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On a p-local stable splitting of $U(n)$

By

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1. Introduction

Let $U(n)$ be the unitary group. H. Miller [5] has introduced a filtration $\{1\} = R^0(\mathbb{C}^n) \subset R^1(\mathbb{C}^n) \subset \cdots \subset R^n(\mathbb{C}^n) = U(n)$ and shown that there is a stable splitting

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+$$

where $R_k(\mathbb{C}^n) = R^k(\mathbb{C}^n) - R^{k-1}(\mathbb{C}^n)$ and $R_k(\mathbb{C}^n)^+ \cong R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$ is the one-point compactification of $R_k(\mathbb{C}^n)$. The filtration is given by counting the dimension of direct sum of eigen subspaces in which eigenvalues are different from 1. Later M. Crabb [1] gave a simpler constrution of the stable splitting.

On the other hand it is known [6] that for a prime p there is a p-local unstable decomposition

$$U(n) \underset{p}{\simeq} X_1(n) \times \cdots \times X_{p-1}(n)$$

into a product of $p-1$ spaces, using the unstable Adams operations with the $X_i(n)$'s satisfying $H^*(X_i(n); \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}(p)}(x_i, x_{i+p-1}, \ldots, x_{i+s(p-1)})$ where $s = [\frac{n-i}{p-1}]$. Then we obtain a p-local stable splitting

$$U(n) \underset{p}{\simeq} \bigvee X_{i_1}(n) \wedge \cdots \wedge X_{i_s}(n), 1 \leq i_1 < \cdots < i_s \leq p-1.$$

Then a natural question arises:

Question. Can one mix those stable splittings and obtain a finer one at an odd prime?

In [2], Crabb, Hubbuck and McCall have shown that $R_k(\mathbb{C}^n)^+$ splits stably as $(\Sigma SR_{k-1}(\mathbb{C}^n)^+) \vee SR_k(\mathbb{C}^n)^+$(written by $SX_k(n)$ in [2]), where $SR_k(\mathbb{C}^n)^+$ is the space in the stable splitting of $SU(n)$, and $SR_2(\mathbb{C}^n)^+$ is stably indecomposable at the prime 2. The purpose of this paper is to give an affirmative answer of the question at an odd prime. Our main theorem is

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Theorem 1.1. (1) *Let n be a positive integer and let p be an odd prime. Then for each pair (t,k) of integers such as $1 \le t \le p-1$ and $1 \le k \le n$, there exists a finite spectrum $X_{t,k}(n)$ satisfying*

$$H_*(X_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$$

as a module where α is the maximal integer such that $t+\alpha(p-1) \le n$

(2) *There exists a stable p-equivalence*

$$U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \cdots \wedge X_{t_m,k_m}(n)$$

where the wedge sum is taken over $1 \le t_1 < \cdots < t_m \le p-1$ and $k_1+\cdots+k_m \le n$. Here $\Lambda^k_{\mathbb{Z}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$ means the module of k-th exterior powers of $y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)}$ and $\deg\ y_i = 2i-1$.

We should mention that in Example 3.12 [3], it is given a simpler splitting without proof. A weaker version of the theorem is also given from the result of Harper and Zabrodsky [4]. They have shown that a mod p H-space Y with $H_*(Y;\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(y_{i_1}, \dots, y_{i_t})$ has a stable splitting $Y \underset{p}{\simeq} Y_1 \vee \cdots \vee Y_{p-1}$ such that $H_*(Y_a;\mathbb{Z}_{(p)}) \cong \bigoplus_{k \equiv a(p-1)} \Lambda^k_{\mathbb{Z}_{(p)}}(y_{i_1}, \dots, y_{i_t})$ for $1 \le a \le p-1$. Applying this to the mod p H-space $X_i(n)$, we obtain spectra $Y_{i,j}(n)$, $1 \le i,j \le p-1$ such that $X_i(n) \underset{p}{\simeq} Y_{i,1}(n) \vee \cdots \vee Y_{i,p-1}(n)$ and

$$H_*(Y_{i,j}(n);\mathbb{Z}_{(p)}) \cong \bigoplus_{k \equiv j(p-1)} \Lambda^k_{\mathbb{Z}_{(p)}}(x_{i_1}, x_{i_1+p-1}, \dots, x_{i_1+\alpha(p-1)}).$$

The idea of proof is roughly as follows. A stable splitting of a space or spectrum Y corresponds to a splitting $1 = e_1 + \cdots + e_k$ of the unity into a sum of orthogonal idempotents in the ring $\{Y, Y\}$ of homotopy classes of stable self-maps of Y. If we are given two splitting $1 = e_1 + \cdots + e_k$ and $1 = f_1 + \cdots + f_l$ and if $e_i f_j = f_j e_i$ for all i, j, then we have a finer splitting $1 = \Sigma e_i f_j$. But this is too strong condition. Let $\psi : \{Y, Y\} \longrightarrow \mathrm{End}\, H_*(Y)$ be a natural ring homomorphism. Write $\psi(e) = e_*$. Then the assumption of the above statement can be weakened as $(e_i)_*(f_j)_* = (f_j)_*(e_i)_*$ for all i, j. The proof of the main theorem is done by checking that this weak condition holds for our two stable splittings of $U(n)$.

2. Multiplicative property of stable splittings

We define an increasing filtration of the unitary group $U(n)$

$$\{1\} = R^0(\mathbb{C}^n) \subset R^1(\mathbb{C}^n) \subset \cdots \subset R^n(\mathbb{C}^n) = U(n)$$

by

$$R^k(\mathbb{C}^n) = \{g \in U(n) | \dim_{\mathbb{C}}(\mathrm{Ker}(g-1)) \ge n-k\}, \quad 0 \le k \le n$$

where 1 denotes the unit element of $U(n)$. The difference $R^k(\mathbb{C}^n)-R^{k-1}(\mathbb{C}^n) = \{g \in U(n) | \dim_{\mathbb{C}}(\mathrm{Ker}(g-1)) = n-k\}$ is written by $R_k(\mathbb{C}^n)$. The one-point compactification $R_k(\mathbb{C}^n)^+$ is identified with the quotient space $R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$. The pointed natural projection map is denoted by $\pi_{n,k} : R^k(\mathbb{C}^n)^+ \longrightarrow R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n) = R_k(\mathbb{C}^n)^+$. Note that $R^k(\mathbb{C}^n)$ consists of unitary matrices A such that the dimension of the sum of eigenspaces of A with eigenvalue different from 1 is equal or less than k.

H. Miller [5] and M. Crabb [1] has shown the following

Theorem 2.1. *There exists a stable splitting*

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+.$$

We recall the construction of the splitting following [1]. Let $\mathrm{End}(\mathbb{C}^k)$ be the space of all $k \times k$ matrices. We regard $\mathrm{End}(\mathbb{C}^k)$ as a $U(k)$- space by the adjoint action. Then we have following $U(k)$-invariant subspaces of $\mathrm{End}(\mathbb{C}^k)$:

(1) $\mathcal{H}(k)$: the space of Hermitian matrices,
(2) $\mathfrak{u}(k)$: the space of skew-Hermitian matrices,
(3) $U(k)$: the unitary group,
(4) $U(k)_0 = \{g \in U(k) | g-1 \text{ is invertible } \}$.

Let $G_{n,k}$ and $V_{n,k}$ be the Grassman manifold of k-planes in $\mathbb{C}^n$ and Stiefel manifold of k-frames in $\mathbb{C}^n$, respectively. Let $U(k) \longrightarrow V_{n,k} \longrightarrow G_{n,k}$ be the standard principal $U(k)$-bundle. We denote this bundle simply by $\zeta_{n,k}$. Let F be a manifold with a $U(k)$-action. We denote the associate fibre bundle by $\zeta_{n,k}(F)$. The Cayley transform

$$\psi : \mathfrak{u}(k) \longrightarrow U(k)_0$$

defined by $\psi(X) = (X-1)(X+1)^{-1}$ is a $U(k)$-equivariant diffeomorphism.

Consider the bundle $\zeta_{n,k}(U(k)) \longrightarrow G_{n,k}$; the fibre over $E \in G_{n,k}$ is the unitary group $U(E)$ of a k-dimensional subspace $E \subseteq \mathbb{C}^n$, and we can regard an elements of $\zeta_{n,k}(U(k))$ as a pair (E, f) where $E \in G_{n,k}, f \in U(E)$. Now we have a surjective map $p_{n,k} : \zeta_{n,k}(U(k)) \longrightarrow R^k(\mathbb{C}^n)$ given by

$$p_{n,k}(E, f) = f \oplus 1 : E \oplus E^{\perp} \longrightarrow E \oplus E^{\perp}.$$

It is clear that the restriction of $p_{n,k}$ gives a homeomorphism

$$\zeta_{n,k}(U(k)_0) \underset{\cong}{\longrightarrow} R_k(\mathbb{C}^n).$$

Now we can use the Cayley transform to identify the bundle $\zeta_{n,k}(U(k)_0)$ over $G_{n,k}$ with the Lie algebra bundle $\zeta_{n,k}(\mathfrak{u}(k))$. Thus we have

Proposition 2.2. *There is a natural diffeomorphism between $R_k(\mathbb{C}^n)$ and the total space of the vector bundle $\zeta_{n,k}(\mathfrak{u}(k))$ over $G_{n,k}$.*

We recall the Iwasawa decomposition $GL(k,\mathbb{C}) \cong U(k) \times P(k)$ where $P(k)$ is the space of positive definite Hermitian matrices. The exponential map $\exp : \mathcal{H}(k) \longrightarrow P(k)$ is a diffeomorphism and hence we have a diffeomorphism

$$1 \times \exp : U(k) \times \mathcal{H}(k) \longrightarrow GL(k,\mathbb{C})$$

which is clearly $U(k)$-equivariant. We now construct a stable map $\sigma_{n,k} : R_k(\mathbb{C}^n)^+ \longrightarrow R^k(\mathbb{C}^n)^+$ as follows. We have a diffeomorphism

$$\zeta_{n,k}(U(k) \times \mathcal{H}(k)) \underset{\cong}{\longrightarrow} \zeta_{n,k}(GL(k,\mathbb{C}))$$

and an open embedding

$$\zeta_{n,k}(GL(k,\mathbb{C})) \subset \zeta_{n,k}(\mathrm{End}(\mathbb{C}^k)) \cong \zeta_{n,k}(\mathfrak{u}(k) \oplus \mathcal{H}(k)).$$

Note that $\zeta_{n,k}(U(k) \times \mathcal{H}(k))$ is identified with the fibre product $\zeta_{n,k}(U(k)) \underset{G_{n,k}}{\times} \zeta_{n,k}(\mathcal{H}(k))$ of the spaces over $G_{n,k}$, and similarly for $\zeta_{n,k}(\mathfrak{u}(k) \oplus \mathcal{H}(k))$. Since $\zeta_{n,k}(\mathcal{H}(k))$ is a real vector bundle over a compact space $G_{n,k}$, there is an embedding $\zeta_{n,k}(\mathcal{H}(k)) \subset G_{n,k} \times \mathbb{R}^N$ into the product bundle for some integer N. Let γ be the orthogonal complement of $\zeta_{n,k}(\mathcal{H}(k))$ so that $\zeta_{n,k}(\mathcal{H}(k)) \oplus \gamma = G_{n,k} \times \mathbb{R}^N$. Then applying $\underset{G_{n,k}}{\times} \gamma$ to the above diffeomorphism and the embedding, we have an open embedding $\zeta_{n,k}(U(k)) \times \mathbb{R}^N \subset \zeta_{n,k}(\mathfrak{u}(k)) \times \mathbb{R}^N$ and applying the Pontrjagin-Thom construction we obtain a stable map $s_{n,k} : \zeta_{n,k}(\mathfrak{u}(k))^+ \longrightarrow \zeta_{n,k}(U(k))^+$. By a usual argument it is easy to see that the homotopy class of the stable map $s_{n,k}$ does not depend on a choice of an embedding $\zeta_{n,k}(\mathcal{H}(k)) \subset G_{n,k} \times \mathbb{R}^N$. Now we define the stable map $\sigma_{n,k}$ as the following composition

$$\sigma_{n,k} : R_k(\mathbb{C}^n)^+ \xrightarrow{(p_{n,k}^{-1})^+} \zeta_{n,k}(\mathfrak{u}(k))^+ \xrightarrow{s_{n,k}} \zeta_{n,k}(U(k))^+ \xrightarrow{(p_{n,k})^+} R^k(\mathbb{C}^n)^+.$$

The next theorem is due to M. Crabb [1].

Theorem 2.3. *$\pi_{n,k} \circ \sigma_{n,k}$ is homotopic to identity.*

We denote the inclusion maps $R^k(\mathbb{C}^n) \longrightarrow R^{k+1}(\mathbb{C}^n)$ by j. Then we have a stable map $j \circ \sigma_{n,k} : R_k(\mathbb{C}^n)^+ \longrightarrow U(n)^+$ and by taking a wedge sum we have

$$\bigvee_{k=0}^{n} R_k(\mathbb{C}^n)^+ \longrightarrow \bigvee U(n)^+ \longrightarrow U(n)^+$$

which is clearly a homotopy equivalence. Since $R_0(\mathbb{C}^n)^+ \simeq S^0$, we finally obtain a stable homotopy equivalence

$$\bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+ \longrightarrow U(n).$$

Now we recall basic facts about the homology of $R^k(\mathbb{C}^n)$ (see, e.g. [7]). Define a map $\rho : \Sigma(\mathbb{C}P_+^{n-1}) \longrightarrow U(n)$ by

$$\rho(\lambda, z) = (\delta_{i,j} + (\lambda - 1)z_i\bar{z}_j), \quad 1 \leq i, j \leq n$$

for $\lambda \in S^1, z = [z_1; \dots ; z_n]$ and $z = (z_1, \dots , z_n) \in S^{2n-1} \subset \mathbb{C}^n$. If $\lambda \neq 1$, then $\rho(\lambda, z)$ is the matrix with λ as a unique non 1 eigenvalue with the eigenvector $(z_1, \dots , z_n)$. Hence we see that $\mathrm{Im}\ \rho = R^1(\mathbb{C}^n)$. Let $x_i = \rho_*(\sigma(s_{i-1})) \in H_{2i-1}(U(n); \mathbb{Z})$ where σ is the homology suspension and $s_{i-1} \in H_{2i-2}(\mathbb{C}P^{n-1}; \mathbb{Z})$ is a generator. Then we have an isomorphism

$$H_*(U(n); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_1, x_2, \dots , x_n)$$

of the homology ring with the exterior algebra generated by $x_1, x_2, \dots , x_n$. Let $j : R^k(\mathbb{C}^n) \longrightarrow U(n)$ be the inclusion. Then we have

Proposition 2.4. *The homomorphism*

$$j_* : H_*(R^k(\mathbb{C}^n); \mathbb{Z}) \longrightarrow H_*(U(n); \mathbb{Z})$$

is injective and $\mathrm{Im}\, j_*$ *is spanned by* $x_{i_1} \cdots x_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq n$, $s \leq k$.

We identify $H_*(R^k(\mathbb{C}^n); \mathbb{Z})$ with its image in $H_*(U(n); \mathbb{Z})$. Now consider the cofibration

$$R^{k-1}(\mathbb{C}^n) \xrightarrow{j} R^k(\mathbb{C}^n) \xrightarrow{\pi_{n,k}} R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n) \cong R_k(\mathbb{C}^n)^+.$$

Then we easily obtain the following

Proposition 2.5. *The homomorphism*

$$(\pi_{n,k})_* : H_*(R^k(\mathbb{C}^n); \mathbb{Z}) \longrightarrow \tilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z})$$

is surjective and $\mathrm{Ker}(\pi_{n,k})_*$ *is spanned by* $x_{i_1} \cdots x_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq n$, $s \leq k - 1$.

We write $(\pi_{n,k})_*(x_{i_1} \cdots x_{i_k}) \in \tilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z})$ by the same symbol $x_{i_1} \cdots x_{i_k}$. Let $\Lambda^k_{\mathbb{Z}}(x_1, \dots , x_n) = \mathbb{Z}\{x_{i_1} \cdots x_{i_k}\}$ be the module of k-th exterior powers. Then we have an isomorphism

$$\tilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(x_1, \dots , x_n)$$

as an abelian group.

Let $n \leq m$ be integers and let $i : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be the inclusion into the first n coordinates. Then we have associated inclusions $U(n) \longrightarrow U(m)$, $G_{n,k} \longrightarrow G_{m,k}$, $R^k(\mathbb{C}^n) \longrightarrow R^k(\mathbb{C}^m)$ and $R_k(\mathbb{C}^n) \longrightarrow R_k(\mathbb{C}^m)$ for $0 \leq k \leq n$. We write all inclusions by the same i. Note that the last map is proper.

Lemma 2.6. *The diagram*

$$\begin{tikzcd}
R_k(\mathbb{C}^n)^+ \arrow[r, "i"'] \arrow[d, "\sigma_{n,k}"] & R_k(\mathbb{C}^m)^+ \arrow[d, "\sigma_{m,k}"] \\
R^k(\mathbb{C}^n)^+ \arrow[r, "i"'] & R^k(\mathbb{C}^m)^+
\end{tikzcd}$$

is homotopy commutative.

Proof. Let F be a $U(k)$-space. Then by definition we have a pull-back diagram

$$\begin{tikzcd}
\zeta_{n,k}(F) \arrow[r, "i"'] \arrow[d] & \zeta_{m,k}(F) \arrow[d] \\
G_{n,k} \arrow[r, "i"'] & G_{m,k}
\end{tikzcd}$$

of associated fibre bundles. Consider the following commutative diagram

$$\begin{tikzcd}
\zeta_{n,k}(U(k) \times \mathcal{H}(k)) \arrow[rr] \arrow[dr] \arrow[dd, "i"] & & \zeta_{n,k}(\mathrm{End}(\mathbb{C}^k)) \arrow[dl] \arrow[dd, "i"] \\
& G_{n,k} \arrow[dd, "i"] & \\
\zeta_{m,k}(U(k) \times \mathcal{H}(k)) \arrow[rr] \arrow[dr] & & \zeta_{m,k}(\mathrm{End}(\mathbb{C}^k)) \arrow[dl] \\
& G_{m,k} &
\end{tikzcd}$$

where the horizontal maps are induced from the exponential map. Then by a simple diagram chasing, we see that the square diagram is pull-back. Now we take a real vector bundle γ_m such that $\zeta_{m,k}(\mathcal{H}(k)) \oplus \gamma_m \cong G_{m,k} \times \mathbb{R}^N$ and let $\gamma_n = \gamma_m \mid_{G_{n,k}}$. Then the associated commutative diagram

$$\begin{tikzcd}
\zeta_{n,k}(U(k)\times\mathcal{H}(k))\underset{G_{n,k}}{\times}\gamma_n \arrow[r] \arrow[d] & \zeta_{n,k}(\mathrm{End}(\mathbb{C}^k))\underset{G_{n,k}}{\times}\gamma_n \arrow[d] \\
\zeta_{m,k}(U(k)\times\mathcal{H}(k))\underset{G_{m,k}}{\times}\gamma_m \arrow[r] & \zeta_{m,k}(\mathrm{End}(\mathbb{C}^k))\underset{G_{m,k}}{\times}\gamma_m
\end{tikzcd}$$

is clearly a pull-back diagram. Hence the diagram

$$\begin{tikzcd}
\zeta_{n,k}(U(k))\times\mathbb{R}^N \arrow[d] & \subset & \zeta_{n,k}(\mathfrak{u}(k))\times\mathbb{R}^N \arrow[d] \\
\zeta_{m,k}(U(k))\times\mathbb{R}^N & \subset & \zeta_{m,k}(\mathfrak{u}(k))\times\mathbb{R}^N
\end{tikzcd}$$

is pull-back and it gives a commutative diagram of the Pontrjagin-Thom constructions. The naturality concerning the map $p_{n,k}$ is obvious and this completes the proof. □

Now let $A\in R^k(\mathbb{C}^n)$ and $B\in R^l(\mathbb{C}^n)$, then it is clear that the composition $AB\in R^{k+l}(\mathbb{C}^n)$. Thus we obtain a pairing

$$\mu: R^k(\mathbb{C}^n)\times R^l(\mathbb{C}^n)\longrightarrow R^{k+l}(\mathbb{C}^n).$$

Note that $\mu(R^{k-1}(\mathbb{C}^n)\times R^l(\mathbb{C}^n))\subset R^{k+l-1}(\mathbb{C}^n)$ and $\mu(R^k(\mathbb{C}^n)\times R^{l-1}(\mathbb{C}^n))\subset R^{k+l-1}(\mathbb{C}^n)$. Therefore, by identifying $R_k(\mathbb{C}^n)^+$ with $R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n)$, we have an induced pairing

$$\mu: R_k(\mathbb{C}^n)^+\wedge R_l(\mathbb{C}^n)^+\longrightarrow R_{k+l}(\mathbb{C}^n)^+.$$

Next let $\varphi: U(n)\times U(n)\longrightarrow U(2n)$ be the map given by

$$\varphi(A,B)=\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}.$$

This induces pairings

$$\tilde{\mu}: R^k(\mathbb{C}^n)\times R^l(\mathbb{C}^n)\longrightarrow R^{k+l}(\mathbb{C}^{2n})$$

and

$$\tilde{\mu}: R_k(\mathbb{C}^n)^+\wedge R_l(\mathbb{C}^n)^+\longrightarrow R_{k+l}(\mathbb{C}^{2n})^+$$

in a natural way. Now we have

Lemma 2.7. *The diagram*

$$\begin{tikzcd} R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \arrow[r, "\tilde{\mu}"'] \arrow[d, "\sigma_{n,k} \wedge \sigma_{n,l}"] & R_{k+l}(\mathbb{C}^{2n})^+ \arrow[d, "\sigma_{2n,k+l}"] \\ R^k(\mathbb{C}^n)^+ \wedge R^l(\mathbb{C}^n)^+ \arrow[r, "\tilde{\mu}"'] & R^{k+l}(\mathbb{C}^{2n})^+ \end{tikzcd}$$

is homotopy commutative.

Proof. First note that the natural diagram

$$\begin{tikzcd} GL(k,\mathbb{C}) \times GL(l,\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{End}(\mathbb{C}^k) \times \mathrm{End}(\mathbb{C}^l) \arrow[d] \\ GL(k+l,\mathbb{C}) \arrow[r] & \mathrm{End}(\mathbb{C}^{k+l}) \end{tikzcd}$$

is a pull-back diagram, where all maps are inclusion maps. The horizontal maps are open embeddings and the vertical maps are proper. The upper horizontal and lower horizontal maps are $U(k)\times U(l)$ and $U(k+l)$ equivariant and vertical maps are equivariant with respect to the $U(k) \times U(l)$-action on $\mathrm{End}(\mathbb{C}^{k+l})$ induced by the inclusion $U(k)\times U(l) \subset U(k+l)$. Therefore we have an induced commutative diagram

$$\begin{tikzcd} \zeta_{n,k}(GL(k,\mathbb{C})) \times \zeta_{n,l}(GL(l,\mathbb{C})) \arrow[r] \arrow[d] & \zeta_{n,k}(\mathrm{End}(\mathbb{C}^k)) \times \zeta_{n,l}(\mathrm{End}(\mathbb{C}^l)) \arrow[d] \\ \zeta_{2n,k+l}(GL(k+l,\mathbb{C})) \arrow[r] & \zeta_{2n,k+l}(\mathrm{End}(\mathbb{C}^{k+l})) \end{tikzcd}$$

We can directly check that the diagram is pull-back. We can identify $\zeta_{n,k}(GL(k,\mathbb{C}))$ with $\zeta_{n,k}(U(k)\times\mathcal{H}(k))$ and similarly for $\zeta_{n,l}(GL(l,\mathbb{C}))$ and $\zeta_{2n,k+l}(GL(k+l,\mathbb{C}))$. Let γ_{k+l} be a vector bundle over $G_{2n,k+l}$ such that $\zeta_{2n,k+l}(\mathcal{H}(k+l)) \oplus \gamma_{k+l} \cong G_{2n,k+l} \times \mathbb{R}^N$. Let $f : G_{n,k} \times G_{n,l} \longrightarrow G_{2n,k+l}$ be the natural map. Note that $\mathcal{H}(k+l) \cong \mathcal{H}(k) \oplus \mathcal{H}(l) \oplus \mathbb{C}^{kl}$ as $U(k) \times U(l)$-module where $\mathbb{C}^{kl}$ is a trivial $U(k) \times U(l)$-module. Hence $f^*(\zeta_{2n,k+l}(\mathcal{H}(k+l))) \cong \zeta_{n,k}(\mathcal{H}(k)) \times \zeta_{n,l}(\mathcal{H}(l)) \oplus (2kl)\epsilon$, where $(2kl)\epsilon$ is the trivial real bundle of dimension $2kl$. Therefore $\zeta_{n,k}(\mathcal{H}(k)) \times \zeta_{n,l}(\mathcal{H}(l)) \oplus f^*\gamma_{k+l} \oplus (2kl)\epsilon$ is isomorphic to a trivial bundle. Hence we have a pull-back diagram

$$\begin{tikzcd}
\zeta_{n,k}(U(k)) \times \zeta_{n,l}(U(l)) \times \mathbb{R}^N \arrow[r, phantom, "\subset"] \arrow[d] & \zeta_{n,k}(\mathfrak{u}(k)) \times \zeta_{n,l}(\mathfrak{u}(l)) \times \mathbb{R}^N \arrow[d] \\
\zeta_{2n,k+l}(U(k+l)) \times \mathbb{R}^N \arrow[r, phantom, "\subset"] & \zeta_{2n,k+l}(\mathfrak{u}(k+l)) \times \mathbb{R}^N
\end{tikzcd}$$

of open embeddings and proper maps. Thus we obtain a homotopy commutative diagram of the Pontrjagin-Thom construction. The rest of the proof is similar to Lemma 2.6. □

Theorem 2.8. *The diagram*

$$\begin{tikzcd}
R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \arrow[r, "\mu"'] \arrow[d, "\sigma_{n,k} \wedge \sigma_{n,l}"] & R_{k+l}(\mathbb{C}^n)^+ \arrow[d, "\sigma_{n,k+l}"] \\
R^k(\mathbb{C}^n)^+ \wedge R^l(\mathbb{C}^n)^+ \arrow[r, "\mu"'] & R^{k+l}(\mathbb{C}^n)^+
\end{tikzcd}$$

induces a commutative diagram in homology groups.

Proof. First consider the homotopy

$$H : U(n) \times U(n) \times I \longrightarrow U(2n)$$

defined by

$$H(A,B,t)$$
$$= \begin{pmatrix} A & 0 \\ 0 & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} \cos t\, \mathrm{I}_n & \sin t\, \mathrm{I}_n \\ -\sin t\, \mathrm{I}_n & \cos t\, \mathrm{I}_n \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos t\, \mathrm{I}_n & -\sin t\, \mathrm{I}_n \\ \sin t\, \mathrm{I}_n & \cos t\, \mathrm{I}_n \end{pmatrix}.$$

It is clear that $H(R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \times I) \subset R^{k+l}(\mathbb{C}^{2n})$ and gives a homotopy $\tilde{\mu} \sim i \circ \mu : R^k(\mathbb{C}^n) \times R^l(\mathbb{C}^n) \longrightarrow R^{k+l}(\mathbb{C}^{2n})$. It is also clear that H induces a map

$$\tilde{H} : (R^k(\mathbb{C}^n)/R^{k-1}(\mathbb{C}^n) \wedge R^l(\mathbb{C}^n)/R^{l-1}(\mathbb{C}^n)) \times I \longrightarrow R^{k+l}(\mathbb{C}^{2n})/R^{k+l-1}(\mathbb{C}^{2n})$$

which gives a homotopy

$$\tilde{\mu} \sim i \circ \mu : R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \longrightarrow R_{k+l}(\mathbb{C}^{2n})^+.$$

Next consider the diagram

$$
\begin{tikzcd}
R_k(\mathbb{C}^n)^+ \wedge R_l(\mathbb{C}^n)^+ \arrow[rr, "\tilde{\mu}"] \arrow[dr, "\mu"'] \arrow[dd, "\sigma \wedge \sigma"'] & & R_{k+l}(\mathbb{C}^{2n})^+ \arrow[dd, "\sigma"] \\
& R_{k+l}(\mathbb{C}^n)^+ \arrow[ur, "i"'] & \\
R^k(\mathbb{C}^n)^+ \wedge R^l(\mathbb{C}^n)^+ \arrow[rr, "\tilde{\mu}"] \arrow[dr, "\mu"] & & R^{k+l}(\mathbb{C}^{2n})^+ \\
& R^{k+l}(\mathbb{C}^n)^+ \arrow[ur, "i"'] \arrow[from=uu, crossing over, "\sigma"] &
\end{tikzcd}
$$

The triangles are homotopy commutative by the above argument. The square and the right parallellogram are homotopy commutative by Lemmas 2.7 and 2.6, respectively. Note that the homomorphism

$$i_* : H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(2n); \mathbb{Z})$$

is a monomorphism and by Lemma 2.6 we see that

$$i_* : H_*(R^{k+l}(\mathbb{C}^n)^+; \mathbb{Z}) \longrightarrow H_*(R^{k+l}(\mathbb{C}^{2n})^+; \mathbb{Z})$$

is also a monomorphism. Then the homology diagram of the left parallellogram is commutative. This shows the theorem. □

Theorem 2.9. *The homomorphism*

$$(\sigma_{n,k})_* : \tilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z}) \longrightarrow H_*(R^k(\mathbb{C}^n); \mathbb{Z})$$

is given by

$$(\sigma_{n,k})_*(x_{i_1} \cdots x_{i_k}) = x_{i_1} \cdots x_{i_k}.$$

Proof. We prove the theorem by induction on k. It is clear for $k = 1$. Suppose that it is true up to k. By Theorem 2.8 we have the following commutative diagram

$$
\begin{tikzcd}
\tilde{H}_*(R_k(\mathbb{C}^n)^+; \mathbb{Z}) \otimes \tilde{H}_*(R_l(\mathbb{C}^n)^+; \mathbb{Z}) \arrow[r, "\mu_*"'] \arrow[d, "\sigma_* \otimes \sigma_*"] & \tilde{H}_*(R_{k+l}(\mathbb{C}^n)^+; \mathbb{Z}) \arrow[d, "\sigma_*"] \\
H_*(R^k(\mathbb{C}^n); \mathbb{Z}) \otimes H_*(R^l(\mathbb{C}^n); \mathbb{Z}) \arrow[r, "\mu_*"'] & H_*(R^{k+l}(\mathbb{C}^n); \mathbb{Z})
\end{tikzcd}
$$

It is clear that

$$\mu_*(x_{i_1} \cdots x_{i_k} \otimes x_{j_1} \cdots x_{j_l}) = \begin{cases} x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_l} & ;\ i_a \neq j_b \text{ for all } a, b \\ 0 & ;\ \text{otherwise} \end{cases}.$$

We take $l = 1$. Then $x_{i_1} \cdots x_{i_k} x_{i_{k+1}} = \mu_*(x_{i_1} \cdots x_{i_k} \otimes x_{i_{k+1}})$. Then $\sigma_*(x_{i_1} \cdots x_{i_k} x_{i_{k+1}}) = x_{i_1} \cdots x_{i_k} x_{i_{k+1}}$ by the assumption of induction. □

3. Adams operation and a decomposition of $U(n)$

Let $\rho : \Sigma(\mathbb{C}P_+^{n-1}) \longrightarrow U(n)$ be the map defined in Section 2 and let $\tilde{\rho} : \Sigma^2(\mathbb{C}P_+^{n-1}) \longrightarrow BU(n)$ be the adjoint of ρ.

Lemma 3.1 ([9]). *Let $c_i \in H^{2i}(BU(n); \mathbb{Z})$ be the i-th Chern class and let $t \in H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$ be a generator. Then*

$$\tilde{\rho}^*(c_i) = \pm\sigma^2(t^{i-1}) \in H^{2i}(\Sigma^2(\mathbb{C}P_+^{n-1}); \mathbb{Z}), \quad i \leq n$$

where σ^2 denotes the double suspension.

Let $y_i \in H^{2i-1}(U(n); \mathbb{Z})$ be a class transgressive to $c_i \in H^{2i}(BU(n); \mathbb{Z})$. Then it is well known that y_i is primitive and we have

$$H^*(U(n); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(y_1, \dots, y_n).$$

Recall that $H_*(U(n); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1, \dots, x_n)$ and $x_i = \rho(\sigma(s_{i-1}))$ where $s_{i-1} \in H_{2i-2}(\mathbb{C}P^{n-1}; \mathbb{Z})$ is a generator.

Lemma 3.2. *For a multi-index $I = (i_1, \dots, i_s)$, we write $x_I = x_{i_1} \cdots x_{i_s}$ and $y_I = y_{i_1} \cdots y_{i_s}$. Then we have*

$$\langle x_I, y_J \rangle = \pm\delta_{I,J}$$

where $\delta_{I,J} = 1$ if $I = J$ and $= 0$ if $I \neq J$.

Proof. By Lemma 3.1, we have

$$\langle x_i, y_j \rangle = \langle \sigma(s_{i-1}), \rho^*(y_j) \rangle = \langle \sigma(s_{i-1}), \pm\sigma(t^{j-1}) \rangle = \pm\delta_{i,j}.$$

Since y_j is primitive, the lemma follows from the standard argument of Hopf algebras. □

Let $f : U(n) \longrightarrow U(n)$ be a continuous map. We say f is homologically diagonal if $f_* : H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(n); \mathbb{Z})$ is represented by a diagonal matrix with respect to the monomial basis $\{x_{i_1} \cdots x_{i_s}\}$. Cohomologically diagonal is similarly defined. Then by Lemma 3.2 we have

Lemma 3.3. *For a map $f : U(n) \longrightarrow U(n)$, the following conditions are equivalent*

(1) *$f^*(y_i) = a_i y_i$, $a_i \in \mathbb{Z}$ for $1 \leq i \leq n$.*
(2) *f is cohomologically diagonal.*
(3) *f is homologically diagonal.*

Let $f, g : U(n) \longrightarrow U(n)$ be based maps. We can define the sum of f and g by the composition

$$U(n) \xrightarrow{\Delta} U(n) \times U(n) \xrightarrow{f \times g} U(n) \times U(n) \xrightarrow{\mu} U(n)$$

where Δ is the diagonal and μ is the product map. We write the map by $[f+g]$ in order to distinguish this to the sum in the sense of stable maps. Similarly the map $\iota \circ f$ is written by $[-f]$ where $\iota : U(n) \longrightarrow U(n)$ is given by $\iota(A) = A^{-1}$.

Lemma 3.4. *Suppose that $f, g : U(n) \longrightarrow U(n)$ are homologically diagonal. Then so are $[f+g]$ and $[-f]$.*

Proof. By assumption and by Lemma 3.3, $f^*(y_i) = a_i y_i$ and $g^*(y_i) = b_i y_i$ for some $a_i, b_i \in \mathbb{Z}$. Since y_i is primitive, we easily see that $[f+g]^*(y_i) = (a_i + b_i) y_i$ and again by Lemma 3.3, $[f+g]$ is homologically diagonal. The case of $[-f]$ is quite similar. □

Now we recall the unstable Adams operation (see, e.g. [8]). Let q be an integer such that $(q, n!) = 1$. Then there is a map $\psi^q : BU(n) \longrightarrow BU(n)$ such that $(\psi^q)^*(c_i) = q^i c_i$, $1 \leq i \leq n$. Now let p be an odd prime and n be a positive integer. Then we can choose a prime l such that $(l, n!) = 1$ and l generates the multiplicative group $(\mathbb{Z}/p)^\times$. For an integer t, $1 \leq t \leq p-1$, we define a map $\phi_t : U(n) \longrightarrow U(n)$ by

$$\phi_t = \prod [\Omega\psi^l - l^i \,\mathrm{id}], \quad 0 \leq i \leq n \quad \text{and} \quad i \not\equiv t \bmod p-1$$

where the product is taken by means of composition.

Proposition 3.5. *$(\phi_t)_* : H_*(U(n); \mathbb{Z}) \longrightarrow H_*(U(n); \mathbb{Z})$ is given by*

$$(\phi_t)_*(x_{i_1} \cdots x_{i_s}) = \begin{cases} a x_{i_1} \cdots x_{i_s} & ; \ i_k \equiv t \bmod p-1 \ \textit{for all } k \\ 0 & ; \ \textit{otherwise} \end{cases}$$

where a is a certain integer $\not\equiv 0 \bmod p$.

Proof. Consider $(\Omega\psi^l)^* : H^*(U(n); \mathbb{Z}) \longrightarrow H^*(U(n); \mathbb{Z})$. Since y_i is transgressive to c_i, we have $(\Omega\psi^l)^*(y_i) = l^i y_i$. Then clearly $[\Omega\psi^l - l^j \,\mathrm{id}]^*(y_i) = (l^i - l^j) y_i$. Hence we have $(\phi_t)^*(y_i) = \prod (l^i - l^j) y_i$, $0 \leq j \leq n$ and $j \not\equiv t \bmod p-1$. Since $l^k - 1 \equiv 0 \bmod p$ if and only if $k \equiv 0 \bmod p-1$, we see that $(\phi_t)^*$ satisfies the required property for the cohomology basis. Then the proposition follows from the duality. □

Now we have a p-local decomposition of $U(n)$ as follows, see [6]. Let $q_1, q_2, \ldots$ be all primes except p and put $d_k = q_1 \cdots q_k$. Consider a sequence

$$U(n) \xrightarrow{d_1} U(n) \xrightarrow{\phi_t} (n) \xrightarrow{d_2} U(n) \xrightarrow{\phi_t} \cdots$$

where d_k means the d_k-times of the identity. We denote by $\tilde{X}_t(n)$ the telescope of the sequence. Note that the map $d_k : U(n) \longrightarrow U(n)$ is homologically diagonal. Let $\mu_t : U(n) \longrightarrow \tilde{X}_t(n)$ be the natural inclusion. Then we have $(\mu_t)_*(x_i) = 0$ for $i \not\equiv t \bmod p-1$ and writing $(\mu_t)_*(x_i)$ also by x_i for $i \equiv t \bmod p-1$, $H_*(\tilde{X}_t(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}_{(p)}}(x_t, x_{t+p-1}, \dots, x_{t+\alpha(p-1)})$ where α is the maximal integer such that $t + \alpha(p-1) \le n$. Thus we obtain

Theorem 3.6 ([6]). *The map*

$$\mu = \prod \mu_t : U(n) \longrightarrow \tilde{X}_1(n) \times \cdots \times \tilde{X}_{p-1}(n)$$

is a p-local equivalence.

4. Proof of the main theorem

In this section we prove our main result.

Theorem 4.1. (1) *Let* n *be a positive integer and let* p *be an odd prime. Then for each pair* (t,k) *of integers such as* $1 \le t \le p-1$ *and* $1 \le k \le n$, *there exists a finite spectrum* $X_{t,k}(n)$ *satisfying*

$$H_*(X_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(y_t, y_{t+p-1}, \dots, y_{t+\alpha(p-1)})$$

as a module where α *is the maximal integer such that* $t + \alpha(p-1) \le n$.

(2) *There exists a stable p-equivalence*

$$U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \cdots \wedge X_{t_m,k_m}(n)$$

where the wedge sum is taken over $1 \le t_1 < \cdots < t_m \le p-1$ *and* $k_1 + \cdots + k_m \le n$.

Proof. The stable splitting

$$U(n) \simeq \bigvee_{k=1}^{n} R_k(\mathbb{C}^n)^+$$

gives a splitting $1 = e_1 + \cdots + e_n$ of the unity by orthogonal idempotents in the ring $\{U(n), U(n)\}$ of homotopy classes of stable self maps so that $R_k(\mathbb{C}^n)^+ \simeq e_k U(n)$ the telescope of the map e_k. By Theorem 2.9, we easily see that $e_{k*} : H_*(U(n);\mathbb{Z}) \longrightarrow H_*(U(n);\mathbb{Z})$ is homologically diagonal and actually given by

$$e_{k*}(x_{i_1} \cdots x_{i_j}) = \begin{cases} 0 & ; j \ne k \\ x_{i_1} \cdots x_{i_j} & ; j = k \end{cases}.$$

In the construction of $\tilde{X}_t(n)$ in Section 3, we can use the stable self map $\phi_t \circ e_k$ instead of ϕ_t and we obtain a p-local spectrum $\tilde{X}_{t,k}(n)$ and a stable map

$\tilde{\lambda}_{t,k} : U(n) \longrightarrow \tilde{X}_{t,k}(n)$. It is clear that $(\tilde{\lambda}_{t,k})_*(x_{i_1}\cdots x_{i_l}) = 0$ if $l \neq k$ or $i_j \not\equiv t \bmod p-1$ for some j. When $l = k$ and $i_j \equiv t \bmod p-1$ for all j, writing $(\tilde{\lambda}_{t,k})_*(x_{i_1}\cdots x_{i_k})$ formaly by $y_{i_1}\cdots y_{i_k}$, we have

$$H_*(\tilde{X}_{t,k}(n);\mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}_{(p)}}(y_t, y_{t+p-1}, \ldots, y_{t+\alpha(p-1)})$$

as a module.

Next given m pairs $(t_1,k_1),\ldots,(t_m,k_m)$ such that $1 \leq t_1 < \cdots < t_m \leq p-1$ and $k = k_1 + \cdots + k_m \leq n$, consider the map

$$\tilde{\lambda}_{\mathbf{t},\mathbf{k}} : U(n) \xrightarrow{\Delta} U(n)\wedge\cdots\wedge U(n) \xrightarrow{\wedge\tilde{\lambda}_{t_i,k_i}} \tilde{X}_{t_1,k_1}(n)\wedge\cdots\wedge\tilde{X}_{t_m,k_m}(n)$$

where Δ is the diagonal map. Now for a set $I = \{i_1,\ldots,i_r\}$ of integers $i_1 < \cdots < i_r$, one can think of a partition $I = J_1 \amalg \cdots \amalg J_m$. We write $x_{i_1}\cdots x_{i_r}$ by x_I. Then $\Delta_* : H_*(U(n);\mathbb{Z}) \longrightarrow H_*(U(n)^{\wedge m};\mathbb{Z})$ is given by

$$\Delta_*(x_I) = \Sigma x_{J_1}\otimes\cdots\otimes x_{J_m}$$

where $I = J_1 \amalg \cdots \amalg J_m$ and $J_i \neq \emptyset$ for all i. Given a pair (t,k), we say $I = \{i_1,\ldots,i_r\}$ is of type (t,k) if $r = k$ and $i_j \equiv t \bmod p-1$ for all j. Then we can describe $(\tilde{\lambda}_{\mathbf{t},\mathbf{k}})_*$ by

$$(\tilde{\lambda}_{\mathbf{t},\mathbf{k}})_*(x_I) = \begin{cases} y_{J_1}\otimes\cdots\otimes y_{J_m} & ;\ J_i \text{ is of type } (t_i,k_i) \text{ for all } i \\ 0 & ;\ \text{otherwise} \end{cases}.$$

Then taking all wedge sum of those $\tilde{\lambda}_{\mathbf{t},\mathbf{k}}$, we have a stable p-equivalence

$$\tilde{\lambda} : U(n) \longrightarrow \bigvee \tilde{X}_{t_1,k_1}(n)\wedge\cdots\wedge\tilde{X}_{t_m,k_m}(n)$$

where $1 \leq t_1 < \cdots < t_m \leq p-1$ and $k_1 + \cdots + k_m \leq n$.

Now by a stable version of Proposition 1.4 [6], we see that there exists a finite spectrum $X_{t,k}(n)$ such that $H_*(X_{t,k}(n);\mathbb{Z})$ is torsion free and $X_{t,k}(n)_{(p)} \simeq \tilde{X}_{t,k}(n)$ where $X_{t,k}(n)_{(p)}$ is the p-localization of $X_{t,k}(n)$. One may think $X_{t,k}(n)$ is a subspectrum of $\tilde{X}_{t,k}(n)$ and the relative homotopy group $\pi_i(\tilde{X}_{t,k}(n), X_{t,k}(n))$ is torsion group whose elements have order prime to p. Then by obstruction theory we can choose an integer d prime to p and show that there exists a stable map $\lambda_{t,k}$ such that the diagram

$$\begin{array}{ccc} U(n) & \xrightarrow[\tilde{\lambda}_{t,k}]{} & \tilde{X}_{t,k}(n) \\ \uparrow d & & \uparrow l \\ U(n) & \xrightarrow[\lambda_{t,k}]{} & X_{t,k}(n) \end{array}$$

is homotopy commutative where d is the d-times map and l is the localization map. Then in a similar way we have a stable p-equivalence

$$\lambda : U(n) \longrightarrow \bigvee X_{t_1,k_1}(n) \wedge \cdots \wedge X_{t_m,k_m}(n).$$

This completes the proof. □

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