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On strong convergence of hyperbolic 3-cone-manifolds whose singular sets have uniformly thick tubular neighborhoods

By

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Abstract

Let C be a compact orientable hyperbolic 3-cone-manifold with cone-type singularity along simple closed geodesics Σ. Let $\{C_i\}_{i=1}^{\infty}$ be a sequence consisting of deformations of C and Σ_i be the singular set of C_i so that the cone angles along Σ_i all are less than 2π. In this paper, we will show that, if tubular neighborhoods of the singular sets Σ_i can be taken to be uniformly thick, then there is a subsequence $\{C_{i_k}\}_{k=1}^{\infty}$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C.

Introduction

By a hyperbolic 3-cone-manifold, we will mean an orientable Riemannian 3-manifold C of constant sectional curvature -1 with cone-type singularity along simple closed geodesics Σ. To each component of the singularity Σ, is associated a cone angle. It is shown in [5] that for any values of cone angles, a non-singular part $C - \Sigma$ carries a complete hyperbolic structure C_{comp} of finite volume, and moreover that if the cone angles of C all are at most π, then there is an angle decreasing continuous family $\{C_t\}_{t\in[0,1)}$ of deformations of $C(= C_0)$ to the complete hyperbolic 3-manifold $C_{comp}(= \lim_{t\to 1} C_t)$. The hyperbolic 3-manifold C_{comp} is regarded as a hyperbolic 3-cone-manifold with cone angles equal to zero at the cusps.

The latter claim is proved by using two machineries, the local rigidity by Hodgson-Kerckhoff [3] and the pointed Hausdorff-Gromov topology [2]. These machineries are fundamental when cone angles are $\leq 2\pi$. In particular, the local rigidity implies the practicability of deformations of a hyperbolic 3-cone-manifold with arbitrary small changes in the cone angles, in the case where the initial cone angles all are at most 2π. Then, if the cone angles of C all are at most π, one obtains deformations of C with decreasing cone angles with an

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arbitrary small amount. In [5], for extending such a small deformation globally, Kojima analyzed phenomena which occur in the two cases, that is, in the case where tubular neighborhoods of the singular loci Σ_t in the deformations C_t ($t \in [0,1)$) are uniformly thick, and in the case where a sequence of radii of the tubular neighborhoods goes to zero. For this analysis, he established three relative constants for hyperbolic 3-cone-manifolds which control the local geometry of cone-manifolds away from the singularities. Lemma 3.1.1 of [5] gives one of them, and is a key lemma to derive the other constants and also to analyze the phenomena above.

In this paper, we will show that the assumption "$\leq \pi$" in Lemma 3.1.1 [5] about the cone angles can be improved to "$< 2\pi$" (see Lemma 2), by using fundamental properties on Dirichlet domains of 3-cone-manifolds (see Lemma 1). Then, it can be seen that, for each sequence $\{C_i\}_{i=1}^{\infty}$ consisting of deformations of C so that tubular neighborhoods of Σ_i ($i \in \mathbf{N}$) are uniformly thick, if the cone angles of C_i ($i \in \mathbf{N}$) are less than 2π, then there is a subsequence $\{C_{i_m}\}_{m=1}^{\infty}$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C (see Theorem). This is a refinement of Cor 5.1.4 [5] and is proved by performing the same argument in the sections 3 and 5 of [5]. By Theorem, even though the initial cone angles of a hyperbolic cone-manifold C are greater than π (but less than 2π), there is an angle decreasing continuous family $\{C_t\}_{t\in[0,1)}$ of deformations of C to C_{comp}, if we can rule out the case where the singular locus Σ_t intersects itself. Kerckhoff announced that Hodgson and Kerckhoff obtained a similar result with ours (see Theorem 2 in [4]).

1. Dirichlet polyhedra and a relative constant for 3-cone-manifolds of constant non-positive curvature

First we will give the definition of cone-manifolds (see [1]). Consider an n-dimensional manifold C which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to the standard sphere and give a complete path metric on C such that the restriction of the metric to each simplex is isomorphic to a geodesic simplex of constant sectional curvature K. The manifold together with the metric above is called an n-cone-manifold of sectional curvature K and denote it again by C. The cone-manifold is hyperbolic, Euclidean or spherical if K is -1, 0 or $+1$. If C is an n-cone-manifold and c is a point in C, the pair (C,c) is called a pointed n-cone-manifold.

The singular locus Σ of a cone-manifold C consists of the points with no neighborhood isometric to a ball in a Riemannian manifold. It is a union of totally geodesic closed simplices of dimension $n-2$. At each point of Σ in an open $(n-2)$-simplex, there is a cone angle which is the sum of dihedral angles of n-simplices containing the point. The subset $C - \Sigma$ has a smooth Riemannian metric of constant curvature K, but this metric is incomplete near Σ.

In this paper we consider hyperbolic 3-cone-manifolds of the following type.

Let M be a closed orientable 3-manifold and

$$\Sigma^1 \cup \cdots \cup \Sigma^n \cup \Sigma^{n+1} \cup \cdots \cup \Sigma^{n+k}$$

be a link in M of $n+k$ components. Let us denote the n components of the link by Σ;

$$\Sigma = \Sigma^1 \cup \cdots \cup \Sigma^n,$$

and the remaining k components by Λ;

$$\Lambda = \Sigma^{n+1} \cup \cdots \cup \Sigma^{n+k}.$$

We assume that $M - \Lambda$ is the underlying space of a hyperbolic 3-cone-manifold C with singular locus Σ and torus cusps at Λ. The subset $N := C - \Sigma$ has a smooth Riemannian metric which is complete near the torus cusps and is incomplete near each component of Σ. The metric completion of the hyperbolic structure on N gives rise to C. The hyperbolic 3-cone-manifold C is compact if Λ is empty. The components of Σ is a totally geodesic submanifold, and in cylindrical coordinates around a component Σ^j of the singular locus, the metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2,$$

where r is the distance from the singular locus, z is the distance along the singular locus, θ is the angular measure around the singular locus defined *modulo* α^j for some $\alpha^j \in (0, \infty)$. The number α^j is a cone angle at Σ^j. To each component of Λ, associated is a cone angle zero. We have a developing map of N from its universal covering space $\tilde{N}$,

$$\mathcal{D}_C : \tilde{N} \to \mathbf{H}^3,$$

and a holonomy representation

$$\rho_C : \pi_1(N) \to \mathrm{PSL}_2(\mathbf{C}).$$

They are called a developing map and a holonomy representation of the cone-manifold C.

Two hyperbolic 3-cone-manifolds C_1, C_2 with underlying spaces $M_1 - \Lambda_1$, $M_2 - \Lambda_2$ respectively are said to be homeomorphic if there is a homeomorphism between $(M_1, \Sigma_1 \cup \Lambda_1)$ and $(M_2, \Sigma_2 \cup \Lambda_2)$.

A deformation of a hyperbolic 3-cone-manifold C is a hyperbolic 3-cone-manifold C' together with a reference homeomorphism $\xi' : (M, \Sigma \cup \Lambda) \to (M', \Sigma' \cup \Lambda')$.

Let L be a number with $L \leq -1$. Let $\mathcal{C}^{<\theta}_{[L,0]}$ be the set of pointed compact orientable 3-cone-manifolds of constant sectional curvature $K \in [L, 0]$ so that the singular loci form links and the cone angles all are less than θ. Let $\mathcal{C}^{<\theta}_{K}$ be a subset of $\mathcal{C}^{<\theta}_{[L,0]}$ consisting of cone-manifolds with a particular curvature constant K.

Now take a cone-manifold $C \in \mathcal{C}^{<2\pi}_{K}$. Given a point p on the singular locus Σ, let $S_r(p)$ denote the set of points at distance r from p. Then for r sufficiently small, $S_r(p)$ with the induced path metric is a 2-cone-manifold of constant positive curvature with two cone points, since Σ is a submanifold of codimension 2 and does not have vertices. The cone angle of $S_r(p)$ at each cone point is equal to that of the component of Σ on which the point p lies.

Take a point $x \in C - \Sigma$. Then define the following subset of C,

$$P_x := \{y \in C \mid y \text{ admits the unique shortest path to } x\},$$

and call it a Dirichlet fundamental domain of C about x.

Lemma 1. *The Dirichlet fundamental domain P_x of $C \in \mathcal{C}_K^{<2\pi}$ about $x \in C - \Sigma$ has the following properties.*

(1) *P_x is isometrically realized as an interior of a star-shaped geodesic polyhedron in the simply connected 3-dimensional space $\mathbf{H}_K$ of constant curvature K. The closure is a star-shaped geodesic polyhedron. We call this embedded compactified polyhedron a Dirichlet polyhedron of C about x, and denote it again by P_x.*

(2) *Let y be a point on Σ and $\gamma_1, \ldots, \gamma_n$ be shortest geodesics from x to y in C. For sufficiently small r, let $v_1, \ldots, v_n$ be the intersections of $\gamma_1, \ldots, \gamma_n$ with $S_r(y)$. Let s_1, s_2 be the two cone points of $S_r(y)$. Let $V(v_i)$ be the Voronoi region at v_i on $S_r(y)$, which is defined as follows:*

$$V(v_i) := \left\{ q \in S_r(y) \;\; ; \begin{array}{l} d(q, v_i) \leq d(q, v_j) \text{ for all } j \neq i, \\ \text{and there is a unique shortest geodesic} \\ \text{from } q \text{ to } \{v_1, \ldots, v_n\} \end{array} \right\}.$$

Then a small neighborhood of y splits into the cones on the Voronoi regions $V(v_i)$'s in the Dirichlet polyhedron P_x. Moreover, the cones which include the point s_j ($j = 1$ or 2) satisfy the following properties.

(2-1) *If there is only one shortest geodesic from s_j to $\{v_1, \ldots, v_n\}$ on $S_r(y)$, say $\overline{s_j v_1}$, then the cone on the Voronoi region $V(v_1)$ is bounded by two faces of P_x sharing a part of Σ as edges, whose dihedral angle equals to the cone angle at y. Moreover, v_1 and x are contained in the bisecting (totally geodesic) surface of these two faces.*

(2-2) *If there are at least two shortest geodesics from s_j to $\{v_1, \ldots, v_n\}$ on $S_r(y)$, say $\overline{s_j v_1}, \ldots, \overline{s_j v_k}$, then for each $i \in \{1, \ldots, k\}$, the cone on $V(v_i)$ is bounded by two faces of P_x sharing a part of Σ as edges whose dihedral angle is at most a half of the cone angle at y, which is smaller than π.*

Proof. See Proposition 3.1.4 (and also its proof) in Cooper-Hodgson-Kerckhoff [1]. □

If $x \in C - \Sigma$, the injectivity radius of C at x is to be the injectivity radius of $C - \Sigma$ at x. Denote it by $\mathrm{inj}_x C$. The key lemma in this paper is the following one which is proved in the same manner as Kojima [5], except for a part where the property (2) of Lemma 1 is used. Kojima showed the following lemma with cone angle condition "$\leq \pi$". In this case, a Dirichlet polyhedron is convex, and then the property (2) of the Lemma 1 is not necessary.

Lemma 2. *Given positive numbers D, I, $R > 0$, and a curvature bound $L \leq -1$, there is a constant $U := U(D, I, R, L) > 0$ so that if $C \in \mathcal{C}_{[L,0]}^{<2\pi}$, $x \in C$*

with $d(x, \Sigma) \geq D$ *and* $\mathrm{inj}_x C \geq I$*, then*

$$\mathrm{inj}_y C \geq U$$

for any $y \in C$ *with* $d(y, \Sigma) \geq D$ *and* $d(y, x) \leq R$*.*

Proof. Suppose that there is not such a uniform bound U. Then, for some numbers D, I, $R > 0$ and $L \leq -1$, there exists a sequence of cone-manifolds $\{C_i\}_{i=1}^{\infty} \subset \mathcal{C}_{[L,0]}^{<2\pi}$ and points x_i, $y_i \in C_i$ such that

(i) $d(x_i, \Sigma_i) \geq D, d(y_i, \Sigma_i) \geq D$,

(ii) $\mathrm{inj}_{x_i} C_i \geq I$,

(iii) $d(y_i, x_i) \leq R$ and

(iv) $\mathrm{inj}_{y_i} C_i \leq 1/i$.

Take a Dirichlet polyhedron P_{y_i} of C_i about y_i in $\mathbf{H}_{K_i}$, where K_i is a curvature of C_i. There are points a_i, b_i on ∂P_{y_i}, which are identified in C_i and attain the shortest distance to y_i from ∂P_{y_i}. The union of these shortest paths $\overline{a_i y_i}$, $\overline{b_i y_i}$ forms a homotopically nontrivial shortest loop l_i in C_i based at y_i.

If i is large enough, then by (i) and (iv), a_i and b_i are not on the singular locus Σ_i. Also they are not points on edges of ∂P_{y_i} and in particular nonsingular. Then they are on the interior of faces of P_{y_i} respectively. Let us denote the faces by A_i and B_i and their extensions in $\mathbf{H}_{K_i}$ by $\tilde{A}_i$ and $\tilde{B}_i$. By Lemma 1, if there are singular points on the boundaries of A_i or B_i, the dihedral angles of P_{y_i} at these points are smaller than π.

If all dihedral angles of P_{y_i} are smaller than or equal to π, P_{y_i} is convex and then P_{y_i} is bounded by $\tilde{A}_i$ and $\tilde{B}_i$.

Consider the case where dihedral angles along some edges of ∂P_{y_i} are greater than π. It follows from Lemma 1 (2) that for each of such edges, there is a totally geodesic surface including it which bisects its neighborhood of P_{y_i}. By Lemma 1 (1), P_{y_i} is star-shaped with respect to y_i. Then we can divide P_{y_i} into convex subregions by cutting along the extensions in $\mathbf{H}_{K_i}$ of such bisecting surfaces.

Let $\phi_i(\leq \pi)$ be the angle between the segments $\overline{a_i y_i}$ and $\overline{b_i y_i}$ at y_i. Now assume that $\phi_i \to \pi$ as $i \to \infty$. Then $\tilde{A}_i$ and $\tilde{B}_i$ tend to be parallel and converge to a totally geodesic surface H by (iv). Moreover, by (i) $d(y_i, \Sigma_i) \geq D > 0$, the bisecting totally geodesic surfaces in $\mathbf{H}_{K_i}$, along which P_{y_i} are divided into the convex subregions, are crushed into H as geodesic segments or subsurfaces. Then the convex subregions are also crushed into H as $i \to \infty$. Therefore $\mathrm{vol}(B_{R+I}(C_i, y_i)) \to 0$ as $i \to \infty$, since P_{y_i} $(\supset B_{R+I}(C_i, y_i))$ is crushed into the surface H. This is a contradiction since $B_I(C_i, x_i) \subset B_{R+I}(C_i, y_i)$ by (iii) and $\mathrm{vol}(B_I(C_i, x_i))$ is uniformly bounded by a positive constant from below by (ii). Thus there is a number ϕ so that $\phi_i \leq \phi < \pi$. Therefore the loop l_i bends at y_i with angle uniformly away from π.

Let us lift l_i to a geodesic segment s_i in $\mathbf{H}_{K_i}$, based at y_i so that a_i is its middle point. Let ρ_i be a holonomy representation of C_i; $\rho_i : \pi_1(C_i - \Sigma_i) \to \mathrm{Isom}\, \mathbf{H}_{K_i}$. Then the action of $\rho_i(l_i)$ on $\mathbf{H}_{K_i}$ is either parabolic, loxodromic or elliptic. In any cases, the orbit of s_i by the action of a group generated by

$\rho_i(l_i)$ forms a piecewise geodesic which bends with angle uniformly away from π, and the length of s_i goes to 0 when $i \to \infty$.

Assume that there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are parabolic. Then the bending angle of the orbit of s_k at the orbit of y_k should approaches π as $k \to \infty$, since the length of s_k goes to 0 as $k \to \infty$. This gives a contradiction.

Assume that there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are loxodromic. Then the orbit of s_k squeezes onto the axis of $\rho_k(l_k)$, since the length of s_k approaches 0 when $k \to \infty$ and since the orbit of s_k bends at the orbit of y_k with corner angle uniformly away from π with respect to k. In particular, the axis of $\rho_k(l_k)$ becomes close to y_k when $k \to \infty$. Thus, if k is large enough, there is a very short simple closed geodesic in C_k near y_k. Then choose a new reference point z_k on this simple closed geodesic, take the Dirichlet polyhedron P_{z_k} about z_k, consider two faces of P_{z_k} and perform the same argument as before. This gives a contradiction.

Therefore $\rho_i(l_i)$ all but finitely many exceptions are elliptic. Take a subsequence $\{j\} \subset \{i\}$ so that $\rho_j(l_j)$ all are elliptic. The orbit of s_j rounds around a geodesic which is an extension of a lift of a component of Σ_j. Then y_j approaches the geodesic, since the length of s_j goes 0 when $j \to \infty$ and since the orbit of s_j bends at the orbit of y_j with corner angle uniformly away from π with respect to j. This contradicts (i). □

2. Strong convergence of hyperbolic 3-cone-manifolds

Let C be a compact orientable hyperbolic 3-cone-manifold with singularity Σ. We assume that the singular set Σ forms a link

$$\Sigma = \Sigma^1 \cup \cdots \cup \Sigma^n$$

as in Section 1, and that the cusp Λ of C is empty. Let $\mathcal{T}$ be the maximal tube about Σ, that is, a union of open tubular neighborhoods $\mathcal{T}^j$'s which has the following properties,

(a) each component $\mathcal{T}^j$ is an equidistant tubular neighborhood to the j-th component Σ^j of Σ,

(b) among ones having the property (a), the set of radii arranged in order of magnitude from the smallest one is maximal in lexicographical order.

Let us denote by $\partial\mathcal{T}^j$ an abstract boundary of $\mathcal{T}^j$. The actual boundary $\partial\mathcal{T}$ of $\mathcal{T}$ in C is a union of isometrically embedded tori with a finite number of contact points. The first contact point on $\partial\mathcal{T}$ is defined to be the point which admits two shortest paths to Σ from $\partial\mathcal{T}$. The finest point on $\partial\mathcal{T}$ is defined to be the point on $\partial\mathcal{T}$ which attains the minimum among $\{\mathrm{inj}_x(C) | x \in \partial\mathcal{T}\}$.

Now take a sequence $\{C_i\}_{i=1}^{\infty}$ of compact orientable hyperbolic 3-cone-manifolds with the following four properties,

(1) each C_i is a deformation of C with a reference homeomorphism $\xi_i : (C, \Sigma) \to (C_i, \Sigma_i)$,

(2) c_i, f_i are the first contact point and the finest point on $\partial\mathcal{T}_i$ respectively,

(3) $\alpha_i^j < 2\pi$ for all $1 \leq j \leq n$ and any $i \in \mathbf{N}$, where α_i^j is a cone angle along the component Σ_i^j,

(4) $\{\alpha_i^j\}_{i=1}^{\infty}$ converges to a number $\beta^j \in [0, 2\pi]$ for all $1 \leq j \leq n$.

We briefly review the definitions of three kinds of convergence; geometric convergence, algebraic convergence and strong convergence.

If X is a metric space and x is a point in X, the pair (X, x) is called a pointed metric space. The sequence $\{(C_i, c_i)\}_{i=1}^{\infty}$ is said to converge geometrically to a pointed metric space (X, x) if it converges to (X, x) on the pointed Hausdorff-Gromov topology. See Gromov [2] or Kojima [5] for the definition of the pointed Hausdorff-Gromov topology.

The sequence $\{C_i\}_{i=1}^{\infty}$ is said to converge algebraically to a hyperbolic 3-cone-manifold Y if Y is homeomorphic to C and a sequence $\{\rho_i\}_{i=1}^{\infty}$ of holonomy representations of C_i converges to a holonomy representation ρ_Y of Y in the space of representations $\mathrm{Hom}(\pi_1(C - \Sigma), \mathrm{PSL}_2(\mathbf{C}))$ with respect to the identification by ξ_i.

The sequence $\{(C_i, c_i)\}_{i=1}^{\infty}$ is said to converge strongly if the sequence $\{(C_i, c_i)\}_{i=1}^{\infty}$ converges geometrically to a pointed hyperbolic 3-cone-manifold (Y, y) and the sequence $\{C_i\}_{i=1}^{\infty}$ converges algebraically to Y.

Theorem. *Let C be a compact hyperbolic 3-cone-manifold and $\{(C_i, c_i)\}_{i=1}^{\infty}$ be a sequence of pointed compact orientable hyperbolic 3-cone-manifolds as above. Suppose that there is a constant $D_1 > 0$ such that $D_1 \leq$ radius $\mathcal{T}_i^j$ for any $1 \leq j \leq n$ and any $i \in \mathbf{N}$. Then there is a subsequence $\{(C_{i_m}, c_{i_m})\}_{m=1}^{\infty}$ which converges strongly to a pointed hyperbolic 3-cone-manifold (C_*, c_*). The limit C_* is homeomorphic to C. If $\beta^j > 0$ for all $1 \leq j \leq n$, then C_* is compact.*

Remark. The property (3) induces the following one,

(5) there is a constant V_{max} such that $\mathrm{vol}(C_i) \leq V_{max}$.

Remark. By the argument on geometric convergence due to Gromov [2], it can be shown that the following property is satisfied,

(6) the sequence $\{(C_i, c_i)\}_{i=1}^{\infty}$ has a subsequence $\{(C_{i_k}, c_{i_k})\}_{k=1}^{\infty}$ which converges geometrically to a complete metric space.

Proof. Take a subsequence $\{i_k\} \subset \{i\}$ which satisfies the properties (1),…,(6). By choosing a further subsequence, we may assume that the sequence $\{C_{i_k}\}_{k=1}^{\infty}$ satisfies the following properties also,

(7) c_{i_k} lies on a component $\partial\mathcal{T}_{i_k}^c$ with a constant reference number c, and

(8) f_{i_k} lies on a component $\partial\mathcal{T}_{i_k}^f$ with a constant reference number f.

Then the sequence $\{C_{i_k}\}_{k=1}^{\infty}$ has the same property as in Kojima [5, Section 4], except for the condition on the range of the cone angles.

By following the arguments described in Sections 3 and 5 of [5], we can verify that Corollary 5.1.4 of [5] holds with replacing the cone angle condition "$\alpha_i^j \leq \pi$" with "$\alpha_i^j < 2\pi$", if Lemma 3.1.1 of [5] holds with the cone angle condition "$< 2\pi$". Lemma 2 is exactly such a version of Lemma 3.1.1 of [5]. Then Corollary 5.1.4 of [5] with the cone angle condition "$\alpha_i^j < 2\pi$" holds. This is what we need. □

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