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Vertical Gromov-Witten invariants of flag bundles

By

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Introduction

Let G be a complex semi-simple connected Lie group and P its parabolic subgroup. Then the maximal torus T of G acts on the homogeneous space G/P and this action can be naturally lifted to the moduli space (stack) of the stable maps $\overline{\mathcal{M}}_{g,n}(G/P,\beta)$, where β is an element in $H_2(G/P,\mathbf{Z})$. In this paper, we investigate the T-equivariant Gromov-Witten invariants of G/P and vertical invariants of flag bundles. We mainly consider the invariants of genus zero. We will discuss on higher genus invariants briefly in Section 6. We will work only over the ground field $\mathbf{C}$ throughout this paper.

Kontsevich used the fixed point localization method for the first time to compute the Gromov-Witten invariants of the projective space and its hypersurfaces in his paper [16]. In Section 5.2 of [16], he mentioned that his computational scheme works well also for homogeneous spaces, and we will follow his method to give a formula of (gravitational) Gromov-Witten invariants of the homogeneous space G/P. The fixed point localization method enables us to obtain the information of the equivariant cohomology $H_T^*(\overline{\mathcal{M}}_{0,n}(G/P,\beta))$ from the data on the fixed points of the action of T. In our case, the set of fixed points $\overline{\mathcal{M}}_{0,n}(G/P,\beta)^T$ can be described in terms of the Bruhat ordering of the Weyl group W of G. Integration of an element in $H_T^*(\overline{\mathcal{M}}_{0,n}(G/P,\beta))$ can be expressed as a sum of the contributions from the components of the fixed locus. Such formula is known as Bott's fixed point formula. Then, the Gromov-Witten invariants of the flag variety G/B can be computed effectively by using Bott's fixed point formula for smooth stacks.

The fixed point localization method was also used in the proof of the Mirror Theorem for complete intersections in projective space by Givental [11]. Kim also obtained the Mirror Theorem when the ambient space is a homogeneous space [15]. The explanation of Kontsevich's method and Givental's proof of the Mirror Theorem can be found in [5].

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In [7], Fulton showed that degeneracy loci associated to a flag bundle over a variety X are expressed as double Schubert polynomials in the Chow group. We give a formula for vertical Gromov-Witten invariants of the flag bundle in terms of double Schubert polynomials.

The localization of the virtual fundamental class was developed by Graber and Pandharipande ([12]), so this method can be applied to the moduli space of stable maps of higher genus and to other target spaces with torus action. In the final section, we briefly see the higher genus invariants.

1. Moduli space of stable maps

We start with the definition of the stable map ([16], [17]). Let C be a connected reduced algebraic curve with at most ordinary double points and pairwise distinct smooth marked points $p_1, \dots, p_n$ $(n \geq 0)$. Then, $(C; p_1, \dots, p_n)$ is called a *prestable curve*. Consider a holomorphic map from a prestable curve to a complex manifold $f\colon C \to V$. Let the arithmetic genus $p_a(C) = g$ and $\beta = f_*([C]) \in H_2(V, \mathbf{Z})$.

Definition 1.1. The data $(C; p_1, \dots, p_n; f)$ is called a stable map of genus g and class β if every irreducible component of C of genus 0 (resp. 1) contracted to a point by f has 3 (resp. 1) marked or singular points (special points) on its normalization. We denote by $\overline{\mathcal{M}}_{g,n}(V, \beta)$ the moduli stack of the stable maps to V of genus g and class β with n marked points.

The stability condition means that the group of automorphisms of the stable map $(C; p_1 \dots, p_n; f)$ which act identically on V is finite.

The following proposition is a fundamental result on $\overline{\mathcal{M}}_{g,n}(V, \beta)$ shown by Kontsevich [16] (see also [9]).

Proposition 1.1. *Let V be a projective algebraic manifold. Then $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is a proper algebraic stack of finite type.*

This holds in general for a projective scheme V of finite type over a field ([16, 1.3.1]).

In this paper, we mainly consider the case $g = 0$ and V is a homogeneous space G/P, where G is a complex connected semisimple Lie group and P is a parabolic subgroup in G. We assume that G is simply-connected. The homogeneous space G/P is a convex manifold in the sense of [17].

Definition 1.2. Denote by $\mathcal{T}_V$ the tangent sheaf of V. The convex manifold V is defined to be a projective manifold such that $H^1(C, f^*\mathcal{T}_V) = 0$ for any stable map of genus 0.

Let $(C; p_1, \dots, p_n; f) \in \overline{\mathcal{M}}_{g,n}(V, \beta)$. Then, the *expected dimension* of $\overline{\mathcal{M}}_{g,n}(V, \beta)$ at $(C; p_1, \dots, p_n; f)$ is

$$\begin{aligned}\chi(C, f^*\mathcal{T}_V) + \dim_{\mathbf{C}} \overline{\mathcal{M}}_{g,n} &= h^0(C, f^*\mathcal{T}_V) - h^1(C, f^*\mathcal{T}_V) + 3g - 3 + n \\ &= (1-g)(\dim_{\mathbf{C}} V - 3) + \int_\beta c_1(V) + n.\end{aligned}$$

We denote by $d_{g,n}(V,\beta)$ the expected dimension of $\overline{\mathcal{M}}_{g,n}(V,\beta)$. In general, the dimension of $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is greater than or equal to $d_{g,n}(V,\beta)$.

Proposition 1.2 ([3], [16]). *If V is a convex manifold, then $\overline{\mathcal{M}}_{0,k}(V,\beta)$ is a smooth stack of dimension $d_{0,n}(V,\beta)$.*

Remark 1.1. From this proposition, the underlying space of the stack $\overline{\mathcal{M}}_{0,n}(V,\beta)$ for a convex manifold V can be regarded as an orbifold.

It is known that there is an element in the Chow group

$$[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt} \in A_{d_{g,n}(V,\beta)}(\overline{\mathcal{M}}_{g,n}(V,\beta))_{\mathbf{Q}}$$

which is called the virtual fundamental class of the moduli stack $\overline{\mathcal{M}}_{g,n}(V,\beta)$ and it can be considered invariant under the deformation of V in a natural sense ([2], [20]). For the definition of the Chow group of algebraic stacks, see [23]. This class is used instead of the usual fundamental class in the definition of the Gromov-Witten invariant. From Proposition 1.2, we know that $[\overline{\mathcal{M}}_{0,n}(G/P,\beta)]^{virt}$ coincides with the usual fundamental class.

Here, we prepare some notation which is used later on. Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G. We choose a Lie subalgebra $\operatorname{Lie} T$ corresponding to a maximal torus T of G as a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by Δ the set of roots in $\mathfrak{h}^* = \operatorname{Hom}(\mathfrak{h}, \mathbf{C})$. For a root $\alpha \in \Delta$, we define an element $\alpha^\vee \in \mathfrak{h}$ by the condition

$$\lambda(\alpha^\vee) = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \quad \text{for } \lambda \in \mathfrak{h}^*,$$

where $(\ ,\ )$ is an invariant inner product on $\mathfrak{h}^*$ induced by the Killing form. We realize the dual root system $\Delta^\vee$ in $\mathfrak{h}$ by setting

$$\Delta^\vee = \{\alpha^\vee \in \mathfrak{h} \mid \alpha \in \Delta\}.$$

The Lie algebra $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \right).$$

We fix a Borel subgroup $B \subset G$ and the set of the positive roots Δ_+ in Δ such that

$$\operatorname{Lie}(B) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha} \right).$$

The root system Δ is decomposed into the disjoint union of Δ_+ and $\Delta_- = -\Delta_+$. The set of positive roots Δ_+ contains a unique simple system $\Pi \subset \Delta_+$.

The positive dual roots $\Delta_+^\vee \subset \Delta^\vee$ and the simple system $\Pi^\vee \subset \Delta_+^\vee$ are also determined by setting $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$. We define $\mathfrak{h}_{\mathbf{Z}} \subset \mathfrak{h}$ and $\mathfrak{h}^*_{\mathbf{Z}} \subset \mathfrak{h}^*$ as

$$\mathfrak{h}_{\mathbf{Z}} = \bigoplus_{\alpha^\vee \in \Pi^\vee} \mathbf{Z} \cdot \alpha^\vee,$$

and

$$\mathfrak{h}^*_{\mathbf{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) \in \mathbf{Z},\ \ {}^\forall\alpha^\vee \in \Delta^\vee\}.$$

A parabolic subgroup $P \supset B$ determines a parabolic subsystem $(\Pi(P), \Delta(P))$, $\Pi(P) \subset \Pi$, $\Delta(P) \subset \Delta$, such that

$$\mathrm{Lie}(P) = \mathrm{Lie}(B) \oplus \bigoplus_{\alpha \in \Delta(P)_+} \mathfrak{g}_\alpha.$$

We also have the dual system $(\Pi^\vee(P), \Delta^\vee(P))$. Reflections s_α with respect to the roots $\alpha \in \Delta(P)$ generate a subgroup W_P in the Weyl group W.

Lemma 1.1 ([13, Proposition 1.10])**.** *Define*

$$W^P = \{w \in W \mid l(ws_\alpha) > l(w),\ \ \forall \alpha \in \Pi(P)\}.$$

Given $w \in W$, there is a unique $w_1 \in W^P$ and a unique $w_2 \in W_P$ such that $w = w_1 w_2$. Their lengths satisfy $l(w) = l(w_1) + l(w_2)$, and w_1 is the unique element of smallest length in the coset wW_P. In particular, we can regard W^P as a set of representatives of W/W_P.

We set

$$\mathfrak{h}_{\mathbf{Z}}(P) = \mathfrak{h}_{\mathbf{Z}} / \bigoplus_{\alpha \in \Pi(P)} \mathbf{Z} \cdot \alpha,$$

and

$$\mathfrak{h}^*_{\mathbf{Z}}(P) = \{\lambda\colon \mathfrak{h}_{\mathbf{Z}}(P) \to \mathbf{Z},\ \text{linear}\} \subset \mathfrak{h}^*_{\mathbf{Z}}.$$

If $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}(P)$ is given, we can define a character of T by

$$\exp(h) \mapsto \exp(\lambda(h)) \in \mathbf{C}^\times$$

for $h \in \mathfrak{h}$. This character can be naturally extended to the character of P and determines a line bundle L_λ over G/P. The following is a well-known fact. (For example, see [4].)

Lemma 1.2. *Let $\iota(\lambda) := c_1(L_\lambda) \in H^2(G/P, \mathbf{Z})$. Then ι gives an isomorphism between $\mathfrak{h}^*_{\mathbf{Z}}(P)$ and $H^2(G/P, \mathbf{Z})$.*

Now we can identify the homology group $H_2(G/P, \mathbf{Z})$ with $\mathfrak{h}_{\mathbf{Z}}(P)$. We denote by $[\gamma]$ the homology class in $H_2(G/P, \mathbf{Z})$ corresponding to an element $\gamma \in \mathfrak{h}_{\mathbf{Z}}(P)$. This identification is compatible with the natural pairing, so we have

$$\int_{[\gamma]} c_1(L_\lambda) = \lambda(\gamma).$$

We can see that the first Chern class $c_1(G/P)$ coincides with $\sum_{\alpha \in \Delta_+ \setminus \Delta(P)_+} \alpha$ under the isomorhism ι, and $\dim G/P = \#\Delta_+ - \#\Delta(P)_+$. Hence, we have the following.

Lemma 1.3. *If $\gamma \in \mathfrak{h}_{\mathbf{Z}}(P)$ is in the image of*

$$\bigoplus_{\alpha^\vee \in \Pi^\vee} \mathbf{Z}_{\geq 0} \cdot \alpha^\vee,$$

then, we have

$$d_{0,n}(G/P, [\gamma]) = \#\Delta_+ - \#\Delta(P)_+ + \sum_{\alpha \in \Delta_+ \setminus \Delta(P)_+} \alpha(\gamma) + n - 3.$$

We also consider the moduli space of vertical stable maps for a flag bundle over a variety. Let X be a variety and $\mathcal{F}$ be a flag bundle $\pi \colon \mathcal{F} \to X$, whose fiber is a homogeneous space G/P. We define the *vertical stable map* to $\mathcal{F}$ to be a stable map $(C; p_1, \dots, p_n; f)$ to $\mathcal{F}$ such that $\mathrm{Im}(\pi \circ f)$ is a point in X. Denote by $\overline{\mathcal{M}}^v_{g,n}(\mathcal{F}, \beta)$ the moduli space of vertical stable maps such that $f_*([C]) = \beta$, $\beta \in H_2(G/P, \mathbf{Z})$ in a fiber of π. Then, the morphism

$$\begin{aligned} \tilde{\pi} \colon \overline{\mathcal{M}}^v_{0,n}(\mathcal{F}, \beta) &\to X \\ (C; (p_i); f) &\mapsto \mathrm{Im}(\pi \circ f) \end{aligned}$$

is proper and smooth of relative dimension $d_{0,n}(G/P, \beta)$ as a morphism between stacks.

2. Fixed points of the action of T

The maximal torus T acts on G/P, and this action induces an action on $\overline{\mathcal{M}}_{0,n}(G/P, \beta)$ as follows. Let $(C; p_1, \dots, p_n; f) \in \overline{\mathcal{M}}_{0,n}(G/P, \beta)$. Then an element $t \in T$ acts on $\overline{\mathcal{M}}_{0,n}(G/P, \beta)$ by

$$t \colon (C; p_1, \dots, p_n; f) \mapsto (C; p_1, \dots, p_n; t \circ f).$$

Assume that $P = B$ for the sake of simplicity. We see how T acts on G/B in detail. The flag variety G/B is decomposed into the disjoint union of the *Schubert cells* $X^\circ_w = BwB/B$,

$$G/B = \coprod_{w \in W} X^\circ_w,$$

where W is the Weyl group. If w has the length $l(w)$, then $X^\circ_w \cong \mathbf{C}^{l(w)}$.

Definition 2.1. (1) Let $w, w' \in W$. Denote by s_γ the reflection on $\mathfrak{h}$ with respect to $\gamma \in \Delta_+^\vee$. Then the arrow $w \xrightarrow{\gamma} w'$ means that $w = w's_\gamma$ and $l(w') = l(w) + 1$. We will omit the label γ over the arrow if we need not to specify it.

(2) When there exists $w_1, \ldots, w_k \in W$ such that w and w' can be connected by a sequence of arrows

$$w \to w_1 \to \cdots \to w_k \to w',$$

or $w = w'$, we write $w \leq w'$, and the relation $\leq$ is called the Bruhat ordering on W. The graph obtained by connecting each element of the Weyl groups by these arrows is also called the Bruhat ordering.

Remark 2.1. Let X_w be the closure of X_w° in G/B. Then,

$$X_w = \coprod_{v \leq w} X_v^\circ.$$

We can choose a linear coordinate $(z_1, \ldots, z_{l(w)})$ on each Schubert cell $X_w^\circ \cong \mathbf{C}^{l(w)}$ such that the action of $t \in T$ is in the form of

$$(z_1, \ldots, z_{l(w)}) \mapsto (a_1(t)z_1, \ldots, a_{l(w)}(t)z_{l(w)}),$$

where $a_1(t), \ldots, a_{l(w)}(t)$ are nonzero complex numbers determined by t. Then the origin is a fixed point of the action of T. Hence, the T-action on G/B has isolated fixed points $p(w)$ in X_w° for the elements of the Weyl group $w \in W$. Moreover, if there is an arrow $w \xrightarrow{\gamma} w'$, then $p(w)$ and $p(w')$ are connected by a T-invariant curve $l_{ww'} \cong \mathbf{P}^1$ in G/B, which belongs to the class $[\gamma] \in H_2(G/B, \mathbf{Z})$. So, the chain of T-invariant curves $l_{ww'}$ in G/B is described by the Bruhat ordering of W.

Proposition 2.1. (1) *There is a one-to-one correspondence between the elements w in W and the fixed points $p(w) \in G/B$ of the action of T.*

(2) *There is a one-to-one correspondence between the arrows $w \xrightarrow{\gamma} w'$ in the Bruhat ordering of W and the T-invariant curves $l_{ww'}$ in G/B isomorphic to $\mathbf{P}^1$ connecting $p(w)$ and $p(w')$, which belongs to the class $[\gamma] \in H_2(G/B, \mathbf{Z})$.*

We can see the situation explicitly in A_r case.

Example 2.1. We consider the case $G = SL(r+1, \mathbf{C})$ and B is the subgroup of G consisting of upper triangle matrices. Consider a $\mathbf{C}$-vector space $U \cong \mathbf{C}^{r+1}$ and fix a standard basis $e_1, \ldots, e_{r+1}$. Then G acts on U, and the flag variety G/B paramerizes the full flags

$$0 \subset U_1 \subset \cdots \subset U_{r+1} = U \quad (\dim U_i = i).$$

The diagonal matrices

$$d(t_1, \ldots, t_{r+1}) = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{r+1} \end{pmatrix}$$

satisfying the condition $t_1 \cdots t_{r+1} = 1$ form a maximal torus T in G. The action of $d(t_1, \dots, t_{r+1})$ on flags is induced by $e_i \mapsto t_i \cdot e_i$. In this case, the Weyl group W is a symmetric group S_{r+1}, which acts on $e_1, \dots, e_{r+1}$ by the permutation of indices. The fixed point $p(w)$ corresponds to a flag

$$F_w \ : \ 0 \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \dots, e_{w(r)} \rangle \subset U.$$

Let s_{ij} $(i < j)$ be a substitution of i and j. If there is an arrow $w \to w' = ws_{ij}$, then $w(i) < w(j)$. The flag obtained by replacing $e_{w(i)}$ by $a_1 e_{w(i)} + a_2 e_{w(j)}$ $(a_1 \neq 0$ or $a_2 \neq 0)$ in the flag F_w determines a point on the T-invariant curve $l_{ww'} \cong \mathbf{P}^1$ connecting $p(w)$ and $p(w')$, and $(a_1 : a_2)$ is regarded as its homogeneous coodinate. Then $d(t_1, \dots, t_{r+1})$ acts on $l_{ww'}$ by

$$(a_1 : a_2) \mapsto (t_{w(i)} a_1 : t_{w(j)} a_2),$$

and $p(w) = (1 : 0)$, $p(w') = (0 : 1)$.

Now we consider the fixed points of the action of T on $\overline{\mathcal{M}}_{0,n}(G/B, \beta)$. From the consideration of the T-action on G/B, we know that if a stable map $(C; p_1, \dots, p_n; f) \in \overline{\mathcal{M}}_{0,n}(G/B, \beta)$ is fixed by the action of T, then each special point of C must be mapped to a fixed point $p(w)$, and the image of each irreducible component of C by f must be a T-invariant curves $l_{ww'}$. We introduce the notion of the *graph* of the stable map fixed by the action of T ([16]).

Definition 2.2. Let $(C; p_1, \dots, p_n; f) \in \overline{\mathcal{M}}_{0,n}(G/B, \beta)$ be a stable curve fixed under the action of T. Then we associate to $(C; p_1, \dots, p_n; f)$ its graph

$$(\Gamma, (w(v))_{v \in \mathrm{Vert}(\Gamma)}, (S_v)_{v \in \mathrm{Vert}(\Gamma)}, (d_e)_{e \in \mathrm{Edge}(\Gamma)})$$

determined by the following conditions.

(i) Γ is a finite one-dimensional CW-complex. Each vertex $v \in \mathrm{Vert}(\Gamma)$ corresponds to the connected component C_v of $f^{-1}(\{p(w) \mid w \in W\})$. Each edge $e \in \mathrm{Edge}(\Gamma)$ corresponds to an irreducible component C^e mapping to a curve $l_{ww'}$. If the irreducible component C^e intersects C_v and $C_{v'}$ $(v \neq v')$, then the vertices v and v' are connected by the edge e.

(ii) The element $w(v) \in W$ is defined by the condition $f(C_v) = p(w(v))$.

(iii) $S_v \subset \{1, \dots, n\}$ is the set of indices of marked points on C_v.

(iv) d_e is the mapping degree of $f|_{C^e} : C^e \to l_{ww'}$.

Remark 2.2. Conversely, we can associate a substack

$$\overline{\mathcal{M}}_\Gamma \subset \overline{\mathcal{M}}_{0,n}(G/B, [\gamma])^T$$

to the equivalence class of a graph $(\Gamma, (w(v))_v, (S_v)_v, (d_e)_e)$ satisfying the conditions:

(a) if an edge e connects vertices v and v', then $w(v) \neq w(v')$,

(b) Γ is a tree, i.e. Γ is connected and $H_1(\Gamma, \mathbf{Z}) = 0$,

(c) if e is an edge connecting v and v', then $|l(w(v)) - l(w(v'))| = 1$ and there exists a positive root $\gamma(e) \in \Delta_+^\vee$ such that $w(v) = w(v')s_{\gamma(e)}$,

(d) $\sum_{e \in \mathrm{Edge}(\Gamma)} d_e \gamma(e) = \gamma$,

(e) $\{1, \dots, n\} = \coprod_{v \in \mathrm{Vert}(\Gamma)} S_v$.

In this paper, $(\Gamma, (w(v)), (S_v), (d_e))$ satisfying these conditions is called a graph of type $(g = 0, n, \beta)$.

Let us see the structure of the substack $\overline{\mathcal{M}}_\Gamma$ (cf. [5, 9.2.1]). Denote by $\mathrm{val}(v)$ the number of the edges attached to a vertex v and $n(v) = \#S_v + \mathrm{val}(v)$. Assume that $\dim C_v = 1$. Then C_v is a connected rational curve contracted to a point $p(w(v))$ by f. There must be more than 3 special points on each irreducible component of such a curve C_v, so $n(v) \geq 3$. If $n \geq 3$, the moduli of the stable n-pointed curves of genus 0 can be represented by a smooth projective algebraic variety $\overline{M}_{0,n}$ ([14]). The curve C_v such that $\dim C_v = 1$ determines a point in $\overline{M}_{0,n}$. If the graph $(\Gamma, (w(v))_v, (S_v)_v, (d_e)_e)$ and the isomorphism class of C_v such that $\dim C_v = 1$ are given, we can construct the corresponding stable map $(C; p_1, \dots, p_n; f)$ uniquely. Now we have a natural morphism

$$\psi_\Gamma \colon \overline{M}_\Gamma := \prod_{\dim C_v = 1} \overline{M}_{0,n(v)} \to \overline{\mathcal{M}}_\Gamma.$$

Let $\mathrm{Aut}(\Gamma)$ be the automorphism group of the graph Γ preserving the labels. The group $\mathbf{Z}/d_e\mathbf{Z}$ acts on $f|_{C^e} : C^e \to l_{ww'}$ by the covering transformation. Hence, the stable map $(C; , p_1, \dots, p_n; f)$ has the automorphism group A_Γ defined by the extension

$$0 \to \prod_{e \in \mathrm{Edge}(\Gamma)} \mathbf{Z}/d_e\mathbf{Z} \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 0.$$

The automorphism group A_Γ acts on $\overline{M}_\Gamma$ and ψ_Γ is the quotient map by A_Γ. Hence, the substack $\overline{\mathcal{M}}_\Gamma$ is also regarded as an orbifold.

3. Equivariant cohomology

Let us recall the definition of equivariant cohomology. Assume that a Lie group $\mathcal{G}$ acts on a topological space X. It is known that there exists a space $B\mathcal{G}$ called the classifying space of $\mathcal{G}$, whose homotopy type is characterized by the existence of a principal $\mathcal{G}$-bundle $E\mathcal{G} \to B\mathcal{G}$ such that $E\mathcal{G}$ is contractible. Then, $\mathcal{G}$ acts on $X \times E\mathcal{G}$ freely, and we put

$$X_\mathcal{G} = (X \times E\mathcal{G})/\mathcal{G}.$$

The *$\mathcal{G}$-equivariant cohomology* $H^*_\mathcal{G}(X)$ is defined to be the cohomology of $X_\mathcal{G}$ with suitable coefficients. In this paper, we consider the cohomology with complex coefficients, namely

$$H^*_\mathcal{G}(X) = H^*(X_\mathcal{G}, \mathbf{C}).$$

The projection $X_{\mathcal{G}} \to B\mathcal{G}$ induces an $H^*(B\mathcal{G})$-module strucure on $H^*_{\mathcal{G}}(X)$. We can take the integration of an element in $H^*_{\mathcal{G}}(X)$ along the fiber of $X_{\mathcal{G}} \to B\mathcal{G}$. Then we obtain a morphism

$$\int_{X_{\mathcal{G}}} : H^*_{\mathcal{G}}(X) \to H^*(B\mathcal{G}).$$

Let E be a $\mathcal{G}$-equivariant vector bundle on X. Then $E_{\mathcal{G}}$ is a vector bundle on $X_{\mathcal{G}}$. The *equivariant Chern classes* $c_i^{\mathcal{G}}(E) \in H^*_{\mathcal{G}}(X)$ are defined to be the Chern classes $c_i(E_{\mathcal{G}})$. The top equivariant Chern class $c^{\mathcal{G}}_{\dim X}(E)$ is called the *equivariant Euler class* and denoted by $\mathrm{Euler}_{\mathcal{G}}(E)$.

We will investigate the case $X = \overline{\mathcal{M}}_{0,n}(G/B, \beta)$ and $\mathcal{G} = T \cong (\mathbf{C}^\times)^r$. We may choose $(\mathbf{P}^\infty)^r$ as the classifying space BT, so that the tautological bundles of the components $\mathbf{P}^\infty$ correspond to the fundamental dominant weights $\lambda_{\alpha^\vee} \in \mathfrak{h}^*$ ($\alpha^\vee \in \Pi^\vee$) defined by the condition

$$\lambda_{\alpha^\vee}(\beta^\vee) = \delta_{\alpha^\vee, \beta^\vee} \quad \text{for } \alpha^\vee, \beta^\vee \in \Pi^\vee.$$

Then we can identify $H^2(BT)$ with $\mathfrak{h}^*$, and we obtain an isomorphism

$$H^*(BT) \cong \mathrm{Sym}_{\mathbf{C}} \mathfrak{h}^*.$$

We denote by R_T the field of fractions of the polynomial ring $\mathrm{Sym}_{\mathbf{C}} \mathfrak{h}^*$,

$$R_T = \mathbf{C}(\lambda_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee).$$

When a graph $(\Gamma, (w(v)), (S_v), (d_e))$ of type $(0, n, \beta)$ is given, we have a morphism

$$j_\Gamma \colon \overline{M}_\Gamma \stackrel{\psi_\Gamma}{\to} \overline{\mathcal{M}}_\Gamma \stackrel{i_\Gamma}{\to} \overline{\mathcal{M}}_{0,n}(G/B, \beta),$$

where ψ_Γ is defined in the previous section, and i_Γ is an inclusion. Denote by $\mathcal{N}_\Gamma$ the normal bundle to $\overline{\mathcal{M}}_\Gamma$ in $\overline{\mathcal{M}}_{0,n}(G/B, \beta)$. The equivariant Euler class of $\mathcal{N}_\Gamma$ determines an element in R_T. We denote it also by $\mathrm{Euler}_T(\mathcal{N}_\Gamma)$.

Proposition 3.1. *The homomorphism*

$$\begin{aligned} j \colon H^*_T(\overline{\mathcal{M}}_{0,n}(G/B, \beta)) \otimes R_T &\to \bigoplus_{\Gamma} \bigotimes_{\dim C_v = 1} H^*(\overline{M}_{0,n(v)}) \otimes R_T \\ \theta \qquad &\mapsto \sum_{\Gamma} j^*_\Gamma(\theta)/(\#A_\Gamma \cdot \mathrm{Euler}_T(\mathcal{N}_\Gamma)) \end{aligned}$$

is an isomorphism. Here Γ *runs over all the graphs of type* $(0, n, \beta)$.

Proof. Since $\overline{\mathcal{M}}_\Gamma$ is a smooth substack of $\overline{\mathcal{M}}_{0,n}(G/B, \beta)$, we can construct the Gysin map

$$(i_\Gamma)_! : H^*_T(\overline{\mathcal{M}}_\Gamma) \to H^*_T(\overline{\mathcal{M}}_{0,n}(G/B, \beta)).$$

For $\varphi \in H_T^*(\overline{\mathcal{M}}_\Gamma)$, we have

$$(i_\Gamma)^* \circ (i_\Gamma)_!(\varphi) = \varphi \cup \mathrm{Euler}_T(\mathcal{N}_\Gamma).$$

As in shown in [1], $\mathrm{Euler}_T(\mathcal{N}_\Gamma)$ is invertible in $H_T^*(\overline{\mathcal{M}}_\Gamma) \otimes R_T$. Consider the homomorphism

$$\begin{aligned} j' : H_T^*(\overline{\mathcal{M}}_{0,n}(G/B,\beta)) &\to \bigoplus_\Gamma \big(H_T^*(\overline{\mathcal{M}}_\Gamma) \otimes R_T\big) \\ \varphi \qquad &\mapsto \sum_\Gamma \frac{i_\Gamma^*(\varphi)}{\mathrm{Euler}_T(\mathcal{N}_\Gamma)}. \end{aligned}$$

The R_T-linear map

$$\sum_\Gamma (i_\Gamma)_! : \bigoplus_\Gamma \big(H_T^*(\overline{\mathcal{M}}_\Gamma) \otimes R_T\big) \to H_T^*(\overline{\mathcal{M}}_{0,n}(G/B,\beta))$$

satisfies $j' \circ (\sum_\Gamma (i_\Gamma)_!) = \mathrm{id}$. In [1], it is shown that the kernel of the restricton map of the equivarinat cohomology of a compact manifold to the fixed points is a torsion module over $H^*(BT)$. Their argument works well also for orbifolds. Since ψ_Γ is a quotient by A_Γ, we obtain the result. □

From Proposition 3.1, it follows that

$$\int_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)_T} \theta \;=\; \sum_\Gamma \int_{\overline{M}_\Gamma} \frac{j_\Gamma^*(\theta)}{\# A_\Gamma \cdot \mathrm{Euler}_T(\mathcal{N}_\Gamma)} \in R_T.$$

In particular, if $E_1, \ldots, E_k$ are equivariant vector bundles over $\overline{\mathcal{M}}_{0,n}(G/B,\beta)$ and Q is a polynomial, then

$$\int_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)} Q(c(E)) = \lim_{\lambda_{\alpha^\vee} \to 0} \sum_\Gamma \int_{\overline{M}_\Gamma} \frac{j_\Gamma^* Q(c^T(E))}{\# A_\Gamma \cdot \mathrm{Euler}_T(\mathcal{N}_\Gamma)},$$

where $c(E) = (c_{i_1}(E_1), \ldots, c_{i_k}(E_k))$ and $c^T(E) = (c_{i_1}^T(E_1), \ldots, c_{i_k}^T(E_k))$. The contribution from the Euler class of $\mathcal{N}_\Gamma$ is given in the next section.

Example 3.1. We define the (orbifold) Euler number of $\overline{\mathcal{M}}_{0.n}(G/B,\beta)$ by

$$\begin{aligned} \chi(\overline{\mathcal{M}}_{0,n}(G/B,\beta)) &= \int_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)_T} c_{\mathrm{top}}^T(\mathcal{T}_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)}) \\ &= \sum_\Gamma \int_{\overline{M}_\Gamma} \frac{c_{\mathrm{top}}(\mathcal{T}_{\overline{M}_\Gamma})}{\# A_\Gamma}. \end{aligned}$$

This can be calculated from the combinatorial data of the graphs Γ. If $\alpha^\vee \in \Delta_+^\vee$, then it cannot happen that more than two arrows labeled by $\alpha^\vee$ are attached to one vertex in the Bruhat ordering of W. Moreover, if $\alpha^\vee \in \Pi^\vee$, then the image of a stable map $(C; (p_i)_i; f) \in \overline{\mathcal{M}}_{0,n}(G/B, d[\alpha^\vee])^T$, $d \in \mathbf{Z}_{>0}$ is a T-invariant curve $l_{ww'}$ which belongs to the class $[\alpha^\vee]$ with multiplicity d. Since

$$l(ws_{\alpha^\vee}) - l(w) = \pm 1$$

for any simple root $\alpha^\vee$, the number of the arrows labeled by $\alpha^\vee$ in the Bruhat ordering is $\#W/2$. Hence, for $\alpha^\vee \in \Pi^\vee$ and $d \in \mathbf{Z}_{>0}$,

$$\chi_{n,d} := \frac{2 \cdot \chi(\overline{\mathcal{M}}_{0,n}(G/B, d[\alpha^\vee]))}{\#W}$$

depends only on d and n. Actually, $\chi_{n,d}$ coincides with $\chi(\overline{\mathcal{M}}_{0,n}(\mathbf{P}^1, d))$. (This holds for an arbitrary genus g.) The first several terms of $\chi_{n,d}$ are

$$\begin{aligned} &\chi_{0,1} = 1, \chi_{0,2} = \frac{3}{2}, \ \chi_{0,3} = 3, \ \dots \\ &\chi_{1,1} = 2, \ \chi_{1,2} = 3, \ \chi_{1,3} = 10, \ \dots \\ &\chi_{2,1} = 4, \chi_{2,2} = \frac{21}{2}, \chi_{2,3} = \frac{179}{3}, \ \dots \end{aligned}$$

and so on.

Example 3.2. Consider the case $n = 3$ and $\beta = 0$. In this case, we can identify $\overline{\mathcal{M}}_{0,3}(G/B, 0)$ with the target space G/B itself. We know that the set of the fixed points of the action of T is $(G/B)^T = \{p(w) \mid w \in W\}$. Hence, we have the isomorphism

$$\begin{aligned} H_T^*(G/B) \otimes R_T &\cong \bigoplus_{w \in W} H_T^*(p(w)) \otimes R_T \\ &\cong (R_T)^{\oplus(\#W)}. \end{aligned}$$

Since the action of T on $\mathcal{N}_\Gamma = T_{p(w)}(G/B)$ is isomorphic to

$$\bigoplus_{\alpha \in \Delta_+} \mathcal{O}(w(\alpha)),$$

where $\mathcal{O}(\lambda)$, $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*$ is a representation of T determined by λ, we have

$$\mathrm{Euler}_T(\mathcal{N}_\Gamma) = c_{\mathrm{top}}^T(T_{p(w)}(G/B)) = w\left(\prod_{\alpha \in \Delta_+} \alpha\right) \in R_T.$$

Let $\lambda_1, \dots, \lambda_k \in \mathfrak{h}_{\mathbf{Z}}^*$ and $\theta_i := c_1^T(L_{\lambda_i})$. If $Q(\theta_1, \dots, \theta_k)$ is a polynomial in $\theta_1, \dots, \theta_k$, then

$$\int_{(G/B)_T} Q(\theta_1, \dots, \theta_k) = \sum_{w \in W} (-1)^{l(w)} \cdot \frac{Q(w\lambda_1, \dots, w\lambda_k)}{\prod_{\alpha \in \Delta_+} \alpha}.$$

Remark 3.1. The results in this section are translated to the statements on the homogeneous space G/P by applying the following substitution:

$$\begin{aligned} \Pi &\to \Pi \setminus \Pi(P) \\ \Delta &\to \Delta \setminus \Delta(P) \\ W &\to W^P. \end{aligned}$$

4. Contribution from $\mathrm{Euler}_T(\mathcal{N}_\Gamma)$

In this section, we calculate the equivariant Euler class $\mathrm{Euler}_T(\mathcal{N}_\Gamma)$ following the way in [16]. We introduce the equivariant K-group with rational coefficients

$$K^0_T(\overline{\mathcal{M}}_\Gamma) \otimes \mathbf{Q} \cong K^0(\overline{\mathcal{M}}_\Gamma) \otimes (\mathrm{Sym}\,\mathfrak{h}^*_{\mathbf{Z}})_{\mathbf{Q}}\,,$$

where we regard $\mathrm{Sym}\,\mathfrak{h}^*_{\mathbf{Z}}$ as the character ring of T. Denote by $[\mathcal{E}]$ the element in $K^0_T(\overline{\mathcal{M}}_\Gamma) \otimes \mathbf{Q}$ corresponding to a T-equivariant vector bundle $\mathcal{E}$. The restricion of a vector bundle $\mathcal{E}$ on $\overline{\mathcal{M}}_{0,n}(G/B,\beta)$ to $\overline{\mathcal{M}}_\Gamma$ is also denoted by the same symbol $\mathcal{E}$. Denote by $[\chi]$ the element corresponding to a character $\chi \in \mathrm{Sym}\,\mathfrak{h}^*_{\mathbf{Z}}$.

The *flag* F of Γ is a pair of an edge e of Γ and a vertex v on e. For a given flag $F = (e, v)$, the *weight* of F is defined by $\omega_F = w(v)\lambda(e)/d_e$, where $\lambda(e)$ is a positive root in Δ such that $\gamma(e) = \lambda(e)^\vee$. The weight ω_F is the character of the action of T on the tangent space to C_v at the point $p(F) = C_v \cap C^e$. Denote by c_F the first Chern class of the line bundle determined by the cotangent line $T^*_{p(F)}(C_v)$ over $\overline{\mathcal{M}}_\Gamma$.

In this section, $\{C^a\}$ is the set of all irreducible components of C, and the index a is not necessarily an edge of Γ.

The class of the normal bundle $\mathcal{N}_\Gamma$ is expressed as

$$[\mathcal{N}_\Gamma] = [\mathcal{T}_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)}] - [\mathcal{T}_{\overline{\mathcal{M}}_\Gamma}].$$

At first, let us consider the case $n = 0$. Let

$$\begin{aligned} T_1 &:= [H^0(C, f^*\mathcal{T}_{G/B})], \\ T_2 &:= \sum_{y \in C^a \cap C^b : a \neq b} [T_y C^a \otimes T_y C^b], \\ T_3 &:= \sum_{y \in C^a \cap C^b : a \neq b} ([T_y C^a] + [T_y C^b]) - \sum_a [H^0(C^a, \mathcal{T}_{C^a})]. \end{aligned}$$

Here, T_1 corresponds to deformations of f of a fixed curve C, T_2 comes from smoothings of singular points of C, and T_3 comes from deformations of singular points of C. Then we have

$$[\mathcal{T}_{\overline{\mathcal{M}}_{0,n}(G/B,\beta)}] = T_1 + T_2 + T_3.$$

Similarly, if we put

$$T_1' := \sum_{y\in C^a\cap C^b : a\neq b : a,b\notin \mathrm{Edge}(\Gamma)} [T_yC^a \otimes T_yC^b]$$
$$T_2' := \sum_{y\in C^a\cap C^b : a\neq b : a\notin \mathrm{Edge}(\Gamma)} [T_yC^a] - \sum_{a\notin \mathrm{Edge}(\Gamma)} [H^0(C^a, \mathcal{T}_{C^a})],$$

then we have

$$[\mathcal{T}_{\overline{\mathcal{M}}_\Gamma}] = T_1' + T_2'.$$

Here, T_1' and T_2' come from smoothings and deformations of the singular points on the contracted components of C respectively. Now we can decompose $[\mathcal{N}_\Gamma]$ into the sum of $[H^0(C, f^*\mathcal{T}_{G/B})]$ and the absolute part $[\mathcal{N}_\Gamma^{abs}]$ which has the expression independent of the target space for given $(\omega_F)_F$. The absolute part is given by

$$\begin{aligned}[\mathcal{N}_\Gamma^{abs}] = &\sum_{y\in C^a\cap C^b : a\neq b : a,b\in \mathrm{Edge}(\Gamma)} [T_yC^a \otimes T_yC^b] \\ &+ \sum_{y\in C^a\cap C^b : a\in \mathrm{Edge}(\Gamma) : b\notin \mathrm{Edge}(\Gamma)} [T_yC^a \otimes T_yC^b] \\ &+ \left(\sum_{y\in C^a\cap C^b : a\neq b, a\in \mathrm{Edge}(\Gamma)} [T_yC^a] - \sum_{a\in \mathrm{Edge}(\Gamma)} [H^0(C^a, \mathcal{T}_{C^a})]\right).\end{aligned}$$

The contribution from the absolute part is obtained by Kontsevich ([16, 3.3.3]). His result is that the contribution of $[\mathcal{N}_\Gamma^{abs}] + \#\mathrm{Edge}(\Gamma)\cdot[0]$ to $\mathrm{Euler}_T\mathcal{N}_\Gamma$ is

$$\begin{aligned}&\prod_{v:\mathrm{val}(v)\geq 3}\left(\prod_{\text{flags } F=(e,v)} (\omega_F - c_F)\right) \\ &\times \prod_{v:\mathrm{val}(v)\leq 2}\left(\left(\sum_{\text{flags } F=(e,v)} \omega_F\right)^{3-\mathrm{val}(v)} \prod_{\text{flags } F=(e,v)} \omega_F^{-1}\right).\end{aligned}$$

When we take the marked points into account, we have to replace $\mathrm{val}(v)$ by $n(v) = \mathrm{val}(v) + \#S_v$ in the above expression.

Next, we calculate the contribution from $H^0(C, f^*\mathcal{T}_{G/B})$. Since we have the exact sequence

$$\begin{aligned}0 \to H^0(C, f^*\mathcal{T}_{G/B}) \to &\bigoplus_{e\in \mathrm{Edge}(\Gamma)} H^0(C^e, f^*\mathcal{T}_{G/B}) \\ &\to \bigoplus_{v\in \mathrm{Vert}(\Gamma)} \left(T_{p(w)}(G/B)\otimes \mathbf{C}^{\mathrm{val}(v)-1}\right) \to 0,\end{aligned}$$

it is enough to know the representations of T on $H^0(C^e, f^*\mathcal{T}_{G/B})$ and $T_{p(w)}(G/B)\otimes \mathbf{C}^{\mathrm{val}(v)-1}$. To see the representation of T on $H^0(C^e, f^*\mathcal{T}_{G/B})$, we consider the restriction of $\mathcal{T}_{G/B}$ to the curve $l_{ww'}$ associated to an edge e. The curve $l_{ww'}$ is defined in Definition 2.2. We have the exact sequence

$$0 \to \mathcal{T}_{l_{ww'}} \to \mathcal{T}_{G/B}\big|_{l_{ww'}} \to \mathcal{N}_{l_{ww'}} \to 0,$$

where $\mathcal{N}_{l_{ww'}}$ is the normal bundle to $l_{ww'}$ in G/B. The tangent sheaf of G is generated by the invariant vector fields which is determined by the elements in $\mathfrak{g}$, and the tangent sheaf of G/B corresponds to $\oplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$. The vector fields in $\mathcal{T}_{l_{ww'}}$ are obtained from the tangent vectors contained in $\mathfrak{g}_{\lambda(e)}$. The vector fields in the direction normal to $l_{ww'}$ come from $\oplus_{\alpha\in\Delta_+\setminus\{\lambda(e)\}}\mathfrak{g}_\alpha$. We can choose vector fields X_α and a homogeneous coordinate $(z_1 : z_2)$ on $l_{ww'} \cong \mathbf{P}^1$ such that the action of $\exp(h) \in T$, $h \in \mathfrak{h}$ is described by

$$z_1X_\alpha \mapsto \exp(w(\alpha)(h))\cdot z_1X_\alpha, \quad z_2X_\alpha \mapsto \exp(w'(\alpha)(h))\cdot z_2X_\alpha$$

on $l_{ww'}$. The restriction of f to C^e is expressed as

$$(z_1 : z_2) \mapsto (z_1^{d_e} : z_2^{d_e}) \in l_{ww'}$$

and $f((1:0)) = p(w)$, $f((0:1)) = p(w')$. The basis of $H^0(C^e, f^*\mathcal{T}_{G/B})$ is given by

$$\begin{aligned}&z_1^{d_e-a}z_2^aX_{\lambda(e)}, \quad -d_e \le a \le d_e\\ &z_1^az_2^bX_\alpha, \quad a+b=d_e,\ a,b\ge 0,\ \alpha\ne\lambda(e).\end{aligned}$$

Hence, we have

$$[H^0(C^e, f^*\mathcal{T}_{l_{ww'}})] = [0] + \sum_{a=1}^{d_e}\left(\left[\frac{a\cdot w\lambda(e)}{d_e}\right] + \left[\frac{a\cdot w'\lambda(e)}{d_e}\right]\right)$$

$$[H^0(C^e, f^*\mathcal{N}_{l_{ww'}})] = \sum_{\substack{\lambda\in\Delta_+\\ \lambda\ne\lambda(e)}}\ \sum_{\substack{a+b=d_e\\ a,b\ge 0}}\left[\frac{a}{d_e}w\lambda + \frac{b}{d_e}w'\lambda\right].$$

Since

$$[T_{p(w)}(G/B)] = \sum_{\alpha\in\Delta_+}[w(\alpha)],$$

we have

$$\begin{aligned}&\left[\bigoplus_{v\in\mathrm{Vert}(\Gamma)}\left(T_{p(w(v))}(G/B)\otimes\mathbf{C}^{\mathrm{val}(v)-1}\right)\right]\\ &= \sum_{v\in\mathrm{Vert}(\Gamma)}\left((\mathrm{val}(v)-1)\sum_{\lambda\in\Delta_+}[w(v)(\lambda)]\right).\end{aligned}$$

Hence, the contribution of $H^0(C, f^*\mathcal{T}_{G/B}) + \#\mathrm{Edge}(\Gamma)\cdot[0]$ is

$$\prod_{\substack{e\in\mathrm{Edge}(\Gamma)\\ v,v':\text{vertices of } e}} \frac{(d_e!)^2(w(v)\lambda(e))^{d_e}(w(v')\lambda(e))^{d_e}}{(d_e)^{2d_e}}$$

$$\times \prod_{\substack{\lambda\in\Delta_+\\ \lambda\neq\lambda(e)}} \prod_{\substack{a,b\geq 0\\ a+b=d_e}} \left(\frac{a}{d_e}w(v)\lambda + \frac{b}{d_e}w(v')\lambda\right) \prod_{v\in\mathrm{Vert}(\Gamma)} \left(\prod_{\lambda\in\Delta_+} w(v)\lambda\right)^{1-\mathrm{val}(v)}.$$

Finally, we have the following.

Proposition 4.1. *The Euler class* $\mathrm{Euler}(\mathcal{N}_\Gamma)$ *is expressed as*

$$\prod_{\substack{e\in\mathrm{Edge}(\Gamma)\\ v,v':\text{vertices of } e}} \frac{(d_e!)^2(w(v)\lambda(e))^{d_e}(w(v')\lambda(e))^{d_e}}{(d_e)^{2d_e}}$$

$$\times \prod_{\substack{\lambda\in\Delta_+\\ \lambda\neq\lambda(e)}} \prod_{\substack{a,b\geq 0\\ a+b=d_e}} \left(\frac{a}{d_e}w(v)\lambda + \frac{b}{d_e}w(v')\lambda\right) \prod_{v\in\mathrm{Vert}(\Gamma)} \left(\prod_{\lambda\in\Delta_+} w(v)\lambda\right)^{1-\mathrm{val}(v)}$$

$$\times \prod_{\substack{\text{flags } F=(e,v)\\ n(v)\geq 3}} (\omega_F - c_F)$$

$$\times \prod_{\substack{v\in\mathrm{Vert}(\Gamma)\\ n(v)\leq 2}} \left(\left(\sum_{\text{flags } F=(e,v)} \omega_F^{-1}\right)^{3-n(v)} \prod_{\text{flags } F=(e,v)} \omega_F^{-1}\right).$$

Remark 4.1. We obtain the result for the homogeneous space G/P by the substitution in Remark 3.1.

The Euler class $\mathrm{Euler}(\mathcal{N}_\Gamma)$ is obtained in [15].

5. Gromov-Witten invariants of genus zero for Schubert classes

Mathematical treatments of the (gravitational) Gromov-Witten invariants or correlators were started by Ruan and Tian [21], [22] and by Kontsevich and Manin [17], [18]. Let V be a projective algebraic manifold. There exist natural morphisms

$$\begin{aligned} \mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(V,\beta) &\to V \\ (C;p_1,\ldots,p_n;f) &\mapsto f(p_i) \end{aligned}$$

which are called the *evaluation maps.* We also have *projections*

$$\begin{aligned} \mathrm{pr}_{n+1}\colon \overline{\mathcal{M}}_{g,n+1}(V,\beta) &\to \overline{\mathcal{M}}_{g,n}(V,\beta) \\ (C;p_1,\ldots,p_{n+1};f) &\mapsto \mathrm{Stab}(C;p_1,\ldots,p_n;f), \end{aligned}$$

where $\mathrm{Stab}(C;p_1,\dots,p_n;f)$ is a stabilization obtained by contractiong unstable components of $(C;p_1,\dots,p_n;f)$. Here, pr_{n+1} is regarded as the universal stable map over $\overline{\mathcal{M}}_{g,n}(V,\beta)$. Consider the sections

$$s_i\colon \overline{\mathcal{M}}_{g,n}(V,\beta)\to\overline{\mathcal{M}}_{g,n+1}(V,\beta)$$

corresponding to the marked points p_i, $1\le i\le n$. Let ω_{n+1} be the relative dualizing sheaf of pr_{n+1}. We define the line bundle L_i as the sheaf $s_i^*\omega_{n+1}$ on $\overline{\mathcal{M}}_{g,n}(V,\beta)$. In other words, L_i is a line bundle whose fiber at $(C;p_1,\dots,p_n;f)$ is the cotangent line of C at p_i.

Definition 5.1. Fix $g,n\in\mathbf{Z}_{\ge0}$ and $\beta\in H_2(V,\mathbf{Z})$. For $d_1,\dots,d_n\in\mathbf{Z}_{\ge0}$, the gravitational correlator is defined to be a linear form

$$\begin{aligned} H^*(V,\mathbf{C})^{\otimes n}&\to\mathbf{C}\\ \phi_1\otimes\cdots\otimes\phi_n&\mapsto\langle\tau_{d_1}\phi_1\cdots\tau_{d_n}\phi_n\rangle_{g,\beta}, \end{aligned}$$

where

$$\langle\tau_{d_1}\phi_1\cdots\tau_{d_n}\phi_n\rangle_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}} c_1(L_1)^{d_1}\cup\mathrm{ev}_1^*(\phi_1)\cup\cdots\cup c_1(L_n)^{d_n}\cup\mathrm{ev}_n^*(\phi_n).$$

In this section, we consider the vertical version of the gravitational correlators of genus zero for the flag bundle $\pi:\mathcal{F}\to X$ whose fiber is expressed as a homogeneous space G/P. Note that L_i induces a line bundle L_i' over $\overline{\mathcal{M}}^v_{g,n}(\mathcal{F},\beta)$.

Definition 5.2. We define the vertical gravitational correlator of genus zero for $\mathcal{F}$ as an $H^*(X)$-linear form

$$\begin{aligned} (H^*(\mathcal{F}))^{\otimes n}&\to H^*(X)\\ \phi_1\otimes\cdots\otimes\phi_n&\mapsto\langle\tau_{d_1}\phi_1\cdots\tau_{d_n}\phi_n\rangle_{0,\beta\ /X}, \end{aligned}$$

where

$$\langle\tau_{d_1}\phi_1\cdots\tau_{d_n}\phi_n\rangle_{0,\beta\ /X}=\tilde{\pi}_!\left(c_1^T(L_1')^{d_1}\cup\mathrm{ev}_1^*(\phi_1)\cup\cdots\cup c_1^T(L_n')^{d_n}\cup\mathrm{ev}_n^*(\phi_n)\right),$$

and

$$\tilde{\pi}\colon\overline{\mathcal{M}}^v_{0,n}(\mathcal{F},\beta)\to X$$

is a morphism induced by π.

This definition makes a sense for $X=BT$ and $\mathcal{F}=(G/P)_T$ if we put $\overline{\mathcal{M}}^v_{0,n}((G/P)_T,\beta)=(\overline{\mathcal{M}}_{0,n}(G/P,\beta))_T$ and $L_i'=(L_i)_T$. In this case, the corresponding correlator is called the *T-equivariant correlator.*

The dual classes of the Schubert varieties of $(G/B)_T$ give an R_T-basis of $H_T^*(G/B) \otimes R_T$. We denote them by Ω_w, $w \in W$. If G is a classical group, an explicit form of the polynomial $\mathfrak{S}_w(x|y)$ representing Ω_w is known. Especially, in case G is of type A_r, the polynomial $\mathfrak{S}_w(x|y)$ is known as the double Schubert polynomial of Lascoux and Schützenberger ([19]). Here, we consider only the case G is of type A_r. For other classical groups, see [10, Section 6.2].

Consider the partial flags

$$0 \subset U_1 \subset U_2 \subset \cdots \subset U_s = E$$

in a fixed vector space $E \cong \mathbf{C}^{r+1}$. Let $r(k) = \dim U_k$. Denote by G/P the variety of such flags. Consider the universal flag over G/P,

$$\mathcal{U}_\bullet \ : \ 0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_s = \mathcal{U} = E \otimes \mathcal{O}_{G/P}.$$

The universal flag $\mathcal{U}_\bullet$ induces a flag of subbundles

$$(\mathcal{U}_\bullet)_T \ : \ 0 \subset (\mathcal{U}_1)_T \subset \cdots \subset (\mathcal{U}_s)_T = (\mathcal{U})_T$$

over $(G/P)_T$. Since T acts on the vector space E, we can construct a vector bundle $\mathcal{E} = E_T$ on BT. Fix a full flag in E,

$$E_\bullet : 0 \subset E_1 \subset E_2 \subset \cdots \subset E_{r+1} = E.$$

Then, $E_\bullet$ induces a flag of subbundles in $\mathcal{E}$,

$$\mathcal{E}_\bullet : 0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_{r+1} = \mathcal{E},$$

where $\mathcal{E}_i = (E_i)_T$. We introduce sets of variables $x = (x_1, \ldots , x_{r+1})$ and $y = (y_1, \ldots , y_{r+1})$. Denote by $e^k_{i_k}$ be the i_k-th elementary symmetric polynomial in x_j, $r(k-1)+1 < j \le r(k)$. Consider the ring homomorphisms

$$\begin{aligned} \mathbf{C}[e^k_{i_k} \mid 1 \le k \le s, 1 \le i_k \le r(k) - r(k-1)] &\to H_T^*(G/P) \\ e^k_{i_k} &\mapsto c_{i_k}((\mathcal{U}_i)_T/(\mathcal{U}_{i-1})_T) \end{aligned}$$

and

$$\begin{aligned} \mathbf{C}[y] &\to H^*(BT) \\ y_i &\mapsto c_1(\mathcal{E}_k/\mathcal{E}_{k-1}). \end{aligned}$$

If $P = B$, these homomorphisms induce an isomorphism

$$\mathbf{C}[x, y]/(e_1(x) - e_1(y), \ldots e_{r+1}(x) - e_{r+1}(y)) \cong H_T^*(G/B),$$

where e_i is the i-th elementary symmetric polynomial. Note that $y_i - y_{i+1}$ $(1 \le i \le r)$ corresponds to a simple root $\alpha_i \in \Pi$ under the identification $H^2(BT) = \mathfrak{h}^*$. For general P, $H_T^*(G/P)$ corresponds to the subalgebra generated by $e^k_{i_k}$'s over $\mathbf{C}[y]$.

Let $w_0 \in W$ be the element of maximal length. The Weyl group $W \cong S_{r+1}$ acts on $\mathbf{C}[x]$ by the permutation of variables. The simple reflections correspond to the transpositions s_i, $1 \le i \le r$, which interchange x_i and x_{i+1}. We define the operator ∂_i, $1 \le i \le r$, acting on polynomials $P = P(x_1, \dots, x_{r+1})$ by

$$\partial_i(P) = \frac{P - s_i(P)}{x_i - x_{i+1}},$$

If $w \in W$ has a irreducible decomposition $w = s_{i_1} \cdots s_{i_l}$, $l = l(w)$, then the *divided difference operator* ∂_w is defined to be the product of the operators

$$\partial_w = \partial_{i_1} \cdots \partial_{i_l}.$$

Since $\partial_i^2 = 0$ and $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$, this definition is independent of the choice of irreducible decompositions of w.

Definition 5.3. The polynomials

$$\mathfrak{S}_w(x|y) = \partial_{w^{-1}w_0} \left(\prod_{i+j \le r+1} (x_i - y_j) \right)$$

are called the double Schubert polynomials. Here, the divided difference operators act on the variables x. The polynomials $\mathfrak{S}_w(x) = \mathfrak{S}_w(x|y)|_{y=0}$ are called the Schubert polynomials.

Let Λ be a vector bundle of rank $(r+1)$ over a variety X. Consider the family of partial flags

$$\Lambda_\bullet : 0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_s = \Lambda,$$

such that $\operatorname{rank} \Lambda_k = r(k)$. Let $\pi : \mathcal{F} \to X$ be the associated G/P-bundle. Suppose that T acts on a variety X and the action of T can be equivariantly lifted to the action on the flag bundle $\mathcal{F}$. We can fix a map $\xi : X \to BT$ such that $\mathcal{F}$ is equivalent to $\xi^*(G/P)_T$. Then, the tautological flag

$$\mathcal{L}_\bullet : 0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots \subset \mathcal{L}_s = \mathcal{L} = \pi^*\Lambda,$$

is equivalent to the pull-back of $(\mathcal{U}_\bullet)_T$ over $(G/P)_T$. Now, we define the Schubert class $\Omega_w \in H_T^*(\mathcal{F})$. Let $\mathcal{E}_i^X = \pi^*\xi^*\mathcal{E}_i$ and $\mathcal{V}_i = \pi^*(\mathcal{L}/\mathcal{L}_{s-i})$. Then we obtain a sequence

$$0 \subset \mathcal{E}_1^X \subset \cdots \subset \mathcal{E}_{r+1}^X = \mathcal{V}_s \to \cdots \to \mathcal{V}_1.$$

Definition 5.4. We define the Schubert class $\Omega_w \in H^*(\mathcal{F})$, $w \in W^P$, to be the dual class of the locus $Y_w \subset \mathcal{F}$ on which the condition

$$\operatorname{rk}(\mathcal{E}_a^X \to \mathcal{V}_b) \le \#\{i \mid i \le \operatorname{rk}(\mathcal{E}_a^X),\ w(i) \le \operatorname{rk}(\mathcal{V}_b)\}$$

holds for all a, b. Here, we choose $E_\bullet$ sufficiently general.

Remark 5.1. This definition works well for the case $\mathcal{E}^X$ is not a full flag. Moreover, further generalization is given in [8].

The following is special case of [7, Theorem 8.2].

Lemma 5.1. *The double Schubert polynomial $\mathfrak{S}_w(x|y)$ represents the Schubert class Ω_w in $H^*(\mathcal{F})$.*

Remark 5.2. For $w \in W^P$, the double Schubert polynomial $\mathfrak{S}_w(x|y)$ can be expressed as a polynomial in $e^k_{i_k}(x)$'s.

The classifying space BT can be obtained as a limit of the products of finite dimensional projective spaces. Hence, by taking the limit, $\mathfrak{S}_w(x|y)$ represents the Schubert class in $H^*_T(G/P)$. On the other hand, it is clear that $\mathfrak{S}_w(x)$ represents the Schubert class in $H^*(G/P)$. Other polynomial representatives of Schubert classes in G/P for general semisimple group G are given in [4] and [6].

Theorem 5.1. *For given $w_1, \ldots, w_n \in W^P$, the gravitational correlator $\langle \tau_{d_1}\Omega_{w_1} \cdots \tau_{d_n}\Omega_{w_n}\rangle_{0,\beta\ /X}$ is given by a sum over the graphs of type $(0, n, \beta)$,*

$$\sum_{\Gamma} \frac{1}{\#A_\Gamma \cdot E_\Gamma} \prod_{\substack{v\in \mathrm{Vert}(\Gamma)\\ n(v)\geq 3}} \left(\frac{(n(v)-3)!}{(n(v)-3-\sum_{i\in S_v} d_i)!} \prod_{i\in S_v} \frac{\mathfrak{S}_{w_i}(w(v)y|y)}{d_i!(\sum_{F=(e,v)} \omega_F^{-1})^{d_i}} \right)$$
$$\times \prod_{\substack{\mathrm{val}(v)=1,\ S_v=\{i\}\\ F=(e,v)}} \omega_F^{d_i} \mathfrak{S}_{w_i}(w(v)y|y),$$

where E_Γ is

$$\prod_{\substack{e\in \mathrm{Edge}(\Gamma)\\ v,v':\text{vertices of } e}} \frac{(d_e!)^2 (w(v)\lambda(e))^{d_e} (w(v')\lambda(e))^{d_e}}{(d_e)^{2d_e}}$$
$$\times \prod_{\substack{\lambda\in\Delta_+\setminus\Delta(P)_+\\ \lambda\neq\lambda(e)}} \prod_{\substack{a,b\geq 0\\ a+b=d_e}} \left(\frac{a}{d_e} w(v)\lambda + \frac{b}{d_e} w(v')\lambda\right)$$
$$\times \prod_{v\in\mathrm{Vert}(\Gamma)} \left(\left(\prod_{\lambda\in\Delta_+\setminus\Delta(P)_+} w(v)\lambda \right)^{1-\mathrm{val}(v)} \right.$$
$$\left. \times \left(\sum_{\text{flags } F=(e,v)} \omega_F^{-1} \right)^{3-n(v)} \prod_{\text{flags } F=(e,v)} \omega_F^{-1} \right).$$

This formula gives a constant independent of y.

Proof. Consider the case $X = BT$ and $\mathcal{F} = (G/P)_T$. From the fixed point formula, we have

$$\langle \tau_{d_1}\Omega_{w_1}\cdots\tau_{d_n}\Omega_{w_n}\rangle_{0,\beta/X} = \sum_{\Gamma}\frac{1}{\#A_\Gamma}\int_{\overline{M}_\Gamma}\prod_v\prod_{i\in S_v} j_\Gamma^*\left(c_1(L_i)^{d_i}\mathrm{ev}_i^*\Omega_{w_i}\right)/\mathrm{Euler}(\mathcal{N}_\Gamma).$$

From Lemma 5.1, it follows that

$$j_\Gamma^*\mathrm{ev}_i^*\Omega_{w_i} = \mathfrak{S}_{w_i}(w(v)y|y) \in H^*(BT),$$

for $i \in S_v$. If p_i $(i \in S_v)$ is on the component C^e that is not contracted, then $\#S_v = \mathrm{val}(v) = 1$. Then $T^*_{p_i}(C)$ gives a trivial line bundle over $\overline{M}_\Gamma$, on which T acts in weight ω_F, $F = (e, v)$. On the other hand, if p_i is on the contracted component, T acts trivially on $T^*_{p_i}(C)$. Let us calculate the integral of

$$\frac{\prod_{i\in S_v} c_1(L_i)^{d_i}}{\prod_{F=(e,v)}(\omega_F - c_F)}$$

over the moduli space $\overline{M}_{0,n(v)}$ parametrizing the contracted component correspoding to a vertex v. As in [16, 3.3.2], it is known that

$$\int_{\overline{M}_{0,n}} c_1(L_1)^{d_1}\cdots c_1(L_n)^{d_n} = \frac{(n-3)!}{d_1!\cdots d_n!}$$

for $n \geq 3$, in general. Hence, if $n(v) \geq 3$, we have

$$\int_{\overline{M}_{0,n(v)}} \frac{\prod_{i\in S_v} c_1(L_i)^{d_i}}{\prod_{F=(e,v)}(\omega_F - c_F)} = \frac{(n(v)-3)!\cdot\left(\sum_{F=(e,v)}\omega_F^{-1}\right)^{n'(v)}}{(n'(v))!\prod_{i\in S_v} d_i!\prod_{F=(e,v)}\omega_F},$$

where $n'(v) = n(v) - 3 - \sum_{i\in S_v} d_i$. Hence, we have the formula for $X = BT$ and $\mathcal{F} = (G/P)_T$. This formula gives a rational function in y. By taking the non-equivariant limit $y \to 0$, the equivariant correlator for Schubert cycles tends to the correlator for Schubert cycles in G/P. From the homogeneity in y, the result must be a constant. For general $\mathcal{F} \to X$, we obtain the formula after applying the pull-back by $\xi : X \to BT$. □

Remark 5.3. This formula holds even if we take a partial flag as $\mathcal{E}^X_\bullet$.

6. Gromov-Witten invariants of higher genus

A main difficulty to deal with the Gromov-Witten invariants of higher genus is that their definition involves virtual fundamental class of the moduli stack $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}(G/P, \beta)$. Graber and Pandharipande [12] developed the virtual localization formula, so we can now apply the fixed point localization method to calculate the Gromov-Witten invariants of higher genus. Let

$i : \overline{\mathcal{M}}^T \to \overline{\mathcal{M}}$ be the inclusion of the fixed locus. Then, their localization formula is as follows:

$$[\overline{\mathcal{M}}]^{virt} = i_* \sum_k \frac{[\overline{\mathcal{M}}_k]^{virt}}{\mathrm{Euler}(\mathcal{N}_k^{virt})},$$

where $\overline{\mathcal{M}}_k$ are connected components of $\overline{\mathcal{M}}^T$, and $\mathcal{N}_k^{virt}$ is the virtual normal bundle to $\overline{\mathcal{M}}_k$. In other words, we have

$$\int_{[\overline{\mathcal{M}}]^{virt}} \theta = \sum_k \int_{[\overline{\mathcal{M}}_k]^{virt}} \frac{i^*\theta}{\mathrm{Euler}(\mathcal{N}_k^{virt})}.$$

Applying this formula to our case $V = G/P$, we can reduce the integration over $\overline{\mathcal{M}}_{g,n}(G/P,\beta)$ to the sum of the integration over $\overline{\mathcal{M}}_{g,n}$. As in the case $g = 0$, the fixed components of $\overline{\mathcal{M}}^T$ are parametrized by graphs Γ. For higher genus case, we modify the condition on the graph $(\Gamma, (w(v))_v, (S_v)_v, (d_e)_e)$ as follows.

(1) Each vertex v has an additional label $g(v) \in \mathbf{Z}_{\geq 0}$. The label $g(v)$ corresponds to the arithmetic genus of the contracted component C_v.

(2) The graph Γ is not necessarily a tree. It is connected and satisfies

$$\sum_{v \in \mathrm{Vert}(\Gamma)} g(v) + \mathrm{rk}\, H_1(\Gamma) = g.$$

Then, the fixed locus $\overline{\mathcal{M}}_\Gamma$ corresponds to a graph Γ is the quotient of the stack

$$\prod_{\substack{v \in \mathrm{Vert}(\Gamma) \\ 3g(v)-3+n(v) \geq 0}} \overline{\mathcal{M}}_{g(v),n(v)}$$

by A_Γ. Since $[\overline{\mathcal{M}}_{g,n}]^{virt} = [\overline{\mathcal{M}}_{g,n}]$, the T-equivariant correlator of genus g is given by

$$\langle \tau_1 \Omega_{w_1} \cdots \tau_n \Omega_{w_n} \rangle_{g,\beta}^T = \sum_\Gamma \int_{\overline{\mathcal{M}}_\Gamma} \frac{i^*(c_1(L_1)^{d_1} \mathrm{ev}_1^*(\Omega_{w_1}) \cdots c_1(L_n)^{d_n} \mathrm{ev}_n^*(\Omega_{w_n}))}{\# A_\Gamma \cdot \mathrm{Euler}(\mathcal{N}_\Gamma^{virt})}.$$

Here, we only give a result on a formula for $\mathrm{Euler}(\mathcal{N}_\Gamma^{virt})$ when the target space is a flag variety G/B. Let $\varpi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve and $E_{g,n} = \varpi_*(\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}})$ the Hodge bundle. Denote by $E_{g,n}^\vee$ the dual of $E_{g,n}$. The Chern polynomial of $E_{g,n}^\vee$ is defined by $c_t(E_{g,n}^\vee) = 1 + \sum_k t^k c_k(E_{g,n}^\vee)$. Following [12], we have

Proposition 6.1. *The Euler class* $\mathrm{Euler}(\mathcal{N}_\Gamma^{virt})$ *is expressed as*

$$\prod_{\substack{e\in \mathrm{Edge}(\Gamma)\\ v,v':\text{vertices of } e}} \frac{(d_e!)^2(w(v)\lambda(e))^{d_e}(w(v')\lambda(e))^{d_e}}{(d_e)^{2d_e}}$$

$$\times \prod_{\substack{\lambda\in\Delta_+\\ \lambda\neq\lambda(e)}} \prod_{\substack{a,b\geq 0\\ a+b=d_e}} \left(\frac{a}{d_e}w(v)\lambda+\frac{b}{d_e}w(v')\lambda\right)$$

$$\times \prod_{v\in\mathrm{Vert}(\Gamma)} \left(\left(\prod_{\lambda\in\Delta_+} w(v)\lambda\right)^{1-\mathrm{val}(v)-g(v)} \left(\prod_{\lambda\in\Delta_+} c_{(w(v)\lambda)^{-1}}(E^\vee_{g(v),n(v)})\right)\right)$$

$$\times \prod_{\substack{\text{flags } F=(e,v)\\ 3g(v)+n(v)\geq 3}} (\omega_F - c_F)$$

$$\times \prod_{\substack{v\in\mathrm{Vert}(\Gamma)\\ 3g(v)+n(v)\leq 2}} \left(\left(\sum_{\text{flags } F=(e,v)} \omega_F^{-1}\right)^{3-n(v)} \prod_{\text{flags } F=(e,v)} \omega_F^{-1}\right).$$

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