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# On a Certain Extended Galois Action on the Space of Arithmetic Modular Forms with respect to a Unitary Group

By

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## Introduction

In his work [8, Theorem 1.5], G. Shimura proved the existence of a certain Galois action on the graded ring of Hilbert modular forms. A holomorphic Hilbert modular form $f$ with respect to $\mathrm{SL}(2, F)$ (where $F$ is a totally real algebraic number field of finite degree) can be expressed as a Fourier series of complex variables $u_1, \ldots, u_l$

$$f(u_1, \ldots, u_l) = \sum_x c_x \exp\left(2\pi\sqrt{-1}\sum_{\mu=1}^{l} x_\mu u_\mu\right), \tag{0.1}$$

where the coefficients $c_x \in \mathbb{C}$ and $x$ runs over a lattice. It is shown first that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a holomorphic modular form $f^\sigma$ whose Fourier expansion is

$$f^\sigma(u_1, \ldots, u_l) = \sum_x c_x^\sigma \exp\left(2\pi\sqrt{-1}\sum_{\mu=1}^{l} x_\mu u_\mu\right). \tag{0.2}$$

A Hilbert modular form with respect to $\mathrm{SL}(2, F)$ has the weight in $\sum_{v\in\mathbf{a}} \mathbb{Z}\cdot v$, where $\mathbf{a}$ is the set of all embeddings of $F$ into $\mathbb{R}$. If $f$ is of weight $k = \sum_{v\in\mathbf{a}} k_v \cdot v$, then $f^\sigma$ is of weight $k^\sigma = \sum_{v\in\mathbf{a}} k_v \cdot v\sigma$. It is also shown that, there exists a certain closed subgroup $\mathfrak{G}$ of $\mathrm{GL}(2, F_A) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which acts on the graded ring of meromorphic Hilbert modular forms which can be expressed as a quotient of holomorphic Hilbert modular forms with $\overline{\mathbb{Q}}$-rational Fourier coefficients. An important aspect here is that the action of $\mathfrak{G}$ on Hilbert modular forms of weight 0 coincides with that of $\mathfrak{G}$ in the theory of canonical models constructed in [2].

In this paper we shall study such a Galois action on modular forms in the case of unitary groups. On unitary groups, modular forms have no Fourier

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expansions. But in the case when the signatures of the group at the infinite places are all equal, they have Fourier-Jacobi expansions. We shall consider such cases in this paper. To be more concrete, let $F$, $\mathbf{a}$ be as above and $K$ be a CM-extension (i.e. a totally imaginary quadratic extension) of $F$, and consider a non-degenerate skew-hermitian matrix $R$ with coefficients in $K$ as

$$R = \begin{pmatrix} & & 1_q \\ & S & \\ -1_q & & \end{pmatrix}, \qquad S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix}$$

where $s_1, \ldots, s_n$ are pure imaginary for any embeddings of $K$ into $\mathbb{C}$ and $s_i/s_j$ $(1 \leq i, j \leq n)$ are totally positive. Then $-\sqrt{-1}R$ is a hermitian matrix of signature $(q, n+q)$ or $(n+q, q)$ for any embedding of $K$ into $\mathbb{C}$. Define the group $G$ of unitary similitudes with respect to $R$ by

$$G(\mathbb{Q}) = \left\{ \gamma \in \mathrm{GL}(2q+n, K) \,\middle|\, {}^t\gamma^\rho R \gamma = \nu(\gamma) R \quad \text{with} \ \ \nu(\gamma) \in F^\times \right\},$$

where $\rho$ stands for the non-trivial element of $\mathrm{Gal}(K/F)$; $\rho$ is the complex conjugation for any embedding of $K$ into $\mathbb{C}$. We can define the natural holomorphic action of $G(\mathbb{R})$ on the symmetric domain

$$D = \prod_{v \in \mathbf{a}} \left\{ \mathfrak{z}_v = \begin{pmatrix} z_v \\ w_v \end{pmatrix} \,\middle|\, \begin{array}{l} z_v \in \mathbb{C}^q_q, \ w_v \in \mathbb{C}^n_q, \\ \sqrt{-1}({}^t\overline{w_v} S^{\Psi_v} w_v + {}^t\overline{z_v} - z_v) > 0 \end{array} \right\}$$

where $\Psi_v$ is the embedding of $K$ into $\mathbb{C}$ which lies above $v$ such that $\mathrm{Im}(S^{\Psi_v}) > 0$. Here $A^n_q$, for a ring $A$, denotes the set of all $n \times q$-matrices with entries in $A$. Then a holomorphic modular form $f$ on $D$ with respect to a congruence subgroup of $G(\mathbb{Q})$ has a Fourier-Jacobi expansion of the form

$$f\left(\begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v \in \mathbf{a}}\right) = \sum_r g_r((w_v)_{v \in \mathbf{a}}) \exp\left(2\pi\sqrt{-1} \sum_{v \in \mathbf{a}} \mathrm{tr}(r^{\Psi_v} z_v)\right), \tag{0.3}$$

where $r$ runs over non-negative hermitian matrices (for any embeddings of $K$ into $\mathbb{C}$) belonging to a $\mathbb{Z}$-lattice in $K^q_q$. Let $\Psi = (\Psi_v)_{v \in \mathbf{a}}$ be the CM-type of $K$ and regard $\Psi$ as an embedding of $K$ into $\mathbb{C}^{\mathbf{a}}$ by $b^\Psi = (b^{\Psi_v})_{v \in \mathbf{a}}$. Then for a lattice $L$ of $K^n_q$, the Fourier coefficients $g_r$ are theta functions on $\prod_{v \in \mathbf{a}} \mathbb{C}^n_q$ with respect to the lattice $L^\Psi$. We call $g_r$ *arithmetic* if the value

$$(g_r)_*((w_v)_{v \in \mathbf{a}}) = \exp\left(\pi\sqrt{-1} \sum_{v \in \mathbf{a}} \mathrm{tr}(r^{\Psi_v}\, {}^t\overline{w_v} S^{\Psi_v} w_v)\right) g_r((w_v)_{v \in \mathbf{a}})$$

is algebraic at each $(w_v)_{v \in \mathbf{a}} \in (K^n_q)^\Psi$. The main theorem of [4] defines a certain Galois action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on the arithmetic theta functions *when $\sigma$ is trivial on the reflex field $K^*$ of $\Psi$.* In the case when $K$ is an imaginary quadratic field and $\sigma$ is trivial on $K$, the Galois action on the modular forms are constructed in [7, Section 4] (but with no proof). In this paper we first generalize the main theorem of [4] to all $\sigma \in \mathrm{Aut}(\mathbb{C})$. To be more concrete, for such $g_r$ and $\sigma$, there exists a certain theta function $g_r^{(\sigma,\Psi,a)}$ (the symbol $a$ denotes an element of the idele group of $K$ depending on $\sigma$ and $\Psi$) with respect to the lattice $(aL)^{\Psi\sigma}$ of $\prod_{v\in\mathbf{a}} \mathbb{C}_q^n$ which satisfies

$$\left(g_r^{(\sigma,\Psi,a)}\right)_* ((au)^{\Psi\sigma}) = \left\{(g_r)_*(u^{\Psi})\right\}^{\sigma}$$

for any $u \in K_q^n$, where $\Psi\sigma$ means the CM-type defined by $\Psi\sigma = \{\Psi_v\sigma | v \in \mathbf{a}\}$.

The main theorem (Theorem 6.1) is as follows.

*Let $f$ be a holomorphic modular form given by (0.3) and $(\sigma, \Psi, a)$ be as above. Then there exists $b \in F^\times$ and another modular form $f^{(\sigma,\Psi,a)}$ with respect to another group $\tilde{G}$ of unitary similitudes and symmetric domain $\tilde{D}$ corresponding to a skew-hermitian form* $\begin{pmatrix} & & 1_q \\ & bS & \\ -1_q & & \end{pmatrix}$ *( $b \in F^\times$ is determined by $\sigma$,$\Psi$ and $a$) whose Fourier-Jacobi expansion is*

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \sum_r g_r^{(\sigma,\Psi,a)}((\tilde{w_v})_{v\in\mathbf{a}}) \exp(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(r^{\Psi_v\sigma^{-1}\sigma}\tilde{z_v})),$$

$$\textit{where} \quad \tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z_v} \\ \tilde{w_v} \end{pmatrix}_{v\in\mathbf{a}} \in \tilde{D}.$$

In the case $\sigma$ is trivial on $K^*$, we can show (cf. (7.1)) that $f^{(\sigma,\Psi,a)}$ can be identified with the modular form $f^{[\sigma,\Psi,a]}$ on $D$ whose Fourier-Jacobi expansion is

$$f^{[\sigma,\Psi,a]}(\mathfrak{z}) = \sum_r g_r^{(\sigma,\Psi,a)}((w_v)_{v\in\mathbf{a}}) \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}((br)^{\Psi_v} z_v)\right),$$

$$\text{where} \quad \mathfrak{z} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v\in\mathbf{a}} \in D.$$

Using the main theorem, we can construct a certain action of the group

$$\mathfrak{G} = \left\{ \begin{array}{l} (x, c, \sigma) \\ \quad \in G_A \times K_A^* \times \mathrm{Gal}(\overline{\mathbb{Q}}/K^*) \end{array} \middle| \begin{array}{l} \det(x)^{-1}(N'_\Psi(c)^\rho)^n N_{K^*/\mathbb{Q}}(c)^q \in K^\times K_\infty^\times, \\ \nu(x)^{-1} N_{K^*/\mathbb{Q}}(c) \in F^\times F_{\infty+}^\times, \\ \left[c^{-1}, K^*\right] = \sigma|_{K^*_{ab}} \end{array} \right\}$$

on the graded ring of $\overline{\mathbb{Q}}$-rational modular forms. The action of $(x, c, \sigma)$ on the arithmetic modular functions in the sense of [1] coincides with that of $x$ in the sense of canonical models. Moreover, the action of $[\sigma, \Psi, a]$ described above coincides with that of $\left(\begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}, c, \sigma\right)$ if $a = N'_\Psi(c)$. When $K$

is an imaginary quadratic field, this action of $\mathfrak{G}$ is a generalization of that given in [7, Section 4].

We shall study basic properties of modular forms with respect to unitary groups in Section 1. In Section 2, we define some equivariant embeddings of algebraic groups and symmetric domains into different ones. We will study the relation of arithmetic modular forms on respective domains for these embeddings to use Shimura's many results in symplectic case. In Section 3, we shall generalize the main theorem of [4] to the case of arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C})$ using the results in [10, Chapter 7] and [11]. In Section 4, we shall consider the embeddings of canonical models and their inverse rational maps precisely. In Section 5, the relation between the arithmeticity defined from Fourier coefficients and that defined from canonical models will be discussed. In Section 6, the main theorem will be proved using all the results till Section 5. In Section 7, we construct the action of $\mathfrak{G}$ on the space of $\overline{\mathbb{Q}}$-rational modular forms using the results in Section 6.

## Notation

For a ring $A$, we define $A_q^n$ as above, and denote $A_1^n$ simply by $A^n$, $1_n$ denotes the identity matrix of degree $n$. The transpose of a matrix $X$ is denoted by ${}^tX$. We denote as usual by $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the set of all positive rational integers, the field of rational numbers, real numbers, and complex numbers, respectively. For any subfields $K_1, K_2$ of an arbitrary field $K$, we denote by $K_1 \vee K_2$ the composite field of $K_1$ and $K_2$. If $K$ is an algebraic number field, $K_{ab}$ denotes the maximal abelian extension of $K$, and we denote by $K_A$ (resp. $K_A^\times$) the adele ring (resp. the idele group) of $K$. By class field theory, every element $x$ of $K_A^\times$ defines an element of $\mathrm{Gal}(K_{ab}/K)$. We denote this by $[x, K]$. We denote by $\mathcal{O}_K$ and $\mathcal{O}_K^\times$ the ring of algebraic integers of $K$ and its unit group. For each finite prime $\mathfrak{p}$ of $K$, we denote the $\mathfrak{p}$-completion of $K$ and its maximal compact subring by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$. In the same way, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the $p$-completion of $\mathbb{Q}$ and $\mathbb{Z}$ for each rational prime number $p$. For an algebraic group $G$ defined over a field $k$, we denote by $G(K)$ the group of $K$-rational elements of $G$ if $K$ is an extension field of $k$. We denote by $G_A$,$G_\infty$, and $G_\mathfrak{f}$ the adelization of $G$, the archimedean component of $G_A$, and the non-archimedean component of $G_A$. By a variety, we understand a Zariski open subset of an absolutely irreducible projective variety.

## 1. Modular forms and the arithmeticity

Let $F$ be a totally real algebraic number field of finite degree and $K$ be its CM-extension (namely, a totally imaginary quadratic extension of $F$). Put $g = [F : \mathbb{Q}]$, then $[K : \mathbb{Q}] = 2g$. As is well known, the non-trivial element of $\mathrm{Gal}(K/F)$ is the complex conjugation for any embedding of $K$ into $\mathbb{C}$. We denote this by $\rho$. For any $b \in K$, we denote $(b + b^\rho)/2 \in F$ by $\mathrm{Re}(b)$. We define a CM-type of $K$ to be a set of $g$ different embeddings of $K$ into $\mathbb{C}$ whose restrictions to $F$ are all the embeddings of $F$ into $\mathbb{R}$.

Define a non-degenerate skew-hermitian matrix $R \in K_m^m$ ($m = n + 2q$, $n, q \geq 1$) by

$$(1.1) \qquad R = \begin{pmatrix} & & 1_q \\ & S & \\ -1_q & & \end{pmatrix} \quad \text{where } S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix},$$

$$s_j^\rho = -s_j \in K^\times \text{ and } s_j/s_k \text{ are totally positive } (1 \leq j, k \leq n).$$

By the Hasse principle for hermitian forms, for any skew-hermitian matrix $R \in K_m^m$, if the signature of $\sqrt{-1}R$ is $(q, n+q)$ or $(n+q, q)$ at each infinite place of $K$, we can write it in the form of (1.1) with some $S$ if we take a suitable basis of $K^m$. Determine the CM-type $\Psi$ of $K$ so that $\mathrm{Im}(s_j^\psi) > 0$ $(1 \leq j \leq n)$ for any $\psi \in \Psi$. Let $G^{(q,n)}(S, \Psi)$ be the group of unitary similitudes with respect to $R$, and we view $G^{(q,n)}(S, \Psi)$ as an algebraic group defined over $\mathbb{Q}$. Then

$$(1.2) \qquad \begin{aligned} &G^{(q,n)}(S, \Psi)(\mathbb{Q}) \\ &\quad = \left\{ \gamma \in \mathrm{GL}(m, K) \,\middle|\, {}^t\gamma^\rho R \gamma = \nu(\gamma) R \quad \text{with } \ \nu(\gamma) \in F^\times \right\}. \end{aligned}$$

We have $\nu(\gamma) \gg 0$ ($\gg 0$ means totally positive from now on) for any $\gamma \in G^{(q,n)}(S, \Psi)(\mathbb{Q})$, since the hermitian form $-\sqrt{-1}\,{}^t\gamma^\rho R \gamma$ must have the same signature as $-\sqrt{-1}R$, that is $(q, n+q)$ or $(n+q, q)$ for any embedding of $K$ into $\mathbb{C}$. Note that for any $\gamma \in G^{(q,n)}(S, \Psi)(\mathbb{Q})$, $\det(\gamma) \det(\gamma)^\rho = \nu(\gamma)^m$. Next, we define an algebraic subgroup $G_1^{(q,n)}(S, \Psi)$ of $G^{(q,n)}(S, \Psi)$ as follows.

$$(1.3) \qquad G_1^{(q,n)}(S, \Psi)(\mathbb{Q}) = \left\{ \gamma \in G^{(q,n)}(S, \Psi)(\mathbb{Q}) \,|\, \nu(\gamma) = \det(\gamma) = 1 \right\}.$$

Then $G_1^{(q,n)}(S, \Psi)$ has the strong approximation property. Hereafter we write $G^{(q,n)}(S, \Psi)$ (resp. $G_1^{(q,n)}(S, \Psi)$) as $G$ (resp. $G_1$) if there is no fear of confusion.

We denote by $\mathbf{a}$ the set of all archimedean primes of $F$. For $v \in \mathbf{a}$ and $b \in F$, we denote by $b_v$ the image of $b$ by the embedding $v : F \hookrightarrow \mathbb{R}$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $v \in \mathbf{a}$, we denote by $v\sigma$ an element of $\mathbf{a}$ so that $b_{v\sigma} = (b_v)^\sigma$. We write $\mathbf{a} = \{v_1, \ldots, v_g\}$ and denote an element of $\mathbf{a}$ by $v$. Given a set $X$, we denote by $X^{\mathbf{a}}$ the set of all indexed elements $(x_v)_{v \in \mathbf{a}}$ with $x_v \in X$. For $x = (x_v)_{v \in \mathbf{a}} \in X^{\mathbf{a}}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, we denote by $x^\sigma$ the element $y = (y_v)_{v \in \mathbf{a}}$ such that $y_{v\sigma} = x_v$.

For a CM-type $\Psi$ of $K$ and $v \in \mathbf{a}$, let $\Psi_v$ be the only element $\psi$ of $\Psi$ whose restriction to $F$ is $v$. Then we can view $\Psi$ as an embedding of $K$ into $\mathbb{C}^{\mathbf{a}}$ such that $b^\Psi = (b^{\Psi_v})_{v \in \mathbf{a}}$ for any $b \in K$. Through $\Psi$, we can view $K$ as a dense subset of $\mathbb{C}^{\mathbf{a}}$. When $b \in F$, we drop the symbol $\Psi$ (since $b^\Psi$ does not depend on $\Psi$) and regard $b$ as the element $(b_v)_{v \in \mathbf{a}}$ in $\mathbb{R}^{\mathbf{a}}$. For $x = (x_v)_{v \in \mathbf{a}} \in \mathbb{C}^{\mathbf{a}}$, we write $\boldsymbol{e}_{\mathbf{a}}(x) = \exp\left(2\pi\sqrt{-1} \sum_{v \in \mathbf{a}} x_v\right)$.

For each $v \in \mathbf{a}$, we can define the $v$-component $G_v = G^{(q,n)}(S, \Psi)_v$ of the algebraic group $G$ as follows.

$$G_v = \left\{ \gamma \in \mathrm{GL}(m, \mathbb{C}) \,\middle|\, \overline{{}^t\gamma} R^{\Psi_v} \gamma = \nu(\gamma) R^{\Psi_v} \quad \text{with } \ \nu(\gamma) \in \mathbb{R}^\times \right\}.$$

Note that for any $\gamma \in G_v$, $\nu(\gamma) > 0$. We can define the corresponding symmetric domain $D_v = D^{(q,n)}(S, \Psi)_v$ as

$$D_v = \left\{ \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{C}_q^{n+q} \;\middle|\; \begin{array}{l} z \in \mathbb{C}_q^q, \quad w \in \mathbb{C}_q^n, \\ \sqrt{-1}\left( \overline{{}^t w} S^{\Psi_v} w + \overline{{}^t z} - z \right) > 0 \end{array} \right\},$$

where $> 0$ means positive definite. Now let us define the action of $G_v$ on $D_v$. For any $\mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in D_v$ and $\alpha = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in G_v$ with blocks corresponding to those of $R$, put

$$\alpha \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} (a_1 z + b_1 w + c_1)(a_3 z + b_3 w + c_3)^{-1} \\ (a_2 z + b_2 w + c_2)(a_3 z + b_3 w + c_3)^{-1} \end{pmatrix}.$$

Then the group $G_v$ acts on $D_v$ as a group of holomorphic automorphism by $\mathfrak{z} \to \alpha(\mathfrak{z})$. The automorphic factors are

$$\lambda_v(\alpha, \mathfrak{z}) = \begin{pmatrix} \overline{a_3}^t z + \overline{c_3} & \overline{a_3}^t w - \overline{b_3}(S^{\Psi_v})^{-1} \\ -S^{\Psi_v} \overline{a_2}^t z - S^{\Psi_v} \overline{c_2} & -S^{\Psi_v} \overline{a_2}^t w + S^{\Psi_v} \overline{b_2} (S^{\Psi_v})^{-1} \end{pmatrix},$$
$$\mu_v(\alpha, \mathfrak{z}) = a_3 z + b_3 w + c_3.$$

By a simple calculation we get

$$\text{(1.4)} \qquad \det(\lambda_v(\alpha, \mathfrak{z})) = \det(\alpha)^{-1} \nu(\alpha)^{n+q} \det(\mu_v(\alpha, \mathfrak{z})),$$
$$\mu_v(\alpha_2 \alpha_1, \mathfrak{z}) = \mu_v(\alpha_2, \alpha_1(\mathfrak{z})) \mu_v(\alpha_1, \mathfrak{z}) \qquad \text{for any } \alpha_1, \alpha_2 \in G_v.$$

Put

$$G^{(q,n)}(S, \Psi)_\infty = \prod_{v \in \mathbf{a}} G^{(q,n)}(S, \Psi)_v,$$
$$D^{(q,n)}(S, \Psi) = \prod_{v \in \mathbf{a}} D^{(q,n)}(S, \Psi)_v,$$

and $G_\infty = G^{(q,n)}(S, \Psi)_\infty$ acts on $D = D^{(q,n)}(S, \Psi)$ componentwise. We can define an embedding of $G(\mathbb{Q}) = G^{(q,n)}(S, \Psi)(\mathbb{Q})$ into $G_\infty$ by $\alpha \to (\alpha^{\Psi_v})_{v \in \mathbf{a}}$ and also define an action of $G(\mathbb{Q})$ onto $D$, $\alpha(\mathfrak{z}) = \left(\alpha^{\Psi_v}(\mathfrak{z}_v)\right)_{v \in \mathbf{a}}$ where $\alpha \in G(\mathbb{Q})$, $\mathfrak{z} = (\mathfrak{z}_v)_{v \in \mathbf{a}} \in D$. Set

$$\mu_v(\alpha, \mathfrak{z}) = \mu_v(\alpha^{\Psi_v}, \mathfrak{z}_v), \quad \lambda_v(\alpha, \mathfrak{z}) = \lambda_v(\alpha^{\Psi_v}, \mathfrak{z}_v).$$

All these conventions are basically same as those of [7].

Now let us define a congruence subgroup of $G(\mathbb{Q})$. For any positive integer $N$, put

$$\text{(1.5)} \qquad \Gamma_N = \{\gamma \in G_1(\mathbb{Q}) \cap \mathrm{SL}(m, \mathcal{O}_K) \,|\, \gamma \equiv 1_m \mod (N\mathcal{O}_K)_m^m\},$$

where $\mathcal{O}_K$ is the ring of integers of $K$. By a congruence subgroup of $G(\mathbb{Q})$, we understand a subgroup $\Gamma$ of $G(\mathbb{Q})$ which contains $\Gamma_N$ for some positive integer

$N$ and $K^\times\Gamma_N$ is a subgroup of $K^\times\Gamma$ of finite index. Any element (except a scalar matrix) of a congruence subgroup $\Gamma$ of $G(\mathbb{Q})$ has no fixed points in $D$ if and only if the group $K^\times\Gamma/K^\times$ is torsion free. As is well known, $K^\times\Gamma_N/K^\times$ is torsion free if $N$ is sufficiently large.

Set $k=(k_v)_{v\in\mathbf{a}}\in\mathbb{Z}^{\mathbf{a}}$. For $\alpha\in G(\mathbb{Q})$ and a $\mathbb{C}$-valued function $f$ on $D$, we define a $\mathbb{C}$-valued function $f|_k\alpha$ on $D$ by

$$(f|_k\alpha)(\mathfrak{z})=\left(\prod_{v\in\mathbf{a}}\det\left(\mu_v(\alpha,\mathfrak{z})\right)^{-k_v}\right)f\left(\alpha(\mathfrak{z})\right)\quad\text{for }\mathfrak{z}\in D.$$

If $f$ is holomorphic on $D$, so is $f|_k\alpha$. For any congruence subgroup $\Gamma$ of $G(\mathbb{Q})=G^{(q,n)}(S,\Psi)(\mathbb{Q})$, we denote by $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$, the set of all holomorphic functions $f$ on $D=D^{(q,n)}(S,\Psi)$ such that $f|_k\gamma=f$ for any $\gamma\in\Gamma$. An element of $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$ is called a holomorphic modular form of weight $k$ with respect to $\Gamma$. We denote by $\mathcal{M}_k^{(q,n)}(S,\Psi)$, the union of $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$ for all congruence subgroups $\Gamma$ of $G(\mathbb{Q})=G^{(q,n)}(S,\Psi)(\mathbb{Q})$. Next we put

$$\mathcal{A}_k^{(q,n)}(S,\Psi)=\bigcup_{l\in\mathbb{Z}^{\mathbf{a}}}\left\{f_1f_2^{-1}\,\middle|\,f_1\in\mathcal{M}_{k+l}^{(q,n)}(S,\Psi),\quad 0\not\equiv f_2\in\mathcal{M}_l^{(q,n)}(S,\Psi)\right\},$$
$$\mathcal{A}_k^{(q,n)}(S,\Psi)(\Gamma)=\left\{f\in\mathcal{A}_k^{(q,n)}(S,\Psi)\ \mid\ f|_k\gamma=f\quad\text{for any }\ \gamma\in\Gamma\right\}.$$

We write simply $\mathcal{M}_k^{(q,n)}(S,\Psi)$, $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma),\mathcal{A}_k^{(q,n)}(S,\Psi)$, $\mathcal{A}_k^{(q,n)}(S,\Psi)(\Gamma)$ by $\mathcal{M}_k$, $\mathcal{M}_k(\Gamma),\mathcal{A}_k$, $\mathcal{A}_k(\Gamma)$, respectively if there is no fear of confusion. An element of $\mathcal{A}_k$ is called a meromorphic modular form of weight $k$.

Hereafter we identify $\mathbb{Z}^{\mathbf{a}}$ with the free module $\sum_{v\in\mathbf{a}}\mathbb{Z}v$ by putting $(k_v)_{v\in\mathbf{a}}=\sum_{v\in\mathbf{a}}k_vv$. Also put $\mathbf{1}=(1)_{v\in\mathbf{a}}=\sum_{v\in\mathbf{a}}v$. We can define the action of $\sigma\in\mathrm{Aut}(\mathbb{C})$ on $\mathbb{Z}^{\mathbf{a}}$ by $\left(\sum_{v\in\mathbf{a}}k_vv\right)^{\sigma}=\sum_{v\in\mathbf{a}}k_v(v\sigma)$. For any $k\in\mathbb{Z}^{\mathbf{a}}$, we denote by $F(k)$ the algebraic number field corresponding to $\left\{\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\,|\,k^{\sigma}=k\right\}$. Then $F(k)$ is contained in the Galois closure of $F$ over $\mathbb{Q}$.

We can define a certain parabolic subgroup of $G=G^{(q,n)}(S,\Psi)$ and consider corresponding Fourier-Jacobi expansions of holomorphic modular forms. Put

$$\mathsf{N}^{(q,n)}(S,\Psi)(\mathbb{Q})=\left\{\mathsf{h}=\begin{pmatrix}1_q & {}^ty^{\rho}S & b+\frac{1}{2}{}^ty^{\rho}Sy\\ 0 & 1_n & y\\ 0 & 0 & 1_q\end{pmatrix}\,\middle|\,\begin{matrix}y\in K_q^n,\ b\in K_q^q,\\ {}^tb^{\rho}=b\end{matrix}\right\},$$
$$\mathsf{H}^{(q,n)}(S,\Psi)(\mathbb{Q})=\mathsf{N}^{(q,n)}(S,\Psi)(\mathbb{Q})\cdot\left\{\begin{pmatrix}\alpha & & \\ & 1_n & \\ & & \left({}^t\alpha^{\rho}\right)^{-1}\end{pmatrix}\,\middle|\,\alpha\in\mathrm{GL}(q,K)\right\},$$

$$\mathsf{P}^{(q,n)}(S,\Psi)(\mathbb{Q}) = \mathsf{N}^{(q,n)}(S,\Psi)(\mathbb{Q}) \cdot \left\{ \begin{pmatrix} \alpha & & \\ & \beta & \\ & & c\left({}^t\alpha^\rho\right)^{-1} \end{pmatrix} \middle| \begin{array}{l} c \in F^\times, \\ \alpha \in \mathrm{GL}(q,K), \\ \beta \in \mathrm{GL}(n,K), \\ {}^t\beta^\rho S\beta = cS \end{array} \right\}.$$

Then $\mathsf{P}^{(q,n)}(S,\Psi)$ is a parabolic subgroup of $G$ and $\mathsf{N}^{(q,n)}(S,\Psi)$ is its unipotent radical. We write simply $\mathsf{N}^{(q,n)}(S,\Psi), \mathsf{H}^{(q,n)}(S,\Psi), \mathsf{P}^{(q,n)}(S,\Psi)$ by $\mathsf{N}, \mathsf{H}, \mathsf{P}$ respectively.

Given a congruence subgroup $\Gamma$, we can find a $\mathbb{Z}$-lattice $L$ in $K_q^n$ and a $\mathbb{Z}$-lattice $L_q$ in a vector space

$$\mathcal{H}_q = \left\{ b \in K_q^q \,\middle|\, {}^t b^\rho = b \right\}$$

such that $\Gamma \cap \mathsf{N}(\mathbb{Q})$ contains all elements of the form $\begin{pmatrix} 1_q & {}^t y^\rho S & b + \frac{1}{2}{}^t y^\rho S y \\ 0 & 1_n & y \\ 0 & 0 & 1_q \end{pmatrix}$ with $y \in L$ and $b \in L_q$. Therefore, if $f \in \mathcal{M}_k(\Gamma)$, we have $f\begin{pmatrix} z + b^\Psi \\ w \end{pmatrix} = f\begin{pmatrix} z \\ w \end{pmatrix}$ $\left(z \in \left(\mathbb{C}_q^q\right)^{\mathbf{a}},\ w \in \left(\mathbb{C}_q^n\right)^{\mathbf{a}}\right)$ for all $b \in L_q$, and hence $f\begin{pmatrix} z \\ w \end{pmatrix}$ has the following expansion.

$$f\begin{pmatrix} z \\ w \end{pmatrix} = \sum_{r \in L_q'} g_r(w) \boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^\Psi z)\right) \tag{1.6}$$

where $L_q'$ is the $\mathbb{Z}$-lattice in $\mathcal{H}_q$ defined by

$$L_q' = \left\{ r \in \mathcal{H}_q \,\middle|\, \mathrm{Tr}_{F/\mathbb{Q}}\left(\mathrm{tr}(rL_q)\right) \subset \mathbb{Z} \right\},$$

and every $g_r$ is a holomorphic function on $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$. Define a hermitian form $H_{r,S,\Psi}$ on $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$ by

$$H_{r,S,\Psi}\left((w_{1v})_{v\in\mathbf{a}}, (w_{2v})_{v\in\mathbf{a}}\right) = -2\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}\left(r^{\Psi_v}\, {}^t\overline{w_{1v}} S^{\Psi_v} w_{2v}\right). \tag{1.7}$$

For any $y \in L$, consider $\mathsf{h} = \begin{pmatrix} 1_q & {}^t y^\rho S & \frac{1}{2}{}^t y^\rho S y \\ 0 & 1_n & y \\ 0 & 0 & 1_q \end{pmatrix} \in \Gamma \cap \mathsf{N}(\mathbb{Q})$. Since $\mathsf{h}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z + \left({}^t y^\rho S\right)^\Psi w + \frac{1}{2}\left({}^t y^\rho S y\right)^\Psi \\ w + y^\Psi \end{pmatrix}$, the function $g_r$ satisfies

$$g_r(w + y^\Psi) = \exp\left(\pi H_{r,S,\Psi}\left(y^\Psi, w + \frac{1}{2}y^\Psi\right)\right) g_r(w) \quad \text{for any } y \in L. \tag{1.8}$$

We denote by $\mathfrak{T}_{(r,S,\Psi)}(L)$ the set of all holomorphic functions $g_r$ on $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$ satisfying (1.8) and by $\mathfrak{T}_{(r,S,\Psi)}$ the union of $\mathfrak{T}_{(r,S,\Psi)}(L)$ for all $\mathbb{Z}$-lattices $L$ in $K_q^n$. Now we have the following lemma.

**Lemma 1.1.** *For $\mathfrak{T}_{(r,S,\Psi)} \neq \{0\}$, it is necessary that $r$ is totally semi-positive definite (i.e. $r$ is semi-positive definite for any embedding of $K$ into $\mathbb{C}$).*

This lemma follows from the classical theory of theta functions as $H_{r,S,\Psi}$ is semi-positive definite on $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$ if and only if $r$ is so on $\mathbb{C}^q$ for any embedding of $K$ into $\mathbb{C}$. Hence we can rewrite (1.6) as

$$f(\mathfrak{z}) = \sum_{0 \leq r \in L'_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}\left(\operatorname{tr}(r^{\Psi} z)\right) \tag{1.9}$$

where $0 \leq r$ means that $r$ is totally semi-positive definite. We often write this expansion without specifying the lattice $L'_q$ employing the convention that $g_r = 0$ if $r \notin L'_q$ :

$$f(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}\left(\operatorname{tr}(r^{\Psi} z)\right). \tag{1.10}$$

Now let us define the arithmeticity of modular forms. Before doing that, we must review the reflex of CM-type. For a CM-field $K$, its CM-type $\Psi$, and any $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we can define another CM-type $\Psi\sigma = \{\psi\sigma \,|\psi \in \Psi\}$ of $K$. We denote by $K^*_\Psi$ (or simply $K^*$ if there is no fear of confusion), the corresponding algebraic number field to $\left\{\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\Psi\sigma = \Psi\right\}$ which is a finite index subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As is well known, $K^*_\Psi$ is a CM-field contained in the Galois closure of $K$. Viewing $\Psi$ as a union of $g$ different right $\operatorname{Gal}(\overline{\mathbb{Q}}/K)$-cosets in $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define a CM-type $\Psi^*$ of $K^*_\Psi$ as follows

$$\operatorname{Gal}(\overline{\mathbb{Q}}/K^*_\Psi)\Psi^* = \left(\operatorname{Gal}(\overline{\mathbb{Q}}/K)\Psi\right)^{-1}.$$

We call $\Psi^*$ by "the reflex of $\Psi$" and the couple $(K^*_\Psi, \Psi^*)$ by "the reflex of $(K, \Psi)$". From the definition, we have $\left(K^*_\Psi\right)^\sigma = K^*_{\Psi\sigma}$ for any $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or $\in \operatorname{Aut}(\mathbb{C})$). By $N'_\Psi$, we denote the group homomorphism $x \to \prod_{\psi^* \in \Psi^*} x^{\psi^*}$ from $K^{*\times}_\Psi$ to $K^\times$. It is a morphism of algebraic groups if we view $K^{*\times}_\Psi$ and $K^\times$ as algebraic groups defined over $\mathbb{Q}$, and so it can naturally be extended to the homomorphism of $(K^*_\Psi)^\times_A$ to $K^\times_A$.

For any $g_r \in \mathfrak{T}_{r,S,\Psi}$, we define a function $(g_r)_*$ on $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$ (which may be non-holomorphic) by

$$(g_r)_*(w) = \exp\left(-\frac{\pi}{2} H_{r,S,\Psi}(w,w)\right) g_r(w) \qquad (w \in \left(\mathbb{C}_q^n\right)^{\mathbf{a}}).$$

Now for every subfield $\Omega$ of $\mathbb{C}$ containing $K^*_{\Psi ab}$ (i.e. the maximal abelian extension of $K^*_\Psi$), we define

$$\mathfrak{T}_{r,S,\Psi}(L,\Omega) = \left\{ g_r \in \mathfrak{T}_{r,S,\Psi}(L) \,\middle|\, (g_r)_*(w) \in \Omega \quad \text{for any } w \in \left(K_q^n\right)^{\Psi} \right\},$$

and put $\mathfrak{T}_{r,S,\Psi}(\Omega)$ the union of $\mathfrak{T}_{r,S,\Psi}(L,\Omega)$ for all $\mathbb{Z}$-lattices $L$ in $K_q^n$. Similarly put $\mathcal{M}_k(\Gamma,\Omega) = \mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma,\Omega)$ the set of all $f \in \mathcal{M}_k(\Gamma)$ whose

Fourier-Jacobi coefficients $g_r$ belong to $\mathfrak{T}_{r,S,\Psi}(\Omega)$ for all $r$, and put $\mathcal{M}_k(\Omega) = \mathcal{M}_k^{(q,n)}(S,\Psi)(\Omega)$ the union of $\mathcal{M}_k(\Gamma,\Omega)$ for all congruence subgroups $\Gamma$ of $G(\mathbb{Q})$. Set

$$\mathcal{A}_k(\Omega) = \mathcal{A}_k^{(q,n)}(S,\Psi)(\Omega) = \bigcup_{l\in\mathbb{Z}^{\mathbf{a}}} \left\{ f_1 f_2^{-1} \;\middle|\; \begin{array}{l} f_1 \in \mathcal{M}_{k+l}^{(q,n)}(S,\Psi)(\Omega), \\ 0 \not\equiv f_2 \in \mathcal{M}_l^{(q,n)}(S,\Psi)(\Omega) \end{array} \right\},$$
$$\mathcal{A}_k(\Gamma,\Omega) = \mathcal{A}_k^{(q,n)}(S,\Psi)(\Gamma,\Omega) = \mathcal{A}_k^{(q,n)}(S,\Psi)(\Omega) \cap \mathcal{A}_k^{(q,n)}(S,\Psi)(\Gamma).$$

**Lemma 1.2.** *Let $\Omega$ be as above.*

(1) *Take $g_r \in \mathfrak{T}_{r,S,\Psi}$. Suppose that there exists a non-empty open subset $U$ of $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$ such that $(g_r)_*(w) \in \Omega$ for any $w \in \left(K_q^n\right)^{\Psi} \cap U$, then we have $g_r \in \mathfrak{T}_{r,S,\Psi}(K_{\Psi ab}^*) \otimes_{K_{\Psi ab}^*} \Omega$.*

(2) $\mathfrak{T}_{r,S,\Psi}(\Omega) = \mathfrak{T}_{r,S,\Psi}(K_{\Psi ab}^*) \otimes_{K_{\Psi ab}^*} \Omega$.

*Proof.* The assertion (2) follows from (1) immediately. Let us prove (1). We get $\mathfrak{T}_{r,S,\Psi}(\mathbb{C}) = \mathfrak{T}_{r,S,\Psi}(K_{\Psi ab}^*) \otimes_{K_{\Psi ab}^*} \mathbb{C}$ by [4, Propositions 1.2, 2.4 and 2.5]. Hence for $g_r \in \mathfrak{T}_{r,S,\Psi}(\mathbb{C})$, we can write $g_r = c_1 g_{r,1} + \cdots + c_l g_{r,l}$ with $g_{r,1}, \ldots, g_{r,l} \in \mathfrak{T}_{r,S,\Psi}(K_{\Psi ab}^*)$ and $c_1, \ldots, c_l \in \mathbb{C}$. Take $g_{r,1}, \ldots, g_{r,l}$ and $c_1, \ldots, c_l \in \mathbb{C}$ so that $g_{r,1}, \ldots, g_{r,l}$ are linearly independent over $\mathbb{C}$. Put $h = (g_r)_*|_{(K_q^n)^\Psi \cap U}$, $h_j = (g_{r,j})_*|_{(K_q^n)^\Psi \cap U}$ $(1 \leq j \leq l)$. Since $h = \sum_{j=1}^{l} c_j^\sigma h_j$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\Omega)$, we have $c_1, \ldots, c_l \in \Omega$. $\square$

From this lemma, we obtain the following proposition.

**Proposition 1.3.** *For any subfield $\Omega$ of $\mathbb{C}$ containing $K_{\Psi ab}^*$, and for any $k \in \mathbb{Z}^{\mathbf{a}}$, we have*

$$\mathcal{A}_k(\Omega) \cap \mathcal{M}_k = \mathcal{M}_k(\Omega).$$

*Proof.* Let $f \in \mathcal{A}_k(\Omega) \cap \mathcal{M}_k$. Then we can write $f = f_1/f_2$ with $f_1 \in \mathcal{M}_{k+l}(\Omega)$, $0 \not\equiv f_2 \in \mathcal{M}_l(\Omega)$ $(l \in \mathbb{Z}^{\mathbf{a}})$. Let their Fourier-Jacobi expansions be

$$f_1(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r^1(w) \boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^\Psi z)\right),$$
$$f_2(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r^2(w) \boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^\Psi z)\right),$$
$$f(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^\Psi z)\right).$$

Then $g_r^1, g_r^2 \in \mathfrak{T}_{r,S,\Psi}(\Omega)$ for all $r$. Take $r_0 \in \mathcal{H}_q$ so that $g_{r_0}^2 \not\equiv 0$, and put

$$U = \left\{ w \in \left(\mathbb{C}_q^n\right)^{\mathbf{a}} \,\middle|\, g_{r_0}^2(w) \neq 0 \right\}.$$

Then $U$ is a non-empty open subset of $\left(\mathbb{C}_q^n\right)^{\mathbf{a}}$. For $w \in (K_q^n)^\Psi \cap U$, we put

$$f_{1*} = \sum_{0 \le r \in \mathcal{H}_q} (g_r^1)_*(w) \boldsymbol{e}_{\mathbf{a}} \left(\mathrm{tr}(r^\Psi z)\right),$$
$$f_{2*} = \sum_{0 \le r \in \mathcal{H}_q} (g_r^2)_*(w) \boldsymbol{e}_{\mathbf{a}} \left(\mathrm{tr}(r^\Psi z)\right),$$
$$f_{*} = \sum_{0 \le r \in \mathcal{H}_q} (g_r)_*(w) \boldsymbol{e}_{\mathbf{a}} \left(\mathrm{tr}(r^\Psi z)\right).$$

By a formal calculation of $ff_2$, we have $\sum_{0 \le t \in \mathcal{H}_q} g_t(w) g_{r-t}^2(w) = g_r(w)$ for each $r \in \mathcal{H}_q$. This implies $\sum_{0 \le t \in \mathcal{H}_q} (g_t)_*(w)(g_{r-t}^2)_*(w) = (g_r)_*(w)$ and hence we obtain $f_* f_{2*} = f_{1*}$. Since $f_*, f_{2*}$ and $f_{1*}$ can be regarded as formal power series of $q^2[F:\mathbb{Q}]$-variables such that $f_{1*}$ and $f_{2*}$ with coefficients in $\Omega$, the coefficients of $f_*$ must also be in $\Omega$. Hence we get $g_r \in \mathfrak{T}_{r,S,\Psi}(\Omega)$ from Lemma 1.2. So we obtain $f \in \mathcal{M}_k(\Omega)$. □

**Lemma 1.4.** (1) *Let $\Omega$ be a subfield of $\mathbb{C}$ containing $K^*_{\Psi ab}$, and $k \in \mathbb{Z}^{\mathbf{a}}$. Then for any $f \in \mathcal{M}_k(\Omega)$ and for any $\mathsf{h} \in \mathsf{N}(\mathbb{Q})$, we have*

$$f|_k \mathsf{h} = f \circ \mathsf{h} \in \mathcal{M}_k(\Omega).$$

(2) *For any $k \in \mathbb{Z}^{\mathbf{a}}$, take any subfield $\Omega$ of $\mathbb{C}$ containing $F(k) \vee K^*_{\Psi ab}$. Then for any $f \in \mathcal{M}_k(\Omega)$ and any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, we have*

$$f|_k \mathsf{h} \in \mathcal{M}_k(\Omega).$$

(3) *Let $f \in \mathcal{M}_k$, whose Fourier-Jacobi expansion is*

$$f(\mathfrak{z}) = \sum_{0 \le r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}} \left(\mathrm{tr}(r^\Psi z)\right).$$

*For* $\mathsf{h} = \begin{pmatrix} 1_q & {}^t y^\rho S & b + \frac{1}{2} {}^t y^\rho S y \\ 0 & 1_n & y \\ 0 & 0 & 1_q \end{pmatrix} \begin{pmatrix} \alpha & & \\ & 1_n & \\ & & ({}^t\alpha^\rho)^{-1} \end{pmatrix} \in \mathsf{H}(\mathbb{Q})$ $(y \in K_q^n,\ b \in \mathcal{H}_q,\ \alpha \in \mathrm{GL}(q,K))$, *we have*

$$(f|_k \mathsf{h}) \begin{pmatrix} z \\ 0 \end{pmatrix}$$
$$= \left( \prod_{v \in \mathbf{a}} (\det(\alpha^\rho)^{\Psi_v})^{k_v} \right) \cdot \sum_{0 \le r \in \mathcal{H}_q} \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(rb)) \left\{ (g_r)_*(y^\Psi) \right\} \boldsymbol{e}_{\mathbf{a}} \left( \mathrm{tr} \left( ({}^t\alpha^\rho r \alpha)^\Psi z \right) \right).$$

*Proof.* By a straightforward calculation, we have

$$(f \circ \mathsf{h}) \begin{pmatrix} z \\ 0 \end{pmatrix} = \sum_{0 \le r \in \mathcal{H}_q} \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(rb)) \left\{ (g_r)_*(y^\Psi) \right\} \boldsymbol{e}_{\mathbf{a}} \left( \mathrm{tr} \left( ({}^t\alpha^\rho r \alpha)^\Psi z \right) \right).$$

This implies (3). Since $\boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(rb)) \in \mathbb{Q}_{ab}$, for any subfield $\Omega$ of $\mathbb{C}$ containing $K^*_{\Psi ab}$, we have

$$\mathcal{M}_k(\Omega) = \left\{ f \in \mathcal{M}_k \,\middle|\, \begin{array}{r} (f \circ \mathsf{h}) \begin{pmatrix} z \\ 0 \end{pmatrix} \text{ has } \Omega\text{-rational Fourier coefficients} \\ \text{for any } \mathsf{h} \in \mathsf{H}(\mathbb{Q}) \end{array} \right\}.$$

Hence we get (1). As the constant $\prod_{v \in \mathbf{a}} (\det(\alpha^\rho)^{\Psi_v})^{k_v}$ is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F(k) \vee K^*_{\Psi ab})$, we get (2). $\square$

For any $0 \ll b \in F^\times$, we can define the isomorphism $I(S,b)$ of algebraic groups $G^{(q,n)}(S,\Psi) \to G^{(q,n)}(bS,\Psi)$ by

$$I(S,b)(\alpha) = \begin{pmatrix} b1_q & & \\ & 1_n & \\ & & 1_q \end{pmatrix} \alpha \begin{pmatrix} b1_q & & \\ & 1_n & \\ & & 1_q \end{pmatrix}^{-1}. \tag{1.11}$$

This is compatible with the biholomorphic bijection $\varepsilon(S,b)$ of $D^{(q,n)}(S,\Psi)$ onto $D^{(q,n)}(bS,\Psi)$ defined by

$$\varepsilon(S,b) \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} bz \\ w \end{pmatrix}.$$

Then we have

$$\mu_v\left(I(S,b)(\alpha), \varepsilon(S,b)(\mathfrak{z})\right) = \mu_v(\alpha,\mathfrak{z}),$$

for $\alpha \in G^{(q,n)}(S,\Psi)(\mathbb{Q})$, $\mathfrak{z} \in D^{(q,n)}(S,\Psi)$ and $v \in \mathbf{a}$. Hence we can identify $\mathcal{M}_k^{(q,n)}(bS,\Psi)$ and $\mathcal{M}_k^{(q,n)}(S,\Psi)$. In terms of Fourier-Jacobi expansion, an element

$$\sum_{0 \le r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(r^\Psi z)) \quad \in \mathcal{M}_k^{(q,n)}(S,\Psi)$$

is identified with

$$\sum_{0 \le r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(b^{-1} r^\Psi z)) \quad \in \mathcal{M}_k^{(q,n)}(bS,\Psi)$$

through $\varepsilon(S,b)$. Using these expressions by Fourier-Jacobi expansions, we can identify $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Omega)$ with $\mathcal{M}_k^{(q,n)}(bS,\Psi)(\Omega)$ (and clearly $\mathcal{A}_k^{(q,n)}(S,\Psi)(\Omega)$ with $\mathcal{A}_k^{(q,n)}(bS,\Psi)(\Omega)$ ) for any subfield $\Omega$ of $\mathbb{C}$ containing $K^*_{\Psi ab}$.

If $b$ ($\in F^\times$) is not totally positive, we can define the group isomorphism $G^{(q,n)}(S,\Psi) \to G^{(q,n)}(bS,\Psi')$ (of course $\Psi \neq \Psi'$) by (1.11). But in this case the corresponding bijection of $D^{(q,n)}(S,\Psi)$ onto $D^{(q,n)}(bS,\Psi')$ is not holomorphic. So we cannot identify modular forms on both symmetric domains.

In [1], the canonical models of $D = D^{(q,n)}(S,\Psi)$ modulo congruence subgroups of $G(\mathbb{Q})$ are constructed. Consider the adelization $G_A = G^{(q,n)}(S,\Psi)_A$ of $G = G^{(q,n)}(S,\Psi)$. That is,

$$G_A = \left\{ x \in \mathrm{GL}(m, K_A) \,\middle|\, {}^t x^\rho R x = \nu(x) R \quad \text{with} \quad \nu(x) \in F_A^\times \right\}.$$

Note that $x_{\mathfrak{p}}$, the $\mathfrak{p}$-component of $x$, belongs to $\mathrm{GL}(m, \mathcal{O}_{\mathfrak{p}})$ for almost all non-archimedean primes $\mathfrak{p}$ of $K$. Put

$$\mathcal{G}_+ = \mathcal{G}_+^{(q,n)}(S,\Psi) = \left\{ x \in G_A \;\middle|\; \begin{array}{l} \exists a \in (K^*)_A^\times \text{ such that} \\ \det(x)\,(N'_\Psi(a))^n \left(N_{K^*/\mathbb{Q}}(a)\right)^q \in \overline{K^\times K_\infty^\times}, \\ \nu(x) N_{K^*/\mathbb{Q}}(a) \in \overline{F^\times F_{\infty+}^\times} \end{array} \right\},$$

where $K_\infty^\times$ (resp. $F_\infty^\times$) denotes the infinite component of the idele group $K_A^\times$ (resp. $F_A^\times$) and $F_{\infty+}^\times$ means the connected component of the identity of $F_\infty^\times$. The overlines mean the topological closures in the idele groups. Clearly we have $\mathcal{G}_+ \supset G(\mathbb{Q})$.

Let $\mathcal{Z} = \mathcal{Z}^{(q,n)}(S,\Psi)$ be the set of all subgroups of $G_A$ which contain $G_\infty$, the infinite component of $G_A$, and whose projections to $G_{\mathfrak{f}}$, the non-archimedean component of $G_A$, are open compact. Then for any $Y \in \mathcal{Z}$, $Y \cap G(\mathbb{Q})$ is a congruence subgroup, which will be denoted by $\Gamma_Y$. For each $Y \in \mathcal{Z}$ we have a variety (more precisely, a Zariski open subset of a projective variety) $V_Y$ defined over $K^*_{\Psi ab}$ and a holomorphic map $\varphi_Y : D \to V_Y$ so that $\varphi_Y$ defines a biregular isomorphism of $\Gamma_Y \setminus D$ onto $V_Y$. For $X, Y \in \mathcal{Z}$ and $x \in \mathcal{G}_+$ so that $X \supset xYx^{-1}$, we take the morphism $J_{XY}(x)$ of $V_Y$ to $V_X^{\sigma(x)}$, where $\sigma(x) \in \mathrm{Aut}(\mathbb{C}/K^*)$ is determined by $x$, as in [1].

Define $W = W^{(q,n)}(S,\Psi)$ by

$$W = \left\{ \mathfrak{z} \in D \,\middle|\, \mathfrak{z} = (b^\Psi) \quad \text{with some } \; b \in K_q^{q+n} \right\}. \tag{1.12}$$

Then $\varphi_Y(\mathfrak{z})$ is $K^*_{\Psi ab}$-rational for any $\mathfrak{z} \in W$. Let $\mathfrak{K} = \mathfrak{K}^{(q,n)}(S,\Psi)$ denote the function field $\mathfrak{L}_{j_0}$ of [1, Section 4.2]. The function field $\mathfrak{K}$ is contained in the union of $K^*_{\Psi ab}(V_Y)$ (i.e. the field of all rational functions on $V_Y$ defined over $K^*_{\Psi ab}$) for all $Y \in \mathcal{Z}$. Now we have $\mathcal{A}_0(\mathbb{C}) = \mathfrak{K} \vee \mathbb{C}$.

The function field $\mathfrak{K}$ determines a certain arithmeticity on $\mathcal{A}_0$. The relation between the arithmeticity defined in this section and that of $\mathfrak{K}$ will be made precise in Section 5.

## 2. On some embeddings of symmetric domains

To analyze the arithmeticity of modular forms with respect to $G = G^{(q,n)}(S,\Psi)$, we need to use Shimura's many results in the symplectic case through some embeddings. So we define three kinds of embeddings of groups and symmetric domains in this section.

First let us review symplectic groups and corresponding symmetric domains. Let $F, K, \mathbf{a}$ be as in Section 1. For any positive integer $l$, put $G^{(l)}(\mathbb{Q}) = \mathrm{GSp}(l,F)$, $G_1^{(l)}(\mathbb{Q}) = \mathrm{Sp}(l,F)$, that is,

$$G^{(l)}(\mathbb{Q}) = \left\{ \gamma \in \mathrm{GL}(2l,F) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \nu(\gamma) \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \quad \text{with } \nu(\gamma) \in F^\times \right\},$$

$$G_1^{(l)}(\mathbb{Q}) = \left\{ \gamma \in \mathrm{GL}(2l,F) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \right\}.$$

We view $G^{(l)}, G_1^{(l)}$ as algebraic groups defined over $\mathbb{Q}$. Then $G_1^{(l)}$ has the strong approximation property. As is well known, we have $\det(\gamma) = 1$ for any $\gamma \in G_1^{(l)}(\mathbb{Q})$. Set

$$G^{(l)}(\mathbb{Q})_+ = \left\{ \gamma \in G^{(l)}(\mathbb{Q}) \,|\nu(\gamma) \gg 0 \right\},$$

where $\gg 0$ means totally positive, and set

$$\mathfrak{H}_l^{\mathbf{a}} = \left\{ z = (z_v)_{v\in\mathbf{a}} \in (\mathbb{C}_l^l)^{\mathbf{a}} \,\middle|\, {}^t z_v = z_v,\ \mathrm{Im}(z_v) > 0 \quad \text{for any } \ v \in \mathbf{a} \right\},$$

where $> 0$ means positive definite. Then $G^{(l)}(\mathbb{Q})_+$ acts on $\mathfrak{H}_l^{\mathbf{a}}$ as $\alpha((z_v)_{v\in\mathbf{a}}) = \left((a_v z_v + b_v)(c_v z_v + d_v)^{-1}\right)_{v\in\mathbf{a}}$ with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^{(l)}(\mathbb{Q})_+$ and $a, b, c, d \in F_l^l$. The automorphic factor is defined by

$$j_v^{(l)}(\alpha, (z_v)_{v\in\mathbf{a}}) = c_v z_v + d_v$$

for each $v \in \mathbf{a}$. We define congruence subgroups of $G^{(l)}(\mathbb{Q})$ and modular forms on $\mathfrak{H}_l^{\mathbf{a}}$ with respect to them as in [9]. Let $\mathcal{M}_k^{(l)}$ ($k \in \mathbb{Z}^{\mathbf{a}}$) denote the space of holomorphic modular forms on $\mathfrak{H}_l^{\mathbf{a}}$ of weight $k$. Set

$$\mathcal{A}_k^{(l)} = \bigcup_{e\in\mathbb{Z}^{\mathbf{a}}} \left\{ f_1 f_2^{-1} \,\middle|\, f_1 \in \mathcal{M}_{k+e}^{(l)}, \quad 0 \not\equiv f_2 \in \mathcal{M}_e^{(l)} \right\}.$$

Now for any subfield $\Omega$ of $\mathbb{C}$, we denote by $\mathcal{M}_k^{(l)}(\Omega)$ the space of all holomorphic modular forms of weight $k$ with $\Omega$-rational Fourier coefficients. (See, [9, Section 25].) Put

$$\mathcal{A}_k^{(l)}(\Omega) = \bigcup_{e\in\mathbb{Z}^{\mathbf{a}}} \left\{ f_1 f_2^{-1} \,\middle|\, f_1 \in \mathcal{M}_{k+e}^{(l)}(\Omega), \quad 0 \not\equiv f_2 \in \mathcal{M}_e^{(l)}(\Omega) \right\}.$$

For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $f \in \mathcal{M}_k^{(l)}$ whose Fourier expansion is

$$f(z) = \sum_{r\in L} c_r \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(rz)), \tag{2.1}$$

where $L$ is a certain lattice in the space of symmetric matrices of degree $l$ with coefficients in $F$, there exists $f^\sigma \in \mathcal{M}_{k^\sigma}^{(l)}$ whose Fourier expansion is

$$f^\sigma(z) = \sum_{r\in L} c_r^\sigma \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(rz)). \tag{2.2}$$

This fact is proved in [3] (cf. also in [9, Section 26]), and this implies $\mathcal{M}_k^{(l)}(\Omega) = \{0\}$ if $\Omega \not\supset F(k)$.

In [2], the canonical models for symplectic cases are constructed. Take $G_A^{(l)} = \mathrm{GSp}(l, F_A)$ and set

$$\mathcal{G}_+^{(l)} = \left\{ x \in G_A \,\middle|\, \nu(x) \in \overline{F^\times F_{\infty+}^\times} \mathbb{Q}_A^\times, \quad \nu(x)_v > 0 \quad \text{for each } \ v \in \mathbf{a} \right\}$$

and denote by $\mathcal{Z}^{(l)}$ the $\mathcal{Z}$ defined in [2]. For any $T \in \mathcal{Z}^{(l)}$, we denote the congruence subgroup $T \cap G^{(l)}(\mathbb{Q})$ by $\Gamma_T^{(l)}$. We denote by $J_{T'T}^{(l)}$, $V_T^{(l)}$, $\varphi_T^{(l)}$, $\mathfrak{K}^{(l)}$, the $J_{T'T}$, $V_T$, $\varphi_T$, $\mathfrak{L}$ defined in [2] respectively, for $T, T' \in \mathcal{Z}^{(l)}$. Take $0 \neq f_1, f_2 \in \mathcal{M}_k^{(l)}$ and $f_1^\sigma, f_2^\sigma$ in the sense of (2.2). Then for any $Y \in \mathcal{Z}^{(l)}$ such that $(f_1/f_2) \circ (\varphi_Y^{(l)})^{-1}$ is defined as a rational function on $V_Y^{(l)}$, we have

$$(f_1^\sigma/f_2^\sigma) = \left[(f_1/f_2) \circ (\varphi_Y^{(l)})^{-1}\right]^\sigma \circ J_{Y\tilde{Y}}^{(l)}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}\right) \circ \varphi_{\tilde{Y}}^{(l)},$$

where $\chi(\sigma) \in \prod_p \mathbb{Z}_p^\times$ so that $[\chi(\sigma)^{-1}, \mathbb{Q}] = \sigma|_{\mathbb{Q}_{ab}}$ and

$$\tilde{Y} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} Y \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}.$$

(See, [8, Theorem 1.5].)

For any CM-extension $K$ of $F$ and its CM-type $\Psi$, put

$$W^{(l)}(\Psi) = \left\{ z \in \mathfrak{H}_l^{\mathbf{a}} \,\middle|\, z = \tau^\Psi \quad \text{for some } \tau \in K_q^q \right\}.$$

Then $\varphi_T^{(l)}(z)$ is $K_{\Psi ab}^*$-rational for any $z \in W^{(l)}(\Psi)$ and $T \in \mathcal{Z}^{(l)}$. For any $z = \tau^\Psi \in W^{(l)}(\Psi)$, we define the group injection $\Phi_z^{(l)} : K_A^\times \to G_A^{(l)}$ as

$$\Phi_z^{(l)}(a) = \begin{pmatrix} (a\tau - a^\rho\tau^\rho)(\tau - \tau^\rho)^{-1} & -(a - a^\rho)\tau^\rho(\tau - \tau^\rho)^{-1}\tau \\ (a - a^\rho)(\tau - \tau^\rho)^{-1} & (\tau - \tau^\rho)^{-1}(a^\rho\tau - a\tau^\rho) \end{pmatrix},$$

namely the $h(\overbrace{a, \ldots, a}^{l \text{ times}})$ in [9, Section 24.10] with $h$ corresponding to $z$. Then it satisfies $\Phi_z^{(l)}(a)\begin{pmatrix} \tau \\ 1_l \end{pmatrix} = \begin{pmatrix} a \cdot \tau \\ a \cdot 1_l \end{pmatrix}$ and $\nu(\Phi_z^{(l)}(a)) = aa^\rho$. If $a \in K^\times$, then $\Phi_z^{(l)}(a) \in G^{(l)}(\mathbb{Q})_+$ and $\Phi_z^{(l)}(a)(z) = z$.

Now let us define the first embedding. For $z = (z_v)_{v \in \mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$, put

$$\varepsilon_0^{(q,n)}(S, \Psi)(z) = \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} z_v \\ 0 \end{pmatrix}_{v \in \mathbf{a}}.$$

Then $\varepsilon_0^{(q,n)}(S, \Psi)$ gives an embedding of $\mathfrak{H}_q^{\mathbf{a}}$ into $D^{(q,n)}(S, \Psi)$. This is compatible with the injection $I_0^{(q,n)}(S, \Psi)$ of $G_1^{(q)}(\mathbb{Q}) = \mathrm{Sp}(q, F)$ into $G_1^{(q,n)}(S, \Psi)(\mathbb{Q})$ defined by

$$I_0^{(q,n)}(S, \Psi)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & 1_n & 0 \\ c & 0 & d \end{pmatrix}, \quad \text{where } a, b, c, d \in F_q^q.$$

As $I_0^{(q,n)}(S, \Psi)$ can be viewed as a homomorphism of algebraic groups, we can extend $I_0^{(q,n)}(S, \Psi)$ to the map $G_{1A}^{(q)} \hookrightarrow G_1^{(q,n)}(S, \Psi)_A$. We denote $I_0^{(q,n)}(S, \Psi)$, $\varepsilon_0^{(q,n)}(S, \Psi)$ by $I_0, \varepsilon_0$ if there is no fear of confusion. We have

$$I_0(\alpha)(\varepsilon_0(z)) = \varepsilon_0(\alpha(z)),$$

$$\mu_v(I_0(\alpha), \varepsilon_0(z)) = j_v^{(q)}(\alpha, z) \quad \text{for any } \alpha \in G_1^{(q)}(\mathbb{Q}),\ z \in \mathfrak{H}_q^{\mathbf{a}}.$$

Hence we can consider the pull-back of modular forms on $D$.

**Lemma 2.1.** (1) *For any $f \in \mathcal{M}_k$, $f \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}$ $(k \in \mathbb{Z}^{\mathbf{a}})$.*

(2) *Let $\Omega$ be a subfield of $\mathbb{C}$ containing $K^*_{\Psi ab}$. Then for any $f \in \mathcal{M}_k(\Omega)$, we have*

$$f \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega) \quad (k \in \mathbb{Z}^{\mathbf{a}}).$$

*Proof.* (1) is clear except for the case when $F = \mathbb{Q}$ and $q = 1$. Put the Fourier-Jacobi expansion of $f$ as

$$f(\mathfrak{z}) = \sum_{0 \le r \in L_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(r^{\Psi} z)) \qquad \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in D,$$

where $L_q$ is a lattice in $\mathcal{H}_q$. Then we have

$$(f \circ \varepsilon_0)(z) = \sum_{0 \le r \in L_q} g_r(0) \boldsymbol{e}_{\mathbf{a}} \left( \mathrm{tr}(\mathrm{Re}(r) z) \right), \qquad z \in \mathfrak{H}_q^{\mathbf{a}}. \tag{2.3}$$

Now $\mathrm{Re}(r)$ is a symmetric matrix contained in $F_q^q$. Equivalently,

$$(f \circ \varepsilon_0)(z) = \sum_b \left( \sum_{\substack{0 \le r \in L_q \\ \mathrm{Re}(r) = b}} g_r(0) \right) \boldsymbol{e}_{\mathbf{a}} \left( \mathrm{tr}(bz) \right). \tag{2.4}$$

If $\mathrm{Re}(r) = b$, we have $r + r^{\rho} = 2b$. This implies $2b - r$ is semi-positive definite for any embedding of $K$ into $\mathbb{C}$. If we embed $\mathcal{H}_q$ into $\left\{ X \in \mathbb{C}_q^q \,\middle|\, \overline{{}^t X} = X \right\}^{\mathbf{a}}$, then $L_q$ is a lattice (hence discrete) in it, and the subset $\prod_{v \in \mathbf{a}} \left\{ X \in \mathbb{C}_q^q \,\middle|\, \overline{{}^t X} = X,\ 2b_v - X \ge 0,\ X \ge 0 \right\}$ is compact. Hence the set $\{ r \in L_q \,| \mathrm{Re}(r) = b \}$ is a finite set. This implies (2). As (2.4) is the Fourier expansion of $f \circ \varepsilon_0$, we get (1) even if $F = \mathbb{Q}$ and $q = 1$. $\square$

Combining this lemma with Lemma 1.4, we have the following lemma.

**Lemma 2.2.** (1) *Let $\Omega$ be a subfield of $\mathbb{C}$ containing $K^*_{\Psi ab}$. For any $f \in \mathcal{M}_k(\Omega)$ and any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, we have*

$$(f \circ \mathsf{h}) \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega) \quad (k \in \mathbb{Z}^{\mathbf{a}}).$$

(2) *Take any $k \in \mathbb{Z}^{\mathbf{a}}$. Let $\Omega$ be a subfield of $\mathbb{C}$ containing $F(k) \vee K^*_{\Psi ab}$. For any $f \in \mathcal{M}_k(\Omega)$ and any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, we have*

$$(f|_k \mathsf{h}) \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega).$$

Further, we have the following lemma.

**Lemma 2.3.** *Take any $k \in \mathbb{Z}^{\mathbf{a}}$. Let $\Omega$ be a subfield of $\mathbb{C}$ containing $F(k) \vee K^*_{\Psi ab}$. Let $f \in \mathcal{M}_k$. Then $f \in \mathcal{M}_k(\Omega)$ if and only if $(f|_k \mathsf{h}) \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega)$ for any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$.*

To prove this, we need the following lemma.

**Lemma 2.4.** *Let $L_q$ be a lattice in $\mathcal{H}_q$. For any $r \in L_q$ which is semi-positive definite in any embedding of $K$ into $\mathbb{C}$, we can take some $\alpha \in \mathrm{GL}(q, K)$ so that*

$$\left\{0 \le r' \in L_q \,\middle|\, \mathrm{Re}({}^t\alpha^\rho r' \alpha) = \mathrm{Re}({}^t\alpha^\rho r \alpha)\right\} = \{r\}.$$

*Proof.* The condition $\mathrm{Re}({}^t\alpha^\rho r' \alpha) = \mathrm{Re}({}^t\alpha^\rho r \alpha)$ implies that $r' \le r + ({}^t\alpha^\rho)^{-1}{}^t\alpha r^\rho \alpha^\rho \alpha^{-1}$. Viewing $\mathrm{GL}(q, K)$ as a dense subset of $\mathrm{GL}(q, \mathbb{C})^{\mathbf{a}}$, we can take a compact neighborhood $C$ of the identity in $\mathrm{GL}(q, \mathbb{C})^{\mathbf{a}}$ such that

$$\left\{0 \le r' \in L_q \,\middle|\, r' \le r + ({}^t\alpha^\rho)^{-1}{}^t\alpha r^\rho \alpha^\rho \alpha^{-1} \quad \text{for some } \alpha \in C\right\}$$

is a finite set. Hence we can choose a suitable $\alpha \in C \cap \mathrm{GL}(q, K)$ satisfying the condition of this lemma. $\square$

*Proof of Lemma* 2.3. The "only if" part has been proved in Lemma 2.2. For any $f \in \mathcal{M}_k$, take the Fourier-Jacobi expansion of $f$ as

$$f(\mathfrak{z}) = \sum_{0 \le r \in L_q} g_r(w) \boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(r^\Psi z))$$

with some lattice $L_q$ of $\mathcal{H}_q$. Assume $f|_k \mathsf{h} \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega)$ for any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$. Fix $r \in L_q, y \in K_q^n$ and let us prove $(g_r)_*(y^\Psi) \in \Omega$. Take $\alpha$ as in Lemma 2.4 and put $\mathsf{h} = \begin{pmatrix} 1_q & {}^t y^\rho S & \frac{1}{2}{}^t y^\rho S y \\ 0 & 1_n & y \\ 0 & 0 & 1_q \end{pmatrix} \begin{pmatrix} \alpha & & \\ & 1_n & \\ & & ({}^t\alpha^\rho)^{-1} \end{pmatrix}$. From Lemma 1.4 (3), (2.4) and Lemma 2.4, $\prod_{v \in \mathbf{a}} \left(\det(\alpha^\rho)^{\Psi_v}\right)^{k_v} (g_r)_*(y^\Psi)$ is a Fourier coefficient of $f|_k \mathsf{h} \circ \varepsilon_0$, hence $(g_r)_*(y^\Psi) \in \Omega$ since $\prod_{v \in \mathbf{a}} \left(\det(\alpha^\rho)^{\Psi_v}\right)^{k_v} \in F(k) \vee K^*_\Psi$. This completes the proof. $\square$

Next we define the embedding of $D = D^{(q,n)}(S, \Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$. Take $\delta \in K^\times$ such that $\delta^\rho = -\delta$. Put

$$\begin{aligned} &\varepsilon_\delta^{(q,n)}(S, \Psi)(\mathfrak{z}) \\ &= \begin{pmatrix} \frac{1}{2}(z + {}^t z - {}^t w S^\Psi w) & {}^t w & \frac{1}{2}\delta^\Psi(z - {}^t z - {}^t w S^\Psi w) \\ w & (-S^{-1})^\Psi & \delta^\Psi w \\ \frac{1}{2}\delta^\Psi({}^t z - z - {}^t w S^\Psi w) & \delta^\Psi {}^t w & -\frac{\delta^2}{2}(z + {}^t z + {}^t w S^\Psi w) \end{pmatrix} \\ &\qquad \text{where } \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \text{ with } z \in \left(\mathbb{C}_q^q\right)^{\mathbf{a}},\ w \in \left(\mathbb{C}_q^n\right)^{\mathbf{a}}. \end{aligned}$$

Then $\varepsilon_\delta = \varepsilon_\delta^{(q,n)}(S,\Psi)$ is an embedding of $D$ into $\mathfrak{H}_m^{\mathbf{a}}$. This is compatible with the injection $I_\delta = I_\delta^{(q,n)}(S,\Psi)$ of $G(\mathbb{Q}) = G^{(q,n)}(S,\Psi)(\mathbb{Q})$ into $G^{(m)}(\mathbb{Q}) = \mathrm{GSp}(m,F)$ defined by

$$I_\delta^{(q,n)}(S,\Psi)(\alpha) = C(S,\delta)\begin{pmatrix} \alpha^\rho & 0 \\ 0 & \alpha \end{pmatrix} C(S,\delta)^{-1},$$

$$\text{where} C(S,\delta) = \begin{pmatrix} 1_q & 0 & 0 & 1_q & 0 & 0 \\ 0 & 1_n & 0 & 0 & 1_n & 0 \\ \delta\cdot 1_q & 0 & 0 & -\delta\cdot 1_q & 0 & 0 \\ 0 & 0 & 1_q & 0 & 0 & 1_q \\ 0 & -S & 0 & 0 & S & 0 \\ 0 & 0 & -\delta^{-1}\cdot 1_q & 0 & 0 & \delta^{-1}\cdot 1_q \end{pmatrix}.$$

Then we have

$$\begin{aligned} &I_\delta(\alpha)(\varepsilon_\delta(\mathfrak{z})) = \varepsilon_\delta(\alpha(\mathfrak{z})), \\ &\nu(I_\delta(\alpha)) = \nu(\alpha), \qquad \det(I_\delta(\alpha)) = \det(\alpha)\det(\alpha)^\rho. \end{aligned} \tag{2.5}$$

Put, for each $v \in \mathbf{a}$

$$\omega_v(\mathfrak{z}) = \begin{pmatrix} 1_q & 0 & \frac{1}{2}\delta^{\Psi_v}\cdot 1_q \\ 0 & 1_n & \frac{1}{2}\delta^{\Psi_v}S^{\Psi_v}w_v \\ (-\delta^{-1})^{\Psi_v}\cdot 1_q & 0 & \frac{1}{2}\cdot 1_q \end{pmatrix} \qquad \text{for } \mathfrak{z} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v\in\mathbf{a}} \in D.$$

Then we have

$$j_v^{(m)}(I_\delta(\alpha),\varepsilon_\delta(\mathfrak{z})) = \omega_v(\alpha(\mathfrak{z}))\begin{pmatrix} \lambda_v(\alpha,\mathfrak{z}) & 0 \\ 0 & \mu_v(\alpha,\mathfrak{z}) \end{pmatrix}\omega_v(\mathfrak{z})^{-1} \tag{2.6}$$

where $\alpha \in G(\mathbb{Q})$, $\mathfrak{z} \in D$. From (1.4) we obtain $\det(\lambda_v(\alpha,\mathfrak{z})) = \det(\mu_v(\alpha,\mathfrak{z}))$ for any $\alpha \in G_1(\mathbb{Q})$, hence $\det\left(j_v^{(m)}(I_\delta(\alpha),\varepsilon_\delta(\mathfrak{z}))\right) = \det(\mu_v(\alpha,\mathfrak{z}))^2$ for each $v \in \mathbf{a}$ if $\alpha \in G_1(\mathbb{Q})$ (since $\det(\omega_v(\mathfrak{z})) = 1$ for any $\mathfrak{z} \in D$ ). Therefore for any $f \in \mathcal{M}_k^{(m)}$, we have $f \circ \varepsilon_\delta^{(q,n)}(S,\Psi) \in \mathcal{M}_{2k}^{(q,n)}(S,\Psi)$ ($k \in \mathbb{Z}^{\mathbf{a}}$). Through the embedding $\varepsilon_\delta$, arithmetic modular forms on $\mathfrak{H}_m^{\mathbf{a}}$ and $D$ are related by a certain proportionality factor, which is essentially a CM-period.

**Lemma 2.5.** *Let $k \in \mathbb{Z}^{\mathbf{a}}$, and $\Omega$ be a subfield of $\mathbb{C}$ containing $F(k) \vee K^*_{\Psi ab}$. Then for any $f \in \mathcal{M}_k^{(m)}(\Omega)$, we have*

$$h^{(1)}(z^{(1)})^{-n}\cdot\left(f\circ\varepsilon_\delta^{(q,n)}(S,\Psi)\right) \in \mathcal{M}_{2k}^{(q,n)}(S,\Psi)(\Omega),$$

*where $h^{(1)} \in \mathcal{M}_k^{(1)}(F(k)\vee K^*_{\Psi ab})$ and $z^{(1)} \in W^{(1)}(\Psi)$ so that $h^{(1)}(z^{(1)}) \neq 0$ .*

*Proof.* For any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, consider $(f\circ\varepsilon_\delta)|_{2k}\mathsf{h}\circ\varepsilon_0 \in \mathcal{M}_{2k}^{(q)}$. Then for any $z \in W^{(q)}(\Psi)$, we have $(f\circ\varepsilon_\delta)|_{2k}\mathsf{h}\circ\varepsilon_0(z) \in h^{(1)}(z^{(1)})^m\cdot\Omega$ since $\varepsilon_\delta(\mathsf{h}\circ$

$\varepsilon_0(z)) \in W^{(m)}(\Psi)$. As stated in [8], $\mathcal{M}_{2k}^{(q)}(\mathbb{C})$ is spanned by $\mathcal{M}_{2k}^{(q)}(F(k))$ (clearly $F(2k) = F(k)$) and so we have

$$h^{(1)}(z^{(1)})^{-n} \cdot (f \circ \varepsilon_\delta)|_{2k}\mathsf{h} \circ \varepsilon_0 = c_1 h_1 + \cdots + c_t h_t$$

with $c_1, \ldots, c_t \in \mathbb{C}$ and $h_1, \ldots, h_t \in \mathcal{M}_{2k}^{(q)}(F(k))$. Now we have $h_j(z) \cdot h^{(1)}(z^{(1)})^{-2q} \in F(k) \vee K_{\Psi ab}^*$ for any $z \in W^{(q)}(\Psi)$ $(1 \leq j \leq t)$. Taking $h_1, \ldots, h_t$ linearly independent over $\mathbb{C}$ and moving $z$ all over $W^{(q)}(\Psi)$, we have $c_1, \ldots, c_t \in \Omega$ and hence $h^{(1)}(z^{(1)})^{-n} \cdot (f \circ \varepsilon_\delta)|_{2k}\mathsf{h} \circ \varepsilon_0 \in \mathcal{M}_{2k}^{(q)}(\Omega)$. Taking any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, we get $h^{(1)}(z^{(1)})^{-n} \cdot (f \circ \varepsilon_\delta) \in \mathcal{M}_{2k}(\Omega)$ from Lemma 2.3. $\square$

We can consider embeddings of canonical models corresponding to $\varepsilon_\delta$. For $Y \in \mathcal{Z}$ and $T \in \mathcal{Z}^{(m)}$, if $I_\delta(Y) \subset T$ then we can define the map from $V_Y$ to $V_T^{(m)}$ which is compatible with $\varepsilon_\delta$. We denote this by $E_{TY}$. Namely, we have the commutative diagram as follows.

$$\begin{array}{ccc} D & \xrightarrow{\quad\varepsilon_\delta\quad} & \mathfrak{H}_m^{\mathbf{a}} \\ {\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_T^{(m)}} \\ V_Y & \xrightarrow[\quad E_{TY}\quad]{} & V_T^{(m)} \end{array}.$$

The map $E_{TY}$ is a rational map from $V_Y$ to $V_T^{(m)}$. Take $W = W^{(q,n)}(S, \Psi)$ as in (1.12). Then we have $\varepsilon_\delta(W) \subset W^{(m)}(\Psi)$, and $\varphi_Y(\mathfrak{z}), \varphi_T^{(m)}(z)$ $(\mathfrak{z} \in W,\ z \in W^{(m)}(\Psi))$ are $K_{\Psi ab}^*$-rational. This implies $E_{TY}$ is defined over $K_{\Psi ab}^*$ since $W$ is dense in $D$. In the same way as in [7, Section 4], we have

$$E_{UX}^{\sigma(x)} \circ J_{XY}(x) = J_{UT}^{(m)}(I_\delta(x)) \circ E_{TY}, \tag{2.7}$$

where $x \in \mathcal{G}_+,\ X, Y \in \mathcal{Z}$ and $T, U \in \mathcal{Z}^{(m)}$, if the both hands sides are defined. The properties of $E_{TY}$ will be mentioned more precisely in Section 4.

The last embedding is that of $\mathrm{Sp}(l, F)$ into $\mathrm{Sp}(lg, \mathbb{Q})$ where $g = [F : \mathbb{Q}]$, stated in [8, Section 1]. Take a basis $\{\beta_1, \ldots, \beta_g\}$ of $F$ over $\mathbb{Q}$ and put

$$B = \begin{pmatrix} (\beta_1)_{v_1} & \cdots & (\beta_g)_{v_1} \\ \cdots & \cdots & \cdots \\ (\beta_1)_{v_g} & \cdots & (\beta_g)_{v_g} \end{pmatrix}, \qquad B^{(l)} = \begin{pmatrix} (\beta_1)_{v_1} 1_l & \cdots & (\beta_g)_{v_1} 1_l \\ \cdots & \cdots & \cdots \\ (\beta_1)_{v_g} 1_l & \cdots & (\beta_g)_{v_g} 1_l \end{pmatrix}.$$

Let $\{\beta_1', \ldots, \beta_g'\}$ be the dual basis of $\{\beta_1, \ldots, \beta_g\}$ with respect to $\mathrm{Tr}_{F/\mathbb{Q}}$, that is,

$$({}^t B)^{-1} = \begin{pmatrix} (\beta_1')_{v_1} & \cdots & (\beta_g')_{v_1} \\ \cdots & \cdots & \cdots \\ (\beta_1')_{v_g} & \cdots & (\beta_g')_{v_g} \end{pmatrix}.$$

We define the embedding $I_B^{(l)}$ of $\mathrm{Sp}(l,F)$ into $\mathrm{Sp}(lg,\mathbb{Q})$ as

$$
\tag{2.8}
I_B^{(l)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} {}^tB^{(l)} & 0 \\ 0 & (B^{(l)})^{-1} \end{pmatrix}
\times \begin{pmatrix} a_{v_1} & & & b_{v_1} & & \\ & \ddots & & & \ddots & \\ & & a_{v_g} & & & b_{v_g} \\ c_{v_1} & & & d_{v_1} & & \\ & \ddots & & & \ddots & \\ & & c_{v_g} & & & d_{v_g} \end{pmatrix} \begin{pmatrix} ({}^tB^{(l)})^{-1} & 0 \\ 0 & B^{(l)} \end{pmatrix},
$$

where $a,b,c,d \in F_l^l$. Then $\mathrm{Sp}(lg,\mathbb{Q})$ acts on

$$
\mathfrak{H}_{lg} = \left\{ Z \in \mathbb{C}_{lg}^{lg} \,\middle|\, {}^tZ = Z, \quad \mathrm{Im}(Z) > 0 \right\}
$$

as

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}(Z) = (aZ+b)(cZ+d)^{-1} \qquad (a,b,c,d \in \mathbb{Q}_{lg}^{lg}).
$$

The corresponding embedding $\varepsilon_B^{(l)}$ of $\mathfrak{H}_l^{\mathbf{a}}$ into $\mathfrak{H}_{lg}$ is defined by

$$
\varepsilon_B^{(l)}((z_v)_{v\in\mathbf{a}}) = {}^tB^{(l)} \begin{pmatrix} z_{v_1} & & \\ & \ddots & \\ & & z_{v_g} \end{pmatrix} B^{(l)}.
$$

This embedding is compatible with $I_B^{(l)}$. For any $\alpha \in \mathrm{Sp}(l,F)$, put $\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$ $= I_B^{(l)}(\alpha)$ with $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in \mathbb{Q}_{lg}^{lg}$. Then we have

$$
\tag{2.9}
c_\alpha \varepsilon_B^{(l)}(z) + d_\alpha = (B^{(l)})^{-1} \begin{pmatrix} j_{v_1}^{(l)}(\alpha,z) & & \\ & \ddots & \\ & & j_{v_g}^{(l)}(\alpha,z) \end{pmatrix} B^{(l)}
$$

for any $z \in \mathfrak{H}_l^{\mathbf{a}}$. Hence we can consider the pull-back of modular forms again in this case. For any modular form $f$ on $\mathfrak{H}_{lg}$ of weight $\kappa$, we have $f \circ \varepsilon_B^{(l)} \in \mathcal{M}_{\kappa\mathbf{1}}^{(l)}$ $(\kappa \in \mathbb{Z})$. If all the Fourier coefficients of $f$ are $\Omega$-rational, then $f \circ \varepsilon_B^{(l)} \in \mathcal{M}_{\kappa\mathbf{1}}^{(l)}(\Omega)$, for each subfield $\Omega$ of $\mathbb{C}$.

## 3. Arbitrary conjugation of theta functions

In [4], Shimura formulated complex multiplication theory in terms of theta functions. In particular, he stated a theorem on conjugation of abelian varieties with complex multiplication by $\sigma \in \mathrm{Aut}(\mathbb{C})$ when $\sigma|_{K^*} = \mathrm{id}_{K^*}$. We shall extend this theorem to the case for an arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C})$ using the results in [10, Chapter 7] and [11].

First we consider the conjugation of polarized abelian varieties. Let $F, K$ be as in Section 1 and take a CM-type $\Psi$ of $K$. Let $(A, \mathcal{C}, \iota)$ be a polarized abelian variety of type $(K, \Psi, L, t)$ with respect to $\Theta$, which is $g$-dimensional and $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K$. The Riemann form $E$ corresponding to $\mathcal{C}$ is given by

$$E(x^\Psi, y^\Psi) = \frac{1}{2}\mathrm{Tr}_{K/\mathbb{Q}}(txy^\rho)$$

for all $x, y \in K$. (See, [9, Section 24] and [10, Chapter 7, Section 3].)

Take any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and consider $(A, \mathcal{C}, \iota)^\sigma$. We denote by $A_{\mathrm{tor}}$ the subgroup of all torsion elements of $A$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, take $\chi(\sigma) \in \prod_p \mathbb{Z}_p^\times$ so that $[\chi(\sigma)^{-1}, \mathbb{Q}] = \sigma|_{\mathbb{Q}_{ab}}$. Then we have the following commutative diagram by virtue of [10, Chapter 7].

$$\begin{array}{ccc} K/L & \xrightarrow{\Theta\circ\Psi} & A_{\mathrm{tor}} \\ {\scriptstyle \times a}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K/aL & \xrightarrow{\Theta_a\circ(\Psi\sigma)} & A_{\mathrm{tor}}^\sigma \end{array}$$

with some $a \in K_A^\times$, and $(A, \mathcal{C}, \iota)^\sigma$ is of type $(K, \Psi\sigma, aL, \iota(\sigma, a)t)$ with respect to $\Theta_a$; $a$ and $\sigma$ are related by $\frac{\chi(\sigma)}{aa^\rho} \in F^\times F_\infty^\times$ and $\iota(\sigma, a) \in F^\times$ so that $\frac{\chi(\sigma)}{aa^\rho} \in \iota(\sigma, a)F_\infty^\times$. Now the coset $aK^\times K_\infty^\times$ is uniquely determined only by $(K, \Psi)$ and $\sigma$ (not depending on $A$ or $L$). We denote $aK^\times K_\infty^\times$ by $g_\Psi(\sigma)$. If $\sigma$ is trivial on $K_\Psi^*$, we have $g_\Psi(\sigma) = N'_\Psi(b)K^\times K_\infty^\times$ with $b \in (K_\Psi^*)_A^\times$ such that $[b^{-1}, K_\Psi^*] = \sigma|_{K^*_{\Psi ab}}$; this fact is a main theorem of complex multiplication theory of [9]. Note that $g_\Psi(\sigma_1)g_{\Psi\sigma_1}(\sigma_2) = g_\Psi(\sigma_1\sigma_2)$. Set

$$C_\Psi(\mathbb{C}) = \{(\sigma, \Psi, a) \,|\sigma \in \mathrm{Aut}(\mathbb{C}), \quad a \in g_\Psi(\sigma)\}.$$

For a polarized abelian variety $(A, \mathcal{C}, \iota)$ of type $(K^l, l\Psi, L, T)$ (for some skew-hermitian form $T$ on $K^l$), we also have the commutative diagram

$$\begin{array}{ccc} K^l/L & \xrightarrow{\Theta\circ\Psi} & A_{\mathrm{tor}} \\ {\scriptstyle \times a}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K^l/aL & \xrightarrow{\Theta_a\circ(\Psi\sigma)} & A_{\mathrm{tor}}^\sigma \end{array} \tag{3.1}$$

for $a \in g_\Psi(\sigma)$. This can be verified by taking $A$ to be a product of $l$ copies of a polarized abelian variety of type $(K, \Psi)$.

Now let us review classical theta functions. Let $\mathcal{V}$ be a finite dimensional $\mathbb{C}$-vector space and $\Lambda$ be a $\mathbb{Z}$-lattice in $\mathcal{V}$. Assume that there exists a semi-positive definite hermitian form $H$ on $\mathcal{V}$ which satisfies $\mathrm{Im}(H(u_1, u_2)) \in \mathbb{Z}$ for any $u_1, u_2 \in \Lambda$. Then we define

$$\mathfrak{T}(\mathcal{V}, \Lambda, H) = \left\{ f : \mathcal{V} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic on } \mathcal{V}, \\ f(u + x) = f(u) \exp\left(\pi H(x, u + \frac{1}{2}x)\right) \\ \text{for each } u \in \mathcal{V},\ x \in \Lambda \end{array} \right\}.$$

For any $f \in \mathfrak{T}(\mathcal{V}, \Lambda, H)$, we define the (non holomorphic) function $f_*$ on $\mathcal{V}$ as

$$f_*(u) = \exp\left(-\frac{\pi}{2}H(u,u)\right) f(u).$$

Consider the case $\mathcal{V} = \mathbb{C}^l$ and $H$ is positive definite. As stated in [4], put

$$\theta^{(l)}(u, Z; p_1, p_2) = \sum_{x \in \mathbb{Z}^l} \exp\left(\pi\sqrt{-1}\left({}^t(x+p_1)Z(x+p_1) + 2{}^t(x+p_1)(u+p_2)\right)\right),$$

$$\varphi^{(l)}(u, Z; p_1, p_2) = \exp\left(\pi\sqrt{-1}{}^t u(Z-\overline{Z})^{-1}u\right)\theta^{(l)}(u, Z; p_1, p_2).$$

Here $u \in \mathbb{C}^l$, $Z \in \mathfrak{H}_l$ and $p_1, p_2 \in \mathbb{Q}^l$. As stated in [4], take $(\omega_1\ \omega_2) \in \mathbb{C}^l_{2l}$ such that $Z = \omega_2^{-1}\omega_1 \in \mathfrak{H}_l \quad (\omega_1,\ \omega_2 \in \mathrm{GL}(l, \mathbb{C}))$ and set

$$\varphi^{(l)}(u, (\omega_1\ \omega_2); p_1, p_2) = \theta^{(l)}(0, Z; p_1', 0)^{-1}\varphi^{(l)}(\omega_2^{-1}u, Z; p_1, p_2) \tag{3.2}$$

where $p_1' \in \mathbb{Q}^l$ so that $\theta^{(l)}(0, Z; p_1', 0) \neq 0$ (it is possible from [4]). Then $\varphi^{(l)}(u, (\omega_1\ \omega_2); p_1, p_2)$ is determined up to the multiplication of non-zero constant. For $\Lambda$ and $H$ above, by the theory of elementary divisors, we can take a $\mathbb{Z}$-basis of $\Lambda$ so that the $\mathbb{Z}$-valued alternating form $\mathrm{Im}(H)$ on $\Lambda$ is expressed as

$$\begin{pmatrix} 0 & -\mu\epsilon \\ \mu\epsilon & 0 \end{pmatrix}, \tag{3.3}$$

where $\epsilon = \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_l \end{pmatrix}, \quad \epsilon_1, \dots, \epsilon_l \in \mathbb{N}, \quad \epsilon_k | \epsilon_{k+1}, \quad \epsilon_1 = 1, \quad \mu \in \mathbb{N}.$

As is well known, if $\mu$ is even and $\mu \geq 3$, we have

$$\dim \mathfrak{T}(\mathbb{C}^l, \Lambda, H) = \mu^l \det(\epsilon).$$

In this case we can take the basis of $\mathfrak{T}(\mathbb{C}^l, \Lambda, H)$ as

$$\left\{\varphi^{(l)}(u, (\omega_1\ \omega_2); j, 0) \,\middle|\, j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^l/\mathbb{Z}^l\right\},$$

where $\Lambda = (\omega_1\ \omega_2)\begin{pmatrix} \mathbb{Z}^l \\ \mu\epsilon\mathbb{Z}^l \end{pmatrix}$.

Now we consider theta functions with complex multiplication. Take a totally real algebraic number field $F$ of finite degree and put $g = [F : \mathbb{Q}]$. Take $B$ and $B^{(l)}$ as in Section 2 and put

$$\theta^{(l)}_{F,B}(u, z; p_1, p_2) = \theta^{(lg)}\left({}^tB^{(l)}\begin{pmatrix} u_{v_1} \\ \vdots \\ u_{v_g} \end{pmatrix}, \varepsilon^{(l)}_B(z); p_1, p_2\right),$$

$$\varphi^{(l)}_{F,B}(u, z; p_1, p_2) = \varphi^{(lg)}\left({}^tB^{(l)}\begin{pmatrix} u_{v_1} \\ \vdots \\ u_{v_g} \end{pmatrix}, \varepsilon^{(l)}_B(z); p_1, p_2\right),$$

for $u \in \left(\mathbb{C}^l\right)^{\mathbf{a}}$, $z \in \mathfrak{H}_l^{\mathbf{a}}$ and $p_1, p_2 \in \mathbb{Q}^{lg}$. Take $(\omega_1\ \omega_2) \in \left(\mathbb{C}_{2l}^l\right)^{\mathbf{a}}$ such that $\omega_2^{-1}\omega_1 \in \mathfrak{H}_l^{\mathbf{a}}$ ( $\omega_1,\ \omega_2 \in \mathrm{GL}(l, \mathbb{C})^{\mathbf{a}}$ ) and set

$$\varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2); p_1, p_2) = \theta_{F,B}^{(l)}(0, z; p_1', 0)^{-1}\varphi_{F,B}^{(l)}(\omega_2^{-1}u, z; p_1, p_2)$$

for $u \in (\mathbb{C}^l)^{\mathbf{a}}$, $z = \omega_2^{-1}\omega_1 \in \mathfrak{H}_l^{\mathbf{a}}$ and $p_1, p_2, p_1' \in \mathbb{Q}^{lg}$ so that $\theta_{F,B}^{(l)}(0, z; p_1', 0) \neq 0$. The theta function $\varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2); p_1, p_2)$ is determined up to the constant multiple. For fixed $p_1, p_2, p_1'$ (by the same reason as in [4]), there exists a congruence subgroup $\Gamma^{(l)}$ of $G^{(l)}(\mathbb{Q}) = \mathrm{GSp}(l, F)$ which satisfies

$$\varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2)^t\gamma; p_1, p_2) = \varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2); p_1, p_2)$$

for any $\gamma \in \Gamma^{(l)}$ and $(\omega_1\ \omega_2)$.

Consider a CM-extension $K$ of $F$ and its CM-type $\Psi$. Take a $\mathbb{Z}$-lattice $L$ in $K^l$ and an $l$-dimensional skew-hermitian matrix $T \in K_l^l$ so that

$$\mathrm{Tr}_{K/\mathbb{Q}}({}^t y_1^\rho T y_2) \in \mathbb{Z} \quad \text{for any } \ y_1, y_2 \in L.$$

We can define the hermitian form $H_{T,\Psi}$ on $(\mathbb{C}^l)^{\mathbf{a}}$ as

$$H_{T,\Psi}(u_1, u_2) = -2\sqrt{-1}\sum_{v \in \mathbf{a}} {}^t\overline{u_{1v}}T^{\Psi_v}u_{2v}.$$

Assume that $H_{T,\Psi}$ is positive definite. This means $-2\sqrt{-1}T^{\Psi_v}$ is positive definite for each $v \in \mathbf{a}$. Then we can consider $\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi})$. Take $L$ sufficiently small so that $\mathrm{Im}(H_{T,\Psi})$ is expressed as (3.3) for a positive even integer $\mu \geq 3$ (in this case we must replace $l$ by $lg$). Then we have $\dim_{\mathbb{C}} \mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi}) = \mu^{lg}\det(\epsilon)$ and its basis is given by

$$\left\{\varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2); j, 0)\,\middle|\, j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}\right\} \tag{3.4}$$

where

$$L^\Psi = (\omega_1\ \omega_2)\begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix}\begin{pmatrix} \mathbb{Z}^{lg} \\ \mu\epsilon\mathbb{Z}^{lg} \end{pmatrix}. \tag{3.5}$$

Hence we have $\omega_1, \omega_2 \in (K_l^l)^\Psi$. This means $z = \omega_2^{-1}\omega_1 \in W^{(l)}(\Psi)$. Put $\omega_1 = \tau_1^\Psi,\ \omega_2 = \tau_2^\Psi$ for $\tau_1, \tau_2 \in K_l^l$. Since $\theta^{(lg)}(0, \varepsilon_B^{(l)}(z); p_1', 0)/\theta^{(lg)}(0, \varepsilon_B^{(l)}(z); p_1'', 0)$ (viewed as a function of $z$ on $\mathfrak{H}_l^{\mathbf{a}}$ ) is contained in $\mathcal{A}_0^{(l)}(\mathbb{Q})$ (for $p_1', p_1'' \in \mathbb{Q}^{lg}$), the functions $\varphi_{F,B}^{(l)}(u, (\omega_1\ \omega_2); j, 0)$ are determined up to the multiplication of $(K_{\Psi ab}^*)^\times$. For any subfield $\Omega$ of $\mathbb{C}$ containing $K_{\Psi ab}^*$, put

$$\begin{aligned}&\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi}, \Omega)\\ &\quad = \left\{f \in \mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi})\,\middle|\, f_*(u) \in \Omega \quad \text{for any } \ u \in (K^l)^\Psi\right\}.\end{aligned}$$

Let us consider $f_*(u)$ when $f(u)=\varphi^{(l)}_{F,B}(u,(\tau_1^\Psi\ \tau_2^\Psi);j,0)$ and $u\in(K^l)^\Psi$. Put

$$u=(\tau_1^\Psi\ \tau_2^\Psi)\begin{pmatrix}\beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0\\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l\end{pmatrix}\begin{pmatrix}\kappa_1\\ \mu\epsilon\kappa_2\end{pmatrix}$$

for $\kappa_1,\kappa_2\in\mathbb{Q}^{lg}$. By [9, Section 27] (or in the same way as in [4]), we have

$$\begin{aligned}&f_*(u)\\ &\quad=\exp(-\pi\sqrt{-1}\,{}^t\kappa_1\mu\epsilon\kappa_2)\theta^{(lg)}(0,\varepsilon_B^{(l)}(z);j+\kappa_1,\mu\epsilon\kappa_2)/\theta^{(lg)}(0,\varepsilon_B^{(l)}(z);p_1',0).\end{aligned}$$

The right hand side belongs to $\mathcal{A}_0^{(l)}(\mathbb{Q}_{ab})$ when viewed as a meromorphic function of $z(\in\mathfrak{H}_l^{\mathbf{a}})$. Hence $f_*(u)\in K^*_{\Psi ab}$ since $z\in W^{(l)}(\Psi)$. This means $\varphi^{(l)}_{F,B}(u,(\tau_1^\Psi\ \tau_2^\Psi);j,0)\in\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},L^\Psi,H_{T,\Psi},K^*_{\Psi ab})$. Hence we obtain

$$\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},L^\Psi,H_{T,\Psi})=\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},L^\Psi,H_{T,\Psi},K^*_{\Psi ab})\otimes_{K^*_{\Psi ab}}\mathbb{C}.$$

For $f\in\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},L^\Psi,H_{T,\Psi})$, consider the restriction of $f_*$ to $(K^l)^\Psi$. Then it satisfies

$$f_*(y^\Psi+x^\Psi)=\mathbf{e}_{\mathbf{a}}(-\mathrm{Re}({}^tx^\rho Ty))f_*(y^\Psi)\quad\text{for any }\ x\in L,\ y\in K^l.$$

Hence for any $\mathbb{Z}$-lattice $M$ of $K^l$, there exists a sublattice $M'$ of $M$ such that

$$f_*(y^\Psi+x^\Psi)=f_*(y^\Psi)\quad\text{for any }\ x\in M',\ y\in M.$$

Therefore for any $y\in(K_A)^l$, taking $M,y_1\in K^l$ so that $y\in M\otimes_{\mathbb{Z}}(\mathbb{Q}_\infty\times\prod_p\mathbb{Z}_p)$ and $y\in y_1+M'\otimes_{\mathbb{Z}}(\mathbb{Q}_\infty\times\prod_p\mathbb{Z}_p)$ (where $\mathbb{Q}_\infty$ denotes the infinite component of $\mathbb{Q}_A$), we can define $f_*(y^\Psi)$ to be equal to $f_*(y_1^\Psi)$.

We have the following theorem which is an extension of the main theorem of [4].

**Theorem 3.1.** *Assume that $H_{T,\Psi}$ is positive definite. Take any $f\in\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},L^\Psi,H_{T,\Psi})$ and $(\sigma,\Psi,a)\in C_\Psi(\mathbb{C})$. Then there exists $f^{(\sigma,\Psi,a)}\in\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}},(aL)^{(\Psi\sigma)},H_{\iota(\sigma,a)T,\Psi\sigma})$ which satisfies*

$$(f^{(\sigma,\Psi,a)})_*\left((ay)^{(\Psi\sigma)}\right)=\left[f_*(y^\Psi)\right]^\sigma\quad\textit{for any }\ y\in K^l.$$

*Proof.* It suffices to prove the case $f(u)=\varphi^{(l)}_{F,B}(u,(\tau_1^\Psi\ \tau_2^\Psi);j,0)$ where the right hand side is as in (3.4). Set $z=(\tau_2^{-1}\tau_1)^\Psi\in\mathfrak{H}_l^{\mathbf{a}}$. Take a congruence subgroup $\Gamma^{(l)}$ of $G_1^{(l)}(\mathbb{Q})$ so that $\varphi^{(l)}_{F,B}(u,(\omega_1\ \omega_2){}^t\gamma;j,0)=\varphi^{(l)}_{F,B}(u,(\omega_1\ \omega_2);j,0)$ holds for any $\gamma\in\Gamma^{(l)}$ and $(\omega_1\ \omega_2)$. Next take $X\in\mathcal{Z}^{(l)}$ so that $\Gamma^{(l)}_X\subset\mathcal{O}_F^\times\Gamma^{(l)}$. Put

$$\tilde{X}=\begin{pmatrix}1_l & 0\\ 0 & \chi(\sigma)1_l\end{pmatrix}^{-1}X\begin{pmatrix}1_l & 0\\ 0 & \chi(\sigma)1_l\end{pmatrix}\quad\in\mathcal{Z}^{(l)},$$

and take $\tilde{z} \in \mathfrak{H}_l^{\mathbf{a}}$ so that

$$\varphi_{\tilde{X}}^{(l)}(\tilde{z}) = \left[ J_{\tilde{X}X}^{(l)} \left( \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} \right) \left( \varphi_X^{(l)}(z) \right) \right]^{\sigma}.$$

Consider $\tilde{f}(u) = \varphi_{F,B}^{(l)}(u, (\tilde{z}\ 1_l); j, 0)$. Take $u \in (\mathbb{C}^l)^{\mathbf{a}}$ by

$$u = (\tilde{z}\ 1_l) \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \mu\epsilon\kappa_2 \end{pmatrix}$$

for $\kappa_1, \kappa_2 \in \mathbb{Q}^{lg}$. Then we have

$$\begin{aligned}
(3.6) \qquad \tilde{f}_*(u) &= \exp(-\pi\sqrt{-1}\,{}^t\kappa_1\mu\epsilon\kappa_2) \\
&\quad \times \theta^{(lg)}(0, \varepsilon_B^{(l)}(\tilde{z}); j + \kappa_1, \mu\epsilon\kappa_2)/\theta^{(lg)}(0, \varepsilon_B^{(l)}(\tilde{z}); p_1', 0) \\
&= \Big[\exp\ (-\pi\sqrt{-1}\,{}^t\kappa_1\mu\epsilon\kappa_2') \\
&\quad \times \theta^{(lg)}(0, \varepsilon_B^{(l)}(z'); j + \kappa_1, \mu\epsilon\kappa_2')/\theta^{(lg)}(0, \varepsilon_B^{(l)}(z'); p_1', 0)\Big]^{\sigma} \\
&= \left[ \left( \varphi_{F,B}^{(l)} \right)_* (u', (z'\ 1_l); j, 0) \right]^{\sigma}.
\end{aligned}$$

Here $\kappa_2' \in \mathbb{Q}^{lg}$ so that $\kappa_2' \equiv \chi(\sigma)_p^{-1}\kappa_2 \bmod (\mathbb{Z}_p)^{lg}$, ${}^t\kappa_1\mu\epsilon(\kappa_2' - \chi(\sigma)_p^{-1}\kappa_2) \in 2(\mathbb{Z}_p)^{lg}$ for each finite prime $p$, where $\chi(\sigma)_p$ denotes the $p$-component of $\chi(\sigma)$, and

$$\varphi_Y^{(l)}(z') = \left[ J_{Y\tilde{Y}}^{(l)} \left( \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix} \right) \left( \varphi_{\tilde{Y}}^{(l)}(\tilde{z}) \right) \right]^{\sigma^{-1}},$$

$$u' = (z'\ 1_l) \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \mu\epsilon\kappa_2' \end{pmatrix},$$

for $\tilde{Y}, Y \in \mathcal{Z}^{(l)}$, $Y = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix} \tilde{Y} \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1}$ such that $Y \subset X$ and the first line of the right hand side of (3.6) is defined as a rational function on $V_{\tilde{Y}}^{(l)}$ (viewed as a modular function of $\tilde{z}$). Now $\Gamma_X^{(l)}(z') = \Gamma_X^{(l)}(z)$. This implies $\Gamma^{(l)}(z') = \Gamma^{(l)}(z)$. Hence the images of $(\mathbb{C}^l)^{\mathbf{a}}$ by the mappings (to the $(\mu^{lg}\det(\epsilon) - 1)$-dimensional projective space)

$$(3.7) \qquad u \to \left[ \varphi_{F,B}^{(l)}(u, (\tau_1^{\Psi}\ \tau_2^{\Psi}); j, 0) \right]_{j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}}$$

and

$$u \to \left[ \varphi_{F,B}^{(l)}(u, (z'\ 1_l); j, 0) \right]_{j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}}$$

are the same abelian varieties (and of course the images of $(K^l)^{\Psi}$ are their subgroups of all torsion elements). We denote this abelian variety by $A$. Then (3.6) means that the image of $(\mathbb{C}^l)^{\mathbf{a}}$ by

$$u \to \left[ \varphi_{F,B}^{(l)}(u, (\tilde{z}\ 1_l); j, 0) \right]_{j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}}$$

is $A^\sigma$.

In the commutative diagram (3.1), we view

$$u \to \left[\varphi_{F,B}^{(l)}(u, (\tau_1^\Psi\ \tau_2^\Psi); j, 0)\right]_{j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}}$$

as $\Theta$. For $X \in \mathcal{Z}^{(l)}$ as above, take

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} \left(X \cap G_{1A}^{(l)}\right) G_\infty^{(l)} \Phi_z^{(l)}(a) \cap G^{(l)}(\mathbb{Q}),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_l^l$. This is possible from the strong approximation property of $G_1^{(l)}$. Then we have

$$\begin{aligned} aL &= (\tau_1{}^t\alpha_1 + \tau_2{}^t\alpha_2 \quad \tau_1{}^t\alpha_3 + \tau_2{}^t\alpha_4) \\ &\quad \cdot \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{lg} \\ \mu\epsilon\mathbb{Z}^{lg} \end{pmatrix}. \end{aligned}$$

For any $\kappa_1, \kappa_2 \in \mathbb{Q}^{lg}$, we can take (sufficiently small) $X \in \mathcal{Z}^{(l)}$ and corresponding $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ above such that

$$\begin{aligned} &\Theta_a \left((\tau_1{}^t\alpha_1 + \tau_2{}^t\alpha_2 \quad \tau_1{}^t\alpha_3 + \tau_2{}^t\alpha_4)^{\Psi\sigma}\right. \\ &\quad \left.\cdot \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \mu\epsilon\kappa_2 \end{pmatrix}\right) \\ &= \left[\left(\varphi_{F,B}^{(l)}\right)_* \left((\tau_1^\Psi\ \tau_2^\Psi) \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \mu\epsilon\kappa_2' \end{pmatrix},\right.\right. \\ &\quad \left.\left.(\tau_1^\Psi\ \tau_2^\Psi); j, 0\right)^\sigma\right]_{j \in \mu^{-1}\epsilon^{-1}\mathbb{Z}^{lg}/\mathbb{Z}^{lg}} \end{aligned}$$

where $\kappa_2' \in \mathbb{Q}^{lg}$ and $\kappa_2' \equiv \chi(\sigma)_p^{-1}\kappa_2 \bmod (\mathbb{Z}_p)^{lg}$ for any finite prime $p$. This fact can be easily verified from (3.1). Consider the structure of polarized abelian variety $(A, \mathcal{C}, \iota, \{t_i\}_{i=1}^r)$ stated in [9], where $\{t_i\}_{i=1}^r$ is a set of torsion elements of $A$. For any $\left\{\begin{pmatrix} \kappa_{1,i} \\ \mu\epsilon\kappa_{2,i} \end{pmatrix}\right\}_{i=1}^r \subset \mathbb{Q}^{2lg}$, (3.6) implies that we can choose (sufficiently small) $X \in \mathcal{Z}^{(l)}$ and corresponding $\tilde{z}$ so that

$$\begin{aligned} &\left(\varphi_{F,B}^{(l)}\right)_* \left((\tilde{z}\ 1_l) \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_{1,i} \\ \mu\epsilon\kappa_{2,i} \end{pmatrix},\right. \\ &\quad \left.(\tilde{z}\ 1_l); j, 0\right) \\ &= \left[\left(\varphi_{F,B}^{(l)}\right)_* \left((\tau_1^\Psi\ \tau_2^\Psi) \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1' 1_l & \cdots & \beta_g' 1_l \end{pmatrix} \begin{pmatrix} \kappa_{1,i} \\ \mu\epsilon\kappa_{2,i}' \end{pmatrix},\right.\right. \\ &\quad \left.\left.(\tau_1^\Psi\ \tau_2^\Psi); j, 0\right)\right]^\sigma \end{aligned}$$

where $\kappa'_{2,i} \equiv \chi(\sigma)_p^{-1}\kappa_{2,i} \bmod (\mathbb{Z}_p)^{lg}$, ${}^t\kappa_{1,i}\mu\epsilon(\kappa'_{2,i} - \chi(\sigma)_p^{-1}\kappa_{2,i}) \in 2(\mathbb{Z}_p)^{lg}$ for each $i$. Assume that $(A, \mathcal{C}, \iota, \{t_i\}_{i=1}^r)$ is of type

$$\left(F^{2l}, \begin{pmatrix} 0 & -1_l \\ 1_l & 0 \end{pmatrix}, \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta'_1 1_l & \cdots & \beta'_g 1_l \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{lg} \\ \mu\epsilon\mathbb{Z}^{lg} \end{pmatrix}, \right.$$
$$\left. \left\{ \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta'_1 1_l & \cdots & \beta'_g 1_l \end{pmatrix} \begin{pmatrix} \kappa_{1,i} \\ \mu\epsilon\kappa'_{2,i} \end{pmatrix} \right\}_{i=1}^r \right)$$

with respect to $z$ in the sense of [9]. Then $(A, \mathcal{C}, \iota, \{t_i\}_{i=1}^r)^\sigma$ is of type

$$\left(F^{2l}, \begin{pmatrix} 0 & -1_l \\ 1_l & 0 \end{pmatrix}, \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta'_1 1_l & \cdots & \beta'_g 1_l \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{lg} \\ \mu\epsilon\mathbb{Z}^{lg} \end{pmatrix}, \right.$$
$$\left. \left\{ \begin{pmatrix} \beta_1 1_l & \cdots & \beta_g 1_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta'_1 1_l & \cdots & \beta'_g 1_l \end{pmatrix} \begin{pmatrix} \kappa_{1,i} \\ \mu\epsilon\kappa_{2,i} \end{pmatrix} \right\}_{i=1}^r \right)$$

with respect to $\tilde{z}$ and $((\alpha_1\tau + \alpha_2)(\alpha_3\tau + \alpha_4)^{-1})^{\Psi\sigma}$ (if we take sufficiently small $X$). Take $\{t_i\}_{i=1}^r$ sufficiently large so that $\Gamma^{(l)}(((\alpha_1\tau + \alpha_2)(\alpha_3\tau + \alpha_4)^{-1})^{\Psi\sigma}) = \Gamma^{(l)}(\tilde{z})$. Then we can take

$$f^{(\sigma,\Psi,a)}(u) = \varphi^{(l)}_{F,B}(u, (\tau_1{}^t\alpha_1 + \tau_2{}^t\alpha_2 \quad \tau_1{}^t\alpha_3 + \tau_2{}^t\alpha_4)^{\Psi\sigma}; j, 0). \qquad \square$$

**Remark.** This theorem holds more generally for the case when $H_{T,\Psi}$ is semi-positive definite (hence even if $T$ is degenerate). To see this, decompose $K^l = V_1 \bigoplus V_2$ with $K$-vector spaces $V_1, V_2$, such that $T$ is non-degenerate on $V_1$ and zero on $V_2$. Then we see that $f$ is $(V_2 \cap L)^\Psi$-periodic and hence $V_2 \otimes_{\mathbb{Q}} \mathbb{R}$-invariant for any $f \in \mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi})$. Therefore $\mathfrak{T}((\mathbb{C}^l)^{\mathbf{a}}, L^\Psi, H_{T,\Psi})$ can be identified with $\mathfrak{T}(V_1 \otimes_{\mathbb{Q}} \mathbb{R}, (V_1 \cap L)^\Psi, H_{T,\Psi}|_{V_1 \otimes_{\mathbb{Q}} \mathbb{R}})$.

In this proof we get the following proposition.

**Proposition 3.2.** *Take any* $z = \tau^\Psi \in W^{(l)}(\Psi)$ $(\tau \in K_l^l,\ {}^t\tau = \tau)$ *and* $\sigma \in \mathrm{Aut}(\mathbb{C})$. *For any* $X \in \mathcal{Z}^{(l)}$, *put*

$$\tilde{X} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} X \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix} \quad \in \mathcal{Z}^{(l)}$$

*and*

$$\left[ J^{(l)}_{\tilde{X}X}\left( \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} \right) \left( \varphi^{(l)}_X(z) \right) \right]^\sigma = \varphi^{(l)}_{\tilde{X}}(\tilde{z}).$$

*Then we have* $\tilde{z} \in W^{(l)}(\Psi\sigma)$ *and*

$$\varphi^{(l)}_{\tilde{X}}(\tilde{z}) = \varphi^{(l)}_{\tilde{X}}\left( \left( (\alpha_1\tau + \alpha_2)(\alpha_3\tau + \alpha_4)^{-1} \right)^{\Psi\sigma} \right),$$

*where* $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in G^{(l)}(\mathbb{Q}) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_l^l)$ *such that*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} (X \cap G_{1A}^{(l)}) G_\infty^{(l)} \Phi_z^{(l)}(a) \cap G^{(l)}(\mathbb{Q}),$$

*with* $a \in g_\Psi(\sigma)$.

The $\mathfrak{T}_{(r,S,\Psi)}(L)$ in Section 1 is clearly equal to $\mathfrak{T}((\mathbb{C}_q^n)^{\mathbf{a}}, L^\Psi, H_{r,S,\Psi})$. Hence we can consider the action of $(\sigma, \Psi, a) \in C_\Psi(\mathbb{C})$ on each Fourier-Jacobi coefficient of modular forms.

Take $\Xi \in \mathcal{M}_{2\cdot\mathbf{1}}^{(q,n)}(S, \Psi)$ by

$$\Xi(\mathfrak{z}) = \left(\theta_{F,B}^{(n)}(0, (-S^{-1})^\Psi; 0, 0)^{-1} \theta_{F,B}^{(m)}(0, \varepsilon_\delta(\mathfrak{z}); p_1, 0)\right)^2 \tag{3.8}$$

for $p_1 \in \mathbb{Q}^{mg}$. Note that $\theta_{F,B}^{(n)}(0, (-S^{-1})^\Psi; 0, 0) \neq 0$. Consider the Fourier-Jacobi expansion of $\Xi$. Put

$$M_{B,\delta} = (\beta_1 1_q \cdots \beta_g 1_q \ (\delta\beta_1) 1_q \cdots (\delta\beta_g) 1_q).$$

Then by a formal calculation, we have

$$\begin{aligned} \Xi(\mathfrak{z}) = \Bigg\{ \sum_{x_1, x_3 \in \mathbb{Z}^{qg}} & \varphi_{F,B}^{(n)} \left( w M_{B,\delta}^\Psi \begin{pmatrix} x_1 + p_{1,1} \\ x_3 + p_{1,3} \end{pmatrix}, \left((-S^{-1})^\Psi \ 1_q\right); p_{1,2}, 0 \right) \\ & \times \mathbf{e}_{\mathbf{a}} \left( \mathrm{tr} \left( r_{(x_1+p_{1,1}),(x_3+p_{1,3})}^\Psi z \right) \right) \Bigg\}^2, \end{aligned}$$

where $p_{1,1} \in \mathbb{Q}^{qg}$ (resp. $p_{1,2} \in \mathbb{Q}^{ng}, p_{1,3} \in \mathbb{Q}^{qg}$) denotes the $\mathbb{Q}$-coefficients column vector consisting of the $1, \ldots, q, m+1, \ldots, m+q, 2m+1, \ldots, (g-1)m+q$-th components (resp. $q+1, \ldots, q+n, m+q+1, \ldots, (g-1)m+q+n$-th components, $q+n+1, \ldots, m, m+q+n+1, \ldots, gm$-th components) of $p_1$ and

$$r_{(x_1+p_{1,1}),(x_3+p_{1,3})} = \frac{1}{2} M_{B,\delta} \begin{pmatrix} x_1 + p_{1,1} \\ x_3 + p_{1,3} \end{pmatrix} {}^t \begin{pmatrix} x_1 + p_{1,1} \\ x_3 + p_{1,3} \end{pmatrix} {}^t M_{B,\delta}^\rho.$$

Note that $\Xi \in \mathcal{M}_{2\cdot\mathbf{1}}^{(q,n)}(S, \Psi)(K_{\Psi ab}^*)$. Set simply

$$\Xi(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} c_r(w) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}(r^\Psi z)).$$

Now $\varphi_{F,B}^{(n)} \left( w M_{B,\delta}^\Psi \begin{pmatrix} x_1 + p_{1,1} \\ x_3 + p_{1,3} \end{pmatrix}, \left((-S^{-1})^\Psi \ 1_q\right); p_{1,2}, 0 \right)$ is a theta function with respect to a degenerate (in case $q > 1$) hermitian form. By the proof of

Theorem 3.1 and a formal calculation, the action of $(\sigma, \Psi, a) \in C_\Psi(\mathbb{C})$ to it is (up to $(K^*_{\Psi\sigma ab})^\times$-times)

$$\varphi^{(n)}_{F,B}\Bigg(wM^{\Psi\sigma}_{B,\delta}\begin{pmatrix} x_1 + p_{1,1} \\ x_3 + p_{1,3}\end{pmatrix}, \\ \big((-S^{-1})^{\Psi\sigma\, t}\alpha_1 + {}^t\alpha_2 \quad (-S^{-1})^{\Psi\sigma\, t}\alpha_3 + {}^t\alpha_4\big)\, ; p_{1,2}, 0\Bigg)$$

where $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \begin{pmatrix} 1_n & 0 \\ 0 & \chi(\sigma)1_n \end{pmatrix}^{-1} (X \cap G^{(n)}_{1A})G^{(n)}_\infty \Phi^{(n)}_{(-S^{-1})^\Psi}(a) \cap G^{(n)}(\mathbb{Q})$ for a sufficiently small $X \in \mathcal{Z}^{(n)}$ $(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F^n_n)$. Note that $\nu(\alpha) = \iota(\sigma, a)^{-1}$. For such $\alpha$, take $\alpha' \in G^{(m)}_1(\mathbb{Q})$ by

$$\alpha' = \begin{pmatrix} 1_q & & & & & \\ & \iota(\sigma,a)\alpha_1 & & & \alpha_2 & \\ & & 1_q & & & \\ & & & 1_q & & \\ & \iota(\sigma,a)\alpha_3 & & & \alpha_4 & \\ & & & & & 1_q \end{pmatrix}$$

and consider

$$\Big[\theta^{(n)}_{F,B}(0, \big[(\alpha_1(-S^{-1}) + \alpha_2)(\alpha_3(-S^{-1}) + \alpha_4)^{-1}\big]^{\Psi\sigma}; p'_1, 0)^{-1} \\ \theta^{(m)}_{F,B}(0, \alpha'\left(\varepsilon^{(q,n)}_\delta\left(\iota(\sigma,a)S, \Psi\sigma\right)(\tilde{\mathfrak{z}})\right); p_1, 0)\Big]^2, \\ \left(\tilde{\mathfrak{z}} \in D^{(q,n)}(\iota(\sigma,a)S, \Psi\sigma)\right). \tag{3.9}$$

It is contained in $\mathcal{M}^{(q,n)}_{2\cdot\mathbf{1}}(\iota(\sigma,a)S, \Psi\sigma)$ since $\det(j^{(m)}_v(\alpha', \varepsilon_\delta(\tilde{\mathfrak{z}})))$ $(v \in \mathbf{a})$ are constants. By a formal calculation, its Fourier-Jacobi expansion is

$$\sum_{0 \le r \in \mathcal{H}_q} c^{(\sigma,\Psi,a)}_r(\tilde{w})\mathbf{e}_\mathbf{a}(\mathrm{tr}(r^{\Psi\sigma}\tilde{z})), \quad \tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} \in D^{(q,n)}\left(\iota(\sigma,a)S, \Psi\sigma\right). \tag{3.10}$$

We denote this modular form by $\Xi^{(\sigma,\Psi,a)}$. For each $\mathfrak{z} \in D^{(q,n)}(S, \Psi)$, we can choose $\Xi$ such that $\Xi(\mathfrak{z}) \neq 0$ by taking a suitable $p_1$ in (3.8).

As stated in [5] (or in [8]), we can define a $\mathbb{C}^{mg}_{mg}$-valued holomorphic function $T$ on $\mathfrak{H}_{mg}$ by

$$T(Z) \\ = \frac{\theta^{(mg)}(0, Z; p^{(0)}_1, 0)}{2\pi\sqrt{-1}} \left.\begin{pmatrix} \frac{\partial}{\partial u_1}\theta^{(mg)}(u, Z; p^{(1)}_1, 0) & \cdots & \frac{\partial}{\partial u_1}\theta^{(mg)}(u, Z; p^{(mg)}_1, 0) \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial u_{mg}}\theta^{(mg)}(u, Z; p^{(1)}_1, 0) & \cdots & \frac{\partial}{\partial u_{mg}}\theta^{(mg)}(u, Z; p^{(mg)}_1, 0) \end{pmatrix}\right|_{u=0}$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_{mg} \end{pmatrix} \in \mathbb{C}^{mg}$ and $p^{(0)}_1, \ldots p^{(mg)}_1 \in \mathbb{Q}^{mg}$. For each $Z \in \mathfrak{H}_{mg}$, we can take suitable $p^{(0)}_1, \ldots p^{(mg)}_1$ such that $\det T(Z) \neq 0$. As stated in [5], $T$ is a vector-valued modular form which satisfies

$$T(\gamma(Z)) = \det(\gamma_3 Z + \gamma_4) \cdot (\gamma_3 Z + \gamma_4) T(Z)$$

for $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}$ contained in a certain congruence subgroup of $\mathrm{Sp}(mg, \mathbb{Q})$ $(\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{Q}^{mg}_{mg})$. Using this $T$, we can define a $\mathbb{C}^{mg}_{mg}$-valued holomorphic function $\Delta$ on $\mathfrak{H}^{\mathbf{a}}_m$ by

$$\Delta(z) = B^{(m)} T(\varepsilon^{(m)}_B(z)) \qquad (z \in \mathfrak{H}^{\mathbf{a}}_m).$$

By (2.9), there exists a congruence subgroup $\Gamma^{(m)}$ of $G^{(m)}_1(\mathbb{Q})$ such that

(3.11)

$$\Delta(\gamma(z)) = \left( \prod_{v \in \mathbf{a}} \det(j^{(m)}_v(\gamma, z)) \right) \begin{pmatrix} j^{(m)}_{v_1}(\gamma, z) & & \\ & \ddots & \\ & & j^{(m)}_{v_g}(\gamma, z) \end{pmatrix} \Delta(z)$$

for any $\gamma \in \Gamma^{(m)}$. Next define a $\mathbb{C}^{mg}_{mg}$-valued holomorphic function $\hat{\Delta}$ on $D^{(q,n)}(S, \Psi)$ by

$$\hat{\Delta}(\mathfrak{z}) = \theta^{(n)}_{F,B}(0, (-S^{-1})^{\Psi}; p'_1, 0)^{-2} \begin{pmatrix} \omega_{v_1}(\mathfrak{z})^{-1} & & \\ & \ddots & \\ & & \omega_{v_g}(\mathfrak{z})^{-1} \end{pmatrix} \Delta(\varepsilon_\delta(\mathfrak{z})).$$

Then by (2.6), there exists a congruence subgroup $\Gamma$ of $G(\mathbb{Q}) = G^{(q,n)}(S, \Psi)(\mathbb{Q})$ such that

(3.12) $\hat{\Delta}(\alpha(\mathfrak{z}))$

$$= \left( \prod_{v \in \mathbf{a}} \det(\mu_v(\alpha, \mathfrak{z}))^2 \right) \begin{pmatrix} \lambda_{v_1}(\alpha, \mathfrak{z}) & & & & \\ & \mu_{v_1}(\alpha, \mathfrak{z}) & & & \\ & & \ddots & & \\ & & & \lambda_{v_g}(\alpha, \mathfrak{z}) & \\ & & & & \mu_{v_g}(\alpha, \mathfrak{z}) \end{pmatrix} \hat{\Delta}(\mathfrak{z})$$

for any $\alpha \in \Gamma$ ( $\mathfrak{z} \in D^{(q,n)}(S, \Psi)$ ). For each $v_k$ $(1 \leq k \leq g)$, take $Q \in \mathbb{Q}^{mg}_q$ and set

$$\xi_{v_k}(\mathfrak{z}) = \det \left( (\overbrace{0 \cdots 0}^{mk-q} 1_q \overbrace{0 \cdots 0}^{m(g-k)}) \hat{\Delta}(\mathfrak{z}) Q \right).$$

Then (3.12) implies that $\xi_v \in \mathcal{M}^{(q,n)}_{v+2q\cdot\mathbf{1}}(S, \Psi)$ for any $v \in \mathbf{a}$. At each $\mathfrak{z} \in D$, we can choose suitable $T$ and $Q$ so that $\xi_v(\mathfrak{z}) \neq 0$ for any $v \in \mathbf{a}$.

Let us consider the Fourier-Jacobi series of $\xi_v$ . For $p_1^{(0)}, \dots, p_1^{(mg)}$, take $p_{1,1}^{(i)}, p_{1,3}^{(i)} \in \mathbb{Q}^{qg}$ and $p_{1,2}^{(i)} \in \mathbb{Q}^{ng}$ $(0 \le i \le mg)$ as above. Then we have

$$\xi_v(\mathfrak{z}) = \left[ \sum_{x_1, x_3 \in \mathbb{Z}^{qg}} \varphi_{F,B}^{(n)} \left( w M_{B,\delta}^{\Psi} \begin{pmatrix} x_1 + p_{1,1}^{(0)} \\ x_3 + p_{1,3}^{(0)} \end{pmatrix}, \left( (-S^{-1})^{\Psi}\ 1_q \right); p_{1,2}^{(0)}, 0 \right) \right.$$
$$\left. \times \mathbf{e_a}(\mathrm{tr}(r^{\Psi}_{(x_1+p_{1,1}^{(0)}),(x_3+p_{1,3}^{(0)})} z)) \right]^q$$
$$\times \det \left( \left( \sum_{x_1, x_3 \in \mathbb{Z}^{qg}} \varphi_{F,B}^{(n)} \left( w M_{B,\delta}^{\Psi} \begin{pmatrix} x_1 + p_{1,1}^{(i)} \\ x_3 + p_{1,3}^{(i)} \end{pmatrix}, \left( (-S^{-1})^{\Psi}\ 1_q \right); p_{1,2}^{(i)}, 0 \right) \right. \right.$$
$$\left. \left. \times \mathbf{e_a}(\mathrm{tr}(r^{\Psi}_{(x_1+p_{1,1}^{(i)}),(x_3+p_{1,3}^{(i)})} z)) (\delta^{-1} M_{B,\delta})^{\Psi_v} \begin{pmatrix} x_1 + p_{1,1}^{(i)} \\ x_3 + p_{1,3}^{(i)} \end{pmatrix} \right)_{1 \le i \le mg} Q \right),$$

where

$$r_{(x_1+p_{1,1}^{(i)}),(x_3+p_{1,3}^{(i)})} = \frac{1}{2} M_{B,\delta} \begin{pmatrix} x_1 + p_{1,1}^{(i)} \\ x_3 + p_{1,3}^{(i)} \end{pmatrix} {}^t \begin{pmatrix} x_1 + p_{1,1}^{(i)} \\ x_3 + p_{1,3}^{(i)} \end{pmatrix} {}^t M_{B,\delta}^{\rho} \qquad (0 \le i \le mg).$$

This implies $\xi_v \in \mathcal{M}_{v+2q\cdot\mathbf{1}}^{(q,n)}(S, \Psi)(K^{\Psi_v} \vee K^*_{\Psi ab})$. Put simply

$$\xi_v(\mathfrak{z}) = \sum_{0 \le r \in \mathcal{H}_q} b_{v,r}(w) \mathbf{e_a}(\mathrm{tr}(r^{\Psi} z)), \qquad \left( \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in D^{(q,n)}(S, \Psi) \right).$$

Now for any $(\sigma, \Psi, a) \in C_{\Psi}(\mathbb{C})$, take $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in G^{(n)}(\mathbb{Q})$ $(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_n^n)$ so that

$$\alpha \in \begin{pmatrix} 1_n & 0 \\ 0 & \chi(\sigma) 1_n \end{pmatrix}^{-1} (X \cap G_{1A}^{(n)}) G_{\infty}^{(n)} \Phi^{(n)}_{(-S^{-1})^{\Psi}}(a) \cap G^{(n)}(\mathbb{Q}),$$

where $X \in \mathcal{Z}^{(n)}$ satisfies the following condition.

$\varphi_{F,B}^{(n)}(u, (\omega_1\ \omega_2)^t \gamma; p_{1,2}^{(i)}, 0) = \varphi_{F,B}^{(n)}(u, (\omega_1\ \omega_2); p_{1,2}^{(i)}, 0)$
*for any* $(\omega_1\ \omega_2)$ *and any*
$\gamma \in \begin{pmatrix} 1_n & 0 \\ 0 & \chi(\sigma) 1_n \end{pmatrix}^{-1} (X \cap G_{1A}^{(n)}) G_{\infty}^{(n)} \begin{pmatrix} 1_n & 0 \\ 0 & \chi(\sigma) 1_n \end{pmatrix} \cap G^{(n)}(\mathbb{Q})$
$(0 \le i \le mg)$.

Consider the $\mathbb{C}^{mg}_{mg}$-valued function $\tilde{\hat{\Delta}}$ on $D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ by

$$\tilde{\hat{\Delta}}(\tilde{\mathfrak{z}})=\theta^{(n)}_{F,B}(0,[(\alpha_1(-S^{-1})+\alpha_2)(\alpha_3(-S^{-1})+\alpha_4)^{-1}]^{\Psi\sigma};p'_1,0)^{-2}$$
$$\times\begin{pmatrix}\omega_{v_1}(\tilde{\mathfrak{z}})^{-1} & & \\ & \ddots & \\ & & \omega_{v_g}(\tilde{\mathfrak{z}})^{-1}\end{pmatrix}\begin{pmatrix}j^{(m)}_{v_1}(\alpha',\varepsilon_\delta(\tilde{\mathfrak{z}})) & & \\ & \ddots & \\ & & j^{(m)}_{v_g}(\alpha',\varepsilon_\delta(\tilde{\mathfrak{z}}))\end{pmatrix}^{-1}$$
$$\times\,\Delta(\alpha'(\varepsilon_\delta(\tilde{\mathfrak{z}}))),$$

where

$$\alpha'=\begin{pmatrix}1_q & & & & & \\ & \iota(\sigma,a)\alpha_1 & & & \alpha_2 & \\ & & 1_q & & & \\ & & & 1_q & & \\ & \iota(\sigma,a)\alpha_3 & & & \alpha_4 & \\ & & & & & 1_q\end{pmatrix}.$$

Note that $\varepsilon_\delta$ means $\varepsilon^{(q,n)}_\delta(\iota(\sigma,a)S,\Psi\sigma)$ and $\omega_v$ is corresponding to $\varepsilon^{(q,n)}_\delta($ $\iota(\sigma,a)S,\Psi\sigma)$ in this case. In the same way put

$$\tilde{\xi_{v_k}}(\tilde{\mathfrak{z}})=\det\left(\left((\overbrace{0\cdots0}^{mk-q}\,1_q\,\overbrace{0\cdots0}^{m(g-k)})\right)\tilde{\hat{\Delta}}(\tilde{\mathfrak{z}})Q\right).$$

Taking $T$ and $Q$ equal to those of $\xi_v$, the Fourier-Jacobi expansion of $\tilde{\xi_{v\sigma}}$ is

$$\sum_{0\le r\in\mathcal{H}_q}b^{(\sigma,\Psi,a)}_{v,r}(\tilde{w})\mathbf{e}_{\mathsf{a}}(\mathrm{tr}(r^{\Psi\sigma}\tilde{z})),\qquad \left(\tilde{\mathfrak{z}}=\begin{pmatrix}\tilde{z}\\ \tilde{w}\end{pmatrix}\in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)\right).$$

We denote this $\tilde{\xi_{v\sigma}}$ by $\xi^{(\sigma,\Psi,a)}_v$. Clearly

$$\xi^{(\sigma,\Psi,a)}_v\in\mathcal{M}_{v\sigma+2q\cdot\mathbf{1}}(\iota(\sigma,a)S,\Psi\sigma).$$

## 4. The embeddings of canonical models

In this section we consider the relation of the embedding $\varepsilon_\delta=\varepsilon^{(q,n)}_\delta(S,\Psi)$ (which was defined in Section 2) and the canonical models.

Take $Y\in\mathcal{Z}=\mathcal{Z}^{(q,n)}(S,\Psi)$ and $T\in\mathcal{Z}^{(m)}$ so that $I_\delta(Y)\subset T$. Assume that $F^\times\Gamma^{(m)}_T/F^\times$ is torsion free. Note that any element (except scalar) of $\Gamma^{(m)}_T$ has no fixed points. Then any element (except scalar) of $\Gamma_Y=Y\cap G(\mathbb{Q})$ also has no fixed points and hence $K^\times\Gamma_Y/K^\times$ is also torsion free. In this case $\varphi_Y,\varphi^{(m)}_T$ are locally biholomorphic, $V_Y,V^{(m)}_T$ are non-singular, and we can define a unique rational map $E_{TY}$ of $V_Y$ into $V^{(m)}_T$ so that $E_{TY}\circ\varphi_Y=\varphi^{(m)}_T\circ\varepsilon_\delta$. The mapping $E_{TY}$ is regular on $V_Y$, and is defined over $K^*_{\Psi ab}$. The Zariski closure $\overline{E_{TY}(V_Y)}$ of $E_{TY}(V_Y)$ in $V^{(m)}_T$ is a subvariety of $V^{(m)}_T$, which may have singular points in general. The purpose of this section is to prove the following theorem.

**Theorem 4.1.** *Let $T \in \mathcal{Z}^{(m)}$ and assume that $T$ satisfies the following conditions* (i) *and* (ii)*.*

(i) $F^\times \Gamma_T^{(m)}/F^\times$ *is torsion free.*

(ii) $C(S,\delta)^{-1}TC(S,\delta) \subset \prod_{v\in\mathbf{a}} \mathrm{GL}(2m,\mathbb{C}) \times \prod_{\mathfrak{p}} \mathrm{GL}(2m,\mathcal{O}_\mathfrak{p})$*, where $\mathfrak{p}$ runs over all non-archimedean primes of $K$.*

*Put $Y = I_\delta^{-1}(T) \in \mathcal{Z}$. Then we can take $\hat{T} \in \mathcal{Z}^{(m)}$ satisfying* (1)–(3)*.*

(1) $I_\delta(Y) \subset \hat{T} \subset T$*.*

(2) $E_{\hat{T}Y}(V_Y)$ *is a non-singular subvariety of* $V_{\hat{T}}^{(m)}$*.*

(3) $E_{\hat{T}Y}$ *is a* (*set theorically*) *injective map on $V_Y$ and its inverse rational map $E_{\hat{T}Y}^{-1}$ is regular on $E_{\hat{T}Y}(V_Y)$.*

**Remark.** For any $\hat{T}_0 \in \mathcal{Z}^{(m)}$ satisfying $I_\delta(Y) \subset \hat{T}_0 \subset \hat{T}$, the assertions (1)–(3) are still valid even if replacing $\hat{T}$ by $\hat{T}_0$. This is because $E_{\hat{T}Y} = J_{\hat{T}\hat{T}_0}^{(m)}(1_{2m}) \circ E_{\hat{T}_0Y}$ and so $E_{\hat{T}_0Y}^{-1} = E_{\hat{T}Y}^{-1} \circ J_{\hat{T}\hat{T}_0}^{(m)}(1_{2m})'$, where " $'$" means the restriction of $J_{\hat{T}\hat{T}_0}^{(m)}(1_{2m})$ to $E_{\hat{T}_0Y}(V_Y)$.

*Proof.* From now on till the end of this section, all the varieties and rational maps are defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. So the word "generic" means generic over $\overline{\mathbb{Q}}$. As is well known, every algebraic set defined over $\overline{\mathbb{Q}}$ is a finite union of varieties defined over $\overline{\mathbb{Q}}$.

For any positive integer $N$, put

$$T_N = \left\{ x \in T \;\middle|\; \begin{array}{l} C(S,\delta)^{-1}x_\mathfrak{p}C(S,\delta) \equiv \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \bmod (N\mathcal{O}_\mathfrak{p})_{2m}^{2m} \\ \text{with some } b_1, b_2 \in (\mathcal{O}_\mathfrak{p})_m^m \text{ for any finite prime } \mathfrak{p} \text{ of } K \end{array} \right\},$$

where $x_\mathfrak{p}$ denotes the $\mathfrak{p}$-component of $x$. Then $T_N \in \mathcal{Z}^{(m)}$ and $T_N$ is a subgroup of $T$. From the definition of $I_\delta$, we have

$$I_\delta(Y) = \bigcap_{N\in\mathbb{N}} T_N, \tag{4.1}$$

$$I_\delta(\Gamma_Y) = \bigcap_{N\in\mathbb{N}} \Gamma_{T_N}^{(m)}. \tag{4.2}$$

For each $P \in V_T^{(m)}$, $E_{TY}^{-1}(P) = \varphi_Y \circ (\varepsilon_\delta^{-1}((\varphi_T^{(m)})^{-1}(P)))$ is at most countable, and clearly an algebraic subset of $V_Y$. Hence $E_{TY}^{-1}(P)$ is a finite set.

Take $Q \in V_Y$, and put $\{Q_1, \dots, Q_l\} = E_{TY}^{-1}(E_{TY}(Q))$. It is clear that $\Gamma_T^{(m)}(\varepsilon_\delta(\varphi_Y^{-1}(Q_1))), \dots, \Gamma_T^{(m)}(\varepsilon_\delta(\varphi_Y^{-1}(Q_l)))$ are the same elements in $\Gamma_T^{(m)} \backslash \mathfrak{H}_m^\mathbf{a}$, and $\Gamma_Y(\varphi_Y^{-1}(Q_1)), \dots, \Gamma_Y(\varphi_Y^{-1}(Q_l))$ are mutually disjoint. Note that any $\gamma \in \Gamma_T^{(m)}$ (except scalar) has no fixed points in $\mathfrak{H}_m^\mathbf{a}$, since $F^\times\Gamma_T^{(m)}/F^\times$ is torsion free. From (4.2) we can find a positive integer $N$ such that $\Gamma_{T_N}^{(m)}(\varepsilon_\delta(\varphi_Y^{-1}(Q_1)))$, $\dots, \Gamma_{T_N}^{(m)}(\varepsilon_\delta(\varphi_Y^{-1}(Q_l)))$ are all different in $\Gamma_{T_N}^{(m)} \backslash \mathfrak{H}_m^\mathbf{a}$. This means $E_{T_NY}(Q_1)$, $\dots, E_{T_NY}(Q_l)$ are all different in $V_{T_N}^{(m)}$. So we can get the following lemma.

**Lemma 4.2.** *Let $T, Y$ be as in Theorem 4.1. For any $Q \in V_Y$, there exists some $T' \in \mathcal{Z}^{(m)}$ satisfying (i), (ii).*
(i) $I_\delta(Y) \subset T' \subset T$.
(ii) $E_{T'Y}^{-1}(E_{T'Y}(Q)) = \{Q\}$.

Put

$$U(Y,T) = \overline{\{Q \in V_Y | \exists Q' \in V_Y \text{ such that } Q' \neq Q \text{ and } E_{TY}(Q') = E_{TY}(Q)\}}.$$

The overline means the ($\overline{\mathbb{Q}}$-)Zariski closure in $V_T^{(m)}$. Set

$$U(Y,T) = \bigcup_{j=1}^{r} U_j,$$

where $U_j (1 \leq j \leq r)$ are subvarieties of $V_Y$ defined over $\overline{\mathbb{Q}}$, and none of them are contained in the other. Assume that

$$\dim U(Y,T) = \dim U_1 = \cdots = \dim U_t > \dim U_{t+1} \geq \cdots \geq \dim U_r.$$

Let $Q_j \quad (1 \leq j \leq t)$ be generic points of $U_j$ (over $\overline{\mathbb{Q}}$ ). Using Lemma 4.2 ($r$ times repeatedly), we can find $T' \in \mathcal{Z}^{(m)}$ so that $I_\delta(Y) \subset T' \subset T$ and $E_{T'Y}^{-1}(E_{T'Y}(Q_j)) = \{Q_j\} \quad (1 \leq j \leq t)$ hold. Then $\overline{E_{T'Y}(U_j)} \quad (1 \leq j \leq t)$ are subvarieties of $V_{T'}^{(m)}$ whose generic points (over $\overline{\mathbb{Q}}$) are $E_{T'Y}(Q_j)$.

As $E_{T'Y}|_{U_j} \quad (1 \leq j \leq t)$ are generically injective, we can define the inverse rational maps $\left(E_{T'Y}|_{U_j}\right)^{-1} \quad (1 \leq j \leq t)$ on $\overline{E_{T'Y}(U_j)} \quad (1 \leq j \leq t)$ which are regular on some non-empty ($\overline{\mathbb{Q}}$-)Zariski open subsets $X_j$ of $\overline{E_{T'Y}(U_j)} \quad (1 \leq j \leq t)$ (hence regular at any $\overline{\mathbb{Q}}$-generic points). Now the restrictions of $E_{T'Y}$ to $E_{T'Y}^{-1}(X_j)$ are injective. Therefore we have

$$\begin{aligned} U(Y,T') \subset & \bigcup_{t+1 \leq j < r} U_j \\ & \cup \bigcup_{j=1}^{t} E_{T'Y}^{-1} \left[ \left( \overline{E_{T'Y}(U_j)} \setminus X_j \right) \cup \bigcup_{1 \leq k \leq r, k \neq j} \left( \overline{E_{T'Y}(U_k)} \cap \overline{E_{T'Y}(U_j)} \right) \right] \end{aligned}$$

As $\overline{E_{T'Y}(U_k)}$ and $\overline{E_{T'Y}(U_j)}$ are different (since $E_{T'Y}^{-1}(E_{T'Y}(Q_j)) = \{Q_j\}$ $(1 \leq j \leq t)$), we have

$$\dim U(Y,T') < \dim U_1 = \cdots = \dim U_t = \dim U(Y,T).$$

By an induction, we can take some $\hat{T} \in \mathcal{Z}^{(m)}$ such that $I_\delta(Y) \subset \hat{T} \subset T$ and $E_{\hat{T}Y}$ is (set theorically) injective on $V_Y$.

So we can define the inverse rational map $E_{\hat{T}Y}^{-1}$ on $\overline{E_{\hat{T}Y}(V_Y)}$ which is regular on some non-empty Zariski open subset of $\overline{E_{\hat{T}Y}(V_Y)}$. Hence $E_{\hat{T}Y}(V_Y)$ contains some non-empty Zariski open subset of $\overline{E_{\hat{T}Y}(V_Y)}$. This implies $E_{\hat{T}Y}(V_Y)$ is dense in $\overline{E_{\hat{T}Y}(V_Y)}$ with respect to the topology of $V_{\hat{T}}^{(m)}$ as a complex manifold.

Clearly

$$E_{\hat{T}Y}(V_Y) = \bigcup_{\gamma \in \Gamma_{\hat{T}}^{(m)}/I_\delta(\Gamma_Y)} \varphi_{\hat{T}}^{(m)}\left(\gamma \circ \varepsilon_\delta(D)\right).$$

Each $\gamma \circ \varepsilon_\delta(D)$ is a complex submanifold of $\mathfrak{H}_m^{\mathbf{a}}$ of dimension $gq(n+q)$. For $\gamma_1, \gamma_2 \in \Gamma_{\hat{T}}^{(m)}$, if $\gamma_1 I_\delta(\Gamma_Y) \neq \gamma_2 I_\delta(\Gamma_Y)$, then $\gamma_1 \circ \varepsilon_\delta(D) \cap \gamma_2 \circ \varepsilon_\delta(D) = \phi$ (from the injectivity of $E_{\hat{T}Y}$ and as $F^\times \Gamma_{\hat{T}}^{(m)}/F^\times$ is torsion free). Now we have

$$(\varphi_{\hat{T}}^{(m)})^{-1}\left(\overline{E_{\hat{T}Y}(V_Y)}\right) = \overline{\bigcup_{\gamma \in \Gamma_{\hat{T}}^{(m)}/I_\delta(\Gamma_Y)} \gamma \circ \varepsilon_\delta(D)},$$

since $\varphi_{\hat{T}}^{(m)}$ is locally biholomorphic. (The overline in the right hand side means the closure with respect to the topology as a complex analytic space.)

As $(\varphi_{\hat{T}}^{(m)})^{-1}\left(\overline{E_{\hat{T}Y}(V_Y)}\right)$ is a $gq(n+q)$-dimensional analytic set in $\mathfrak{H}_m^{\mathbf{a}}$, and so is each $\gamma \circ \varepsilon_\delta(D) \quad (\gamma \in \Gamma_{\hat{T}}^{(m)}/I_\delta(\Gamma_Y))$, there is no limit point of infinite numbers of different $\gamma \circ \varepsilon_\delta(D) \quad (\gamma \in \Gamma_{\hat{T}}^{(m)}/I_\delta(\Gamma_Y))$. Hence we have

$$(\varphi_{\hat{T}}^{(m)})^{-1}\left(\overline{E_{\hat{T}Y}(V_Y)}\right) = \bigcup_{\gamma \in \Gamma_{\hat{T}}^{(m)}/I_\delta(\Gamma_Y)} \gamma \circ \varepsilon_\delta(D) = (\varphi_{\hat{T}}^{(m)})^{-1}\left(E_{\hat{T}Y}(V_Y)\right).$$

As $\varphi_{\hat{T}}^{(m)}$ is locally biholomorphic,we have $\overline{E_{\hat{T}Y}(V_Y)} = E_{\hat{T}Y}(V_Y)$ and $E_{\hat{T}Y}(V_Y)$ is a non-singular subvariety of $V_{\hat{T}}^{(m)}$. As the Jacobians of $\varphi_{\hat{T}}^{(m)}$ and $\varepsilon_\delta$ are non-zero and $E_{\hat{T}Y}$ is injective, we can define $E_{\hat{T}Y}^{-1}$ to be $\varphi_Y \circ \left(\varphi_{\hat{T}}^{(m)} \circ \varepsilon_\delta\right)^{-1}$ as a holomorphic map on $E_{\hat{T}Y}(V_Y)$, and as $E_{\hat{T}Y}(V_Y)$ is non-singular, $E_{\hat{T}Y}^{-1}$ is regular on $E_{\hat{T}Y}(V_Y)$ as a rational map. $\square$

## 5. Canonical models and arithmeticity

In this section we consider the relation of arithmeticity defined in Section 1, and the canonical models. In case of the modular forms with respect to a symplectic group, we have $\mathfrak{K}^{(l)} = \mathcal{A}_0^{(l)}(\mathbb{Q}_{ab})$ as shown in [9, Section 26.4] (or [3]). In case of unitary similitude $G = G^{(q,n)}(S, \Psi)$, we have the following theorem.

**Theorem 5.1.** *For any subfield $\Omega$ of $\mathbb{C}$ containing $K_{\Psi ab}^*$, we have*

$$\mathcal{A}_0^{(q,n)}(S, \Psi)(\Omega) = \mathfrak{K}^{(q,n)}(S, \Psi) \vee \Omega.$$

*Proof.* (1) *proof of* $\mathcal{A}_0(\Omega) \subset \mathfrak{K} \vee \Omega$.

Take $W^{(q)}(\Psi)$ as in Section 2. For any $f \in \mathcal{A}_0(\Omega)$, write $f = f_1/f_2$ with some $f_1, f_2 \in \mathcal{M}_k(\Omega) \quad (f_2 \not\equiv 0)$. For any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$, we have $f_1 \circ \mathsf{h} \circ \varepsilon_0, f_2 \circ \mathsf{h} \circ \varepsilon_0 \in$

$\mathcal{M}_k^{(q)}(\Omega)$ from Lemma 2.2. Take $W$ as in (1.12). Clearly $W$ is stable under the action of $G(\mathbb{Q})$ and is dense in $D$. Now we have $W = \mathsf{H}(\mathbb{Q})\left(\varepsilon_0(W^{(q)}(\Psi))\right)$. Put

$$W' = \{\mathfrak{z} \in W \,|\, f_2(\mathfrak{z}) \neq 0\}.$$

Clearly $W'$ is dense in $D$, and as $f_1 \circ \mathsf{h} \circ \varepsilon_0 / f_2 \circ \mathsf{h} \circ \varepsilon_0 \in \mathcal{A}_0^{(q)}(\Omega) = \mathfrak{K}^{(q)} \vee \Omega$ if $f_2 \circ \mathsf{h} \circ \varepsilon_0 \not\equiv 0$, we have $f(\mathfrak{z}) \in \Omega$ for any $\mathfrak{z} \in W'$.

As $\mathcal{A}_0(\mathbb{C}) = \mathfrak{K} \vee \mathbb{C}$, we can view $f$ as a rational function on some canonical model. Put $f = p \circ \varphi_Y$ for some $Y \in \mathcal{Z}$ and some rational function $p$ on $V_Y$. Let us prove $p$ is $\Omega$-rational. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\Omega)$, take $p^\sigma$, which is a rational function on $V_Y$. As $\varphi_Y(\mathfrak{z})$ is $K^*_{\Psi ab}$-rational for any $\mathfrak{z} \in W'$, $p^\sigma \circ \varphi_Y(\mathfrak{z}) = (p \circ \varphi_Y(\mathfrak{z}))^\sigma = p \circ \varphi_Y(\mathfrak{z})$ holds for $\mathfrak{z} \in W'$. As $W'$ is dense in $D$, $p$ and $p^\sigma$ must be equal. Hence $p$ is $\Omega$-rational. This means $f \in \mathfrak{K} \vee \Omega$.

(2) *proof of* $\mathcal{A}_0(\Omega) \supset \mathfrak{K} \vee \Omega$.

It suffices to prove $\mathcal{A}_0(K^*_{\Psi ab}) \supset \mathfrak{K}$. For any $f \in \mathfrak{K}$, we can take (sufficiently small) $Y \in \mathcal{Z}$ and $T \in \mathcal{Z}^{(m)}$ as in Theorem 4.1 so that $f \circ \varphi_Y^{-1}$ is a rational function on $V_Y$ defined over $K^*_{\Psi ab}$. Then for $\hat{T} \in \mathcal{Z}^{(m)}$ (in Theorem 4.1), $E_{\hat{T}Y}$ is injective (hence of course generically injective) rational map defined over $K^*_{\Psi ab}$. Therefore $f \circ \varphi_Y^{-1}$ must be a pull-back of a certain rational function on $V_{\hat{T}}^{(m)}$ defined over $K^*_{\Psi ab}$. This rational function can be written in the form $(h_1/h_2) \circ \left(\varphi_{\hat{T}}^{(m)}\right)^{-1}$ with some $h_1, h_2 \in \mathcal{M}_k^{(m)}(K^*_{\Psi ab})$ such that $h_2|_{\varepsilon_\delta(D)} \not\equiv 0$. Since $F(k) \subset K^*_{\Psi ab}$, we obtain $f = (h_1/h_2) \circ \varepsilon_\delta \in \mathcal{A}_0(K^*_{\Psi ab})$ using Lemma 2.5. $\square$

We have the following proposition about the relation of the arithmeticity and the action of $G(\mathbb{Q})$.

**Proposition 5.2.** *Let $k \in \mathbb{Z}^{\mathbf{a}}$. Take any subfield $\Omega$ of $\mathbb{C}$ containing $F(k) \vee K^*_{\Psi ab}$. Then for any $f \in \mathcal{M}_k(\Omega)$ and any $\alpha \in G(\mathbb{Q})$, we have*

$$f|_k\alpha \in \mathcal{M}_k(\Omega).$$

*Proof.* Take any $\mathsf{h} \in \mathsf{H}(\mathbb{Q})$ and consider $f|_k(\alpha\mathsf{h}) \circ \varepsilon_0$. For each $z_0 \in W^{(q)}(\Psi)$, we can take $z_0' \in W^{(q)}(\Psi)$ and $\mathsf{h}' \in \mathsf{H}(\mathbb{Q})$ so that $\alpha\mathsf{h} \circ \varepsilon_0(z_0) = \mathsf{h}' \circ \varepsilon_0(z_0')$ since $W = \mathsf{H}(\mathbb{Q})\left(\varepsilon_0(W^{(q)}(\Psi))\right)$. Then we have

$$(f|_k\alpha\mathsf{h})\,(\varepsilon_0(z_0)) = \left[\prod_{v\in\mathbf{a}} \det\left(\mu_v((\mathsf{h}')^{-1}\alpha\mathsf{h}, \varepsilon_0(z_0))\right)^{-k_v}\right](f|_k\mathsf{h}')(\varepsilon_0(z_0')).$$

Note that $\left[\prod_{v\in\mathbf{a}} \det\left(\mu_v((\mathsf{h}')^{-1}\alpha\mathsf{h}, \varepsilon_0(z_0))\right)^{-k_v}\right] \in F(k) \vee K^*_{\Psi}$. From Lemma 2.2 we obtain $(f|_k\mathsf{h}') \circ \varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega)$. Hence we have $(f|_k\mathsf{h}')(\varepsilon_0(z_0')) \in h^{(1)}(z^{(1)})^q \cdot \Omega$

with $h^{(1)} \in \mathcal{M}_k^{(1)}(F(k) \vee K^*_{\Psi ab})$ and $z^{(1)} \in W^{(1)}(\Psi)$ which are $h^{(1)}(z^{(1)}) \neq 0$. Combining these, we have

$$(f|_k \alpha \mathsf{h})\left(\varepsilon_0(z_0)\right) \in h^{(1)}(z^{(1)})^q \cdot \Omega.$$

We can prove $f|_k(\alpha\mathsf{h})\circ\varepsilon_0 \in \mathcal{M}_k^{(q)}(\Omega)$ in the same way as the proof of Lemma 2.5. Moving $\mathsf{h}$ all over $\mathsf{H}(\mathbb{Q})$, we can get $f|_k\alpha \in \mathcal{M}_k(\Omega)$ by using Lemma 2.3. $\square$

## 6. A certain Galois action

In this section we construct a certain Galois action on the space of modular forms. The purpose of this section is to prove the following theorem.

**Theorem 6.1.** *Let $f \in \mathcal{M}_k^{(q,n)}(S,\Psi)$ and let*

$$f(\mathfrak{z}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r(w)\mathbf{e_a}(\mathrm{tr}(r^{\Psi} z)) \quad \text{where } \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in D^{(q,n)}(S,\Psi),$$

*be its Fourier-Jacobi expansion. For any $(\sigma,\Psi,a) \in C_\Psi(\mathbb{C})$, there exists $f^{(\sigma,\Psi,a)} \in \mathcal{M}_{k^\sigma}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ whose Fourier-Jacobi expansion is*

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \sum_{0 \leq r \in \mathcal{H}_q} g_r^{(\sigma,\Psi,a)}(\tilde{w})\mathbf{e_a}(\mathrm{tr}(r^{(\Psi\sigma)}\tilde{z}))$$

*where $\tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} \in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$.*

To prove this, we first consider the relation of two embeddings, $\varepsilon_\delta^{(q,n)}(S,\Psi)$ and $\varepsilon_\delta^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$. We have the following lemma.

**Lemma 6.2.** *Let $(\sigma,\Psi,a) \in C_\Psi(\mathbb{C})$ and $T \in \mathcal{Z}^{(m)}$. Assume $F^\times\Gamma_T^{(m)}/F^\times$ is torsion free. Put $Y = \left(I_\delta^{(q,n)}(S,\Psi)\right)^{-1}(T) \in \mathcal{Z}^{(q,n)}(S,\Psi)$. Set*

$$A(\sigma,\Psi,a) = \begin{pmatrix} 1_q & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{\chi(\sigma)(a+a^\rho)}{2aa^\rho}\right)1_n & 0 & 0 & (\frac{-a+a^\rho}{2})S^{-1} & 0 \\ 0 & 0 & 1_q & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi(\sigma)1_q & 0 & 0 \\ 0 & \left(\frac{\chi(\sigma)(-a+a^\rho)}{2aa^\rho}\right)S & 0 & 0 & (\frac{a+a^\rho}{2})1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \chi(\sigma)1_q \end{pmatrix}.$$

*Then $A(\sigma,\Psi,a) \in \mathcal{G}_+^{(m)}$ and the following assertions hold.*

(1) $\quad J_{T\tilde{T}}^{(m)}(A(\sigma,\Psi,a))\left(\overline{E_{\tilde{T}\tilde{Y}}(V_{\tilde{Y}})}\right) = \left(\overline{E_{TY}(V_Y)}\right)^\sigma$

*where*

$$\begin{aligned}\tilde{T} &= A(\sigma,\Psi,a)^{-1}TA(\sigma,\Psi,a) \quad \in \mathcal{Z}^{(m)},\\ \tilde{Y} &= \left(I_\delta^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)\right)^{-1}(\tilde{T})\\ &= \begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q\end{pmatrix}^{-1} Y \begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q\end{pmatrix}\\ &\quad \in \mathcal{Z}^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right),\end{aligned}$$

*and the overlines denote the Zariski closures in $V_T^{(m)}$ and in $V_{\tilde{T}}^{(m)}$.*

(2) *For any $\tilde{z}_0 \in W^{(q)}(\Psi\sigma)$, $\tilde{\mathsf{h}} \in \mathsf{H}^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)(\mathbb{Q})$ and $\tilde{X} \in \mathcal{Z}^{(q)}$ which satisfies $\Gamma_{\tilde{X}}^{(q)} \subset F^\times\left(I_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)\right)^{-1}(\tilde{\mathsf{h}}^{-1}\tilde{Y}\tilde{\mathsf{h}})$, put*

$$X = \begin{pmatrix} 1_q & 0\\ 0 & \chi(\sigma)1_q\end{pmatrix}\tilde{X}\begin{pmatrix} 1_q & 0\\ 0 & \chi(\sigma)1_q\end{pmatrix}^{-1} \in \mathcal{Z}^{(q)},$$

*and take $z_0 \in W^{(q)}(\Psi)$ so that*

$$\varphi_X^{(q)}(z_0) = \left[J_{X\tilde{X}}^{(q)}\left(\begin{pmatrix} 1_q & 0\\ 0 & \chi(\sigma)1_q\end{pmatrix}\right)\left(\varphi_{\tilde{X}}^{(q)}(\tilde{z}_0)\right)\right]^{\sigma^{-1}}.$$

*Then we have*

$$(*)\qquad J_{T\tilde{T}}^{(m)}(A(\sigma,\Psi,a))\left(\varphi_{\tilde{T}}^{(m)}\left(\varepsilon_\delta\left(\tilde{\mathsf{h}}\circ\varepsilon_0(\tilde{z}_0)\right)\right)\right) = \left[\varphi_T^{(m)}\left(\varepsilon_\delta\left(\mathsf{h}\circ\varepsilon_0(z_0)\right)\right)\right]^\sigma$$

*where*

$$(\star)\quad \mathsf{h} \in \mathsf{H}^{(q,n)}(S,\Psi)(\mathbb{Q}) \cap Y\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q\end{pmatrix}\tilde{\mathsf{h}}\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q\end{pmatrix}^{-1}.$$

(Note that the right hand side of $(*)$ is independent of the choice of $z_0$ since $\Gamma_X^{(q)} \subset F^\times\left(I_0^{(q,n)}(S,\Psi)\right)^{-1}(\mathsf{h}^{-1}Y\mathsf{h})$. The $\varepsilon_0,\varepsilon_\delta$ in the left hand side mean $\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right),\varepsilon_\delta^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)$ and those in the right hand side mean $\varepsilon_0^{(q,n)}(S,\Psi),\varepsilon_\delta^{(q,n)}(S,\Psi)$.)

*Proof.* The assertion (1) follows immediately from (2) since the set

$$\left\{\tilde{\mathsf{h}}\circ\varepsilon_0(\tilde{z}_0)\,\middle|\,\tilde{z}_0 \in W^{(q)}(\Psi\sigma),\quad \tilde{\mathsf{h}} \in \mathsf{H}^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)(\mathbb{Q})\right\}$$

is dense in $D^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)$ and $\overline{E_{\tilde{T}\tilde{Y}}(V_{\tilde{Y}})},\overline{E_{TY}(V_Y)}$ are subvarieties of $V_{\tilde{T}}^{(m)},V_T^{(m)}$.

It suffices to prove (2). First let us consider the case when $\tilde{\mathsf{h}} = 1_m$. Since

$$\varepsilon_\delta^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)\circ\varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)\,(\tilde{z_0}) = \begin{pmatrix} \tilde{z_0} & & \\ & \left(-\iota(\sigma,a)^{-1}S^{-1}\right)^{(\Psi\sigma)} & \\ & & \tilde{z_0} \end{pmatrix}$$

(where $\tilde{z_0} = ((\tilde{z_0})_v)_{v\in\mathbf{a}}$), and as

$$A(\sigma,\Psi,a)$$
$$= \begin{pmatrix} 1_q & & & & & \\ & (\frac{\chi(\sigma)(a+a^\rho)}{2aa^\rho})1_n & & & (\frac{-a+a^\rho}{2\chi(\sigma)})S^{-1} & \\ & & 1_q & & & \\ & & & 1_q & & \\ & (\frac{\chi(\sigma)(-a+a^\rho)}{2aa^\rho})S & & & (\frac{a+a^\rho}{2\chi(\sigma)})1_n & \\ & & & & & 1_q \end{pmatrix} \begin{pmatrix} 1_m & \\ & \chi(\sigma)1_m \end{pmatrix},$$

we can get (2) by formal calculations of $\Phi^{(m)}_{\varepsilon_\delta\circ\varepsilon_0(\tilde{z_0})}(a), \Phi^{(q)}_{\tilde{z_0}}(a)$ and Proposition 3.2.

For an arbitrary $\tilde{\mathsf{h}}$, the left hand side of $(*)$ can be rewritten as

$$(**)\qquad J^{(m)}_{[T][I_\delta(\tilde{\mathsf{h}})^{-1}\tilde{T}I_\delta(\tilde{\mathsf{h}})]}\left(A(\sigma,\Psi,a)\cdot I_\delta(\tilde{\mathsf{h}})\right)\left(\varphi^{(m)}_{I_\delta(\tilde{\mathsf{h}})^{-1}\tilde{T}I_\delta(\tilde{\mathsf{h}})}\left(\varepsilon_\delta\circ\varepsilon_0(\tilde{z_0})\right)\right).$$

By a computation, we have

$$A(\sigma,\Psi,a)\cdot I_\delta^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)(\tilde{\mathsf{h}}) \in T\cdot I_\delta^{(q,n)}(S,\Psi)(\mathsf{h})\cdot A(\sigma,\Psi,a).$$

Hence $(**)$ is equal to

$$\left(J^{(m)}_{[T][I_\delta(\mathsf{h})^{-1}TI_\delta(\mathsf{h})]}\left(I_\delta(\mathsf{h})\right)\right)^\sigma$$
$$\circ\, J^{(m)}_{[I_\delta(\mathsf{h})^{-1}TI_\delta(\mathsf{h})][I_\delta(\tilde{\mathsf{h}})^{-1}\tilde{T}I_\delta(\tilde{\mathsf{h}})]}\left(A(\sigma,\Psi,a)\right)\left(\varphi^{(m)}_{I_\delta(\tilde{\mathsf{h}})^{-1}\tilde{T}I_\delta(\tilde{\mathsf{h}})}\left(\varepsilon_\delta\circ\varepsilon_0(\tilde{z_0})\right)\right).$$

By the result when $\tilde{\mathsf{h}}$ is identity, replacing $T$ by $I_\delta(\mathsf{h})^{-1}TI_\delta(\mathsf{h})$, it is equal to

$$\left[J^{(m)}_{[T][I_\delta(\mathsf{h})^{-1}TI_\delta(\mathsf{h})]}\left(I_\delta(\mathsf{h})\right)\left(\varphi^{(m)}_{I_\delta(\mathsf{h})^{-1}TI_\delta(\mathsf{h})}\left(\varepsilon_\delta\circ\varepsilon_0(z_0)\right)\right)\right]^\sigma,$$

hence equal to the right hand side of $(*)$. □

*Proof of Theorem* 6.1. First we consider the case when $k = \kappa\mathbf{1}$ with a positive even integer $\kappa$. For $f \in \mathcal{M}^{(q,n)}_{\kappa\mathbf{1}}(S,\Psi)$ and $(\sigma,\Psi,a) \in C_\Psi(\mathbb{C})$, we define

(6.1)
$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \left[\left(f\Xi^{-\kappa/2}\right)\circ\varphi_Y^{-1}\circ E_{TY}^{-1}\right]^\sigma \circ J^{(m)}_{T\tilde{T}}\left(A(\sigma,\Psi,a)\right)\left(\varphi^{(m)}_{\tilde{T}}\circ\varepsilon_\delta(\tilde{\mathfrak{z}})\right)$$
$$\times\left(\Xi^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})\right)^{\kappa/2}$$

where $\tilde{\mathfrak{z}} \in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma), T \in \mathcal{Z}^{(m)}, Y = \left(I_\delta^{(q,n)}(S,\Psi)\right)^{-1}(T) \in \mathcal{Z}^{(q,n)}(S,$ $\Psi)$ so that $\left(f\Xi^{-\kappa/2}\right)\circ\varphi_Y^{-1}$ can be defined as a rational function on $V_Y$, $F^\times\Gamma_T^{(m)}/F^\times$ is torsion free, and $E_{TY}^{-1}$ is a regular rational map on $\overline{E_{TY}(V_Y)}$. (By Theorem 4.1, we can take such $T, Y$.) Here

$$
\begin{aligned}
\tilde{T} &= A(\sigma,\Psi,a)^{-1} T A(\sigma,\Psi,a),\\
\tilde{Y} &= \left(I_\delta^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)\right)^{-1}(\tilde{T})\\
&= \begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q \end{pmatrix}^{-1} Y \begin{pmatrix} \iota(\sigma,a)^{-1}1_q & 0 & 0\\ 0 & a^\rho 1_n & 0\\ 0 & 0 & aa^\rho 1_q \end{pmatrix}\\
&\in \mathcal{Z}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma).
\end{aligned}
$$

This definition is independent of the choice of $T, Y$. Next let us prove that it does not depend on $\Xi$. For different $\Xi_1, \Xi_2$ we have only to prove

$$
\begin{aligned}
(***) \qquad & \left[\left(\Xi_1\Xi_2^{-1}\right)\circ\varphi_Y^{-1}\circ E_{TY}^{-1}\right]^\sigma \circ J_{T\tilde{T}}^{(m)}(A(\sigma,\Psi,a))\left(\varphi_{\tilde{T}}^{(m)}\circ\varepsilon_\delta(\tilde{\mathfrak{z}})\right)\\
&= \Xi_1^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})\left(\Xi_2^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})\right)^{-1}
\end{aligned}
$$

viewing both sides as meromorphic functions on $D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$, where $\left(\Xi_1\Xi_2^{-1}\right)\circ\varphi_Y^{-1}$ (resp. $\Xi_1^{(\sigma,\Psi,a)}/\Xi_2^{(\sigma,\Psi,a)}\circ(\varphi_{\tilde{Y}})^{-1}$) can be defined as a rational function on $V_Y$ (resp. $V_{\tilde{Y}}$). Consider the case when $\tilde{\mathfrak{z}} = \tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}(\iota(\sigma,a)S,$ $\Psi\sigma)(\tilde{z_0})$ with $\tilde{z_0} \in W^{(q)}(\Psi\sigma)$ and $\tilde{\mathsf{h}} \in \mathsf{H}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\mathbb{Q})$. Take $\mathsf{h}, z_0$ as in Lemma 6.2. Then the left hand side of $(***)$ is equal to

$$
\begin{aligned}
&\left[(\Xi_1\Xi_2^{-1})\circ\varphi_Y^{-1}\circ E_{TY}^{-1}\circ\varphi_T^{(m)}\circ\varepsilon_\delta\left(\mathsf{h}\circ\varepsilon_0(z_0)\right)\right]^\sigma\\
&\quad = \left[\left(\Xi_1\Xi_2^{-1}\right)\left(\mathsf{h}\circ\varepsilon_0(z_0)\right)\right]^\sigma
\end{aligned}
$$

if $\Xi_1\Xi_2^{-1}$ is holomorphic at $\mathsf{h}\circ\varepsilon_0(z_0)$. Taking suitable $\mathsf{h}$, we have

$$
\Xi_i^{(\sigma,\Psi,a)}\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma) = \left(\Xi_i\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\right)^\sigma \qquad (i=1,2),
$$

where the action of $\sigma$ is in the sense of (2.2). Hence for such $\mathsf{h}$, we have

$$
\begin{aligned}
&\left[\Xi_1^{(\sigma,\Psi,a)}\left(\Xi_2^{(\sigma,\Psi,a)}\right)^{-1}\right]\left(\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\tilde{z_0})\right)\\
&\quad = \left[\left(\Xi_1\Xi_2^{-1}\right)\left(\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)(z_0)\right)\right]^\sigma
\end{aligned}
$$

if $\Xi_2^{(\sigma,\Psi,a)}(\tilde{\mathsf{h}}\circ\varepsilon_0(\tilde{z_0})) \neq 0$. (In this case we have $\Xi_2(\mathsf{h}\circ\varepsilon_0(z_0)) \neq 0$ if we take a suitable $z_0$, and so $\Xi_1\Xi_2^{-1}$ is holomorphic at each $\mathsf{h}\circ\varepsilon_0(z_0)$ as above.) Since the set

$$
\left\{\tilde{\mathsf{h}}\circ\varepsilon_0(\tilde{z_0}) \;\middle|\; \begin{array}{l} \tilde{\mathsf{h}} \in \mathsf{H}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma), \quad \tilde{z_0} \in W^{(q)}(\Psi\sigma),\\ \Xi_2^{(\sigma,\Psi,a)}(\tilde{\mathsf{h}}\circ\varepsilon_0(\tilde{z_0})) \neq 0 \end{array}\right\}
$$

is dense in $D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$, we get $(***)$. Hence the definition (6.1) is well-defined.

Now let us consider $f^{(\sigma,\Psi,a)}$. Clearly $f^{(\sigma,\Psi,a)} \in \mathcal{A}^{(q,n)}_{\kappa\mathbf{1}}(\iota(\sigma,a)S,\Psi\sigma)$. By Theorem 4.1 we can take suitable $T \in \mathcal{Z}^{(m)}$ and $Y = I_\delta^{-1}(T) \in \mathcal{Z}^{(q,n)}(\iota(\sigma,a)S, \Psi\sigma)$ satisfying the following conditions (1)–(4).

(1) $\varphi^{(m)}_T, \varphi_Y$ are locally biholomorphic.

(2) $V^{(m)}_T, V_Y$ are non-singular.

(3) $E_{TY}(V_Y)$ is a non-singular subvariety of $V^{(m)}_T$.

(4) $E^{-1}_{TY}$ is a regular rational map on $E_{TY}(V_Y)$.

For each $\tilde{\mathfrak{z}} \in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$, take $\Xi$ which is non-zero at $\varphi_Y^{-1} \circ E^{-1}_{TY} \circ \left[J^{(m)}_{T\tilde{T}}(A(\sigma,\Psi,a))\left(\varphi^{(m)}_{\tilde{T}} \circ \varepsilon_\delta(\tilde{\mathfrak{z}})\right)\right]^{\sigma^{-1}}$. Then $f^{(\sigma,\Psi,a)}$ is holomorphic at $\tilde{\mathfrak{z}}$. As $f^{(\sigma,\Psi,a)}$ is independent of the choice of $\Xi$, it is holomorphic on the whole $D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$. Hence $f^{(\sigma,\Psi,a)} \in \mathcal{M}^{(q,n)}_{\kappa\mathbf{1}}(\iota(\sigma,a)S,\Psi\sigma)$. Set

$$f(\mathfrak{z}) = \sum_{0\le r\in\mathcal{H}_q} g_r(w)\boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^\Psi z)\right),$$

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \sum_{0\le r\in\mathcal{H}_q} \tilde{g_r}(\tilde{w})\boldsymbol{e}_{\mathbf{a}}\left(\mathrm{tr}(r^{\Psi\sigma} \tilde{z})\right),$$

where $\mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v\in\mathbf{a}} \in D^{(q,n)}(S,\Psi)$ and $\tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{z_v} \\ \tilde{w_v} \end{pmatrix}_{v\in\mathbf{a}}$ $\in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$. Now it suffices to prove $\tilde{g_r} = g_r^{(\sigma,\Psi,a)}$ for each $0 \le r \in \mathcal{H}_q$. Fix $r$ and take a $\mathbb{Z}$-lattice $L_q$ of $\mathcal{H}_q$ satisfying the following (1), (2).

(1) $r \in L_q$.

(2) For any $t \in \mathcal{H}_q - L_q$, we have $\tilde{g_t}, g_t \equiv 0$.

Take any $u \in K_q^n$ and fix it. From Lemma 2.4 we can choose some $\alpha \in \mathrm{GL}(q,K)$ so that

$$\left\{0 \le r' \in L_q \,\middle|\, \mathrm{Re}\left({}^t\alpha^\rho r'\alpha\right) = \mathrm{Re}\left({}^t\alpha^\rho r\alpha\right)\right\} = \{r\} \tag{6.2}$$

holds. Put

$$\mathsf{h} = \begin{pmatrix} 1_q & {}^tu^\rho S & \frac{1}{2}{}^tu^\rho S u \\ 0 & 1_n & u \\ 0 & 0 & 1_q \end{pmatrix} \begin{pmatrix} \alpha & & \\ & 1_n & \\ & & ({}^t\alpha^\rho)^{-1} \end{pmatrix} \in \mathsf{H}^{(q,n)}(S,\Psi)(\mathbb{Q}).$$

Take $\Xi$ so that $\Xi \circ \mathsf{h} \circ \varepsilon_0^{(q,n)}(S,\Psi) \not\equiv 0$ in $\mathcal{M}^{(q)}_{2\cdot\mathbf{1}}$ and choose $Y, T$ as in (6.1). Take $\tilde{Y}, \tilde{T}$ as in Lemma 6.2. Set the Fourier-Jacobi expansion of $\Xi^{\kappa/2}$ as

$$(\Xi(\mathfrak{z}))^{\kappa/2} = \sum_{0\le t\in\mathcal{H}_q} c_t(w)\boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(t^\Psi z)).$$

Then of course

$$\left(\Xi^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})\right)^{\kappa/2} = \sum_{0\le t\in\mathcal{H}_q} c_t^{(\sigma,\Psi,a)}(\tilde{w})\boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(t^{\Psi\sigma}\tilde{z})).$$

Take a congruence subgroup $\Gamma_N$ of $G^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\mathbb{Q})$ as in (1.5) so that $f^{(\sigma,\Psi,a)} \in \mathcal{M}^{(q,n)}_{\kappa\cdot\mathbf{1}}(\iota(\sigma,a)S,\Psi\sigma)(\Gamma_N)$ and $\Xi^{(\sigma,\Psi,a)} \in \mathcal{M}^{(q,n)}_{2\cdot\mathbf{1}}(\iota(\sigma,a)S,\Psi\sigma)(\Gamma_N)$. For a (sufficiently small) integral ideal $\mathfrak{a}$ of $K$, take $y \in K_q^n$ satisfying $y \equiv au \bmod (\mathfrak{a}\mathcal{O}_\mathfrak{p})_q^n$ for each finite prime $\mathfrak{p}$ of $K$ so that

$$\begin{aligned}(1)\ \tilde{\mathsf{h}} = &\begin{pmatrix} 1_q & \iota(\sigma,a)^t y^\rho S & \frac{1}{2}\iota(\sigma,a)^t y^\rho S y \\ 0 & 1_n & y \\ 0 & 0 & 1_q \end{pmatrix}\begin{pmatrix} \alpha & & \\ & 1_n & \\ & & ({}^t\alpha^\rho)^{-1} \end{pmatrix} \\ &\in \mathsf{H}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\mathbb{Q}) \\ &\cap \tilde{Y}\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}^{-1}\mathsf{h}\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}.\end{aligned}$$

(This condition is equivalent to $(\star)$ in Lemma 6.2(2).)
(2) For above $\tilde{\mathsf{h}}$,

$$\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}^{-1}\mathsf{h}\begin{pmatrix} \iota(\sigma,a)^{-1}1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}\tilde{\mathsf{h}}^{-1}$$

is contained in

$$\left\{ x \in G^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)_A \,\middle|\, \begin{array}{l} x_\mathfrak{p} \in \mathrm{SL}(m,\mathcal{O}_\mathfrak{p}), \\ \nu(x_\mathfrak{p}) = 1, \\ x_\mathfrak{p} \equiv 1_m \mod (N\mathcal{O}_\mathfrak{p})^m_m, \\ \text{for any finite prime } \mathfrak{p} \text{ of } K \end{array} \right\}.$$

For such $y$ and $\tilde{\mathsf{h}}$, we consider $\left(\Xi^{(\sigma,\Psi,a)}\right)^{\kappa/2} \circ \tilde{\mathsf{h}} \circ \varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ and $f^{(\sigma,\Psi,a)} \circ \tilde{\mathsf{h}} \circ \varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ as elements in $\mathcal{M}^{(q)}_{\kappa\mathbf{1}}$. These modular forms are independent of the choice of $y$ or $\tilde{\mathsf{h}}$ as the coset $\left(\Gamma_N \cap \mathsf{N}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\mathbb{Q})\right)\tilde{\mathsf{h}}$ is determined. We have

$$\begin{aligned}&\left(\Xi^{(\sigma,\Psi,a)}\right)^{\kappa/2} \circ \tilde{\mathsf{h}} \circ \varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(z) \\ &\quad = \sum_{0 \le t \in \mathcal{H}_q} \left(c_t^{(\sigma,\Psi,a)}\right)_*(y^{\Psi\sigma})\boldsymbol{e}_\mathbf{a}\left(\mathrm{tr}(\mathrm{Re}({}^t\alpha^\rho t\alpha)z)\right) \qquad (z \in \mathfrak{H}_q^\mathbf{a}),\end{aligned}$$

since the left hand side does not depend on the choice of $y$ or $\tilde{\mathsf{h}}$. Hence we obtain

$$(6.3) \qquad \left[(\Xi)^{\kappa/2} \circ \mathsf{h} \circ \varepsilon_0^{(q,n)}(S,\Psi)\right]^\sigma = \left(\Xi^{(\sigma,\Psi,a)}\right)^{\kappa/2} \circ \tilde{\mathsf{h}} \circ \varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$$

where the action of $\sigma$ is in the sense of (2.2) as an element of $\mathcal{M}^{(q)}_{\kappa\mathbf{1}}$. Choose any $\tilde{z_0} \in W^{(q)}(\Psi\sigma)$ such that $\left(\Xi^{(\sigma,\Psi,a)}\right)^{\kappa/2} \circ \tilde{\mathsf{h}} \circ \varepsilon_0^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)(\tilde{z_0}) \neq 0$.

Take $X, \tilde{X} \in \mathcal{Z}^{(q)}$ and $z_0 \in W^{(q)}(\Psi)$ as in Lemma 6.2(2). From Lemma 6.2 and (6.1), we have

$$\begin{aligned}&\left(f^{(\sigma,\Psi,a)}\left(\Xi^{(\sigma,\Psi,a)}\right)^{-\kappa/2}\right)\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)(\tilde{z_0})\\&\quad=\left[\left(f\Xi^{-\kappa/2}\right)\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)(z_0)\right]^{\sigma},\end{aligned}$$

if $\left(f\Xi^{-\kappa/2}\right)$ is holomorphic at $\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)(z_0)$. In this case the right hand side is equal to

$$\left(f\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\right)^{\sigma}\left\{\left(\Xi^{\kappa/2}\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\right)^{\sigma}\right\}^{-1}(\tilde{z_0}), \tag{6.4}$$

where the action of $\sigma$ is as above. This is holomorphic at $\tilde{z_0}$ as a meromorphic modular form on $\mathfrak{H}_q^{\mathbf{a}}$. Now the set

$$\left\{\tilde{z_0}\in W^{(q)}(\Psi\sigma)\;\middle|\;\begin{array}{l}\Xi^{(\sigma,\Psi,a)}\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)(\tilde{z_0})\neq 0,\\ \Xi^{\kappa/2}\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\text{ is non-zero}\\ \text{at }\left(\varphi_X^{(q)}\right)^{-1}\left[J_{X\tilde{X}}^{(q)}\left(\begin{pmatrix}1_q & 0\\ 0 & \chi(\sigma)1_q\end{pmatrix}\right)\left(\varphi_{\tilde{X}}^{(q)}(\tilde{z_0})\right)\right]^{\sigma^{-1}}\end{array}\right\}$$

is dense in $\mathfrak{H}_q^{\mathbf{a}}$. Hence combining (6.3) and (6.4), we have

$$\begin{aligned}&\left[f^{(\sigma,\Psi,a)}(\Xi^{(\sigma,\Psi,a)})^{-\kappa/2}\right]\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)\\&\quad=\left(f\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\right)^{\sigma}\Big/\left(\left(\Xi^{(\sigma,\Psi,a)}\right)^{\kappa/2}\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)\right)\end{aligned}$$

as meromorphic functions on $\mathfrak{H}_q^{\mathbf{a}}$. This means

$$f^{(\sigma,\Psi,a)}\circ\tilde{\mathsf{h}}\circ\varepsilon_0^{(q,n)}\left(\iota(\sigma,a)S,\Psi\sigma\right)=\left(f\circ\mathsf{h}\circ\varepsilon_0^{(q,n)}(S,\Psi)\right)^{\sigma}$$

as holomorphic functions on $\mathfrak{H}_q^{\mathbf{a}}$. The Fourier expansion of the left hand side (as an element of $\mathcal{M}_{\kappa\mathbf{1}}^{(q)}$) is

$$\sum_b\left(\sum_{\substack{0\le t\in L_q\\ \mathrm{Re}({}^t\alpha^\rho t\alpha)=b}}(\tilde{g_t})_*\left(y^{(\Psi\sigma)}\right)\right)\boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(bz))\qquad(z=(z_v)_{v\in\mathbf{a}}\in\mathfrak{H}_q^{\mathbf{a}}),$$

and that of the right hand side is

$$\sum_b\left(\sum_{\substack{0\le t\in L_q\\ \mathrm{Re}({}^t\alpha^\rho t\alpha)=b}}\left[(g_t)_*(u^{\Psi})\right]^{\sigma}\right)\boldsymbol{e}_{\mathbf{a}}(\mathrm{tr}(bz))\qquad(z=(z_v)_{v\in\mathbf{a}}\in\mathfrak{H}_q^{\mathbf{a}}).$$

From (6.2) we have

$$(\tilde{g_r})_*\left(y^{(\Psi\sigma)}\right) = \left[(g_r)_*(u^\Psi)\right]^\sigma.$$

As the right hand side does not depend on the choice of $y$, we can obtain $\tilde{g_r} = g_r^{(\sigma,\Psi,a)}$ if we take each $u \in K_q^n$. This completes the proof when $k = \kappa\mathbf{1}$ for a positive even integer $\kappa$.

For an arbitrary $k \in \mathbb{Z}^{\mathbf{a}}$, take $\xi_v \in \mathcal{M}_{v+2q\cdot\mathbf{1}}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ as in Section 3 (for each $v \in \mathbf{a}$). Then $\xi_v^{(\sigma^{-1},\Psi\sigma,a^{-1})} \in \mathcal{M}_{v\sigma^{-1}+2q\cdot\mathbf{1}}^{(q,n)}(S,\Psi)$. For $f \in \mathcal{M}_k^{(q,n)}(S,\Psi)$, take $l = (l_v)_{v\in\mathbf{a}} \in \mathbb{N}^{\mathbf{a}}$ so that

$$f\prod_{v\in\mathbf{a}}\left(\xi_v^{(\sigma^{-1},\Psi\sigma,a^{-1})}\right)^{l_v} \in \mathcal{M}_{\kappa\mathbf{1}}^{(q,n)}(S,\Psi)$$

for a positive even integer $\kappa$. Put

$$f^{(\sigma,\Psi,a)} = \left(f\prod_{v\in\mathbf{a}}\left(\xi_v^{(\sigma^{-1},\Psi\sigma,a^{-1})}\right)^{l_v}\right)^{(\sigma,\Psi,a)}\prod_{v\in\mathbf{a}}\xi_v^{-l_v}. \tag{6.5}$$

As $\kappa\mathbf{1} = k + \sum_{v\in\mathbf{a}} l_v(v\sigma^{-1} + 2q\cdot\mathbf{1})$, we have $f^{(\sigma,\Psi,a)} \in \mathcal{A}_{k^\sigma}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$. By a formal calculation of Fourier-Jacobi series, (6.5) does not depend on the choice of $(\xi_v)_{v\in\mathbf{a}}$. Now for each $\tilde{\mathfrak{z}} \in D^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$, take $(\xi_v)_{v\in\mathbf{a}}$ so that $\xi_v(\tilde{\mathfrak{z}}) \neq 0$ for every $v \in \mathbf{a}$. Then $f^{(\sigma,\Psi,a)}$ is holomorphic at $\tilde{\mathfrak{z}}$. This means $f^{(\sigma,\Psi,a)} \in \mathcal{M}_{k^\sigma}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$. By a formal computation at $\tilde{w} = y^{(\Psi\sigma)}$ (for $y \in K_q^n$), the Fourier-Jacobi expansion of $f^{(\sigma,\Psi,a)}$ is as in Theorem 6.1. □

For $f \in \mathcal{A}_k^{(q,n)}(S,\Psi)$, we can also define $f^{(\sigma,\Psi,a)}$. Put $f = f_1/f_2$ by $f_1 \in \mathcal{M}_{k+l}^{(q,n)}(S,\Psi)$, $0 \neq f_2 \in \mathcal{M}_l^{(q,n)}(S,\Psi)$ $(l \in \mathbb{Z}^{\mathbf{a}})$ and define $f^{(\sigma,\Psi,a)} = f_1^{(\sigma,\Psi,a)}/f_2^{(\sigma,\Psi,a)}$. Then this does not depend on the choice of $f_1, f_2$ and $f^{(\sigma,\Psi,a)} \in \mathcal{A}_{k^\sigma}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$.

Using the previous theorem, we can get the following proposition.

**Proposition 6.3.** *For any $k \in \mathbb{Z}^{\mathbf{a}}$, we have*

$$\mathcal{M}_k^{(q,n)}(S,\Psi)(\mathbb{C}) = \mathcal{M}_k^{(q,n)}(S,\Psi)\left(F(k)\vee K_{\Psi ab}^*\right)\otimes_{F(k)\vee K_{\Psi ab}^*}\mathbb{C}.$$

*Proof.* For any $f \in \mathcal{M}_k^{(q,n)}(S,\Psi)(\mathbb{C})$ and any $\sigma \in \mathrm{Aut}(\mathbb{C}/F(k)\vee K_{\Psi ab}^*)$, consider $f^{(\sigma,\Psi,1)} \in \mathcal{M}_k^{(q,n)}(S,\Psi)(\mathbb{C})$. Fix $(\xi_v)_{v\in\mathbf{a}}$ and $\Xi$. Take $Y \in \mathcal{Z}^{(q,n)}(S,\Psi)$ so that $\left(f\prod_{v\in\mathbf{a}}\left(\xi_v^{(\sigma^{-1},\Psi,1)}\right)^{l_v}\right)\Xi^{-\kappa/2}$ (of (6.5)) can be viewed as a rational function on $V_Y$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/F(k)\vee K_{\Psi ab}^*)$. Take a congruence subgroup $\Gamma$ of $G^{(q,n)}(S,\Psi)(\mathbb{Q})$ satisfying the following conditions (1), (2).

(1) $\Gamma \subset \Gamma_Y$.

(2) $f, \xi_v^{(\sigma^{-1},\Psi,1)}$ $(v \in \mathbf{a},\ \sigma \in \mathrm{Aut}(\mathbb{C}/F(k) \vee K^*_{\Psi ab}))$ and $\Xi$ are all modular forms with respect to $\Gamma$.

As the set $\left\{ \xi_v^{(\sigma^{-1},\Psi,1)} \middle| v \in \mathbf{a},\ \sigma \in \mathrm{Aut}(\mathbb{C}/F(k) \vee K^*_{\Psi ab}) \right\}$ is a finite set, we can take such $Y$ and $\Gamma$.

From the definition, we have $f^{(\sigma,\Psi,1)} \in \mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/F(k) \vee K^*_{\Psi ab})$. As is well known, $\mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$ is finite dimensional. Put $d = \dim_{\mathbb{C}} \mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma)$ and take the Fourier-Jacobi expansion of $f$,

$$f(\mathfrak{z}) = \sum_{0 \le r \in \mathcal{H}_q} g_r(w) \boldsymbol{e}_{\mathbf{a}} \left( \mathrm{tr}(r^{\Psi} z) \right), \qquad \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in D^{(q,n)}(S,\Psi).$$

Now set

$$M = \langle \left\{ (g_r)_*(y^{\Psi}) \,\middle|\, y \in K_q^n, \quad 0 \le r \in \mathcal{H}_q \right\} \rangle_{F(k) \vee K^*_{\Psi ab}}.$$

Then $M$ is a $F(k) \vee K^*_{\Psi ab}$-vector space contained in $\mathbb{C}$. Let us prove that $\dim_{F(k) \vee K^*_{\Psi ab}} M \le d$. If not, we can take $y_1, \dots, y_{d+1} \in K_q^n$ and $r_1, \dots, r_{d+1} \in \mathcal{H}_q$ such that $(g_{r_1})_*(y_1^{\Psi}), \dots, (g_{r_{d+1}})_*(y_{d+1}^{\Psi})$ are linearly independent over $F(k) \vee K^*_{\Psi ab}$. So we can take $\sigma_1, \dots, \sigma_{d+1} \in \mathrm{Aut}(\mathbb{C}/F(k) \vee K^*_{\Psi ab})$ such that

$$\det \begin{pmatrix} \left[ (g_{r_1})_*(y_1^{\Psi}) \right]^{\sigma_1} & \cdots & \left[ (g_{r_{d+1}})_*(y_{d+1}^{\Psi}) \right]^{\sigma_1} \\ \cdots & \cdots & \cdots \\ \left[ (g_{r_1})_*(y_1^{\Psi}) \right]^{\sigma_{d+1}} & \cdots & \left[ (g_{r_{d+1}})_*(y_{d+1}^{\Psi}) \right]^{\sigma_{d+1}} \end{pmatrix} \ne 0.$$

This means

$$\det \begin{pmatrix} g_{r_1}^{(\sigma_1,\Psi,1)}(y_1^{\Psi}) & \cdots & g_{r_{d+1}}^{(\sigma_1,\Psi,1)}(y_{d+1}^{\Psi}) \\ \cdots & \cdots & \cdots \\ g_{r_1}^{(\sigma_{d+1},\Psi,1)}(y_1^{\Psi}) & \cdots & g_{r_{d+1}}^{(\sigma_{d+1},\Psi,1)}(y_{d+1}^{\Psi}) \end{pmatrix} \ne 0.$$

This implies that $f^{(\sigma_1,\Psi,1)}, \dots, f^{(\sigma_{d+1},\Psi,1)}$ are linearly independent over $\mathbb{C}$. Hence it contradicts to $\dim_{\mathbb{C}} \mathcal{M}_k^{(q,n)}(S,\Psi)(\Gamma) = d$. Therefore $M$ is at most $d$-dimensional over $F(k) \vee K^*_{\Psi ab}$. Let $\{c_1, \dots, c_t\}$ be a basis of $M$ over $F(k) \vee K^*_{\Psi ab}$. Then we can take $\sigma_1 = \mathrm{id}_{\mathbb{C}}, \sigma_2, \dots, \sigma_t \in \mathrm{Aut}(\mathbb{C}/F(k) \vee K^*_{\Psi ab})$ such that

$$\det \begin{pmatrix} c_1^{\sigma_1} & \cdots & c_t^{\sigma_1} \\ \cdots & \cdots & \cdots \\ c_1^{\sigma_t} & \cdots & c_t^{\sigma_t} \end{pmatrix} \ne 0.$$

Put

$$\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix} = \begin{pmatrix} c_1^{\sigma_1} & \cdots & c_t^{\sigma_1} \\ \cdots & \cdots & \cdots \\ c_1^{\sigma_t} & \cdots & c_t^{\sigma_t} \end{pmatrix}^{-1} \begin{pmatrix} f^{(\sigma_1,\Psi,1)} \\ \vdots \\ f^{(\sigma_t,\Psi,1)} \end{pmatrix}.$$

By a formal calculation, we have $f_1, \dots, f_t \in \mathcal{M}_k^{(q,n)}(S,\Psi)\left(\Gamma, F(k) \vee K^*_{\Psi ab}\right)$. Hence we get $f = c_1 f_1 + \cdots + c_t f_t \in \langle \mathcal{M}_k^{(q,n)}(S,\Psi)\left(F(k) \vee K^*_{\Psi ab}\right) \rangle_{\mathbb{C}}$. This shows the proposition. □

## 7. An extended Galois group

In this section we shall fix the unitary similitude group $G = G^{(q,n)}(S,\Psi)$ and define the action of a certain extended Galois group using the results in Section 6.

Take $f(\mathfrak{z}) = \sum_{0 \le r \in \mathcal{H}_q} g_r(w)\mathbf{e_a}\left(\mathrm{tr}(r^\Psi z)\right) \in \mathcal{M}_k^{(q,n)}(S,\Psi)$, $(\sigma,\Psi,a) \in C_\Psi(\mathbb{C})$ and $f^{(\sigma,\Psi,a)} \in \mathcal{M}_{k^\sigma}^{(q,n)}(\iota(\sigma,a)S,\Psi\sigma)$ as in Section 6. For $\sigma \in \mathrm{Aut}(\mathbb{C}/K_\Psi^*)$, put

$$f^{[\sigma,\Psi,a]} = f^{(\sigma,\Psi,a)} \circ \varepsilon(S,\iota(\sigma,a)). \tag{7.1}$$

Then $f^{[\sigma,\Psi,a]} \in \mathcal{M}_{k^\sigma}^{(q,n)}(S,\Psi)$ and its Fourier-Jacobi expansion is

$$f^{[\sigma,\Psi,a]}(\mathfrak{z}) = \sum_{0 \le r \in \mathcal{H}_q} g_r^{(\sigma,\Psi,a)}(w)\mathbf{e_a}\left(\mathrm{tr}(\iota(\sigma,a)r^\Psi z)\right).$$

For $f \in \mathcal{A}_k^{(q,n)}(S,\Psi)$, we also define $f^{[\sigma,\Psi,a]}$ by (7.1). Clearly $f^{[\sigma,\Psi,a]} \in \mathcal{A}_{k^\sigma}^{(q,n)}(S,\Psi)$. Note that $\left(f^{[\sigma_1,\Psi,a_1]}\right)^{[\sigma_2,\Psi,a_2]} = f^{[\sigma_1\sigma_2,\Psi,a_1a_2]}$ and $aK^\times K_\infty^\times = N'_\Psi(b)K^\times K_\infty^\times$ with $b \in K_A^{*\times}$ so that $[b^{-1},K^*] = \sigma|_{K_{ab}^*}$.

Now we have the following lemma.

**Lemma 7.1.** *For $0 \ne f_1, f_2 \in \mathcal{M}_k^{(q,n)}(S,\Psi)$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K^*)$, take $Y \in \mathcal{Z}$ so that $(f_1/f_2)\circ\varphi_Y^{-1}$ is defined as a rational function on $V_Y$. Then we have*

$$f_1^{[\sigma,\Psi,a]}/f_2^{[\sigma,\Psi,a]} = \left[(f_1/f_2)\circ\varphi_Y^{-1}\right]^\sigma \circ J_{YX}\left(\begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}\right)\circ\varphi_X,$$

*where*

$$X = \begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}^{-1} Y \begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix} \in \mathcal{Z}^{(q,n)}(S,\Psi).$$

**Remark.** Note that the right hand side is independent of the choice of $Y$ and $X$.

*Proof.* By an easy computation, we have the following commutative diagram.

$$\begin{array}{ccc} D^{(q,n)}(S,\Psi) & \xrightarrow{\varepsilon_\delta^{(q,n)}(S,\Psi)} & \mathfrak{H}_m^{\mathbf{a}} \\ {\scriptstyle\varepsilon(S,b)}\downarrow & & \downarrow{\scriptstyle\beta(b)} \\ D^{(q,n)}(bS,\Psi) & \xrightarrow{\varepsilon_\delta^{(q,n)}(bS,\Psi)} & \mathfrak{H}_m^{\mathbf{a}} \end{array}, \tag{7.2}$$

where $\beta(b) = \begin{pmatrix} 1_q & & & & & \\ & b^{-1}1_n & & & & \\ & & 1_q & & & \\ & & & b^{-1}1_q & & \\ & & & & 1_n & \\ & & & & & b^{-1}1_q \end{pmatrix}$.

By (6.1), we have

$$f_1^{(\sigma,\Psi,a)}/f_2^{(\sigma,\Psi,a)} = \left[(f_1/f_2)\circ\varphi_Y^{-1}\circ E_{TY}^{-1}\right]^{\sigma}\circ J_{T\tilde{T}}^{(m)}(A(\sigma,\Psi,a))\circ E_{\tilde{T}\tilde{Y}}\circ\varphi_{\tilde{Y}},$$

with $T, \tilde{T}$ and $\tilde{Y}$ as in (6.1). Combining (7.1) and (7.2), we have

$$\begin{aligned} &f_1^{[\sigma,\Psi,a]}/f_2^{[\sigma,\Psi,a]} \\ &\quad = \left[(f_1/f_2)\circ\varphi_Y^{-1}\circ E_{TY}^{-1}\right]^{\sigma}\circ J_{TU}^{(m)}\left(A(\sigma,\Psi,a)\beta\left(\iota(\sigma,a)\right)\right)\circ E_{UX}\circ\varphi_X, \end{aligned}$$

where $U = \beta\left(\iota(\sigma,a)\right)^{-1} A(\sigma,\Psi,a)^{-1}TA(\sigma,\Psi,a)\beta\left(\iota(\sigma,a)\right)$ (then $U \supset I_\delta(X)$). Since $A(\sigma,\Psi,a)\beta\left(\iota(\sigma,a)\right) \in I_\delta\left(\begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix} G_\infty\right)$, we can get this lemma from (2.7). □

Now we define a subgroup $\mathfrak{G} = \mathfrak{G}^{(q,n)}(S,\Psi)$ of $G_A \times K_A^{*\times} \times \mathrm{Gal}(\overline{\mathbb{Q}}/K^*)$ as follows.

$$\mathfrak{G} = \left\{ \begin{array}{l|l} (x,c,\sigma) & \det(x)^{-1}(N'_\Psi(c)^\rho)^n N_{K^*/\mathbb{Q}}(c)^q \in K^\times K_\infty^\times, \\ \quad \in G_A \times K_A^* \times \mathrm{Gal}(\overline{\mathbb{Q}}/K^*) & \nu(x)^{-1}N_{K^*/\mathbb{Q}}(c) \in F^\times F_{\infty+}^\times, \\ & \left[c^{-1},K^*\right] = \sigma|_{K^*_{ab}} \end{array} \right\}.$$

Then we can define an action of $\mathfrak{G}$ on the space of modular forms as follows.

**Theorem 7.2.** *There is an action of* $\mathfrak{G}$ *on the graded ring* $\sum_{k\in\mathbb{Z}^{\mathbf{a}}} \mathcal{A}_k(\overline{\mathbb{Q}})$ *written as*

$$((x,c,\sigma),f) \to f^{(x,c,\sigma)} \quad \textit{for } (x,c,\sigma) \in \mathfrak{G} \textit{ and } f \in \sum_{k\in\mathbb{Z}^{\mathbf{a}}} \mathcal{A}_k(\overline{\mathbb{Q}}),$$

*satisfying the following conditions* (i)–(vii).

(i) $(b_1f_1 + b_2f_2)^{(x,c,\sigma)} = b_1^\sigma f_1^{(x,c,\sigma)} + b_2^\sigma f_2^{(x,c,\sigma)}$ *for* $b_1, b_2 \in \overline{\mathbb{Q}}$.

(ii) $(f_1f_2)^{(x,c,\sigma)} = f_1^{(x,c,\sigma)} f_2^{(x,c,\sigma)}$.

(iii) $(f^{(x_1,c_1,\sigma_1)})^{(x_2,c_2,\sigma_2)} = f^{(x_1x_2,c_1c_2,\sigma_1\sigma_2)}$.

(iv) $f^{(\alpha,1,1)} = f|_k\alpha$ *if* $\alpha \in G(\mathbb{Q})$ *and* $f \in \mathcal{A}_k(\overline{\mathbb{Q}})$.

(v) $f^{(x,c,\sigma)} = f^{[\sigma,\Psi,a]}$ *if* $a = N'_\Psi(c)$ *and* $x = \begin{pmatrix} 1_q & & \\ & a^\rho 1_n & \\ & & aa^\rho 1_q \end{pmatrix}$.

(vi) $\mathcal{A}_k(\overline{\mathbb{Q}})^{(x,c,\sigma)} = \mathcal{A}_{k^\sigma}(\overline{\mathbb{Q}})$ *and* $\mathcal{M}_k(\overline{\mathbb{Q}})^{(x,c,\sigma)} = \mathcal{M}_{k^\sigma}(\overline{\mathbb{Q}})$.

(vii) *If* $f \in \mathcal{A}_0(\overline{\mathbb{Q}}) = \mathfrak{K} \vee \overline{\mathbb{Q}}$, *then*

$$f^{(x,c,\sigma)} = \left[(f \circ \varphi_Y^{-1})\right]^{\sigma} J_{YX}(x) \circ \varphi_X,$$

*where* $Y \in \mathcal{Z}$ *is sufficiently small so that* $f \circ \varphi_Y^{-1}$ *is defined as a rational function on* $V_Y$ *and* $X = x^{-1}Yx$.

*Proof.* For any positive integer $N$, put

$$Y_N = \left\{ x \in G_A \cap \left( G_\infty \times \prod_{\mathfrak{p}} \mathrm{GL}(m, \mathcal{O}_{\mathfrak{p}}) \right) \middle| \begin{array}{l} x_{\mathfrak{p}} \equiv 1_m \mod (N\mathcal{O}_{\mathfrak{p}})_m^m \\ \text{for any finite prime } \mathfrak{p} \text{ of } K \end{array} \right\}.$$

Then we have

$$G_\infty(Y_N \cap G_{1A}) \cap G(\mathbb{Q}) = \Gamma_N,$$

where $\Gamma_N$ is as in (1.5).

For any $(x, c, \sigma) \in \mathfrak{G}$, set $\tilde{x} = \begin{pmatrix} 1_q & & \\ & N'_\Psi(c)^\rho 1_n & \\ & & N_{k^*/\mathbb{Q}}(c) 1_q \end{pmatrix}^{-1} x$. Since $\det(\tilde{x}) \in K^\times K_\infty^\times$ and $\nu(\tilde{x}) \in F^\times F_{\infty+}^\times$, we can take $b_1 \in K^\times$ and $b_2 \in F^\times$ so that $\det(\tilde{x}) \in b_1 K_\infty^\times$ and $\nu(\tilde{x}) \in b_2 F_{\infty+}^\times$. As $\det(\tilde{x}) \det(\tilde{x})^\rho = \nu(\tilde{x})^m$, we can get $b_1 b_1^\rho = b_2^m$ by comparing the non-archimedean components. By the Hasse principle, we can take $\alpha \in G(\mathbb{Q})$ such that $\nu(\alpha) = b_2$. Hence $\tilde{x} \begin{pmatrix} 1_q & & & \\ & \det(\alpha) b_1^{-1} & & \\ & & 1_{n-1} & \\ & & & 1_q \end{pmatrix} \alpha^{-1} \in G_\infty G_{1A}$ is contained in $G_\infty(Y_N \cap G_{1A}) \cdot G_1(\mathbb{Q})$ for any positive integer $N$ because of the strong approximation property of $G_1$.

For $f \in \mathcal{A}_k(\overline{\mathbb{Q}})$ and $(x, c, \sigma) \in \mathfrak{G}$, take a positive integer $N$ so that $f^{[\sigma, \Psi, N'_\Psi(c)]} \in \mathcal{A}_{k^\sigma}(\Gamma_N)$. For such $N$, take $u_N \in G_\infty(Y_N \cap G_{1A})$ and $\alpha_N \in G(\mathbb{Q})$ so that $\tilde{x} = u_N \alpha_N$ (where $\tilde{x}$ is as above), and define

$$f^{(x,c,\sigma)} = f^{[\sigma, \Psi, N'_\Psi(c)]}|_{k^\sigma} \alpha_N.$$

Clearly $f^{(x,c,\sigma)}$ is independent of the choice of $u_N$ and $\alpha_N$. We can easily verify the conditions (i),(ii),(iv),(v) and (vi). Using Lemma 7.1, we can get (vii).

Now we have only to prove (iii). In case $f \in \mathcal{A}_0(\overline{\mathbb{Q}})$, we can get (iii) from (vii). In case $f = \Xi$, the condition (iii) can be verified by computations using (3.9) and (3.10). In the same way we can also get (iii) when $f = \xi_v$ ($v \in \mathbf{a}$). Since any element of $\mathcal{A}_k(\overline{\mathbb{Q}})$ can be expressed as a multiple of positive or negative powers of $\Xi$, $\xi_v$ and an element of $\mathcal{A}_0(\overline{\mathbb{Q}})$, we can get (iii) for any $f \in \mathcal{A}_k(\overline{\mathbb{Q}})$ by using (ii). □

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