On a *p*-local stable splitting of U(n)

By

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1. Introduction

Let U(n) be the unitary group. H. Miller [5] has introduced a filtration $\{1\} = R^0(\mathbb{C}^n) \subset R^1(\mathbb{C}^n) \subset \cdots \subset R^n(\mathbb{C}^n) = U(n)$ and shown that there is a stable splitting

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+$$

where $R_k(\mathbb{C}^n) = R^k(\mathbb{C}^n) - R^{k-1}(\mathbb{C}^n)$ and $R_k(\mathbb{C}^n)^+ \cong R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$ is the one-point compactification of $R_k(\mathbb{C}^n)$. The filtration is given by counting the dimension of direct sum of eigen subspaces in which eigenvalues are different from 1. Later M. Crabb [1] gave a simpler construction of the stable splitting.

On the other hand it is known [6] that for a prime p there is a p-local unstable decomposition

$$U(n) \simeq X_1(n) \times \cdots \times X_{p-1}(n)$$

into a product of p-1 spaces, using the unstable Adams operations with the $X_i(n)$'s satisfying $H^*(X_i(n); \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(x_i, x_{i+p-1}, \dots, x_{i+s(p-1)})$ where $s = [\frac{n-i}{p-1}]$. Then we obtain a *p*-local stable splitting

$$U(n) \simeq \bigvee X_{i_1}(n) \wedge \dots \wedge X_{i_s}(n), 1 \le i_1 < \dots < i_s \le p-1.$$

Then a natural question arises:

Question. Can one mix those stable splittings and obtain a finer one at an odd prime?

In [2], Crabb, Hubbuck and McCall have shown that $R_k(\mathbb{C}^n)^+$ splits stably as $(\Sigma SR_{k-1}(\mathbb{C}^n)^+) \vee SR_k(\mathbb{C}^n)^+$ (written by $SX_k(n)$ in [2]), where $SR_k(\mathbb{C}^n)^+$ is the space in the stable splitting of SU(n), and $SR_2(\mathbb{C}^n)^+$ is stably indecomposable at the prime 2. The purpose of this paper is to give an affirmative answer of the question at an odd prime. Our main theorem is

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Theorem 1.1. (1) Let n be a positive integer and let p be an odd prime. Then for each pair (t,k) of integers such as $1 \le t \le p-1$ and $1 \le k \le n$, there exists a finite spectrum $X_{t,k}(n)$ satisfying

$$H_*(X_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$$

as a module where α is the maximal integer such that $t + \alpha(p-1) \leq n$

(2) There exists a stable p-equivalence

$$U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \dots \wedge X_{t_m,k_m}(n)$$

where the wedge sum is taken over $1 \leq t_1 < \cdots < t_m \leq p-1$ and $k_1 + \cdots + k_m \leq n$. Here $\Lambda_{\mathbb{Z}}^k(y_t, y_{t+p-1}, \ldots, y_{t+\alpha(p-1)})$ means the module of k-th exterior powers of $y_t, y_{t+p-1}, \ldots, y_{t+\alpha(p-1)}$ and deg $y_i = 2i - 1$.

We should mention that in Example 3.12 [3], it is given a simpler splitting without proof. A weaker version of the theorem is also given from the result of Harper and Zabrodsky [4]. They have shown that a mod p H-space Y with $H_*(Y; \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(y_{i_1}, \ldots, y_{i_t})$ has a stable splitting $Y \simeq Y_1 \vee \cdots \vee Y_{p-1}$ such that $H_*(Y_a; \mathbb{Z}_{(p)}) \cong \bigoplus_{k \equiv a(p-1)} \Lambda_{\mathbb{Z}_{(p)}}^k(y_{i_1}, \ldots, y_{i_t})$ for $1 \le a \le p-1$. Applying this to the mod p H-space $X_i(n)$, we obtain spectra $Y_{i,j}(n), 1 \le i, j \le p-1$ such that $X_i(n) \simeq Y_{i,1}(n) \vee \cdots \vee Y_{i,p-1}(n)$ and

$$H_*(Y_{i,j}(n);\mathbb{Z}_{(p)}) \cong \bigoplus_{k\equiv j(p-1)} \Lambda^k_{\mathbb{Z}_{(p)}}(x_{i_1}, x_{i_1+p-1}, \dots, x_{i_1+\alpha(p-1)}).$$

The idea of proof is roughly as follows. A stable splitting of a space or spectrum Y corresponds to a splitting $1 = e_1 + \cdots + e_k$ of the unity into a sum of orthogonal idempotents in the ring $\{Y, Y\}$ of homotopy classes of stable selfmaps of Y. If we are given two splitting $1 = e_1 + \cdots + e_k$ and $1 = f_1 + \cdots + f_l$ and if $e_i f_j = f_j e_i$ for all i, j, then we have a finer splitting $1 = \Sigma e_i f_j$. But this is too strong condition. Let $\psi : \{Y, Y\} \longrightarrow \text{End } H_*(Y)$ be a natural ring homomorphism. Write $\psi(e) = e_*$. Then the assumption of the above statement can be weakened as $(e_i)_*(f_j)_* = (f_j)_*(e_i)_*$ for all i, j. The proof of the main theorem is done by checking that this weak condition holds for our two stable splittings of U(n).

2. Multiplicative property of stable splittings

We define an increasing filtration of the unitary group U(n)

$$\{1\} = R^0(\mathbb{C}^n) \subset R^1(\mathbb{C}^n) \subset \dots \subset R^n(\mathbb{C}^n) = U(n)$$

by

$$R^{k}(\mathbb{C}^{n}) = \{g \in U(n) | \dim_{\mathbb{C}}(\operatorname{Ker}(g-1)) \ge n-k\}, \quad 0 \le k \le n$$

where 1 denotes the unit element of U(n). The difference $R^k(\mathbb{C}^n) - R^{k-1}(\mathbb{C}^n) = \{g \in U(n) | \dim_{\mathbb{C}}(\operatorname{Ker}(g-1)) = n-k\}$ is written by $R_k(\mathbb{C}^n)$. The one-point compactification $R_k(\mathbb{C}^n)^+$ is identified with the quotient space $R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$. The pointed natural projection map is denoted by $\pi_{n,k} : R^k(\mathbb{C}^n)^+ \longrightarrow R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n) = R_k(\mathbb{C}^n)^+$. Note that $R^k(\mathbb{C}^n)$ consists of unitary matrices A such that the dimension of the sum of eigenspaces of A with eigenvalue different from 1 is equal or less than k.

H. Miller [5] and M. Crabb [1] has shown the following

Theorem 2.1. There exists a stable splitting

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+.$$

We recall the construction of the splitting following [1]. Let $\operatorname{End}(\mathbb{C}^k)$ be the space of all $k \times k$ matrices. We regard $\operatorname{End}(\mathbb{C}^k)$ as a U(k)- space by the adjoint action. Then we have following U(k)-invariant subspaces of $\operatorname{End}(\mathbb{C}^k)$:

- (1) $\mathcal{H}(k)$: the space of Hermitian matrices,
- (2) $\mathfrak{u}(k)$: the space of skew-Hermitian matrices,
- (3) U(k): the unitary group,
- (4) $U(k)_0 = \{g \in U(k) | g 1 \text{ is invertible } \}.$

Let $G_{n,k}$ and $V_{n,k}$ be the Grassman manifold of k-planes in \mathbb{C}^n and Stiefel manifold of k-frames in \mathbb{C}^n , respectively. Let $U(k) \longrightarrow V_{n,k} \longrightarrow G_{n,k}$ be the standard principal U(k)-bundle. We denote this bundle simply by $\zeta_{n,k}$. Let F be a manifold with a U(k)-action. We denote the associate fibre bundle by $\zeta_{n,k}(F)$. The Cayley transform

$$\psi:\mathfrak{u}(k)\longrightarrow U(k)_0$$

defined by $\psi(X) = (X-1)(X+1)^{-1}$ is a U(k)-equivariant diffeomorphism.

Consider the bundle $\zeta_{n,k}(U(k)) \longrightarrow G_{n,k}$; the fibre over $E \in G_{n,k}$ is the unitary group U(E) of a k-dimensional subspace $E \subseteq \mathbb{C}^n$, and we can regard an elements of $\zeta_{n,k}(U(k))$ as a pair (E, f) where $E \in G_{n,k}, f \in U(E)$. Now we have a surjective map $p_{n,k} : \zeta_{n,k}(U(k)) \longrightarrow R^k(\mathbb{C}^n)$ given by

$$p_{n,k}(E,f) = f \oplus 1 : E \oplus E^{\perp} \longrightarrow E \oplus E^{\perp}$$

It is clear that the restriction of $p_{n,k}$ gives a homeomorphism

$$\zeta_{n,k}(U(k)_0) \xrightarrow{\sim} R_k(\mathbb{C}^n).$$

Now we can use the Cayley transform to identify the bundle $\zeta_{n,k}(U(k)_0)$ over $G_{n,k}$ with the Lie algebra bundle $\zeta_{n,k}(\mathfrak{u}(k))$. Thus we have

Proposition 2.2. There is a natural diffeomorphism between $R_k(\mathbb{C}^n)$ and the total space of the vector bundle $\zeta_{n,k}(\mathfrak{u}(k))$ over $G_{n,k}$. We recall the Iwasawa decomposition $GL(k, \mathbb{C}) \cong U(k) \times P(k)$ where P(k) is the space of positive definite Hermitian matrices. The exponential map $\exp : \mathcal{H}(k) \longrightarrow P(k)$ is a diffeomorphism and hence we have a diffeomorphism

$$1 \times \exp: U(k) \times \mathcal{H}(k) \longrightarrow GL(k, \mathbb{C})$$

which is clearly U(k)-equivariant. We now construct a stable map $\sigma_{n,k}$: $R_k(\mathbb{C}^n)^+ \longrightarrow R^k(\mathbb{C}^n)^+$ as follows. We have a diffeomorphism

$$\zeta_{n,k}(U(k) \times \mathcal{H}(k)) \xrightarrow{\simeq} \zeta_{n,k}(GL(k,\mathbb{C}))$$

and an open embedding

$$\zeta_{n,k}(GL(k,\mathbb{C})) \subset \zeta_{n,k}(\operatorname{End}(\mathbb{C}^k)) \cong \zeta_{n,k}(\mathfrak{u}(k) \oplus \mathcal{H}(k)).$$

Note that $\zeta_{n,k}(U(k) \times \mathcal{H}(k))$ is identified with the fibre product $\zeta_{n,k}(U(k)) \\ \times \zeta_{n,k}(\mathcal{H}(k))$ of the spaces over $G_{n,k}$, and similarly for $\zeta_{n,k}(\mathfrak{u}(k) \oplus \mathcal{H}(k))$. Since $\zeta_{n,k}(\mathcal{H}(k))$ is a real vector bundle over a compact space $G_{n,k}$, there is an embedding $\zeta_{n,k}(\mathcal{H}(k)) \subset G_{n,k} \times \mathbb{R}^N$ into the product bundle for some integer N. Let γ be the orthogonal complement of $\zeta_{n,k}(\mathcal{H}(k))$ so that $\zeta_{n,k}(\mathcal{H}(k)) \oplus \gamma = G_{n,k} \times \mathbb{R}^N$. Then applying $\underset{G_{n,k}}{\times} \gamma$ to the above diffeomorphism and the embedding, we have an open embedding $\zeta_{n,k}(U(k)) \times \mathbb{R}^N \subset \zeta_{n,k}(\mathfrak{u}(k)) \times \mathbb{R}^N$ and applying the Pontrjagin-Thom construction we obtain a stable map $s_{n,k}$: $\zeta_{n,k}(\mathfrak{u}(k))^+ \longrightarrow \zeta_{n,k}(U(k))^+$. By a usual argument it is easy to see that the homotopy class of the stable map $s_{n,k}$ does not depend on a choice of an embedding $\zeta_{n,k}(\mathcal{H}(k)) \subset G_{n,k} \times \mathbb{R}^N$. Now we define the stable map $\sigma_{n,k}$ as the following composition

$$\sigma_{n,k}: R_k(\mathbb{C}^n)^+ \xrightarrow{(p_{n,k}^{-1})^+} \zeta_{n,k}(\mathfrak{u}(k))^+ \xrightarrow{s_{n,k}} \zeta_{n,k}(U(k))^+ \xrightarrow{(p_{n,k})^+} R^k(\mathbb{C}^n)^+.$$

The next theorem is due to M. Crabb [1].

Theorem 2.3. $\pi_{n,k} \circ \sigma_{n,k}$ is homotopic to identity.

We denote the inclusion maps $R^k(\mathbb{C}^n) \longrightarrow R^{k+1}(\mathbb{C}^n)$ by j. Then we have a stable map $j \circ \sigma_{n,k} : R_k(\mathbb{C}^n)^+ \longrightarrow U(n)^+$ and by taking a wedge sum we have

$$\bigvee_{k=0}^{n} R_k(\mathbb{C}^n)^+ \longrightarrow \bigvee U(n)^+ \longrightarrow U(n)^+$$

which is clearly a homotopy equivalence. Since $R_0(\mathbb{C}^n)^+ \simeq S^0$, we finally obtain a stable homotopy equivalence

$$\bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+ \longrightarrow U(n).$$

Now we recall basic facts about the homology of $R^k(\mathbb{C}^n)$ (see, e.g. [7]). Define a map $\rho: \Sigma(\mathbb{C}P^{n-1}_+) \longrightarrow U(n)$ by

$$\rho(\lambda, z) = (\delta_{i,j} + (\lambda - 1)z_i\bar{z}_j), \quad 1 \le i, j \le n$$

for $\lambda \in S^1, z = [z_1; \ldots; z_n]$ and $z = (z_1, \ldots, z_n) \in S^{2n-1} \subset \mathbb{C}^n$. If $\lambda \neq 1$, then $\rho(\lambda, z)$ is the matrix with λ as a unique non 1 eigenvalue with the eigenvector (z_1, \ldots, z_n) . Hence we see that Im $\rho = R^1(\mathbb{C}^n)$. Let $x_i = \rho_*(\sigma(s_{i-1})) \in H_{2i-1}(U(n); \mathbb{Z})$ where σ is the homology suspension and $s_{i-1} \in H_{2i-2}(\mathbb{C}P^{n-1}; \mathbb{Z})$ is a generator. Then we have an isomorphism

$$H_*(U(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_1, x_2, \dots, x_n)$$

of the homology ring with the exterior algebra generated by x_1, x_2, \ldots, x_n . Let $j: R^k(\mathbb{C}^n) \longrightarrow U(n)$ be the inclusion. Then we have

Proposition 2.4. The homomorphism

$$j_*: H_*(R^k(\mathbb{C}^n); \mathbb{Z}) \longrightarrow H_*(U(n); \mathbb{Z})$$

is injective and $\text{Im } j_*$ is spanned by $x_{i_1} \cdots x_{i_s}$, $1 \le i_1 < \cdots < i_s \le n$, $s \le k$.

We identify $H_*(\mathbb{R}^k(\mathbb{C}^n);\mathbb{Z})$ with its image in $H_*(U(n);\mathbb{Z})$. Now consider the cofibration

$$R^{k-1}(\mathbb{C}^n) \xrightarrow{j} R^k(\mathbb{C}^n) \xrightarrow{\pi_{n,k}} R^k(\mathbb{C}^n) / R^{k-1}(\mathbb{C}^n) \cong R_k(\mathbb{C}^n)^+.$$

Then we easily obtain the following

Proposition 2.5. The homomorphism

$$(\pi_{n,k})_*: H_*(R^k(\mathbb{C}^n); \mathbb{Z}) \longrightarrow H_*(R_k(\mathbb{C}^n)^+; \mathbb{Z})$$

is surjective and $\operatorname{Ker}(\pi_{n,k})_*$ is spanned by $x_{i_1} \cdots x_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq n$, $s \leq k-1$.

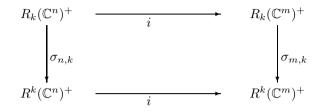
We write $(\pi_{n,k})_*(x_{i_1}\cdots x_{i_k}) \in \tilde{H}_*(R_k(\mathbb{C}^n)^+;\mathbb{Z})$ by the same symbol $x_{i_1}\cdots x_{i_k}$. Let $\Lambda^k_{\mathbb{Z}}(x_1,\ldots,x_n) = \mathbb{Z}\{x_{i_1}\cdots x_{i_k}\}$ be the module of k-th exterior powers. Then we have an isomorphism

$$H_*(R_k(\mathbb{C}^n)^+;\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(x_1,\ldots,x_n)$$

as an abelian group.

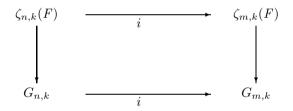
Let $n \leq m$ be integers and let $i : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be the inclusion into the first n coordinates. Then we have associated inclusions $U(n) \longrightarrow U(m), G_{n,k} \longrightarrow G_{m,k}, R^k(\mathbb{C}^n) \longrightarrow R^k(\mathbb{C}^m)$ and $R_k(\mathbb{C}^n) \longrightarrow R_k(\mathbb{C}^m)$ for $0 \leq k \leq n$. We write all inclusions by the same i. Note that the last map is proper.

Lemma 2.6. The diagram

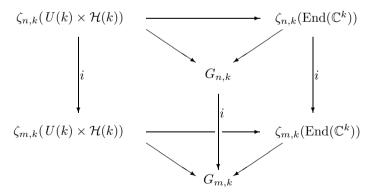


is homotopy commutative.

 $\mathit{Proof.}\;$ Let F be a $U(k)\text{-space.}\;$ Then by definition we have a pull-back diagram



of associated fibre bundles. Consider the following commutative diagram



where the horizontal maps are induced from the exponential map. Then by a simple diagram chasing, we see that the square diagram is pull-back. Now we take a real vector bundle γ_m such that $\zeta_{m,k}(\mathcal{H}(k)) \oplus \gamma_m \cong G_{m,k} \times \mathbb{R}^N$ and let $\gamma_n = \gamma_m \mid_{G_{n,k}}$. Then the associated commutative diagram

is clearly a pull-back diagram. Hence the diagram

$$\begin{aligned} \zeta_{n,k}(U(k)) \times \mathbb{R}^N & \subset & \zeta_{n,k}(\mathfrak{u}(k)) \times \mathbb{R}^N \\ & & \downarrow \\ \zeta_{m,k}(U(k)) \times \mathbb{R}^N & \subset & \zeta_{m,k}(\mathfrak{u}(k)) \times \mathbb{R}^N \end{aligned}$$

is pull-back and it gives a commutative diagram of the Pontrjagin-Thom constructions. The naturality concerning the map $p_{n,k}$ is obvious and this completes the proof.

Now let $A \in R^k(\mathbb{C}^n)$ and $B \in R^l(\mathbb{C}^n)$, then it is clear that the composition $AB \in R^{k+l}(\mathbb{C}^n)$. Thus we obtain a pairing

$$\mu: R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \longrightarrow R^{k+l}(\mathbb{C}^n).$$

Note that $\mu(R^{k-1}(\mathbb{C}^n) \times R^l(\mathbb{C}^n)) \subset R^{k+l-1}(\mathbb{C}^n)$ and $\mu(R^k(\mathbb{C}^n) \times R^{l-1}(\mathbb{C}^n)) \subset R^{k+l-1}(\mathbb{C}^n)$. Therefore, by identifying $R_k(\mathbb{C}^n)^+$ with $R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$, we have an induced pairing

$$\mu: R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \longrightarrow R_{k+l}(\mathbb{C}^n)^+.$$

Next let $\varphi: U(n) \times U(n) \longrightarrow U(2n)$ be the map given by

$$\varphi(A,B) = \left(\begin{array}{cc} A & 0\\ 0 & B \end{array}\right).$$

This induces pairings

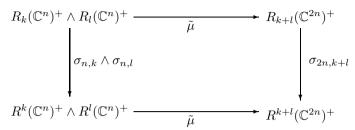
$$\tilde{\mu}: R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \longrightarrow R^{k+l}(\mathbb{C}^{2n})$$

and

$$\tilde{\mu}: R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \longrightarrow R_{k+l}(\mathbb{C}^{2n})^+$$

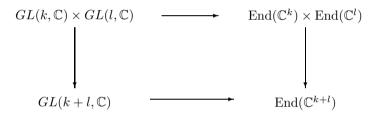
in a natural way. Now we have

Lemma 2.7. The diagram

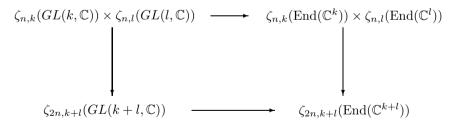


is homotopy commutative.

Proof. First note that the natural diagram

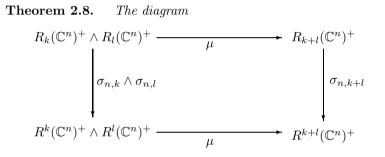


is a pull-back diagram, where all maps are inclusion maps. The horizontal maps are open embeddings and the vertical maps are proper. The upper horizontal and lower horizontal maps are $U(k) \times U(l)$ and U(k+l) equivariant and vertical maps are equivariant with respect to the $U(k) \times U(l)$ -action on $\operatorname{End}(\mathbb{C}^{k+l})$ induced by the inclusion $U(k) \times U(l) \subset U(k+l)$. Therefore we have an induced commutative diagram



We can directly check that the diagram is pull-back. We can identify $\zeta_{n,k}(GL(k, \mathbb{C}))$ with $\zeta_{n,k}(U(k) \times \mathcal{H}(k))$ and similarly for $\zeta_{n,l}(GL(l, \mathbb{C}))$ and $\zeta_{2n,k+l}(GL(k+l, \mathbb{C}))$. Let γ_{k+l} be a vector bundle over $G_{2n,k+l}$ such that $\zeta_{2n,k+l}(\mathcal{H}(k+l)) \oplus \gamma_{k+l} \cong G_{2n,k+l} \times \mathbb{R}^N$. Let $f: G_{n,k} \times G_{n,l} \longrightarrow G_{2n,k+l}$ be the natural map. Note that $\mathcal{H}(k+l) \cong \mathcal{H}(k) \oplus \mathcal{H}(l) \oplus \mathbb{C}^{kl}$ as $U(k) \times U(l)$ -module where \mathbb{C}^{kl} is a trivial $U(k) \times U(l)$ -module. Hence $f^*(\zeta_{2n,k+l}(\mathcal{H}(k+l))) \cong \zeta_{n,k}(\mathcal{H}(k)) \times \zeta_{n,l}(\mathcal{H}(l)) \oplus (2kl)\epsilon$, where $(2kl)\epsilon$ is the trivial real bundle of dimension 2kl. Therefore $\zeta_{n,k}(\mathcal{H}(k)) \times \zeta_{n,l}(\mathcal{H}(l)) \oplus f^*\gamma_{k+l} \oplus (2kl)\epsilon$ is isomorphic to a trivial bundle. Hence we have a pull-back diagram

of open embeddings and proper maps. Thus we obtain a homotopy commutative diagram of the Pontrjagin-Thom construction. The rest of the proof is similar to Lemma 2.6. $\hfill \square$



induces a commutative diagram in homology groups.

Proof. First consider the homotopy

$$H: U(n) \times U(n) \times I \longrightarrow U(2n)$$

defined by

$$H(A, B, t) = \begin{pmatrix} A & 0 \\ 0 & \mathbf{I}_n \end{pmatrix} \begin{pmatrix} \cos t \, \mathbf{I}_n & \sin t \, \mathbf{I}_n \\ -\sin t \, \mathbf{I}_n & \cos t \, \mathbf{I}_n \end{pmatrix} \begin{pmatrix} \mathbf{I}_n & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos t \, \mathbf{I}_n & -\sin t \, \mathbf{I}_n \\ \sin t \, \mathbf{I}_n & \cos t \, \mathbf{I}_n \end{pmatrix} + \mathbf{I}_n = \mathbf{I}_n \mathbf{I}_n$$

It is clear that $H(R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \times I) \subset R^{k+l}(\mathbb{C}^{2n})$ and gives a homotopy $\tilde{\mu} \sim i \circ \mu : R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \longrightarrow R^{k+l}(\mathbb{C}^{2n})$. It is also clear that H induces a map

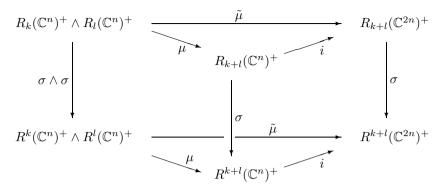
$$\tilde{H}: (R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n) \wedge R^l(\mathbb{C}^n)/R^{l-1}(\mathbb{C}^n)) \times I \longrightarrow R^{k+l}(\mathbb{C}^{2n})/R^{k+l-1}(\mathbb{C}^{2n})$$

which gives a homotopy

$$\tilde{\mu} \sim i \circ \mu : R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \longrightarrow R_{k+l}(\mathbb{C}^{2n})^+.$$

Next consider the diagram

Goro Nishida and Yeong-mee Yang



The triangles are homotopy commutative by the above argument. The square and the right parallellogram are homotopy commutative by Lemmas 2.7 and 2.6, respectively. Note that the homomorphism

$$i_*: H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(2n); \mathbb{Z})$$

is a monomorphism and by Lemma 2.6 we see that

$$i_*: H_*(\mathbb{R}^{k+l}(\mathbb{C}^n)^+; \mathbb{Z}) \longrightarrow H_*(\mathbb{R}^{k+l}(\mathbb{C}^{2n})^+; \mathbb{Z})$$

is also a monomorphism. Then the homology diagram of the left parallellogram is commutative. This shows the theorem. $\hfill \square$

Theorem 2.9. The homomorphism

$$(\sigma_{n,k})_*: \widetilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z}) \longrightarrow H_*(R^k(\mathbb{C}^n); \mathbb{Z})$$

is given by

$$(\sigma_{n,k})_*(x_{i_1}\cdots x_{i_k})=x_{i_1}\cdots x_{i_k}.$$

Proof. We prove the theorem by induction on k. It is clear for k = 1. Suppose that it is true up to k. By Theorem 2.8 we have the following commutative diagram

It is clear that

$$\mu_*(x_{i_1}\cdots x_{i_k}\otimes x_{j_1}\cdots x_{j_l}) = \begin{cases} x_{i_1}\cdots x_{i_k}x_{j_1}\cdots x_{j_l} & ; i_a \neq j_b \text{ for all } a, b\\ 0 & ; \text{ otherwise} \end{cases}.$$

We take l = 1. Then $x_{i_1} \cdots x_{i_k} x_{i_{k+1}} = \mu_*(x_{i_1} \cdots x_{i_k} \otimes x_{i_{k+1}})$. Then $\sigma_*(x_{i_1} \cdots x_{i_k} x_{i_{k+1}}) = x_{i_1} \cdots x_{i_k} x_{i_{k+1}}$ by the assumption of induction.

3. Adams operation and a decomposition of U(n)

Let $\rho : \Sigma(\mathbb{C}P^{n-1}_+) \longrightarrow U(n)$ be the map defined in Section 2 and let $\tilde{\rho} : \Sigma^2(\mathbb{C}P^{n-1}_+) \longrightarrow BU(n)$ be the adjoint of ρ .

Lemma 3.1 ([9]). Let $c_i \in H^{2i}(BU(n);\mathbb{Z})$ be the *i*-th Chern class and let $t \in H^2(\mathbb{C}P^{n-1};\mathbb{Z})$ be a generator. Then

$$\tilde{\rho}^*(c_i) = \pm \sigma^2(t^{i-1}) \in H^{2i}(\Sigma^2(\mathbb{C}P^{n-1}_+);\mathbb{Z}), \quad i \le n$$

where σ^2 denotes the double suspension.

Let $y_i \in H^{2i-1}(U(n);\mathbb{Z})$ be a class transgressive to $c_i \in H^{2i}(BU(n);\mathbb{Z})$. Then it is well known that y_i is primitive and we have

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_1,\ldots,y_n).$$

Recall that $H_*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1,\ldots,x_n)$ and $x_i = \rho(\sigma(s_{i-1}))$ where $s_{i-1} \in H_{2i-2}(\mathbb{C}P^{n-1};\mathbb{Z})$ is a generator.

Lemma 3.2. For a multi-index $I = (i_1, \ldots, i_s)$, we write $x_I = x_{i_1} \cdots x_{i_s}$ and $y_I = y_{i_1} \cdots y_{i_s}$. Then we have

$$\langle x_I, y_J \rangle = \pm \delta_{I,J}$$

where $\delta_{I,J} = 1$ if I = J and = 0 if $I \neq J$.

Proof. By Lemma 3.1, we have

$$\langle x_i, y_j \rangle = \langle \sigma(s_{i-1}), \rho^*(y_j) \rangle = \langle \sigma(s_{i-1}), \pm \sigma(t^{j-1}) \rangle = \pm \delta_{i,j}.$$

Since y_j is primitive, the lemma follows from the standard argument of Hopf algebras.

Let $f: U(n) \longrightarrow U(n)$ be a continuous map. We say f is homologically diagonal if $f_*: H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(n); \mathbb{Z})$ is represented by a diagonal matrix with respect to the monomial basis $\{x_{i_1} \cdots x_{i_s}\}$. Cohomologically diagonal is similarly defined. Then by Lemma 3.2 we have

Lemma 3.3. For a map $f : U(n) \longrightarrow U(n)$, the following conditions are equivalent

- (1) $f^*(y_i) = a_i y_i, a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq n.$
- (2) f is cohomologically diagonal.
- (3) f is homologically diagonal.

Let $f, g: U(n) \longrightarrow U(n)$ be based maps. We can define the sum of f and g by the composition

$$U(n) \stackrel{\Delta}{\longrightarrow} U(n) \times U(n) \stackrel{f \times g}{\longrightarrow} U(n) \times U(n) \stackrel{\mu}{\longrightarrow} U(n)$$

where Δ is the diagonal and μ is the product map. We write the map by [f+g] in order to distinguish this to the sum in the sense of stable maps. Similarly the map $\iota \circ f$ is written by [-f] where $\iota : U(n) \longrightarrow U(n)$ is given by $\iota(A) = A^{-1}$.

Lemma 3.4. Suppose that $f, g : U(n) \longrightarrow U(n)$ are homologically diagonal. Then so are [f + g] and [-f].

Proof. By assumption and by Lemma 3.3, $f^*(y_i) = a_i y_i$ and $g^*(y_i) = b_i y_i$ for some $a_i, b_i \in \mathbb{Z}$. Since y_i is primitive, we easily see that $[f + g]^*(y_i) = (a_i + b_i)y_i$ and again by Lemma 3.3, [f + g] is homologically diagonal. The case of [-f] is quite similar.

Now we recall the unstable Adams operation (see, e.g. [8]). Let q be an integer such that (q, n!) = 1. Then there is a map $\psi^q : BU(n) \longrightarrow BU(n)$ such that $(\psi^q)^*(c_i) = q^i c_i, 1 \le i \le n$. Now let p be an odd prime and n be a positive integer. Then we can choose a prime l such that (l, n!) = 1 and l generates the multiplicative group $(\mathbb{Z}/p)^{\times}$. For an integer $t, 1 \le t \le p-1$, we define a map $\phi_t : U(n) \longrightarrow U(n)$ by

$$\phi_t = \prod [\Omega \psi^l - l^i \operatorname{id}], \quad 0 \le i \le n \quad \text{and} \quad i \not\equiv t \mod p - 1$$

where the product is taken by means of composition.

 $\begin{aligned} \mathbf{Proposition \ 3.5.} \quad (\phi_t)_* : H_*(U(n);\mathbb{Z}) &\longrightarrow H_*(U(n);\mathbb{Z}) \text{ is given by} \\ (\phi_t)_*(x_{i_1}\cdots x_{i_s}) &= \begin{cases} ax_{i_1}\cdots x_{i_s} & ; \ i_k \equiv t \mod p-1 \text{ for all } k \\ 0 & ; \ otherwise \end{cases} \end{aligned}$

where a is a certain integer $\not\equiv 0 \mod p$.

Proof. Consider $(\Omega \psi^l)^* : H^*(U(n); \mathbb{Z}) \longrightarrow H^*(U(n); \mathbb{Z})$. Since y_i is transgressive to c_i , we have $(\Omega \psi^l)^*(y_i) = l^i y_i$. Then clearly $[\Omega \psi^l - l^j \operatorname{id}]^*(y_i) = (l^i - l^j)y_i$. Hence we have $(\phi_t)^*(y_i) = \prod (l^i - l^j)y_i, 0 \le j \le n \text{ and } j \ne t \mod p - 1$. Since $l^k - 1 \equiv 0 \mod p$ if and only if $k \equiv 0 \mod p - 1$, we see that $(\phi_t)^*$ satisfies the required property for the cohomology basis. Then the proposition follows from the duality.

Now we have a *p*-local decomposition of U(n) as follows, see [6]. Let q_1, q_2, \ldots be all primes except *p* and put $d_k = q_1 \cdots q_k$. Consider a sequence

$$U(n) \xrightarrow{d_1} U(n) \xrightarrow{\phi_t} (n) \xrightarrow{d_2} U(n) \xrightarrow{\phi_t} \cdots$$

where d_k means the d_k -times of the identity. We denote by $\tilde{X}_t(n)$ the telescope of the sequence. Note that the map $d_k : U(n) \longrightarrow U(n)$ is homologically diagonal. Let $\mu_t : U(n) \longrightarrow \tilde{X}_t(n)$ be the natural inclusion. Then we have $(\mu_t)_*(x_i) = 0$ for $i \not\equiv t \mod p - 1$ and writing $(\mu_t)_*(x_i)$ also by x_i for $i \equiv t \mod p - 1$, $H_*(\tilde{X}_t(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}_{(p)}}(x_t, x_{t+p-1}, \ldots, x_{t+\alpha(p-1)})$ where α is the maximal integer such that $t + \alpha(p-1) \leq n$. Thus we obtain

Theorem 3.6 ([6]). The map

$$\mu = \prod \mu_t : U(n) \longrightarrow \tilde{X}_1(n) \times \dots \times \tilde{X}_{p-1}(n)$$

is a *p*-local equivalence.

4. Proof of the main theorem

In this section we prove our main result.

Theorem 4.1. (1) Let n be a positive integer and let p be an odd prime. Then for each pair (t,k) of integers such as $1 \le t \le p-1$ and $1 \le k \le n$, there exists a finite spectrum $X_{t,k}(n)$ satisfying

$$H_*(X_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$$

as a module where α is the maximal integer such that $t + \alpha(p-1) \leq n$.

(2) There exists a stable p-equivalence

$$U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \dots \wedge X_{t_m,k_m}(n)$$

where the wedge sum is taken over $1 \le t_1 < \cdots < t_m \le p-1$ and $k_1 + \cdots + k_m \le n$.

Proof. The stable splitting

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+$$

gives a splitting $1 = e_1 + \cdots + e_n$ of the unity by orthogonal idempotents in the ring $\{U(n), U(n)\}$ of homotopy classes of stable self maps so that $R_k(\mathbb{C}^n)^+ \simeq e_k U(n)$ the telescope of the map e_k . By Theorem 2.9, we easily see that e_{k_*} : $H_*(U(n);\mathbb{Z}) \longrightarrow H_*(U(n);\mathbb{Z})$ is homologically diagonal and actually given by

$$e_{k_*}(x_{i_1}\cdots x_{i_j}) = \begin{cases} 0 & ; j \neq k \\ x_{i_1}\cdots x_{i_j} & ; j = k \end{cases}$$

In the construction of $\tilde{X}_t(n)$ in Section 3, we can use the stable self map $\phi_t \circ e_k$ instead of ϕ_t and we obtain a *p*-local spectrum $\tilde{X}_{t,k}(n)$ and a stable map

 $\tilde{\lambda}_{t,k} : U(n) \longrightarrow \tilde{X}_{t,k}(n)$. It is clear that $(\tilde{\lambda}_{t,k})_*(x_{i_1}\cdots x_{i_l}) = 0$ if $l \neq k$ or $i_j \not\equiv t \mod p - 1$ for some j. When l = k and $i_j \equiv t \mod p - 1$ for all j, writing $(\tilde{\lambda}_{t,k})_*(x_{i_1}\cdots x_{i_k})$ formally by $y_{i_1}\cdots y_{i_k}$, we have

$$H_*(\tilde{X}_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}_{(p)}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$$

as a module.

Next given m pairs $(t_1, k_1), \ldots, (t_m, k_m)$ such that $1 \le t_1 < \cdots < t_m \le p-1$ and $k = k_1 + \cdots + k_m \le n$, consider the map

$$\tilde{\lambda}_{\mathbf{t},\mathbf{k}}: U(n) \xrightarrow{\Delta} U(n) \wedge \dots \wedge U(n) \xrightarrow{\wedge \tilde{\lambda}_{t_i,k_i}} \tilde{X}_{t_1,k_1}(n) \wedge \dots \wedge \tilde{X}_{t_m,k_m}(n)$$

where Δ is the diagonal map. Now for a set $I = \{i_1, \ldots, i_r\}$ of integers $i_1 < \cdots < i_r$, one can think of a partition $I = J_1 \amalg \cdots \amalg J_m$. We write $x_{i_1} \cdots x_{i_r}$ by x_I . Then $\Delta_* : H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(n)^{\wedge m}; \mathbb{Z})$ is given by

$$\Delta_*(x_I) = \Sigma x_{J_1} \otimes \cdots \otimes x_{J_m}$$

where $I = J_1 \amalg \cdots \amalg J_m$ and $J_i \neq \emptyset$ for all *i*. Given a pair (t, k), we say $I = \{i_1, \ldots, i_r\}$ is of type (t, k) if r = k and $i_j \equiv t \mod p - 1$ for all *j*. Then we can describe $(\tilde{\lambda}_{\mathbf{t},\mathbf{k}})_*$ by

$$(\tilde{\lambda}_{\mathbf{t},\mathbf{k}})_*(x_I) = \begin{cases} y_{J_1} \otimes \dots \otimes y_{J_m} & ; \ J_i \text{ is of type } (t_i,k_i) \text{ for all } i \\ 0 & ; \text{ otherwise} \end{cases}$$

Then taking all wedge sum of those $\tilde{\lambda}_{\mathbf{t},\mathbf{k}}$, we have a stable *p*-equivalence

$$\tilde{\lambda}: U(n) \longrightarrow \bigvee \tilde{X}_{t_1,k_1}(n) \wedge \dots \wedge \tilde{X}_{t_m,k_m}(n)$$

where $1 \le t_1 < \dots < t_m \le p - 1$ and $k_1 + \dots + k_m \le n$.

Now by a stable version of Proposition 1.4 [6], we see that there exists a finite spectrum $X_{t,k}(n)$ such that $H_*(X_{t,k}(n);\mathbb{Z})$ is torsion free and $X_{t,k}(n)_{(p)} \simeq \tilde{X}_{t,k}(n)$ where $X_{t,k}(n)_{(p)}$ is the p-localization of $X_{t,k}(n)$. One may think $X_{t,k}(n)$ is a subspectrum of $\tilde{X}_{t,k}(n)$ and the relative homotopy group $\pi_i(\tilde{X}_{t,k}(n), X_{t,k}(n))$ is torsion group whose elements have order prime to p. Then by obstruction theory we can choose an integer d prime to p and show that there exists a stable map $\lambda_{t,k}$ such that the diagram

is homotopy commutative where d is the d-times map and l is the localization map. Then in a similar way we have a stable p-equivalence

$$\lambda: U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \cdots \wedge X_{t_m,k_m}(n).$$

This completes the proof.

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