:

$$\widehat{E(n)}^0(F_n(p))\otimes\mathbf{Q}\xrightarrow{\eta_1^*}\widehat{E(n)}^0(F_n(p^2))\otimes\mathbf{Q}\xrightarrow{\eta_2^*}\cdots$$

is identified with the expanding sequence

$$D_1\otimes\mathbf{Q}\subsetneq D_2\otimes\mathbf{Q}\subsetneq D_3\otimes\mathbf{Q}\subsetneq\cdots$$

where D_r is a representing ring of the functor Level_r.

Proof. This follows from the fact that the homomorphism

$$\widehat{E(n)}^0(B(\boldsymbol{Z}/p^{r-1})^n)\to\widehat{E(n)}^0(B(\boldsymbol{Z}/p^r)^n)$$

induced by the projection $(\boldsymbol{Z}/p^r)^n\to(\boldsymbol{Z}/p^{r-1})^n$ is given by $x_i\mapsto[p](x_i)$.

Department of Mathematics
Kyoto University

References

[1] M. F. Atiyah, Characters and cohomology of finite groups, Inst. Hautes Études Sci. Publ. Math., **9** (1961), 23–64.
[2] V. G. Drinfel'd, Elliptic modules, (Russian) Mat. Sb. (N.S.) [94(136)] (1974), 594–627, 656. English translation: Math. USSR-Sb., **23**–4 (1974), 561–592 (1976).
[3] W. G. Dwyer and D. M. Kan, Simplicial localizations of categories, J.Pure Appl. Algebra, **17**–3 (1980), 267–284.
[4] W. G. Dwyer, D. M. Kan and J. H. Smith, Homotopy commutative diagrams and their realizations, J. Pure Appl. Algebra, **57**–1 (1989), 5–24.
[5] M. J. Hopkins, N. J. Kuhn and D. C. Ravenel, Generalised group characters and complex oriented cohomology theories, preprint.
[6] M. J. Hopkins, N. J. Kuhn and D. C. Ravenel, Morava K-theories of classifying spaces and generalized characters for finite groups, Algebraic topology (San Feliu de Guixols, 1990), 186–209, Lecture Notes in Math. 1509, Springer, Berlin, 1992.
[7] J. P. C. Greenless and N. P. Strickland, Varieties and local cohomology for chromatic group cohomology rings, preprint.
[8] M. Hazewinkel, Formal groups and applications, Pure and Applied Mathematics, 78. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
[9] N. M. Katz and B. Mazur, Arithmetic moduli of elliptic curves, Annals of Mathematics Studies 108. Princeton University Press, Princeton, N.J., 1985.

[10] J. Lubin and J. Tate, Formal moduli for one-parameter formal Lie groups, Bull. Soc. Math. France, **94** (1966), 49–59.

[11] N. P. Strickland, Finite subgroups of formal groups, J.Pure Appl. Algebra, **121**–2 (1997), 161–208.

[12] T. Torii, Topological realization of the integer ring of local field, J.Math. Kyoto Univ., **38**–4 (1998), 781–788.