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The intersection pairings on the configuration spaces of points in the projective line

By

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Abstract

We compute the intersection pairings on the configuration spaces of points in $\mathbf{CP}^1$ with arbitrary weight from an algebro-geometric point of view. We also derive their generating function, which was originally proved by Takakura based on symplectic technique and representation theory.

1. Introduction

Let $m \geq 3$ be an integer and $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbf{R}_{>0})^m$ be an ordered set of positive real numbers. Let $M(\alpha)$ be the space of stable configurations of m points in $\mathbf{CP}^1$ with weight α. This space is constructed as an algebro-geometric quotient of $(\mathbf{CP}^1)^m$ by $PSL(2;\mathbf{C})$. On the other hand, $M(\alpha)$ can be also constructed as a symplectic quotient of $(\mathbf{CP}^1)^m$ by $SO(3)$. Many authors studied its topology and geometry from both algebro-geometric and symplectic points of view. For example, the Betti numbers [Ki1], the rational cohomology ring [B], [Ki2], and the intersection pairings [KT], [M] are computed in the case $\alpha = (1, \ldots, 1)$ for odd m. For general α, the Betti numbers [Kl], [HK] and the integral cohomology ring [HK] are also computed.

Recently Takakura computed the intersection pairings on $M(\alpha)$ for general α [T]. He also gave their generating function by applying the 'quantization commutes with reduction' theorem due to [GS] and representation theory. One may also compute them by other symplectic technique due to [JK], [M].

In this paper we compute the intersection pairings on $M(\alpha)$ from an algebro-geometric point of view. We do it by using the information on α-unstable points. We also derive their generating function, which was originally proved in [T] from a symplectic point of view. Our result is enough to give all Chern numbers and the symplectic volume of $M(\alpha)$ as pointed out in [T]. Our method can be considered as a generalization of the method due to [KT],

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and it is quite simple and geometric in a sense that it enables us to understand natural cycles in $M(\alpha)$ and their intersections directly.

In Section 2 we discuss basic properties of $M(\alpha)$, especially natural cycles and their intersections. In Section 3 we compute the intersection pairings and derive their generating function.

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2. The configuration spaces of points in $\mathbf{CP}^1$

In this section we fix our notations and discuss basic properties of the configuration spaces of points in $\mathbf{CP}^1$.

Let $m \geq 3$ be an integer and $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbf{R}_{>0})^m$ be an ordered set of positive real numbers. We recall the notion of α-stability for the configuration of points in $\mathbf{CP}^1$ due to [DM] as follows;

Definition. $(x_1, \ldots, x_m) \in (\mathbf{CP}^1)^m$ is α-*stable* if and only if

$$\sum_{i=1}^{m} \delta_y(x_i)\alpha_i < \frac{1}{2}\sum_{i=1}^{m} \alpha_i \quad \text{for any} \quad y \in \mathbf{CP}^1,$$

where $\delta_y \colon \mathbf{CP}^1 \to \mathbf{R}$ is defined by

$$\delta_y(x) = \begin{cases} 1 & \text{if} \quad x = y, \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

Moreover we say that α is *generic* if $\sum_{i=1}^{m} \epsilon(i)\alpha_i \neq 0$ for any $\epsilon \colon \{1, \ldots, m\} \to \{\pm 1\}$.

Supose that α is generic. We define the configuration space $M(\alpha)$ with weight α by

$$M(\alpha) = \{(x_1, \ldots, x_m) \in (\mathbf{CP}^1)^m | \alpha\text{-stable}\}/PSL(2; \mathbf{C}),$$

where $PSL(2; \mathbf{C})$ acts on $(\mathbf{CP}^1)^m$ diagonally. It is well known that $M(\alpha)$ is a compact complex manifold of dimension $m - 3$ [DM]. Throughout this paper we assume that α is generic.

Let $p_i \colon (\mathbf{CP}^1)^m \to \mathbf{CP}^1$ be the projection to the i-th component for $i = 1, \ldots, m$. Let $H \to \mathbf{CP}^1$ be the hyperplane bundle. The action of $PSL(2; \mathbf{C})$ on $\mathbf{CP}^1$ cannot lift to the action on H, but the action on $(\mathbf{CP}^1)^m$ lifts to the action on $p_i^* H \otimes p_j^* H$ holomorphically. This induces the holomorphic line bundles $L_{i,j}$ on $M(\alpha)$. We set

$$z_i = c_1(L_{i,i}) \in H^2(M(\alpha); \mathbf{Z}) \quad \text{for} \quad i = 1, \ldots, m.$$

Then we have

$$\frac{z_i + z_j}{2} = c_1(L_{i,j}) \in H^2(M(\alpha); \mathbf{Z}).$$

Here we recall the observation due to [KT].

Lemma 2.1. *The Poincaré dual of* $(z_i + z_j)/2$ *is represented by the divisor*

$$D_{i,j} = \{[x_1, \ldots, x_m] \in M(\alpha) | x_i = x_j\} \quad \textit{for} \quad i \neq j.$$

Next we discuss the generators of the cohomology ring of $M(\alpha)$ and their relations. It is well known that $H^*(M(\alpha); \mathbf{R})$ is generated by $z_1, \ldots, z_m$ as a ring and that they satisfy the relations [B], [Ki2];

$$z_1^2 = z_2^2 = \cdots = z_m^2. \tag{1}$$

To see other relations, we introduce the following notations. Let $\mathcal{L}$ be a set of subsets of $\{1, \ldots, m\}$, which is defined by

$$L \in \mathcal{L} \quad \text{if and only if} \quad \sum_{i \in L} \alpha_i > \frac{1}{2} \sum_{i=1}^{m} \alpha_i \ .$$

Then we have

$$\prod_{i \in L \setminus \{j\}} \frac{z_i + z_j}{2} = 0 \quad \text{for} \quad L \in \mathcal{L}, \quad j \in L. \tag{2}$$

In fact, thanks to Lemma 2.1, the Poincaré dual of the left hand side of (2) is the homology class represented by $\bigcap_{i \in L \setminus \{j\}} D_{i,j}$, which is empty because of α-stability. The relation (1) can be also proved similarly.

We rewrite the above relations with respect to another basis. We set

$$v_i = \frac{z_i + z_m}{2} \quad \text{for} \quad i = 1, \ldots, m \ .$$

Then the relation (1) is written as

$$v_i^2 - v_i v_m = 0 \quad \text{for} \quad i = 1, \ldots, m-1 \ . \tag{3}$$

If $L \in \mathcal{L}$, $m \in L$, then the relation (2) is written as

$$\prod_{i \in L \setminus \{m\}} v_i = 0 \quad \text{for} \quad L \in \mathcal{L}, \quad m \in L \ . \tag{4}$$

Next we consider the relation (2) in the case $L \in \mathcal{L}$, $m \notin L$. If $j \in L$, since $(z_i + z_j)/2 = v_i + v_j - v_m$, then we have

$$0 = \prod_{i \in L \setminus \{j\}} (v_i + v_j - v_m) = \sum_{S' \subset L \setminus \{j\}} \left(\prod_{i \in S'} v_i \right) (v_j - v_m)^{\#(L \setminus S') - 1} \ .$$

Since $v_j(v_j - v_m) = 0$, we have

$$\begin{aligned} 0 &= \prod_{i \in L \setminus \{j\}} v_i + \sum_{S' \subsetneq L \setminus \{j\}} \left(\prod_{i \in S'} v_i \right) \{ v_j (-v_m)^{\#(L \setminus S') - 2} + (-v_m)^{\#(L \setminus S') - 1} \} \\ &= \sum_{S \subsetneq L} \left(\prod_{i \in S} v_i \right) (-v_m)^{\#(L \setminus S) - 1} \ . \end{aligned}$$

Thus the relation (2) in this case turns out to be

$$\sum_{S \subsetneq L} \left(\prod_{i \in S} v_i \right) (-v_m)^{\#(L \setminus S)-1} = 0 \quad \text{for} \quad L \in \mathcal{L}, \quad m \notin L. \tag{5}$$

Here we should mention the following result due to [HK], although we don't use it in this paper.

Theorem 2.2. *Suppose that α is generic. Then we have*

$$H^*(M(\alpha); \mathbf{Z}) \cong \mathbf{Z}[v_1, \ldots, v_m]/I \quad \textit{as a ring,}$$

where the ideal I is generated by all polynomials in the left hand side of (3), (4) *and* (5).

Since in [HK] the relations (3), (4) and (5) are derived in a totally different way from ours, we explain their geometric meaning precicely here.

3. The intersection pairings

In this section we compute the intersection pairings on the configuration space $M(\alpha)$. To state our result, we introduce the following notations. Let $\mathcal{S}$ be a set of subsets of $\{1, \ldots, m\}$, which is defined by

$$S \in \mathcal{S} \quad \text{if and only if} \quad \sum_{i \in S} \alpha_i < \frac{1}{2} \sum_{i=1}^{m} \alpha_i \ .$$

Then our result is described as follows.

Theorem 3.1. *Suppose that $\alpha \in (\mathbf{R}_{>0})^m$ is generic. Let $(d_1, \ldots, d_m) \in (\mathbf{Z}_{\geq 0})^m$ be an ordered set of non-negative integers with $d_1 + \cdots + d_m = m-3$. Then we have*

$$\int_{M(\alpha)} v_1^{d_1} \ldots v_m^{d_m} = (-1)^m \sum_{S \subset T \in \mathcal{S}} (-1)^{\#T},$$

where $S = \{i | d_i > 0\} \cup \{m\}$ and $\#T$ is the number of elements of T.

Proof. By the relation (3), we have

$$\int_{M(\alpha)} v_1^{d_1} \ldots v_m^{d_m} = \int_{M(\alpha)} \left(\prod_{i \in S \setminus \{m\}} v_i \right) v_m^{m-2-\#S}.$$

So we have to show

$$\int_{M(\alpha)} \left(\prod_{i \in S \setminus \{m\}} v_i \right) v_m^{m-2-\#S} = (-1)^m \sum_{S \subset T \in \mathcal{S}} (-1)^{\#T}. \tag{6}$$

We prove (6) by induction on m. First of all we show it in the case $m = 3$. Since $S = \{3\}$ in this case, (6) is equivalent to

$$\#M(\alpha) = -\sum_{3 \in T \in \mathcal{S}} (-1)^{\#T}.$$

It is easy to see that both hand sides are equal to 1 if and only if $|\alpha_1 - \alpha_2| < \alpha_3 < \alpha_1 + \alpha_2$, and 0 otherwise. So we showed (6) in the case $m = 3$.

Next we show (6) for general m. First we show it in the case $\{m\} \subsetneqq S$. We may assume $m - 1 \in S$. If we set

$$\beta = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_{m-1} + \alpha_m) \in (\mathbf{R}_{>0})^{m-1},$$

then β is generic. Moreover the Poincaré dual of v_{m-1} is represented by the divisor $D_{m-1,m}$, which is isomorphic to $M(\beta)$. Therefore we have

$$\int_{M(\alpha)} \left(\prod_{i \in S \setminus \{m\}} v_i \right) v_m^{m-2-\#S} = \int_{M(\beta)} \left(\prod_{i \in S' \setminus \{m\}} v_i \right) v_m^{m-3-\#S'},$$

where $S' = S \setminus \{m-1\}$. Since we can compute the right hand side by the assumption of the induction, we finish the proof in the case $\{m\} \subsetneqq S$.

Next we show (6) in the case $S = \{m\}$, that is, we show

$$\int_{M(\alpha)} v_m^{m-3} = (-1)^m \sum_{m \in T \in \mathcal{S}} (-1)^{\#T} \quad \text{for} \quad m \geq 4. \tag{7}$$

Suppose that $\{i, m\} \in \mathcal{L}$ for any $i = 1, \ldots, m-1$. By the relation (4) we have

$$z_1 = \cdots = z_{m-1} = -z_m = -v_m.$$

Therefore we have

$$\int_{M(\alpha)} v_m^{m-3} = (-1)^{m-3} \int_{M(\alpha)} \prod_{i=1}^{m-3} \frac{z_i + z_{m-2}}{2} = (-1)^{m-3} \#M(\gamma),$$

where $\gamma = (\alpha_1 + \cdots + \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in (\mathbf{R}_{>0})^3$. In the second equality we used Lemma 2.1 repeatedly. Therefore we see that both hand sides of (7) is equal to $(-1)^{m+1}$ if and only if $\{m\} \in \mathcal{S}$, and 0 otherwise. Thus we finish the proof of (7) in this case.

Finally we prove (7) in the case that there exists $j \in \{1, \ldots, m-1\}$ such that $\{j, m\} \in \mathcal{S}$. We may assume $\{m-1, m\} \in \mathcal{S}$. If we set $L' = \{1, \ldots, m-2\}$, then $L' \in \mathcal{L}$. Therefore, by the relation (5) for L', we have

$$\int_{M(\alpha)} (-v_m)^{m-3} = -\sum_{\emptyset \neq S' \subsetneqq L'} \int_{M(\alpha)} \left(\prod_{i \in S'} v_i \right) (-v_m)^{m-3-\#S'}.$$

If we set $L = L' \cup \{m\}$ and $S = S' \cup \{m\}$, then we have

$$\int_{M(\alpha)} v_m^{m-3} = \sum_{\{m\} \subsetneq S \subsetneq L} (-1)^{\#S} \int_{M(\alpha)} \left(\prod_{i \in S \setminus \{m\}} v_i \right) v_m^{m-2-\#S}.$$

Since we can compute the right hand side by the above argument, we have

$$\begin{aligned}\int_{M(\alpha)} v_m^{m-3} &= (-1)^m \sum_{\{m\} \subsetneq S \subsetneq L} (-1)^{\#S} \sum_{S \subset T \in \mathcal{S}} (-1)^{\#T} \\ &= (-1)^m \sum_{m \in T \in \mathcal{S}'} (-1)^{\#T} \sum_{\{m\} \subsetneq S \subset (T \cap L)} (-1)^{\#S},\end{aligned}$$

where $\mathcal{S}' = \mathcal{S} \setminus \{\{m\}, \{m-1, m\}\}$. Since we have

$$\sum_{\{m\} \subsetneq S \subset (T \cap L)} (-1)^{\#S} = 1 \quad \text{for} \quad T \in \mathcal{S}', \quad m \in T,$$

we finish the proof of (7) in this case. Thus we completed the proof of Theorem 3.1. □

We can derive the generating function of the intersection pairings from Theorem 3.1 as follows.

Theorem 3.2. *Suppose that α is generic. Then we have*

$$\int_{M(\alpha)} \exp(t_1 v_1 + \cdots + t_m v_m) = \frac{(-1)^m}{(m-3)!} \sum_{m \in T \in \mathcal{S}} (-1)^{\#T} \left(\sum_{i \in T} t_i \right)^{m-3}. \tag{8}$$

Proof. It is obvious that both hand sides of (8) are equal to the generating function of the intersection pairings

$$F(t_1, \ldots, t_m) = \sum_{d_1 + \cdots + d_m = m-3} \frac{t_1^{d_1} \ldots t_m^{d_m}}{d_1! \ldots d_m!} \int_{M(\alpha)} v_1^{d_1} \ldots v_m^{d_m},$$

where $d_1, \ldots, d_m$ are non-negative integers. □

If we set $(t_1, \ldots, t_m) = (2s_1, \ldots, 2s_{m-1}, s_m - s_1 - \cdots - s_{m-1})$, then (8) turns out to be

$$\int_{M(\alpha)} \exp(s_1 z_1 + \cdots + s_m z_m) = \frac{-1}{(m-3)!} \sum_{m \in T \in \mathcal{S}} (-1)^{\#T} \left(\sum_{i=1}^m s_i - 2 \sum_{i \in T} s_i \right)^{m-3}.$$

Since it is easy to see that

$$\begin{aligned}&\sum_{m \in T \in \mathcal{S}} (-1)^{\#T} \left(\sum_{i=1}^m s_i - 2 \sum_{i \in T} s_i \right)^{m-3} - \sum_{m \notin T \in \mathcal{S}} (-1)^{\#T} \left(\sum_{i=1}^m s_i - 2 \sum_{i \in T} s_i \right)^{m-3} \\ &\qquad = \sum_{m \in T} (-1)^{\#T} \left(\sum_{i=1}^m s_i - 2 \sum_{i \in T} s_i \right)^{m-3} = 0,\end{aligned}$$

we reprove the following Takakura's formula [T], which gives the generating function of the intersection pairings with respect to the basis $\{z_1, \dots, z_m\}$.

Corollary 3.3. *Suppose that α is generic. Then we have*

$$\int_{M(\alpha)} \exp(s_1 z_1 + \cdots + s_m z_m) = \frac{-1}{2(m-3)!} \sum_{T \in \mathcal{S}} (-1)^{\#T} \left(\sum_{i=1}^{m} s_i - 2 \sum_{i \in T} s_i \right)^{m-3}.$$

Takakura showed the above formula by symplectic technique, more precisely, the 'quantization commutes with reduction' theorem due to [GS] and representation theory. Here we gave an algebro-geometric proof of his formula.

Finally let us discuss the following case as an example.

Corollary 3.4. *Let $\alpha_0 = (1, \dots, 1) \in (\mathbf{R}_{>0})^m$ for odd $m = 2l+1 \geq 3$. Let $(d_1, \dots, d_m) \in (\mathbf{Z}_{\geq 0})^m$ be an ordered set of non-negative integers with $d_1 + \cdots + d_m = m-3$. Then we have*

$$\int_{M(\alpha_0)} v_1^{d_1} \dots v_m^{d_m} = (-1)^{l+1} \binom{2l - \#S}{l},$$

where $S = \{i | d_i > 0\} \cup \{m\}$.

Proof. Since m is odd, α_0 is generic. Since $T \in \mathcal{S}$ is equivalent to $\#T \leq l$, both hand sides are equal to 0 in the case $\#S > l$. So we may assume $\#S \leq l$. Then, by Theorem 3.1, we have

$$\begin{aligned} \int_{M(\alpha_0)} v_1^{d_1} \dots v_m^{d_m} &= - \sum_{S \subset T \in \mathcal{S}} (-1)^{\#T} \\ &= - \sum_{i=\#S}^{l} (-1)^i \binom{m - \#S}{i - \#S} = (-1)^{l+1} \binom{2l - \#S}{l}. \end{aligned}$$

□

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