J. Math. Kyoto Univ. (JMKYAZ)
41-2 (2001), 233–250

Self-intersection local time of fractional Brownian motions—via chaos expansion

By

Yaozhong Hu*

Abstract

Let $B^H_{1,t}, \cdots, B^H_{d,t}$ be d independent fractional Brownian motions with Hurst parameter $H \in (0,1)$. Denote $X_t = (B^H_{1,t}, \cdots, B^H_{d,t})$ and let δ be the Dirac delta function. It is shown that when $H < \min(3/(2d), 2/(d+2))$, the (renormalized) self-intersection local time of fractional Brownian motion, $\int_0^T \int_0^t \delta(X_t - X_s) dsdt - \mathbb{E} \int_0^T \int_0^t \delta(X_t - X_s) dsdt$, is in $D_{1,2}$, where $D_{1,2}$ is the Meyer-Watanabe test functional space, i.e. the L^2 space of "differentiable" functionals, whose precise meaning is given in Section 2.

1. Introduction

Since the work of Varadhan [22], the *self-intersection local times* of Brownian motion has been studied by many authors. Chaos expansion approach is useful in determining their smoothness (see [1], [8], [9], [10], [15], [18] and the references therein). In particular, the exact smoothness in the sense of Meyer-Watanabe is discussed in [1], [10]. Let us mention relevant result: It is shown in [1] that when $d = 2$, the self-intersection local time of Brownian motion is in $D_{\alpha,2}$ for all $\alpha < 1$ and it is not in $D_{1,2}$ (see (2.2) for the definition of $D_{\alpha,2}$). As illustrated in [1], the smoothness is important in stochastic quantization.

On the other hand, fractional Brownian motions have recently been studied extensively. It is natural to extend the results on the self-intersection local time of Brownian motion to fractional Brownian motion cases. Let $H \in (0,1)$. A (real valued) Gaussian process B^H_t, $0 \le t \le T$, is called a fractional Brownian motion with the Hurst parameter H if its mean is 0 and its covariance is given by

$$\mathrm{Cov}(B^H_t, B^H_s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] . \tag{1.1}$$

Communicated by Prof. K. Ueno, August 9, 1999
Revised August 9, 2000
*This work is supported in part by the National Science Foundation under Grant No. EPS-9874732, matching support from the State of Kansas and General Research Fund of the University of Kansas.

Let $B^H_{1,t}, \cdots, B^H_{d,t}$ be d independent fractional Brownian motions, let $X_t = (B^H_{1,t}, \cdots, B^H_{d,t})$, and let $X = (X_t, 0 \le t \le T)$. Let $\delta(x), x \in \mathbb{R}^d$, be the Dirac delta function, i.e. formally $\int_{\mathbb{R}^d} \delta(x) f(x) dx = f(0)$. The following formal expression

$$I(d,H,T) = \int_0^T \int_0^t \delta(X_t - X_s) ds dt$$

is called the self-intersection local time of fractional Brownian motion X. In [19], it is shown that when $d = 2$ and when $1/2 < H < 3/4$, $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is square integrable. There are also studies on self-intersection local time for more general Gaussian processes using the local nondeterminism property (see [2]–[6]. This property is also used in [19]). In this paper we shall study the smoothness of $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ in any dimension and for any Hurst parameter $H \in (0,1)$.

As indicated in [7], any square integrable functional F of a fractional Brownian motion can be written as $F = \sum_{n=0}^\infty F_n$, where F_n is the n-th chaos of F (see also the explanation in the next section). Define $D_1 := \left\{F; \sum_{n=0}^\infty n\mathbb{E}\ |F_n|^2 < \infty\right\}$. ($D_1$ is usually denoted by $D_{1,2}$ in the Malliavin calculus). The main result of this paper is as follows (see Theorem 3.2 below).

Main Result. *When $H < \min(3/(2d), 2/(d+2))$, $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is in D_1.*

The proof of this main result will utilize the local nondeterminism of the fractional Brownian motions. This approach is also applicable to more general Gaussian processes. Therefore it is stated a result for general Gaussian processes in Section 2. In Section 3, this result is applied to the fractional Brownian motions.

The condition $H < \min(3/(2d), 2/(d+2))$ is also "optimal" in the sense that when $H \ge \min(3/(2d), 2/(d+2))$, $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ might not be in D_1. In fact, it is shown in [1] that when $d = 2$, $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is not in D_1 (Note that when $d = 2$, $\min(3/(2d), 2/(d+2)) = 1/2$.) It is also interesting to note that when $d = 2$, although $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is not in D_1 for $H = 1/2$, however, once $H < 1/2$, $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is in D_1. Hence, $H = 1/2$ is a critical value.

While this paper is in revision, its idea is being applied to the local time of fractional Brownian motions in [14].

2. The general approach

Let Ω be the space of continuous $\mathbb{R}^d$-valued functions ω on $[0,T]$. Then Ω is a Banach space with respect to the sup norm. Let $\mathcal{F}$ be the Borel σ-algebra on Ω. Let P be a probability measure on the measurable space $(\Omega, \mathcal{F})$. Let $\mathbb{E}$ denote the expectation on this probability space. Let $X = (X_t, 0 \le t \le T)$ be a d-dimensional Gaussian processes on $(\Omega, \mathcal{F}, P)$ with mean 0 and covariance matrices

$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}\ (X_s X_t^T), \quad 0 \le s, \quad t \le T,$$

where A^T denotes the transpose of a matrix (or vector) A. The Gaussian process X_t can also be considered as d real valued Gaussian processes $X_t^1, \cdots, X_t^d$, $0 \le t \le T$.

Denote the variance matrices of a random vector X by $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$.

We define a *square integrable nonlinear functional* F of the Gaussian process X as a real (or complex) valued functional on Ω such that

$$\mathbb{E}\ (F^2) = \int_\Omega |F(\omega)|^2 P(d\omega) < \infty\,.$$

The set of all square integrable functionals is denoted by L^2.

Let $p(x_1, \cdots, x_k)$ be a polynomial of degree n of k variables $x_1, \cdots, x_k$. Then $p(X_{t_1}^{i_1}, \cdots, X_{t_k}^{i_k})$ is called a *polynomial functional* of X, where $t_1, \cdots, t_k \in [0, T]$ and $1 \le i_1, \cdots, i_k \le d$. Let $\mathcal{P}_n$ be the completion with respect to the L^2 norm of the set of all polynomials of degree less than or equal to n. Then $\mathcal{P}_n$ is a subspace of L^2. Let $\mathcal{C}_n$ be the orthogonal complement of $\mathcal{P}_{n-1}$ in $\mathcal{P}_n$. Then L^2 is the direct sum of $\mathcal{C}_n$, i.e.

$$L^2 = \oplus_{n=0}^\infty \mathcal{C}_n\,.$$

Namely, for any functional F in L^2, there are F_n in $\mathcal{C}_n$, $n = 0, 1, 2, \cdots$, such that

$$F = \sum_{n=0}^\infty F_n\,. \tag{2.1}$$

This decomposition is called the *chaos expansion* of F. F_n is called the *n-th chaos* of F. It is easy to see that $F_0 = \mathbb{E}\ (F)$. From the orthogonality it follows

$$\mathbb{E}\ |F|^2 = \sum_{n=0}^\infty \mathbb{E}\ |F_n|^2\,.$$

To simplify notation we also denote $\|F\| = \left(\mathbb{E}\ |F|^2\right)^{1/2}$. We refer to [11], [12], [16], [17], [21], and the references therein for a more detailed study of chaos expansion.

As in the Malliavin calculus, we introduce the spaces of "smooth" functionals in the sense of Meyer-Watanabe [23]:

$$D_\alpha^H = D_{\alpha,2}^H = \left\{ F \in L^2 :\ \|F\|_\alpha^2 = \sum_{n=0}^\infty (n+1)^\alpha \|F_n\|^2 < \infty \right\}. \tag{2.2}$$

For simplicity the super index will be omitted when there is no ambiguity. Introduce the second quantization operator $\Gamma(u)$ on L^2 by the following identity:

$$\Gamma(u)F = \sum_{n=0}^\infty u^n F_n \tag{2.3}$$

if F is given by (2.1), $|u| \le 1$. Denote

$$F(u) = \Gamma(\sqrt{u})F\,.$$

Thus $F = F(1)$. In the following we denote $\gamma_F(u) = d/du\left(\|F(u)\|^2\right)$. The following lemma is easy to verify.

Lemma 2.1. (a) *Let F be in L^2. Then $F \in D_1$ if and only if $\gamma_F(1) < \infty$.*

(b) *If $F = \sum_{n=1}^{\infty} F_n$, where $F_n \in \mathcal{C}_n$ and $\gamma_F(1) < \infty$, then $F \in D_1$.*

In addition to the polynomial functionals, the functional of the form

$$e^{a_1 X_{t_1} + \cdots + a_n X_{t_n}}\,, \quad \text{where} \quad a_1, \cdots, a_n \in \mathbb{R}^d\,, \quad 0 \le t_1, \cdots, t_n \le T$$

will also be used in what follows. They are called *exponential functionals.* We shall find the chaos expansion of some exponential functionals.

Define the Hermite polynomials

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{\partial^n}{\partial x^n} e^{-x^2/2}\,.$$

In this paper we denote the scalar product of two vectors x and y in $\mathbb{R}^d$ by xy or $\langle x, y\rangle$. It is easy to find the chaos expansion of $e^{iu\xi(X_t - X_s) + (1/2)u^2\langle \xi, \mathrm{Var}(X_t - X_s)\xi\rangle}$, where $i = \sqrt{-1}$. In fact it is easy to check that

$$\begin{aligned} & e^{iu\xi(X_t - X_s) + \frac{1}{2}u^2\langle \xi, \mathrm{Var}(X_t - X_s)\xi\rangle} \\ & \quad = \sum_{n=0}^{\infty} (iu)^n \sigma(s,t,\xi)^n H_n\left(\frac{\xi\,(X_t - X_s)}{\sigma(s,t,\xi)}\right)\,, \end{aligned} \tag{2.4}$$

where $\sigma(s,t,\xi) = \sqrt{\langle \xi\,, \mathrm{Var}(X_t - X_s)\xi\rangle}$. One can verify that $(iu)^n \sigma(s,t,\xi)^n \times H_n(\xi\,(X_t - X_s)/\sigma(s,t,\xi))$ is the n-th chaos of $e^{iu\xi(X_t - X_s) + 1/2u^2\langle \xi, \mathrm{Var}(X_t - X_s)\xi\rangle}$.

In this section we shall study the self-intersection local time of X. It is defined formally by the following expression:

$$I(T, X) := \int_0^T \int_0^t \delta(X_t - X_s)\,ds\,dt\,, \tag{2.5}$$

where δ is the Dirac delta function at 0. We will give a general condition so that $I(T,X) - \mathbb{E}\, I(T,X)$ is in D_1. This condition will be applied to fractional Brownian motions in the next section.

As in [10], [19] , we approximate the Dirac delta function by the heat kernel (as $\varepsilon \to 0$)

$$P_\varepsilon(x) = \frac{e^{-|x|^2/2\varepsilon}}{(2\pi\varepsilon)^{d/2}} = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} e^{ix\xi} e^{-\varepsilon|\xi|^2/2} d\xi\,.$$

Thus we shall study

$$I_\varepsilon(T,X) = \int_0^T \int_0^t P_\varepsilon(X_t - X_s)dsdt \tag{2.6}$$
$$= \frac{1}{(2\pi)^d}\int_0^T \int_0^t \int_{\mathbb{R}^d} e^{i\xi(X_t-X_s)}e^{-\varepsilon|\xi|^2/2}d\xi dsdt\,.$$

It is easy to verify that for any $\varepsilon > 0$, $I_\varepsilon(T,X)$ is an element of D_1. We will give conditions so that $I_\varepsilon(T,X) - \mathbb{E}\, I_\varepsilon(T,X)$ is convergent in D_1 as $\varepsilon \to 0$. As in [1], [19], [18], [15], in order to show the convergence in D_1, we know that the most important point is to show the boundedness of $I_\varepsilon(T,X) - \mathbb{E}\, I_\varepsilon(T,X)$ in D_1 as $\varepsilon \to 0$. Therefore as in the above mentioned papers, we will give detailed argument to show the boundedness and leave the convergence issue to the readers.

First we want to find the chaos expansion of $I_\varepsilon(T,X)$. To this end we need to find the chaos expansion of

$$e^{i\xi(X_t-X_s)} = e^{-\frac{1}{2}\langle\xi,\mathrm{Var}(X_t-X_s)\xi\rangle}e^{i\xi(X_t-X_s)+\frac{1}{2}\langle\xi,\mathrm{Var}(X_t-X_s)\xi\rangle}\,. \tag{2.7}$$

Let us define

$$\mathcal{E}_u(X) = e^{uX-\frac{1}{2}u^2\,\mathrm{Var}(X)}\,.$$

By (2.4), the second factor in (2.7) is

$$\mathcal{E}_1(i\xi(X_t-X_s)) = \sum_{n=0}^\infty i^n H_n(\xi, X_t - X_s)\,,$$

where

$$H_n(\xi, X_t-X_s) = \sigma(s,t,\xi)^n H_n\left(\frac{\xi\,(X_t-X_s)}{\sigma(s,t,\xi)}\right)\,,$$

with $\sigma(s,t,\xi) = \sqrt{\langle\xi\,,\mathrm{Var}(X_t-X_s)\xi\rangle}$. That means, the n-th chaos of $\mathcal{E}_1(i\xi(X_t - X_s))$ is the coefficients of u^n of $\mathcal{E}_u(i\xi(X_t - X_s))$. Thus the n-th chaos of $e^{i\xi(X_t-X_s)}$ is the coefficients of u^n of $\mathcal{F}_u$, where

$$\mathcal{F}_u(s,t,\xi) = e^{-\frac{1}{2}\langle\xi,\mathrm{Var}(X_t-X_s)\xi\rangle}\mathcal{E}_u(i\xi(X_t-X_s))\,.$$

Therefore we have

$$\Gamma(u)I_\varepsilon(T,X) = \frac{1}{(2\pi)^d}\int_0^T\int_0^t\int_{\mathbb{R}^d}\mathcal{F}_u(s,t,\xi)e^{-\varepsilon|\xi|^2/2}d\xi dsdt\,.$$

Denote

$$\kappa_\varepsilon(u,T,X) = \mathbb{E}\,|\Gamma(\sqrt{u})I_\varepsilon(T,X)|^2\,. \tag{2.8}$$

Now we are going to estimate $\kappa_\varepsilon(u,T,X)$. Let $\mathcal{T}=\{0<s<t<T\ ;\ 0<s'<t'<T\}$. It is easy to see

$$\kappa_\varepsilon(u,T,X) = \frac{1}{(2\pi)^{2d}}\int_{\mathcal{T}}\int_{\mathbb{R}^{2d}} \mathbb{E}\ \left\{\mathcal{F}_{\sqrt{u}}(s,t,\xi)\mathcal{F}_{\sqrt{u}}(s',t',\eta)\right\} e^{-\varepsilon(|\xi|^2+|\eta|^2)/2} d\xi d\eta ds dt ds' dt' .$$

To compute the above expectation the following identity will be useful:

$$\mathbb{E}\ (\mathcal{E}_u(X)\mathcal{E}_v(Y)) = e^{uv\,\mathrm{Cov}(X,Y)} .$$

It is also easy to check that

$$\mathrm{Cov}(\xi(X_t-X_s),\eta(X_{t'}-X_{s'})) = \langle \xi, \mathrm{Cov}(X_t-X_s, X_{t'}-X_{s'})\eta\rangle .$$

Thus

$$\mathbb{E}\ \left(\mathcal{E}_{\sqrt{u}}(i\xi(X_t-X_s))\mathcal{E}_{\sqrt{u}}(i\eta(X_{t'}-X_{s'}))\right) = e^{-u\langle\xi,\mathrm{Cov}(X_t-X_s,X_{t'}-X_{s'})\eta\rangle} .$$

Consequently,

$$\begin{aligned}\mathbb{E}\ &\left(\mathcal{F}_{\sqrt{u}}(s,t,\xi)\mathcal{F}_{\sqrt{u}}(s',t',\eta)\right) \\ &= e^{-\frac{1}{2}\langle\xi,\mathrm{Var}(X_t-X_s)\xi\rangle - u\langle\xi,\mathrm{Cov}(X_t-X_s,X_{t'}-X_{s'})\eta\rangle-\frac{1}{2}\langle\eta,\mathrm{Var}(X_{t'}-X_{s'})\eta\rangle} .\end{aligned} \tag{2.9}$$

Since the above expectation is positive,

$$\int_{\mathbb{R}^{2d}} \mathbb{E}\ \left\{\mathcal{F}_{\sqrt{u}}(s,t,\xi)\mathcal{F}_{\sqrt{u}}(s',t',\eta)\right\} e^{-\varepsilon(|\xi|^2+|\eta|^2)/2} d\xi d\eta$$

is bounded by

$$\int_{\mathbb{R}^{2d}} \mathbb{E}\ \left\{\mathcal{F}_{\sqrt{u}}(s,t,\xi)\mathcal{F}_{\sqrt{u}}(s',t',\eta)\right\} d\xi d\eta . \tag{2.10}$$

From (2.9) it follows that (2.10) is bounded by

$$(2\pi)^d \det(A(u;s,t,s',t'))^{-1/2} , \tag{2.11}$$

where

$$\begin{aligned}&A(u;s,t,s',t') \\ &= \begin{pmatrix} \mathrm{Var}(X_t-X_s) & u\,\mathrm{Cov}(X_t-X_s,X_{t'}-X_{s'}) \\ u\,\mathrm{Cov}(X_t-X_s,X_{t'}-X_{s'}) & \mathrm{Var}(X_{t'}-X_{s'})\end{pmatrix} .\end{aligned} \tag{2.12}$$

Thus we obtain

$$\kappa_\varepsilon(u,T,X) \le \frac{1}{(2\pi)^d}\int_{\mathcal{T}} (\det A(u;s,t,s',t'))^{-1/2} ds dt ds' dt' . \tag{2.13}$$

From now on we assume that $X_t^1, \cdots, X_t^d$ are independent Gaussian process. Denote

$$\Lambda = \mathrm{diag}(\alpha_1, \cdots, \alpha_d) = \mathrm{Var}(X_t - X_s)\,,$$

$$\tilde{\Lambda} = \mathrm{diag}(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_d) = \mathrm{Var}(X_{t'} - X_{s'})\,,$$

$$\Xi = \mathrm{diag}(\rho_1, \cdots, \rho_d) = \mathrm{Cov}(X_t - X_s, X_{t'} - X_{s'})\,,$$

and

$$A_\varepsilon(u) = \begin{pmatrix} \Lambda + \varepsilon I & u\Xi \\ u\Xi & \tilde{\Lambda} + \varepsilon I \end{pmatrix}\,,$$

where diag means the diagonal matrix. Then it follows from (2.9) that

$$\begin{aligned}\kappa_\varepsilon(u, T, X) &= \frac{1}{(2\pi)^{2d}} \int_{\mathcal{T}} \int_{\mathbb{R}^{2d}} e^{-\frac{1}{2}\langle \xi, \mathrm{Var}(X_t - X_s)\xi\rangle - u\langle \xi, \mathrm{Cov}(X_t - X_s, X_{t'} - X_{s'})\eta\rangle} \\ &\quad e^{-\frac{1}{2}\langle \eta, \mathrm{Var}(X_{t'} - X_{s'})\eta\rangle} e^{-\varepsilon(|\xi|^2 + |\eta|^2)/2} d\xi d\eta ds dt ds' dt' \\ &= \frac{1}{(2\pi)^d} \int_{\mathcal{T}} (\det A_\varepsilon(u; s, t, s', t'))^{-1/2} ds dt ds' dt'\,.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{d}{du}\kappa_\varepsilon(u, T, X) \\ &\quad = -C \int_{\mathcal{T}} (\det A_\varepsilon(u; s, t, s', t'))^{-3/2} \frac{d}{du} \det A_\varepsilon(u; s, t, s', t') ds dt ds' dt'\,,\end{aligned}$$

where C is a positive constant.

It is easy to verify that

$$\det(A_\varepsilon(u)) = \prod_{i=1}^d \left[(\alpha_i + \varepsilon)(\tilde{\alpha}_i + \varepsilon) - u^2\rho_i^2\right]\,.$$

By differentiating the above expression with respect to u, we obtain

$$\frac{d}{du}\det(A_\varepsilon(u)) = -2u \sum_{j=1}^d \prod_{i \neq j} \left[(\alpha_i + \varepsilon)(\tilde{\alpha}_i + \varepsilon) - u^2\rho_i^2\right] \rho_j^2\,.$$

Therefore

$$\begin{aligned}&\det(A_\varepsilon(u))^{-3/2} \frac{d}{du}\det(A_\varepsilon(u)) \\ &\quad = -2u \prod_{i=1}^d \left[(\alpha_i + \varepsilon)(\tilde{\alpha}_i + \varepsilon) - u^2\rho_i^2\right]^{-1/2} \sum_{j=1}^d \left[(\alpha_j + \varepsilon)(\tilde{\alpha}_j + \varepsilon) - u^2\rho_j^2\right]^{-1} \rho_j^2 \\ &\quad \geq -2u \prod_{i=1}^d \left[\alpha_i\tilde{\alpha}_i - u^2\rho_i^2\right]^{-1/2} \sum_{j=1}^d \left[\alpha_j\tilde{\alpha}_j - u^2\rho_j^2\right]^{-1} \rho_j^2 \\ &\quad = \det(A(u, s, t, s', t'))^{-3/2} \frac{d}{du}\det(A(u, s, t, s', t'))\end{aligned}$$

Thus we have

$$\frac{d}{du}\kappa_\varepsilon(u,T,X) \le -C\int_{\mathcal{T}}(\det A(u;s,t,s',t'))^{-3/2}\frac{d}{du}\det A(u;s,t,s',t')dsdtds'dt'\,, \tag{2.14}$$

Remark 1. (a) It is easy to see that $\det A(u;s,t,s',t')$ is a decreasing function of u. For example, in the 1-dimensional case, $\det\begin{pmatrix} a & ub \\ ub & c\end{pmatrix} = ac - u^2b^2$ which is decreasing.

(b) When the RHS of (2.13) or (2.14) is infinity, then the corresponding inequality is understood trivial.

Now put $\Theta_\varepsilon = I_\varepsilon(T,X) - \mathbb{E}\ I_\varepsilon(T,X)$. Then by (2.14), we have that

$$\gamma_{\Theta_\varepsilon}(u) \le -C\int_{\mathcal{T}}\Delta(u,X,s,t,s',t')^{-3/2}\tilde\Delta(u,X,s,t,s',t')dsdtds'dt'\,,$$

where C is a positive constant and

$$\Delta(u,X,s,t,s',t') = \det(A(u;s,t,s',t'))$$

and

$$\tilde\Delta(u,X,s,t,s',t') = \frac{d}{du}\det(A(u;s,t,s',t'))\,.$$

Thus from Lemma 2.1 it follows

Theorem 2.2. *Let $X = (X_t, 0\le t\le T)$ be a $\mathbb{R}^d$-valued Gaussian processes with independent components. Let $A(u;s,t,s',t')$, $\Delta(u,X,s,t,s',t')$, and $\tilde\Delta(u,X,s,t,s',t')$ be defined as above. Denote*

$$\Delta(X,s,t,s',t') = \Delta(1,X,s,t,s',t')\,,\ \text{and}\ \tilde\Delta(X,s,t,s',t') = \tilde\Delta(1,X,s,t,s',t')\,.$$

Then $I(T,X) - \mathbb{E}\ [I(T,X)]$ is in D_1 if

$$\int_{\mathcal{T}}\Delta(X,s,t,s',t')^{-3/2}\tilde\Delta(X,s,t,s',t')dsdtds'dt' > -\infty\,. \tag{2.15}$$

Remark 2. We shall call the determinant $\Delta(u,X,s,t,s',t')$ the increment correlation determinant for the Gaussian process X.

3. Fractional Brownian motions

In this section we apply Theorem 2.2 (i.e. (2.15)) to the fractional Brownian motions. The main result of this paper will be proved.

Let $H\in(0,1)$ be a parameter. A (real valued) Gaussian process B_t^H, $0\le t\le T$, is called a fractional Brownian motion with the Hurst parameter H if its mean is 0 and its covariance is given by

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]\,. \tag{3.1}$$

From (3.1) the following identities follow easily:

$$\mathrm{Var}(B_t^H - B_s^H) = |t-s|^{2H} \tag{3.2}$$

and

$$\begin{aligned}&\mathrm{Cov}(B_t^H - B_s^H, B_{t'}^H - B_{s'}^H)\\ &= \frac{1}{2}\left[|s-t'|^{2H} + |t-s'|^{2H} - |t-t'|^{2H} - |s-s'|^{2H}\right] .\end{aligned} \tag{3.3}$$

We will use the property of the local nondeterminism of the fractional Brownian motions (see e.g. [2], [3], [4], [5], [6]), which states: if $0 \le t_1 < t_2 < \cdots < t_n \le T$, then there is a constant $k > 0$ so that

$$\mathrm{Var}\left(\sum_{i=2}^n u_i (B_{t_i}^H - B_{t_{i-1}}^H)\right) \ge k \sum_{i=2}^n u_i^2 |t_i - t_{i-1}|^{2H} \tag{3.4}$$

for any vector $(u_2, u_3, \cdots, u_n)$. This property is also used in [19] for the study of self-intersection local time of two dimensional fractional Brownian motions. In the remaining part of this paper k will be a generic positive constant whose values may differ from line to line. We also assume that k is sufficiently small, however, positive.

Let us now estimate the following increment correlation determinant

$$\begin{aligned}&d_H(s,t,s',t')\\ &:= \det\begin{pmatrix} \mathrm{Var}(B_t^H - B_s^H) & \mathrm{Cov}(B_t^H - B_s^H, B_{t'}^H - B_{s'}^H) \\ \mathrm{Cov}(B_t^H - B_s^H, B_{t'}^H - B_{s'}^H) & \mathrm{Var}(B_{t'}^H - B_{s'}^H)\end{pmatrix}\\ &= \mathrm{Var}(B_t^H - B_s^H)\,\mathrm{Var}(B_{t'}^H - B_{s'}^H) - \mathrm{Cov}(B_t^H - B_s^H, B_{t'}^H - B_{s'}^H)^2 ,\end{aligned} \tag{3.5}$$

where $0 < s < t < T, 0 < s' < t' < T$. There are three possibilities of the position of the two intervals (s,t) and (s',t'). They are disjoint; the one contains the another; or they are overlapped but no one contains the another. For different cases, we will have different estimates.

In the following lemma, the letters a, b, and c denote the lengths of the left interval, the middle interval, and the right interval determined by s, t, s', t'. We also denote

$$\lambda = \mathrm{Var}(B_t^H - B_s^H)^2 = |t-s|^{2H} ; \tag{3.6}$$

$$\rho = \mathrm{Var}(B_{t'}^H - B_{s'}^H)^2 = |t'-s'|^{2H} ; \tag{3.7}$$

and

$$\begin{aligned}\mu &= \mathrm{Cov}(B_t^H - B_s^H, B_{t'}^H - B_{s'}^H)\\ &= \frac{1}{2}\left[|t'-s|^{2H} + |t-s'|^{2H} - |t'-t|^{2H} - |s'-s|^{2H}\right] .\end{aligned} \tag{3.8}$$

Lemma 3.1. *Let B_t^H, $0 \le t \le T$ be a (real valued) fractional Brownian motion with the Hurst parameter H. Let $d_H(s,t,s',t')$ be defined by (3.5). Namely,*

$$d_H(s,t,s',t') = \lambda\rho - \mu^2\,.$$

(1) *Let $0 < s < s' < t < t' < T$. Denote $a = s' - s$, $b = t - s'$, and $c = t' - t$. Then*

$$d_H(s,t,s',t') \ge k\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right]\,. \tag{3.9}$$

(2) *Let $0 < s' < s < t < t' < T$. Denote $a = s - s'$, $b = t - s$, and $c = t' - t$. Then*

$$d_H(s,t,s',t') \ge k\,(a+b+c)^{2H}\,b^{2H}\,. \tag{3.10}$$

(3) *Let $0 < s < t < s' < t' < T$. Denote $a = t - s$ and $c = t' - s'$. Then*

$$d_H(s,t,s',t') \ge ka^{2H}c^{2H}\,. \tag{3.11}$$

Proof. First we prove (1). By the local nondeterminism (3.4), for all u and v,

$$\begin{aligned}
&\mathrm{Var}(u(B_t^H - B_s^H) + v(B_{t'}^H - B_{s'}^H)) \\
&\quad = \mathrm{Var}(u(B_{s'}^H - B_s^H) + (u+v)(B_t^H - B_{s'}^H) + v(B_{t'}^H - B_t^H)) \\
&\quad \ge k(a^{2H}u^2 + b^{2H}(u+v)^2 + c^{2H}v^2)\,.
\end{aligned}$$

This implies that

$$\lambda u^2 + 2\mu uv + \rho v^2 \ge k(a^{2H}u^2 + b^{2H}(u+v)^2 + c^{2H}v^2)\,.$$

Namely

$$(\lambda - ka^{2H} - kb^{2H})u^2 + 2(\mu - kb^{2H})uv + (\rho - kb^{2H} - kc^{2H})v^2 \ge 0$$

for all u and v. Hence the discriminant of the left hand side of the above inequality must satisfy

$$(\lambda - ka^{2H} - kb^{2H})(\rho - kb^{2H} - kc^{2H}) - (\mu - kb^{2H})^2 \ge 0\,.$$

So

$$\begin{aligned}
d_H(s,t,s',t') &= \lambda\rho - \mu^2 \\
&\ge k\lambda(b^{2H} + c^{2H}) + k\rho(a^{2H} + b^{2H}) - 2k\mu b^{2H} \\
&\quad - k^2(a^{2H}b^{2H} + b^{2H}c^{2H} + a^{2H}c^{2H})\,.
\end{aligned}$$

Since $\mu^2 \le \lambda\rho$,

$$\mu \le \sqrt{\lambda\rho} \le \frac{1}{2}(\lambda+\rho)\,.$$

Therefore

$$d_H(s,t,s',t') \ge k\left(\lambda c^{2H} + \rho a^{2H}\right) - k^2(a^{2H}b^{2H} + b^{2H}c^{2H} + a^{2H}c^{2H})\,.$$

But in this case

$$\lambda = (a+b)^{2H}\,, \quad \rho = (b+c)^{2H}\,.$$

Thus

$$\lambda c^{2H} + \rho a^{2H} \ge \frac{1}{2}\left(a^{2H}b^{2H} + b^{2H}c^{2H} + a^{2H}c^{2H}\right)\,.$$

Therefore when k is small enough we have

$$d_H(s,t,s',t') \ge k\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right]\,.$$

Secondly, we prove (2). In this case $\lambda = b^{2H}$ and $\rho = (a+b+c)^{2H}$. By the local nondeterminism (3.4), for all u and v,

$$\begin{aligned}
&\mathrm{Var}(u(B_t^H - B_s^H) + v(B_{t'}^H - B_{s'}^H))\\
&\quad = \mathrm{Var}(u(B_t^H - B_s^H) + v(B_s^H - B_{s'}^H) + v(B_t^H - B_s^H) + v(B_{t'}^H - B_t^H))\\
&\quad \ge k(b^{2H}u^2 + (a^{2H} + b^{2H} + c^{2H})v^2)\,.
\end{aligned}$$

This implies that

$$\lambda u^2 + 2\mu uv + \rho v^2 \ge k(b^{2H}u^2 + (a^{2H} + b^{2H} + c^{2H})v^2)\,.$$

Consequently,

$$(\lambda - kb^{2H})(\rho - ka^{2H} - kb^{2H} - kc^{2H}) - \mu^2 \ge 0\,.$$

Thus

$$\begin{aligned}
d_H(s,t,s',t') &= \lambda\rho - \mu^2\\
&\ge k\lambda(a^{2H} + b^{2H} + c^{2H}) + k\rho b^{2H} - k^2(a^{2H} + b^{2H} + c^{2H})b^{2H}\\
&\ge k(a+b+c)^{2H}b^{2H}
\end{aligned}$$

when k is sufficiently small. This proves (2).

Lastly we show (3). In this case $\lambda = a^{2H}$ and $\rho = c^{2H}$. By (3.4),

$$\mathrm{Var}(u(B_t^H - B_s^H) + v(B_{t'}^H - B_{s'}^H)) \ge k(a^{2H}u^2 + c^{2H}v^2)\,.$$

Thus

$$\lambda u^2 + 2\mu uv + \rho v^2 \ge k(a^{2H}u^2 + c^{2H}v^2)\,.$$

Hence

$$\left(\lambda - ka^{2H}\right)\left(\rho - kc^{2H}\right) - \mu^2 \geq 0\,.$$

This implies that

$$\lambda\rho - \mu^2 \geq ka^{2H}c^{2H}\,.$$

This completes the proof of the lemma. □

Now let $X_t^H = (B_{1,t}^H, \cdots, B_{d,t}^H)$, $0 \leq t \leq T$ be d independent fractional Brownian motions. We shall study the self-intersection local time of the d-dimensional fractional Brownian motions X_t^H:

$$I(d,H,T) = \int_0^T \int_0^t \delta(X_s^H - X_s^H) ds dt\,.$$

We are interested in the smoothness of $I(d,H,T) - \mathbb{E}\ I(d,H,T)$. We will continue to use the notations introduced earlier. Recall

$$d_H(u,s,t,s',t') = \lambda\rho - u\mu^2\,,$$

where λ, ρ, and μ are defined by (3.6)–(3.8). Since $B_{1,t}^H, \cdots, B_{d,t}^H$ are independent,

$$\Delta(u, X^H, s, t, s', t') = d_H(u,s,t,s',t')^d\,.$$

It is then easy to see that

$$\begin{aligned} &\Delta(u, X^H, s,t,s',t')^{-3/2}\tilde{\Delta}(u, X^H, s,t,s',t') \\ &\quad = -d_H(u,s,t,s',t')^{-\frac{d}{2}-1}\mu^2\,. \end{aligned} \tag{3.12}$$

Now we state the main result of this paper.

Theorem 3.2. *If $H < \min(3/(2d), 2/(d+2))$, then $I(d,H,T) - \mathbb{E}\ I(d,H,T)$ is in D_1.*

Proof. By (3.12) and Theorem 2.2 it suffices to show that

$$\int_{\mathcal{T}} d_H(s,t,s',t')^{-d/2-1}\mu^2 ds dt ds' dt' < \infty\,,$$

where $\mathcal{T} = \{0 < s < t < T, 0 < s' < t' < T\}$. Therefore it suffices to show that for $j = 1,2,3$,

$$\int_{\mathcal{T}_j} d_H(s,t,s',t')^{-d/2-1}\mu^2 ds dt ds' dt' < \infty\,, \tag{3.13}$$

where $\mathcal{T}_1 = \{s,t,s',t'; 0 < s < s' < t < t' < T\}$, $\mathcal{T}_2 = \{s,t,s',t'; 0 < s' < s < t < t' < T\}$, and $\mathcal{T}_3 = \{s,t,s',t'; 0 < s < t < s' < t' < T\}$.

As in Lemma 3.1, in what follows the symbols a, b, and c always denote the lengths of the left interval, the middle interval, and the right interval determined by the four points s, t, s', and t'. It is elementary to see that (3.13) is true if there are $\alpha, \beta, \gamma > -1$ such that

$$d_H(s,t,s',t')^{-d/2-1}\mu^2 \le C a^\alpha b^\beta c^\gamma \,, \tag{3.14}$$

where and in what follows C will denote a generic constant whose value may differ in different occasions. Our strategy to show (3.13) is to derive estimates of the type (3.14) in three different cases corresponding to $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$.
(1) When $0 < s < s' < t < t' < T$, we denote $a = s' - s$, $b = t - s'$, and $c = t' - t$. By Lemma 3.1, we have

$$d_H(s,t,s',t') \ge k\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right] .$$

In this case

$$\begin{aligned}\mu &= (a+b+c)^{2H} + b^{2H} - a^{2H} - c^{2H}\\ &= (a^2+b^2+c^2+2ab+2ac+2bc)^H + b^{2H} - a^{2H} - c^{2H}\\ &\le 2b^{2H} + 2^H a^H b^H + 2^H a^H c^H + 2^H b^H c^H .\end{aligned}$$

Thus

$$\mu^2 \le C(a^{2H}b^{2H} + a^{2H}c^{2H} + b^{2H}c^{2H}) + Cb^{4H} .$$

Since each term in the above bracket is dominated by $d_H(s,t,s',t')$. Thus there is a constant C such that

$$\begin{aligned}&d_H(s,t,s',t')^{-d/2-1}\mu^2\\ &\qquad\le C d_H(s,t,s',t')^{-d/2} + C d_H(s,t,s',t')^{-d/2-1}b^{4H} .\end{aligned} \tag{3.15}$$

The first term of the right hand side of (3.15) is estimated by

$$\begin{aligned}d_H(s,t,s',t')^{-d/2} &\le C\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right]^{-d/2}\\ &\le C\left[(a+b)^H(b+c)^H a^H c^H\right]^{-d/2} .\end{aligned}$$

Using the inequalities $a + b \ge Ca^{2/3}b^{1/3}$, we have

$$d_H(s,t,s',t')^{-d/2} \le C a^{-2dH/3} b^{-2dH/3} c^{-2dH/3} .$$

The exponent $-2dH/3$ is greater than -1 if the condition of the theorem is satisfied. To estimate the second term in (3.15), we have

$$\begin{aligned}d_H(s,t,s',t')^{-d/2-1}b^{4H} &\le C\left((a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right)^{-\frac{d+2}{2}} b^{4H}\\ &\le C\left(b^{2H}c^{2H} + b^{2H}a^{2H}\right)^{-\frac{d+2}{2}} b^{4H}\\ &\le C a^{-\frac{d+2}{2}H} c^{-\frac{d+2}{2}H} b^{-(d+2)H+4H} .\end{aligned}$$

If $d \le 6$ and if the condition of the theorem is satisfied, then we see that all the exponents are greater than -1.

When $d > 6$, for $\alpha, \beta \ge 0$, $\alpha + \beta = 1$,

$$\begin{aligned} d_H(s,t,s',t')^{-d/2-1} b^{4H} &\le C\left((a+b)^H c^H a^H (b+c)^H\right)^{-\frac{d+2}{2}} b^{4H} \\ &= C a^{-(\alpha+1)\frac{d+2}{2}H} c^{-(\alpha+1)\frac{d+2}{2}H} b^{-\beta(d+2)H+4H} . \end{aligned}$$

Let $\alpha = (d-6)/(3d+6)$, $\beta = (2d+12)/(3d+6)$. Then

$$d_H(s,t,s',t')^{-d/2-1} b^{4H} \le C a^{-\frac{2dH}{3}} b^{-\frac{2dH}{3}} c^{-\frac{2dH}{3}} .$$

This implies (3.13) for $j = 1$.
(2) Let us consider the case $0 < s' < s < t < t'$. We will consider the cases $H \ge 1/2$ and $H < 1/2$ separately. If $H \ge 1/2$, then

$$\begin{aligned} \mu &= (b+c)^{2H} + (a+b)^{2H} - a^{2H} - c^{2H} \\ &= 2Hb \int_0^1 [(a+bu)^{2H-1} + (c+bu)^{2H-1}] du \\ &\le Cb\,(a+b+c)^{2H-1} . \end{aligned}$$

Thus

$$\mu^2 \le C b^2 (a+b+c)^{4H-2} .$$

Therefore by (3.10)

$$\begin{aligned} d_H(s,t,s',t')^{-d/2-1} \mu^2 &\le C\left((a+b+c)^{2H} b^{2H}\right)^{-d/2-1} b^2 (a+b+c)^{4H-2} \\ &\le C(a+b+c)^{-(d+2)H+4H-2} b^{-(d+2)H+2} \\ &= C(a+b+c)^{-(d-2)H-2} b^{-(d+2)H+2} . \end{aligned}$$

Let $\alpha = 2Hd/(3dH + 6 - 6H) \in (0,1)$. Since $(6+d)H \le 6$, which is implied by the condition of the theorem, we have that $1 - 2\alpha \ge 0$. Using the inequality

$$a + b + c \ge C a^\alpha c^\alpha b^{1-2\alpha} ,$$

we obtain

$$d_H(s,t,s',t')^{-d/2-1} \mu^2 \le C a^{-\frac{2dH}{3}} b^{-\frac{2dH}{3}} c^{-\frac{2dH}{3}} .$$

This proves that when $H \ge 1/2$ and when the conditions of the theorem is satisfied,

$$\int_{\mathcal{T}_2} d_H(s,t,s',t')^{-d/2-1} \mu^2 ds dt ds' dt' < \infty . \tag{3.16}$$

The case $H < 1/2$ is slightly more complicated. We shall consider the case $d \le 6$ and $d > 6$. When $d \le 6$ and $H < 1/2$,

$$\mu \le C b a^{\alpha(2H-1)} b^{(2H-1)\beta} + C b c^{\alpha(2H-1)} b^{(2H-1)\beta}$$

i.e.

$$\mu^2 \le C a^{\alpha(4H-2)} b^{(4H-2)\beta+2} + C c^{\alpha(4H-2)} b^{(4H-2)\beta+2},$$

where $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$. Thus

$$\begin{aligned} d_H(s,t,s',t')^{-d/2-1}\mu^2 &\le C\left((a+b+c)^{2H} b^{2H}\right)^{-d/2-1} a^{\alpha(4H-2)} b^{(4H-2)\beta+2} \\ &\quad + C\left((a+b+c)^{2H} b^{2H}\right)^{-d/2-1} c^{\alpha(4H-2)} b^{(4H-2)\beta+2} \\ &= I_1 + I_2 \,. \end{aligned}$$

Now let us estimate I_1. By $a+b+c \ge a^{1/2}c^{1/2}$,

$$\begin{aligned} I_1 &\le C a^{-(d+2)H/2} c^{-(d+2)H/2} a^{\alpha(4H-2)} b^{\beta(4H-2)-(d+2)H+2} \\ &= C a^{\alpha(4H-2)-(d+2)H/2} b^{\beta(4H-2)-(d+2)H+2} c^{-(d+2)H/2} \,. \end{aligned}$$

Let $\alpha = 0$ and $\beta = 1$. Then

$$I_1 \le C a^{-(d+2)H/2} c^{-(d+2)H/2} b^{4H-2-(d+2)H+2} \,.$$

When $d \le 6$ and the condition of the theorem is satisfied, $-(d+2)H/2 > -1$ and $4H - (d+2)H > -1$. Thus when $d \le 6$, $H < 1/2$, and the condition of the theorem is satisfied, $\int_{\mathcal{T}_2} I_1 ds dt ds' dt' < \infty$. When $d > 6$ and $H < 1/2$, let $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Then

$$I_1 \le C a^{-\gamma_1(d+2)H+\alpha(4H-2)} b^{-(\gamma_2+1)(d+2)H+(4H-2)\beta+2} c^{-\gamma_3(d+2)H} \,.$$

Let

$$\gamma_3 = \frac{2d}{3d+6}\,, \quad \gamma_1 = \frac{d+6}{3d+6}\,, \quad \gamma_2 = 0\,,$$

and let

$$\alpha = \frac{(d-6)H}{3(2-4H)}\,, \quad \beta = \frac{6-6H-dH}{3(2-4H)}\,.$$

Then we have

$$I_1 \le a^{\frac{-2dH}{3}} b^{\frac{-2dH}{3}} c^{\frac{-2dH}{3}} \,.$$

Hence, we have shown that when $d \le 6$, $H < 1/2$, and the condition of the theorem is satisfied, $\int_{\mathcal{T}_2} I_1 ds dt ds' dt' < \infty$. In a similar way we can show that $\int_{\mathcal{T}_2} I_2 ds dt ds' dt' < \infty$. Therefore it follows that when $H < 1/2$ and when the condition of the theorem is satisfied,

$$\int_{\mathcal{T}_2} d_H(s,t,s',t')^{-d/2-1}\mu^2 ds dt ds' dt' < \infty \,.$$

(3) Now let us consider the case $0 < s < t < s' < t' < T$. Denote $a = t - s$, $b = s' - t$, and $c = t' - s'$. In this case we have

$$\begin{aligned}
\mu &= (a+b+c)^{2H} + b^{2H} - (a+b)^{2H} - (b+c)^{2H} \\
&= 2Ha \int_0^1 \left[(b+c+ua)^{2H-1} - (b+ua)^{2H-1} \right] du \\
&= 2H(2H-1)ac \int_0^1 \int_0^1 (b+vc+ua)^{2H-2} du dv \,.
\end{aligned}$$

Using

$$b + vc + ua \ge C v^\beta u^\beta b^\alpha c^\beta a^\beta \,,$$

where $\alpha, \beta > 0$ and $\alpha + 2\beta = 1$, we have

$$\begin{aligned}
\mu^2 &\le C a^2 c^2 \int_0^1 \int_0^1 (b+vc+ua)^{4H-4} du dv \\
&\le C a^2 c^2 \left(b^\alpha c^\beta a^\beta \right)^{4H-4} \\
&= C b^{4\alpha H - 4\alpha} (ac)^{4\beta H - 4\beta + 2} \,.
\end{aligned}$$

Let

$$\alpha = \frac{dH}{6-6H} \,, \quad \beta = \frac{1-\alpha}{2} = \frac{6-6H-Hd}{12-12H} \,.$$

If $(6+d)H \le 6$, then $\alpha, \beta \ge 0$ and $\alpha + 2\beta = 1$. Thus

$$\begin{aligned}
d_H(s,t,s',t')^{-d/2-1} \mu^2 &\le C \left(a^{2H} c^{2H} \right)^{-d/2-1} b^{4\alpha H - 4\alpha} (ac)^{4\beta H - 4\beta + 2} \\
&\le C a^{\frac{-2dH}{3}} b^{\frac{-2dH}{3}} c^{\frac{-2dH}{3}} \,.
\end{aligned}$$

This shows that when $dH < 3/2$,

$$\int_{\mathcal{T}_3} d_H(s,t,s',t')^{-d/2-1} \mu^2 ds dt ds' dt' < \infty \,. \tag{3.17}$$

Thus we have completed the proof of the theorem. □

Acknowledgements. The author is grateful to the referee for the careful reading of the paper.

Department of Mathematics
University of Kansas
405 Snow Hall, Lawrence, KS 66045-2142
e-mail: hu@math.ukans.edu

References

[1] S. Albeverio, Y. Z. Hu and X. Y. Zhou, A remark on non smoothness of self intersection local time of planar Brownian motion, Statist. Probab. Letter, **32** (1997), 57–65.

[2] S. M. Berman, A central limit theorem for the renormalized self-intersection local time of a stationary process. Probability in Banach spaces, 8 (Brunswick, ME, 1991), 351–363, Progr. Probab. 30, Birkhäuser Boston, Boston, MA, 1992.

[3] S. M. Berman, Self-intersections and local nondeterminism of Gaussian processes, Ann. Probab. **19**-1 (1991), 160–191.

[4] S. M. Berman, Local nondeterminism and local times of general stochastic processes. Ann. Inst. H. Poincaré Sect. B (N.S.) **19**-2 (1983), 189–207.

[5] S. M. Berman, Local nondeterminism and local times of Gaussian processes, Indiana Univ. Math. J., **23** (1973/74), 69–94.

[6] S. M. Berman, Local nondeterminism and local times of Gaussian processes, Bull. Amer. Math. Soc., **79** (1973), 475–477.

[7] T. E. Duncan, Y. Z. Hu and B. Pasik-Duncan, Itô calculus for fractional Brownian motions I-Theory, SIAM J. Control Optim., **38** (2000), 582–612.

[8] M. De Faria, T. Hida, L. Streit and H. Watanabe, Intersection local times as generalized white noise functionals, Acta Appl. Math., **46** (1997), 351–362.

[9] S. W. He, W. Q. Yang, R. Q. Yao and J. G. Wang, Local times of self-intersection for multidimensional Brownian motion, Nagoya Math. J., **138** (1995), 51–64.

[10] Y. Z. Hu, On the self-intersection local time of Brownian motion – via chaos expansion, Publ. Mat., **40** (1996), 337–350.

[11] Y. Z. Hu, Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities, J. of Theoret. Probab., **10** (1997), 835–848.

[12] Y. Z. Hu and G. Kallianpur, Exponential integrability and application to stochastic quantization, J. Appl. Math. and Optimization, **37** (1998), 295–353.

[13] Y. Z. Hu and P. A. Meyer, Chaos de Wiener et intégrales de Feynman. Séminaire de Probabilités XXII, ed. by J. Azema, P. A. Meyer and M. Yor, Lecture Notes in Math. 1321, Springer, 1988, 51–71.

[14] Y. Z. Hu and B. Øksendal, Chaos expansion of local time of fractional Brownian motions, preprint 2000.

[15] P. Imkeller, V. Pérez-Abreu and J. Vives, Chaos expansions of double intersection local time of Brownian motion in $\mathbb{R}^d$ and renormalization, Stochastic Process. Appl., **56** (1995), 1–34.

[16] P. A. Meyer, Quantum Probability for Probabilists, Lecture, Notes in Math. 1538, Springer, 1993.

[17] J. Neveu, Processus Aléatoires Gaussiens. Les Presses De L'Université De Montréal, 1968.

[18] D. Nualart and J. Vives, Chaos expansion and local time, Publ. Mat., **36**-2 (1992), 827–836.

[19] J. Rosen, The intersection local time of fractional Brownian motion in the plane, J. Multivariate Anal., **23**-1 (1987), 37–46.

[20] B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Rev., **10** (1968) 422–437.

[21] B. Simon, The $P(\Phi)_2$ Euclidean Field Theory, Princeton University Press, Princeton, NJ., 1974.

[22] S. R. S. Varadhan, Appendix to "Euclidean quantum field theory" by K. Symanzik, in "Local Quantum Theory" (R. Jost ed.), Academic, New York, 1969.

[23] S. Watanabe, Stochastic Differential Equation and Malliavin Calculus. Tata institute of fundamental research, Springer, 1984.