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Hofer's symplectic energy and lagrangian intersections in contact geometry

By

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Abstract

There is a version of Lagrangian intersection theory in contact geometry [2]. But it works well only with very restrictive contact manifolds. For example, it does not work well with overtwisted contact 3-manifolds. Here we show the following. If we have an estimate on Hamiltonian functions of contact flow, then we can apply the theory to a much wider class of contact manifolds.

1. Introduction

In this paper, we show a version of Lagrangian intersection theory in contact geometry. Especially under an estimate of Hamiltonian functions of contact flow, we can construct Floer homology. Then we can show a version of the Arnold conjecture in a wide class of contact manifolds under the new condition.

Let M be a $(2n+1)$-dimensional contact manifold, i.e., M has a 1-form γ which satisfies the condition $\gamma \wedge (d\gamma)^n \neq 0$. For simplicity, we consider the case of a global 1-form γ. Then the hyperplane distribution, ξ, defined by the kernel of γ is called a *contact structure* on M, and the 1-form γ is called a *contact form*. For each contact form γ there is a unique vector field Y which satisfies the conditions $\iota(Y)\gamma = 1$ and $\iota(Y)d\gamma = 0$. We call this vector field Y the *Reeb vector field* of γ.

We may regard the distribution ξ as a rank 2 vector subbundle of TM, the tangent bundle of M. Then the restriction of $d\gamma$ on ξ, $d\gamma|_\xi$, defines a non-degenerate 2-form on ξ. Hence we can define a complex structure J and a Hermitian metric on ξ which has $d\gamma|_\xi$ as a fundamental 2-form.

Next we can associate a symplectic manifold with a contact manifold M, so called the *symplectization of* M. Consider a product $R \times M$. Denote the pull-back of γ by the projection from $R \times M$ to M, also by γ, and denote the coordinate of R by θ. Then a 2-form $d(e^\theta\gamma)$ is an exact symplectic form on $R \times M$. And we define an almost complex structure $\tilde{J}$ on $R \times M$ by the following.

$$\tilde{J}|_\xi := J, \quad \tilde{J}\frac{\partial}{\partial\theta} := Y \quad \text{and} \quad \tilde{J}Y := -\frac{\partial}{\partial\theta},$$

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where the pull-backs by the projection are also denoted by the same notation.

Now we introduce some important objects in this paper. Let L_1 be an n-dimensional submanifold of M which satisfies the condition that the tangent bundle of L_1 is contained in the ξ, i.e., $TL_1 \subset \xi$. Then we call such a submanifold a *Legandrian submanifold*. It is easy to see that $(n+1)$-dimensional submanifold $R \times L_1$ is a Lagrangian submanifold in the symplectization of M. Let L_0 be an $(n+1)$-dimensional submanifold of M. If there is a Lagrangian submanifold $\widehat{L_0}$ of the $R \times M$ which satisfies the condition that $\widehat{L_0}$ is diffeomorphic to L_0 by the projection, then we call L_0 a *pre-Lagrangian submanifold* and call $\widehat{L_0}$ a *Lagrangian lift of* L_0. A shift of the $\widehat{L_0}$ along the R-direction is also a Lagrangian lift of L_0. Then we define the following quantity and call it the *height* of $\widehat{L_0}$.

$$h_\gamma(\widehat{L_0}) :=] \max_{p \in \widehat{L_0}} \theta(p) - \min_{p \in \widehat{L_0}} \theta(p),$$

where θ is the coordinate of R of the symplectization $R \times M$. Moreover we put $h_\gamma(L_0)$ to be the infimum of $h_\gamma(\widehat{L_0})$ over the all Lagrangian lifts and call it the *height* of L_0.

Here we fix a contact form γ of a contact structure ξ. For any function $H : M \to R$, there is a unique vector field X_H which satisfies the conditions

$$\iota(X_H)\gamma = -H \quad \text{and} \quad \iota(X_H)d\gamma = dH - (\iota(Y)dH)\gamma,$$

where Y is the Reeb vector field of γ. It is easy to see that the X_H is a contact vector field, i.e., the flow generated by X_H preserves the contact structure ξ. This map, from the space of functions to the space of contact vector fields, is bijective. Moreover it holds for time-dependent functions and time-dependent contact vector fields. Let X_{H_s} be the contact vector field generated by a time-dependent function H_s. We define a function $e^\theta H_s : R \times M \to R$ by $(\theta, q) \mapsto e^\theta H_s(q)$. Then the Hamiltonian vector field generated by the $e^\theta H_s$ on the symplectization of M is

$$(\iota(Y)dH)\frac{\partial}{\partial\theta} + X_{H_s},$$

where we regard X_{H_s} as a vector field on the symplectization of M which is constant along the R-direction.

Finally we introduce some quantities. Let γ be a contact form, Y the Reeb vector field of γ and L_1 a Legandrian submanifold. We put

$$\sigma_\gamma := \inf \left\{ \int_l \gamma \;\middle|\; l \text{ is a contractible closed orbit of } Y \right\}.$$

If there is no contractible closed orbit, we put $\sigma_\gamma := \infty$.

$$\sigma_\gamma(L_1) := \inf \left\{ \int_l \gamma \;\middle|\; \begin{array}{l} l \text{ is an orbit of } Y \text{ which satisfies that } l(0), l(1) \in L_1 \\ \text{and } l \text{ represents the zero element of } \pi_1(M, L_1) \end{array} \right\}.$$

If there is no such an orbit as above, we put $\sigma_\gamma(L_1) := \infty$. We put

$$C_\gamma(L_1) := \min\{\sigma_\gamma, \sigma_\gamma(L_1)\}.$$

For time-dependent function H_s on M, we put

$$\|H_s\| := \int_0^1 \left\{\max_{p\in M} H_s(p) - \min_{p\in M} H_s(p)\right\} ds,$$
$$d_\gamma := \max_{s\in[0,1],p\in M} |dH_s(Y)|,$$

where Y is the Reeb vector field of γ.

Theorem 1.1. *Let M be a closed contact manifold with a contact form γ. Fix this contact form γ. Let L_0 be a closed pre-Lagrangian submanifold and L_1 a closed Legandrian submanifold which satisfy the conditions $L_0 \supset L_1$ and the boundary homomorphism $\pi_2(M, L_0) \to \pi_1(L_0)$ is trivial.*

Let φ_1 be a time-1 map of the contact flow generated by a time-dependent function H_s such that $\varphi_1(L_1)$ intersects L_0 transversaly. Assume that the following estimate holds.

$$\|H_s\| \cdot \exp\{2d_\gamma + h_\gamma(L_0) + \varepsilon\} < C_\gamma(L_1).$$

Then we obtain the following estimate.

$$\sharp\{L_0 \cap \varphi_1(L_1)\} \geq \text{rank } H_*(L_1; Z_2).$$

Notice that we do not need a C^1-small estimate of H_s except for the Reeb direction. Eliashberg, Hofer and Salamon constructed a version of Lagrangian intersection theory in contact geometry [2]. Their theorems work well only with restrictive contact manifolds. For example, overtwisted contact 3-manifolds are not suitable, see [2] and [4]. Their theorem is the case of $C_\gamma(L_1) = \infty$ in our theorem.

Theorem 1.2 (Eliashberg, Hofer and Salamon). *Let M be a closed contact manifold with a contact form γ. Fix this contact form γ. Let L_0 be a closed pre-Lagrangian submanifold and L_1 a closed Legandrian submanifold which satisfy the conditions $L_0 \supset L_1$ and the boundary homomorphism $\pi_2(M, L_0) \to \pi_1(L_0)$ is trivial.*

Let φ_1 be a time-1 map of the contact flow generated by a time-dependent function H_s such that $\varphi_1(L_1)$ intersects L_0 transversaly. Assume that there is no contractible closed orbit of the Reeb flow Y and there is no orbit of Y which satisfies that $l(0), l(1) \in L_1$ and l represents the zero element of $\pi_1(M, L_1)$. Then we obtain the following estimate.

$$\sharp\{L_0 \cap \varphi_1(L_1)\} \geq \text{rank } H_*(L_1; Z_2).$$

The key points of the proof of our theorem is to show the compactness of the moduli space of pseudo-holomorphic disks with an estimate of the energy and to construct the homomorphism between Floer homologies of different contactomorphisms by using the technique of Chekanov [1].

2. Path spaces and functionals

In this section, we introduce path spaces and functionals. From now on, we denote the symplectization of a contact manifold M by P. Let $\widetilde{H}_s : [0,1] \times P \to R$ be a time-dependent function on P and $\widetilde{X}_{\widetilde{H}_s}$ the Hamiltonian venter field generated by the Hamiltonian function $\widetilde{H}_s$. Namely the vector field $\widetilde{X}_{\widetilde{H}_s}$ satisfies the condition that

$$d\widetilde{H}_s = \omega(\cdot, \widetilde{X}_{\widetilde{H}_s}),$$

where ω is the symplectic form on P. Moreover, we put $G^s : P \to P$ to be the time-s Hamiltonian flow generated by $\widetilde{X}_{\widetilde{H}_s}$. Namely G^s satisfies the condition that

$$\begin{cases} \frac{d}{ds}G^s &= \widetilde{X}_{\widetilde{H}_s} \circ G^s, \\ G^0 &= \mathrm{id}. \end{cases}$$

For a closed Lagrangian submanifold $\widehat{L_0}$ of P and a closed Legandrian submanifold L_1 of M satisfying that $\widehat{L_0} \cap (R \times L_1)$ is not empty, we put

$$\Omega'_s := \bigl\{ l : [0,1] \to P \bigm| l(0) \in R \times L_1,\ l(1) \in G^s(\widehat{L_0}) \bigr\}.$$

Moreover we denote a component which contains a path $l(t) := G^{ts}(x_0)$, for $x_0 \in \widehat{L_0} \cap (R \times L_1)$, by Ω_s. Put $\Omega := \bigcup_{s\in[0,1]}(s, \Omega_s)$ and, for a fixed path $l_0 \in \Omega$,

$$\widetilde{\Omega} := \left\{ l_\tau(t) : [0,1] \times [0,1] \to P \;\middle|\; \begin{array}{l} l_{\tau=0}(t) = l_0(t), \\ l_\tau(0) \in R \times L_1, \text{ and } l_\tau(1) \in G^{s(\tau)}(\widehat{L_0}) \end{array} \right\}.$$

Next we introduce a functional F on $\widetilde{\Omega}$. For $\widetilde{l} := l_\tau(t) \in \widetilde{\Omega}$, we put

$$F(\widetilde{l}) := \int_0^1 d\tau \int_0^1 dt\ \omega\left(\frac{\partial}{\partial t} l, \frac{\partial}{\partial \tau} l\right) - \int_{s(0)}^{s(1)} ds(\tau)\ \widetilde{H}_{s(\tau)}(l_\tau(1)).$$

Lemma 2.1. *The value of F depends only on the homotopy type of l which fixes an end path. Namely, put $\widetilde{l}_\sigma : [0,1] \mapsto \widetilde{\Omega}; \sigma \to l(\sigma;\tau,t)$ which satisfies that $l(\sigma;1,t) = l(\sigma';1,t)$, for any $\sigma, \sigma' \in [0,1]$, then $F(\widetilde{l}_{\sigma=0}) = F(\widetilde{l}_{\sigma=1})$.*

Lemma 2.2. *Let L_0 be a closed pre-Lagrangian submanifold of M with a Lagrangian lift $\widehat{L_0}$ and L_1 be a closed Legandrian submanifold of M. Suppose that L_1 is contained in L_0, i.e., $L_1 \subset L_0$, and the boundary homomorphism $\pi_2(M, L_0) \to \pi_1(L_0)$ is trivial. Then, for $\widetilde{l} \in \widetilde{\Omega}$ which satisfies that $\widetilde{l}_{\tau=1} = \widetilde{l}_{\tau=0}$, we obtain $F(\widetilde{l}) = 0$.*

From the above lemmas, we can regard the functional F on the $\widetilde{\Omega}$ as a functional on the Ω, i.e., the value of F is determined by an end point $l_{\tau=1}$ and the base point l_0. Note that the restriction of F on Ω_s coincides with the usual Floer's functional and denote $F|_{\Omega_s}$ by F_s.

3. Floer homology

In this section, let $\tilde{J}$ be the almost complex structure on P as mentioned in introduction. To put it more precisely, we have to consider perturbations of almost complex structures. It is a little complicated. Hence we omit it here. See [2].

We define a metric on Ω_s by

$$(\xi_1, \xi_2) := \int_0^1 \omega(\xi_1(t), \tilde{J}\xi_2(t))dt,$$

where $\xi_1, \xi_2 \in T_l\Omega_s$. For this metric, the gradient vector field ∇F_s of the F_s is

$$\nabla F_s(l)(t) = \tilde{J}(l(t))\dot{l}(t).$$

The set of critical points of F_s consists of the intersection points of $R \times L_1$ and $G^s(\widehat{L_0})$. Suppose that $G^s(\widehat{L_0})$ intersects $R \times L_1$ transversaly. For critical points x_+, x_- of F_s, we put the moduli space of descending gradient trajectories as

$$\begin{aligned}&\mathcal{M}_s(x_-, x_+)\\ &\qquad := \left\{ u : R \to \Omega_s \;\middle|\; \begin{array}{l} \frac{du(\tau)}{d\tau} = -\nabla F_s(u(\tau)), u \text{ is not constant and} \\ \lim_{\tau\to\pm\infty} u(\tau) = x_\pm \end{array} \right\}.\end{aligned}$$

For a suitable perturbation of almost complex structures we can assume that the regularity condition holds and this space is a manifold, see [2]. And R acts on $\mathcal{M}_s(x_-, x_+)$ by translation, $u(\cdot) \mapsto u(\cdot + a), a \in R$. We denote the quotient by $\widehat{\mathcal{M}}_s(x_-, x_+)$.

From now on, let $\widehat{L_0}$ be the Lagrangian lift of a closed pre-Lagrangian submanifold L_0 with the condition $\min_{p\in\widehat{L_0}} \theta(p) = 0$ and L_1 be a closed Legandrian submanifold with the condition $L_1 \subset L_0$. Moreover assume that the boundary homomorphism $\pi_2(M, L_0) \to \pi_1(L_0)$ is trivial. From these assumptions we can say the followings. First we may regard the functional F on $\widetilde{\Omega}$ as a functional on Ω from Lemma 2.2. Second the bubbling off phenomena can't occur at the boundary points of pseudo-holomorphic disks. Because the symplectic form of P is exact the bubbling off phenomena always can't occur at the interior points of pseudo-holomorphic disks.

Fix constants b_- and C'. Let $p : R \to [b_-, b_- + C')$ be a projection. For a critical point x of F_s, we put $\widetilde{F}_s(x) := p \circ F_s(x)$. Next we put the length of a descending gradient trajectory u by

$$l(u) := -\int_{-\infty}^{\infty} u^* dF_s = \int_{-\infty}^{\infty} \left(\frac{du(\tau)}{d\tau}, \frac{du(\tau)}{d\tau} \right) d\tau > 0.$$

Then we define the moduli space of distinguished gradient trajectories by

$$\mathcal{M}_s^d(x_-,x_+) := \{u \in \mathcal{M}_s(x_-,x_+) \mid l(u) = \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+)\}.$$

If $\widetilde{F}_s(x_-) - \widetilde{F}_s(x_+)$ is negative, then $\mathcal{M}_s^d(x_-,x_+)$ is empty. And the quotient of $\mathcal{M}_s^d(x_-,x_+)$ by the action of R is denoted by $\widehat{\mathcal{M}}_s^d(x_-,x_+)$.

Theorem 3.1. *Assume that $C'e^{d+\varepsilon} < C_\gamma(L_1)$ for some positive number ε, where $d = \int_0^s \max_{p\in G^t(\widehat{L_0})} |d\theta(X_{\widetilde{H}_t})|dt$. Then the images of all distinguished gradient trajectories are contained in some compact set of P.*

Corollary 3.2. *There is no bubble in the distinguished gradient trajectories.*

Corollary 3.3. *The set of isolated points of $\widehat{\mathcal{M}}_s^d(x_-,x_+)$ is compact.*

We show the proof of Theorem 3.1 in the last section. Owing to the compactness, we can define Floer homology. Let $Y(s)$ be the set of critical points of F_s and $C(s)$ be the vector space over Z_2 spanned by elements of $Y(s)$. Then we define a boundary operator $\partial_s : C(s) \to C(s)$ by

$$\partial_s x := \sum_{y\in Y(s)} \sharp\{\text{isolated points of } \widehat{\mathcal{M}}_s^d(x,y)\} y,$$

where $x \in Y(s)$.

Proposition 3.4. *We have $\partial_s^2 = 0$.*

Proof. We prove this proposition by the standard gluing argument in the Floer theory. From the definition

$$\partial_s^2 x := \sum_{z,y\in Y(s)} \sharp\{\text{isolated points of } \widehat{\mathcal{M}}_s^d(x,y)\}\sharp\{\text{isolated points of } \widehat{\mathcal{M}}_s^d(y,z)\} z.$$

Hence we show that the coefficient of each z is even. Take isolated points $u_1 \in \widehat{\mathcal{M}}_s^d(x,y)$ and $u_2 \in \widehat{\mathcal{M}}_s^d(y,z)$, then there is a 1-dimensional component N of $\widehat{\mathcal{M}}_s(x,z)$ so that (u_1,u_2) is an end of the compactification of N. Because the length is additive under the gluing procedure, it holds that $l(u) = l(u_1) + l(u_2)$ $= \widetilde{F}_s(x) - \widetilde{F}_s(z)$ for $u \in N$. Then $N \subset \widehat{\mathcal{M}}_s^d(x,z)$. From Corollary 3.2 there is no bubble in the sequence of points of N. And there are isolated points $u_1' \in \widehat{\mathcal{M}}_s(x,y')$ and $u_2' \in \widehat{\mathcal{M}}_s(y',z)$ so that (u_1',u_2') is the other end point of the compactification of N. Finally we show that u_1' and u_2' are distinguished. We put $l(u_1') = \widetilde{F}_s(x) - \widetilde{F}_s(y') + nC'$ and $l(u_2') = \widetilde{F}_s(y') - \widetilde{F}_s(z) - nC'$, $n \in Z$. Since $l(u_1') > 0$ and $C' > \widetilde{F}_s(x) - \widetilde{F}_s(y') > -C'$, n have to be non-negative. Similarly, since $l(u_2') > 0$ and $C' > \widetilde{F}_s(y') - \widetilde{F}_s(z) > -C'$, n have to be non-positive. Then n is zero and $l(u_1') = \widetilde{F}_s(x) - \widetilde{F}_s(y')$ and $l(u_2') = \widetilde{F}_s(y') - \widetilde{F}_s(z)$.

Namely, u_1' and u_2' are distinguished. Hence the coefficient of each z is even and $\partial_s^2 = 0$. □

We construct a homology group $H(C(s), \partial_s)$ from this chain complex $(C(s), \partial_s)$. In the next section, we show the following. If we have a suitable estimate of a Hamiltonian function, then there is an injective homomorphism $V_1^s : H(C(s), \partial_s) \to H(C(1), \partial_1)$ for small s. Hence we have $\operatorname{rank} H(C(1), \partial_1) \geq \operatorname{rank} H(C(s), \partial_s)$. Of course, if $R \times L_1$ intersects $G^1(\widehat{L_0})$ transversaly, then $\sharp\big\{(R \times L_1) \cap G^1(\widehat{L_0})\big\} \geq \operatorname{rank} H(C(1), \partial_1)$.

Proposition 3.5. *For some small s, $H(C(s), \partial_s)$ is isomorphic to $H_*(L_1; Z_2)$ as a vector space.*

Sketch of proof. There is a contact diffeomorphism from a small neighborhood of L_1 in M to a small neighborhood of the zero section of 1-jet of L_1 such that the image of L_0 is the 0-wall. Under the assumption that $\varphi_s(L_1)$ intersects W transversaly, for small s there is a Morse function such that the set of its critical points is isomorphic to the set of intersection points of $\varphi_s(L_1)$ and W. And take a suitable metric on L_1, the set of gradient trajectories of the Morse function between critical points x_- and x_+ is isomorphic to the set of gradient trajectories of the Floer's functional between the intersection points x_- and x_+. Moreover for some small s the gradient trajectories of the Floer's functional are distinguished. Then the Morse complex is isomorphic to the complex $(C(s), \partial_s)$. □

Hence if we have a suitable estimate of Hamiltonian function, then we obtain $\sharp\big\{(R \times L_1) \cap G^1(\widehat{L_0})\big\} \geq \operatorname{rank} H_*(L_1; Z_2)$.

4. Continuations and homotopy of continuations

In this section, we describe the technique of Chekanov [1]. We put

$$a_+ := \int_0^1 \max_{p \,\in\, G^s(\widehat{L_0})} \widetilde{H}_s(p)ds \quad \text{and} \quad a_- := \int_0^1 \min_{p \,\in\, G^s(\widehat{L_0})} \widetilde{H}_s(p)ds,$$

and we may think that $a_- \leq 0 \leq a_+$. From now we assume that $a_+ - a_- < C'$. At the beginning we can take a generic base point of $\widetilde{\Omega}$ and $\varepsilon > 0$ such that $F_s(y) > 0$ for any $s < \varepsilon$ and any $y \in Y(s)$. Moreover we retake the generic base point so that $\widetilde{F}_1(y) \neq a_-$ for any $y \in Y(1)$. Since the number of elements of $Y(1)$ is finite we can choose the number b_- so that $\widetilde{F}_1(y) \notin [b_-, a_-]$ for any $y \in Y(1)$. Then we take an interval $(c_-, c_+) \subset (b_-, b_+)$, where $b_+ = b_- + C'$, such that $a_- < c_-$, $a_+ < c_+$ and $\widetilde{F}_1(y) \in [c_-, c_+]$ for any $y \in Y(1)$. Moreover we take c_- enough close to a_- and c_+ enough close to b_+ and retake $\varepsilon > 0$ small such that $\widetilde{F}_s(x) \in [c_- - a_-, c_+ - a_+]$ for any $s < \varepsilon$ and $x \in Y(s)$.

We introduce a continuation map $Q_{s_+}^{s_-} \in \operatorname{Hom}(C(s_-), C(s_+))$. Let $\rho : R \to [0, 1]$ be a function which satisfies that there are some constants $K > 0$ such that

$\rho(\tau) = s_-$ for $\tau < -K$ and $\rho(\tau) = s_+$ for $\tau > K$. We call such a function an (s_-, s_+)-*continuation function*. Moreover, if ρ is a monotone function, we call it a *monotone* (s_-, s_+)-*continuation function*. For critical points $x_- \in Y(s_-)$ and $x_+ \in Y(s_+)$, we put the moduli spaces of continuation trajectories as

$$\mathcal{M}_\rho(x_-, x_+) := \left\{ u : R \to \Omega \;\middle|\; \frac{du(\tau)}{d\tau} = -\nabla F_{\rho(\tau)}(u(\tau)) \text{ and } \lim_{\tau \to \pm\infty} u(\tau) = x_\pm \right\}.$$

For a suitable perturbation of almost complex structures we can assume that the regularity condition holds and this space is manifold.

For a continuation trajectory u, we put the length $l(u)$ by

$$l(u) := -\int_{-\infty}^{\infty} u^* dF$$

and the symplectic area $A(u)$ by

$$A(u) := -\int_{R\times[0,1]} u^*\omega = \int_{-\infty}^{\infty} \left(\frac{du(\tau)}{d\tau}, \frac{du(\tau)}{d\tau} \right) d\tau \geq 0.$$

And we put

$$h(u) := \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{d\tau} \widetilde{H}_{\rho(\tau)}(u(\tau, 1)) d\tau,$$

then $A(u) = l(u) + h(u)$. We define the moduli space of distinguished continuation trajectories by

$$\mathcal{M}^d_\rho(x_-, x_+) := \left\{ u \in \mathcal{M}_\rho(x_-, x_+) \;\middle|\; l(u) = \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) \right\}.$$

Lemma 4.1. *Let $s_- < \varepsilon$, $s_+ = 1$ or $s_- = 1$, $s_+ < \varepsilon$. And let ρ be a monotone (s_-, s_+)-continuation function, then we obtain*

$$A(u) \leq c_+ - c_- (< C'),$$

where $u \in \mathcal{M}^d_\rho(x_-, x_+)$.

Proof. First let $s_- < \varepsilon$ and $s_+ = 1$. Notice that ρ is monotone. We obtain

$$\begin{aligned} h(u) &\leq \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{d\tau} \max_{p \in G^{\rho(\tau)}(\widehat{L_0})} \widetilde{H}_{\rho(\tau)}(p) d\tau \\ &= \int_{s}^{1} \max_{p \in G^s(\widehat{L_0})} \widetilde{H}_s(p) ds \leq a_+, \end{aligned}$$

where $u \in \mathcal{M}^d_\rho(x_-, x_+)$. Since $l(u) = \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) \leq (c_+ - a_+) - c_-$, we obtain $A(u) = l(u) + h(u) \leq c_+ - c_-$.

Second let $s_- = 1$ and $s_+ < \varepsilon$. Notice that ρ is monotone. We obtain

$$\begin{aligned} h(u) &\le \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{d\tau} \min_{p \in G^{\rho(\tau)}(\widehat{L_0})} \widetilde{H}_{\rho(\tau)}(p) d\tau \\ &= \int_1^s \min_{p \in G^s(\widehat{L_0})} \widetilde{H}_s(p) ds \le -a_-, \end{aligned}$$

where $u \in \mathcal{M}^d_\rho(x_-, x_+)$. Since $l(u) = \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) \le c_+ - (c_- - a_-)$, we obtain $A(u) = l(u) + h(u) \le c_+ - c_-$. □

Theorem 4.2. *Let s_-, s_+ and ρ be the same as Lemma* 4.1. *Assume that $C' e^{d+\varepsilon} < C_\gamma(L_1)$ for some positive number ε, where $d = \int_0^1 \max_{p \in G^t(\widehat{L_0})} |d\theta(X_{\widetilde{H}_t})| dt$. Then the images of all continuation trajectories are contained in some compact set of P.*

Corollary 4.3. *There is no bubble in the continuation trajectories.*

The proofs of them is the same ones as Theorem 3.1 and Corollary 3.2.

For a monotone (s_-, s_+)-continuation function ρ, we define a continuation map $Q^{s_+}_{s_-} : C(s_-) \to C(s_+)$ by

$$Q^{s_+}_{s_-}(x) := \sum_{y \in Y(s_+)} \sharp \{\text{isolated points of } \mathcal{M}^d_\rho(x, y)\} y,$$

where $x \in Y(s_-)$.

Proposition 4.4. *Let $s_- < \varepsilon,\ s_+ = 1$ or $s_- = 1,\ s_+ < \varepsilon$. And let ρ be a monotone (s_-, s_+)-continuation function, then we obtain*

$$Q^{s_-}_{s_+} \circ \partial_{s_-} = \partial_{s_+} \circ Q^{s_-}_{s_+}.$$

Proof. Let $x \in Y(s_-)$ and $z \in Y(s_+)$. Take a pair of isolated points (u_1, u_2) where $u_1 \in \widehat{\mathcal{M}}^d_{s_-}(x, y)$, $u_2 \in \mathcal{M}^d_\rho(y, z)$, $y \in Y(s_-)$ or $u_1 \in \mathcal{M}^d_\rho(x, y)$, $u_2 \in \widehat{\mathcal{M}}^d_{s_+}(y, z)$, $y \in Y(s_+)$. Then there is a 1-dimensional component N of $\mathcal{M}_\rho(x, z)$ so that (u_1, u_2) is an end of the compactification of N. Because the length is additive under the gluing procedure, it holds that $l(u) = l(u_1) + l(u_2) = \widetilde{F}_{s_-}(x) - \widetilde{F}_{s_+}(z)$ for $u \in N$. Then $N \subset \mathcal{M}^d_\rho(x, z)$. From Corollary 4.3 there is no bubble in the sequence of points of N. And there is a pair of isolated points (u_1', u_2') where $u_1' \in \widehat{\mathcal{M}}_{s_-}(x, y)$, $u_2' \in \mathcal{M}_\rho(y, z)$, $y \in Y(s_-)$ or $u_1' \in \mathcal{M}_\rho(x, y)$, $u_2' \in \widehat{\mathcal{M}}_{s_+}(y, z)$, $y \in Y(s_+)$ so that (u_1', u_2') is the other end point of the compactification of N. Finally we need to show that u_1' and u_2' are distinguished.

Lemma 4.5. *If $u \in \mathcal{M}_\rho(x_-, x_+)$, then we have $l(u) \ge \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+)$.*

Proof. First let $s_- < \varepsilon$ and $s_+ = 1$. Then we have

$$\begin{aligned} l(u) = A(u) - h(u) &\geq -a_+ \\ &> (c_+ - a_+) - c_- - C' \\ &\geq \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) - C'. \end{aligned}$$

Second let $s_- = 1$ and $s_+ < \varepsilon$. Then we have

$$\begin{aligned} l(u) = A(u) - h(u) &\geq a_- \\ &> c_+ - (c_- - a_-) - C' \\ &\geq \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) - C'. \end{aligned}$$

In both cases, we obtain $l(u) > \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) - C'$. Moreover, $l(u) = \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+) + nC'$, $n \in Z$. Hence we obtain $l(u) = \widetilde{F}_{s_-}(x_-) - \widetilde{F}_{s_+}(x_+)$. □

Let $(u_1, u_2) \in \widehat{\mathcal{M}}_{s_-}(x, y) \times \mathcal{M}_\rho(y, z)$, $y \in Y(s_-)$, be an end point of the compactification of a 1-dimensional component $N \subset \mathcal{M}^d_\rho(x, z)$. Put $l(u_1) = \widetilde{F}_{s_-}(x) - \widetilde{F}_{s_-}(y) + nC'$ and $l(u_2) = \widetilde{F}_{s_-}(y) - \widetilde{F}_{s_+}(z) - nC'$. Because $l(u_1) = A(u_1) \geq 0$, we obtain $n \geq 0$. From Lemma 4.5, $l(u_2) \geq \widetilde{F}_{s_-}(y) - \widetilde{F}_{s_+}(z)$, we obtain $n \leq 0$. Hence $n = 0$ and u_1, u_2 are distinguished. Similarly, let $(u_1, u_2) \in \mathcal{M}_\rho(x, y) \times \widehat{\mathcal{M}}_{s_+}(y, z)$, $y \in Y(s_+)$, be an end point of the compactification of a 1-dimensional component $N \subset \mathcal{M}^d_\rho(x, z)$. Put $l(u_1) = \widetilde{F}_{s_-}(x) - \widetilde{F}_{s_+}(y) + nC'$ and $l(u_2) = \widetilde{F}_{s_+}(y) - \widetilde{F}_{s_+}(z) - nC'$. Because $l(u_2) = A(u_2) \geq 0$, we obtain $n \leq 0$. From Lemma 4.5, $l(u_1) \geq \widetilde{F}_{s_+}(x) - \widetilde{F}_{s_+}(y)$, we obtain $n \geq 0$. Hence $n = 0$ and u_1, u_2 are distinguished. Then we obtain

$$(Q^{s_-}_{s_+} \circ \partial_{s_-} - \partial_{s_+} \circ Q^{s_-}_{s_+})x = 0,$$

where $x \in Y(s_-)$. □

From Q^s_1 and Q^1_s, $s < \varepsilon$, we construct homomorphisms

$$V^s_1 : H(C(s), \partial_s) \to H(C(1), \partial_1) \quad \text{and} \quad V^1_s : H(C(1), \partial_1) \to H(C(s), \partial_s).$$

Proposition 4.6. *We have* $V^1_s \circ V^s_1 = \mathrm{id}$.

Proof. Consider a family of (s, s)-continuation functions π_ω, $\omega \in [0, \infty)$, which satisfies the following conditions.

- $\pi_0(\tau) \equiv s$.
- $\omega \mapsto \pi_\omega(0)$ is monotone and surjective onto $[s, 1]$.
- $d\pi_\omega(\tau)/d\tau \geq 0$, for $\tau < 0$, and $d\pi_\omega(\tau)/d\tau \leq 0$, for $\tau > 0$.
- For large ω,

$$\pi_\omega(\tau) = \begin{cases} \rho^-(\tau + \omega) & \text{for } \tau \leq 0, \\ \rho^+(\tau - \omega) & \text{for } \tau \geq 0, \end{cases}$$

where ρ^+ and ρ^- are monotone $(1,s)$-continuation function and monotone $(s,1)$-continuation function which we use to construct Q_s^1 and Q_1^s respectively. For critical points x_- and $x_+ \in Y(s)$, we put

$$\mathcal{M}_\pi(x_-,x_+) := \big\{(\omega,u) \bigm| u \in \mathcal{M}_{\pi_\omega}(x_-,x_+)\big\}.$$

For a suitable perturbation of almost complex structures we can assume that the regularity condition holds and this space is manifold. Moreover we put

$$\mathcal{M}^d_\pi(x_-,x_+) := \big\{(\omega,u) \bigm| u \in \mathcal{M}^d_{\pi_\omega}(x_-,x_+)\big\}.$$

Lemma 4.7. *Let $u \in \mathcal{M}^d_\pi(x_-,x_+)$, then we obtain*

$$A(u) \le c_+ - c_-(< C').$$

Proof. For $u \in \mathcal{M}^d_{\pi_\omega}(x_-,x_+)$,

$$\begin{aligned} h(u) &\le \int_{-\infty}^0 \frac{d\pi_\omega(\tau)}{d\tau} \max_{p\in G^{\pi_\omega(\tau)}(\widehat{L_0})} \widetilde{H}_{\pi_\omega(\tau)}(p)d\tau \\ &\quad + \int_0^\infty \frac{d\pi_\omega(\tau)}{d\tau} \min_{p\in G^{\pi_\omega(\tau)}(\widehat{L_0})} \widetilde{H}_{\pi_\omega(\tau)}(p)d\tau \\ &= \int_s^0 \max_{p\in G^s(\widehat{L_0})} \widetilde{H}_s(p)ds + \int_0^s \min_{p\in G^s(\widehat{L_0})} \widetilde{H}_s(p)ds \\ &\le a_+ - a_-. \end{aligned}$$

Since $l(u) = \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+) \le (c_+ - a_+) - (c_- - a_-)$, we obtain $A(u) = l(u) + h(u) \le c_+ - c_-$. $\square$

From Lemma 4.7 the set of isolated points of $\mathcal{M}^d_\pi(x,z)$ is compact and there is no bubble in $\mathcal{M}^d_\pi(x,z)$ in the same way of Corollary 4.3. Hence the number of 1-dimensional components of $\mathcal{M}^d_\pi(x,z)$ is finite. And there are four types of the end points of the compactification of a 1-dimensional component as follows.

1. a pair (u_1,u_2) of isolated points $u_1 \in \mathcal{M}_{\rho^-}(x,y)$ and $u_2 \in \mathcal{M}_{\rho^+}(y,z)$, for $y \in Y(1)$.
2. an isolated point $u \in \mathcal{M}_{\pi_0}(x,z)$.
3. a pair (u_1,u_2) of isolated points $u_1 \in \widehat{\mathcal{M}}_s(x,y)$ and $u_2 \in \mathcal{M}_{\pi_\omega}(y,z)$, for $y \in Y(s)$.
4. a pair (u_1,u_2) of isolated points $u_1 \in \mathcal{M}_{\pi_\omega}(x,y)$ and $u_2 \in \widehat{\mathcal{M}}_s(y,z)$, for $y \in Y(s)$.

We put

$$h_s(x) := \sum_{y\in Y(s)} \sharp\big\{\text{isolated points of } \mathcal{M}^d_\pi(x,y)\big\}y,$$

where $x \in Y(s)$. If all the end points as mentioned are distinguished, we obtain

$$Q_s^1 \circ Q_1^s + \mathrm{id} + h_s \circ \partial_s + \partial_s \circ h_s = 0.$$

Then we can say $V_s^1 \circ V_1^s = \mathrm{id}$. First the end points of type 1 are distinguished from Lemma 4.5. Second the end points of type 2 are constant maps because if they are not constant then they have non-zero dimension by the R-action. And constant maps are obviously distinguished. Finally we show that the end points of type 3 and 4 are distinguished.

Lemma 4.8. *If $u \in \mathcal{M}_{\pi_\omega}(x_-, x_+)$, then we obtain $l(u) \geq \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+)$.*

Proof. We have $h(u) \leq a_+ - a_-$ in the same way as Lemma 4.7. And

$$\begin{aligned} l(u) = A(u) - l(u) &\geq a_+ - a_- \\ &> (c_+ - a_+) - (c_- - a_-) - C' \\ &\geq \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+) - C'. \end{aligned}$$

Moreover $l(u) = \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+) + nC'$, $n \in Z$. Hence $l(u) \geq \widetilde{F}_s(x_-) - \widetilde{F}_s(x_+)$. □

In the case of type 3, put $l(u_1) = \widetilde{F}_s(x) - \widetilde{F}_s(y) + nC'$ and $l(u_2) = \widetilde{F}_s(y) - \widetilde{F}_s(z) - nC'$. Since $l(u_1) = A(u_1) \geq 0$, we obtain $n \geq 0$. From Lemma 4.8, $l(u_2) \geq \widetilde{F}_s(y) - \widetilde{F}_s(z)$, we obtain $n \leq 0$. Hence $n = 0$ and u_1, u_2 are distinguished. Similarly, in the case of type 4, put $l(u_1) = \widetilde{F}_s(x) - \widetilde{F}_s(y) + nC'$ and $l(u_2) = \widetilde{F}_s(y) - \widetilde{F}_s(z) - nC'$. Since $l(u_2) = A(u_2) \geq 0$, we obtain $n \leq 0$. From Lemma 4.8, $l(u_1) \geq \widetilde{F}_s(x) - \widetilde{F}_s(y)$, we obtain $n \geq 0$. Hence $n = 0$ and u_1, u_2 are distinguished.

Then we obtain $V_s^1 \circ V_1^s = \mathrm{id}$. □

5. Proof of main theorem

We summarize our story. Let M be a closed contact manifold with a contact form γ. Let L_0 be a closed pre-Lagrangian submanifold of M and L_1 a closed Legandrian submanifold of M which satisfy the conditions $L_0 \supset L_1$ and the boundary homomorphism $\pi_2(M, L_0) \to \pi_1(L_0)$ is trivial. We take $\widehat{L_0}$ the Lagrangian lift of L_0 with the condition $\min_{p \in \widehat{L_0}} \theta(p) = 0$.

We denote the symplectization of M by P. Let $\widetilde{H}_s$ be a time-dependent Hamiltonian function on P, and $G^s : P \to P$ the time-s flow generated by $\widetilde{H}_s$. We put

$$a_+ := \int_0^1 \max_{p \in G^s(\widehat{L_0})} \widetilde{H}_s(p)ds \quad \text{and} \quad a_- := \int_0^1 \min_{p \in G^s(\widehat{L_0})} \widetilde{H}_s(p)ds.$$

Suppose that $C'e^{d+\varepsilon} < C_\gamma(L_1)$, where $d = \int_0^1 \max_{p \in G^t(\widehat{L_0})} |d\theta(X_{\widetilde{H}_t})|dt$, then we can define Floer homology. Moreover if we have $a_+ - a_- < C'$, then there are homomorphisms $V_1^s : H(C(s), \partial_s) \to H(C(1), \partial_1)$ and $V_s^1 : H(C(1), \partial_1) \to$

$H(C(s),\partial_s)$, for small s, such that $V_s^1 \circ V_1^s = \mathrm{id}$. Hence if $G^1(\widehat{L_0})$ intersects $R \times L_1$ transversaly, we obtain

$$\sharp\{G^1(\widehat{L_0}) \cap (R \times L_1)\} \geq \text{rank } H(C(1),\partial_1) \\ \geq \text{rank } H(C(s),\partial_s).$$

And, for small s, $H(C(s),\partial_s)$ is isomorphic to $H_*(L_1;Z_2)$ as a vector space. Hence we obtain

$$\sharp\{G^1(\widehat{L_0}) \cap (R \times L_1)\} \geq \text{rank } H(L_1;Z_2).$$

In the next step, let H_s be a time-dependent function on M and X_{H_s} the contact vector field generated by H_s. Let $e^\theta H_s$ be a Hamiltonian function on P, then the Hamiltonian vector field generated by $e^\theta H_s$ is $(dH_s(Y), X_{H_s})$ on $P = R \times M$, where Y is the Reeb vector field. We put $d = \max_{s\in[0,1],p\in M} |dH_s(Y)|$ and moreover

$$a_+ := \int_0^1 \max_{p\in M} H_s(p)ds \cdot \exp\{d + h(\widehat{L_0})\},$$

and

$$a_- := \int_0^1 \min_{p\in M} H_s(p)ds \cdot \exp\{d + h(\widehat{L_0})\}.$$

Finally, if we have $C'e^{d+\varepsilon} < C_\gamma(L_1)$ and $a_+ - a_- < C'$, then Theorem 1.1 holds.

6. Compactness

For simplicity we fix the almost complex structure $\tilde{J}$ on P defined in the introduction. To put it more precisely, we have to consider perturbations of almost complex structures. It is a little complicated. Hence we omit it here. See [2].

We introduce notation. Put $K_{a,b} := [a,b] \times M \subset P$ and $M_\theta := \{\theta\} \times M$. If we denote $d = \int_0^s \max_{p\in G^t(\widehat{L_0})} |d\theta(X_{\widetilde{H}_t})|dt$, then we may have $G^s(\widehat{L_0}) \subset K_{-d,h(L_0)+d+\varepsilon}$.

Lemma 6.1. *For any $u \in \mathcal{M}_s(x_-,x_+)$, the image of u is contained in $K_{-\infty,h(L_0)+d+\varepsilon}$.*

Proof. Assume that the image of u is not contained in $K_{-\infty,h(L_0)+d+\varepsilon}$, then we have $\sup(\theta \circ u) > h(L_0) + d + \varepsilon$. Because u converges to $x_\pm$ at infinity and $\theta(x_\pm) \leq h(L_0) + d + \varepsilon$, there are some points of the image of u where $\theta \circ u$ takes the maximum. Let p_0 be one of these points. From the pseudo-convexity of M_θ and maximum principle, p_0 is not an interior point of the image of u. Assume that p_0 is a boundary point. Let v be a tangent vector along the boundary at p_0. From $p_0 \in K_{h(L_0)+d+\varepsilon,\infty}$, we have $p_0 \in R \times L_1$ and v is tangent to L_1. Then $v \in \xi_{p_0}$. Because $\tilde{J}v \in \tilde{J}\xi_{p_0} = \xi_{p_0}$ and u is pseudo-holomorphic, the image of u is tangent to $M_{\theta(p_0)}$ at p_0. But this contradicts the pseudo-convexity of M_θ and strong maximum principle. □

We denote the pull-back of γ by the projection $\pi : R \times M \to M$, also by γ. Let $\overline{\omega} := d\gamma$, then we have $\overline{\omega}|_{T_p(R\times M)} = e^{-\theta}(d\pi)^*(\omega|_{T_pM_\theta})$, where $p \in M_\theta$.

Lemma 6.2. *Put $C^i := u^{-1}(K_{-\infty,-i})$ for $u \in \mathcal{M}_s(x_-, x_+)$. If $i \geq d$, then we have*

$$0 \leq \int_{C^i} u^*\overline{\omega} = e^i \int_{C^i} u^*\omega.$$

Hence, if $\int_{R\times[0,1]} u^*\omega < C'$, then we obtain $0 \leq \int_{C^i} u^*\overline{\omega} < C'e^i$.

Proof. From $i \geq d$, $\partial C^i = u^{-1}(M_{-i}) \cup u^{-1}((R \times L_1) \cap K_{-\infty,-i})$. Notice that L_1 is a Legandrian submanifold. Hence

$$\int_{C^i} u^*\overline{\omega} = \int_{\partial C^i} u^*\gamma = \int_{u^{-1}(M_{-i})} u^*\gamma = e^i \int_{u^{-1}(M_{-i})} u^*(e^\theta\gamma) = e^i \int_{C^i} u^*\omega. \qquad \square$$

Corollary 6.3. *If $i \geq d$, then we have*

$$\int_{C^i} u^*\omega \leq e^{d-i} \int_{C^d} u^*\omega.$$

Hence, if $\int_{R\times[0,1]} u^*\omega < C'$, then we obtain $\int_{C^i} u^*\omega < C'e^{d-i}$.

Proof. Since u is pseudo-holomorphic, $\int_{C^{i+1}} u^*\overline{\omega} \leq \int_{C^i} u^*\overline{\omega}$. Then from Lemma 6.2

$$e^{i+1} \int_{C^{i+1}} u^*\omega = \int_{C^{i+1}} u^*\overline{\omega} \leq \int_{C^i} u^*\overline{\omega} = e^i \int_{C^i} u^*\omega.$$

Hence we obtain

$$\int_{C^{i+1}} u^*\omega \leq e^{-1} \int_{C^i} u^*\omega$$

and repeat this inequality. □

For a map $u : R\times[0,1] \to P$, if a domain G of $R\times M$ satisfies the following conditions, we call G a *special domain for u of level k and width l.*

- G is either a disk or an annulus.
- $u|_G$ intersects $M_{-k} \cup M_{-k-l}$ transversaly.
- $u(\partial G) \subset M_{-k} \cup M_{-k-l} \cup (R \times L_1)$ and $u(\partial G \cap \partial(R \times [0,1])) \subset R \times L_1$.
- $u(\partial G) \cap M_{-k} \neq \emptyset$ and $u(\partial G) \cap M_{-k-l} \neq \emptyset$.
- $u(G) \subset K_{-\infty,-d}$.

Lemma 6.4. *For $u \in \mathcal{M}_s(x_-, x_+)$, C^i is a disjoint union of disks.*

Proof. It follows from the pseudo-convexity of M_θ and maximum principle. □

Again note that, for simplicity we fix the almost complex structure $\tilde{J}$ on P defined in the introduction. To put it more precisely, we have to consider perturbations of almost complex structures. It is a little complicated. Hence we omit it here. See [2].

Lemma 6.5. *Let $\{u_n\} \subset \mathcal{M}_s(x_-, x_+)$ be a sequence of pseudo-holomorphic maps. Assume that the union of the images of $u_n{}'s$ is not contained in any compact subset of P. Then there are a subsequence $\{u_{n_k}\}$ and a sequence of domains $\{G_k\}$, $G_k \subset R \times [0,1]$, which satisfy the following.*

- *G_k is a special domain for u_{n_k}.*
- *G_k is of width l and level j for $d \leq j$ and $u_{n_k}(G_k) \subset K_{-j-2l,-j}$.*
- *If $\int_{R\times[0,1]} u_n^*\omega < C'$, then we have $\int_{G_k} u_{n_k}^*\overline{\omega} \to 0$ as $k \to \infty$.*

Proof. From Lemma 6.1, the union of the images of $u_n{}'s$ is bounded above along the R-direction of P. Hence we may assume that there is a subsequence $\{u_{n_k}\}$ such that $u_{n_k}(R \times [0,1]) \cap M_{-(k+1)l,-d} \neq \emptyset$. For $d \leq i \leq d + kl$ we put

$$B_k^i := u_{n_k}^{-1}(K_{-\infty,-i}) \setminus u_{n_k}^{-1}(K_{-\infty,-i-l}).$$

Let B be a connected component of B_k^i such that $u_{n_k}^{-1}(M_{-i}) \cap B \neq \emptyset$ and $u_{n_k}^{-1}(M_{-i-l}) \cap B \neq \emptyset$. From Lemma 6.4, B is a disk with some holes. We patch some disks back to these holes so that B turns to either a disk or an annulus. We denote this disk or annulus by $\widehat{B}$. Especially we can do this procedure so that $\partial\widehat{B} \cap u_{n_k}^{-1}(M_{-i}) \neq \emptyset$ and $\partial\widehat{B} \cap u_{n_k}^{-1}(M_{-i-l}) \neq \emptyset$. Then $\widehat{B}$ is a special domain for u_{n_k} of width l and level i. In this way we can find special domains.

For each k, we can find special domains $\widehat{B}_k^j$ for u_{n_k} of width l and level j, $j = d, d+l, \ldots, d+kl$, such that Int $\widehat{B}_k^i \cap$ Int $\widehat{B}_k^j = \emptyset$ for $i \neq j$. From $\bigcup_j \widehat{B}_k^j \subset u_{n_k}^{-1}(K_{-\infty,-d})$,

$$\sum_j \int_{\widehat{B}_k^j} u_{n_k}^*\overline{\omega} \leq \int_{C_k^d} u_{n_k}^*\overline{\omega} = e^d \int_{C_k^d} u_{n_k}^*\omega,$$

where $C_k^d = u_{n_k}^{-1}(K_{-\infty,-d})$. Hence, if $\int_{R\times[0,1]} u_n^*\omega < C'$, we have

$$\sum_j \int_{\widehat{B}_k^j} u_{n_k}^*\overline{\omega} < C'e^d.$$

Because each term of the sum is positive, there is at least one special domain $\widehat{B}_k^j$ such that $\int_{\widehat{B}_k^j} u_{n_k}^*\overline{\omega} < C'e^d/k$.

We put a special domain G_k for u_{n_k} such that $\int_{G_k} u_{n_k}^*\overline{\omega}$ is the minimum over all the special domains of width l and level $d, d+l, \ldots, d+kl$. From the construction, we have $G_k \cap M_{-j+l} = \emptyset$, where j is the level of G_k. Moreover we have $G_k \cap M_{-j-2l} = \emptyset$, because if $G_k \cap M_{-j-2l} \neq \emptyset$ we can find another special domain G'_k such that $\int_{G'_k} u_{n_k}^*\overline{\omega} < \int_{G_k} u_{n_k}^*\overline{\omega}$. This contradicts the minimum of G_k. □

Fix a width l. We put $\widetilde{u}_{n_k}$ to be the $(j-d)$-shift, along the R-direction of P, of a pseudo-holomorphic map $u_{n_k} : G_k \to P$, where j is the level of G_k. Then we have

$$\int_{G_k} \widetilde{u}_{n_k}^*\omega = e^{j-d}\int_{G_k} u_{n_k}^*\omega.$$

From the inequality $\int_{u_{n_k}^{-1}(K_{-\infty,-j})} u_{n_k}^*\omega \leq e^{d-j}\int_{u_{n_k}^{-1}(K_{-\infty,-d})} u_{n_k}^*\omega$, in Corollary 6.3, we obtain

$$\int_{G_k} \widetilde{u}_{n_k}^*\omega \leq e^{j-d}\int_{u_{n_k}^{-1}(K_{-\infty,-j})} u_{n_k}^*\omega \leq \int_{u_{n_k}^{-1}(K_{-\infty,-d})} u_{n_k}^*\omega.$$

Hence, if $\int_{R\times[0,1]} u_{n_k}^*\omega < C'$, then we have

$$\int_{G_k} \widetilde{u}_{n_k}^*\omega < C'.$$

Consider the pseudo-holomorphic maps $\widetilde{u}_{n_k} : G_k \to (-d-2l, -d] \times M$, and apply the Gromov's compactness theorem. See [2].

Proposition 6.6. *There is a subsequence $\{\widetilde{u}_{n_k}\}$ which converges uniformly on compact sets to a non-constant pseudo-holomorphic map $\widetilde{u}_\infty$. The boundary of this image is contained in $M_{-d} \cup M_{-d-l} \cup (R \times L_1)$ and smoothness of the boundary holds at points in $R \times L_1$.*

Notice that, if $\int_{R\times[0,1]} u_{n_k}^*\omega < C'$, we have $\int_{G_k} \widetilde{u}_{n_k}^*\overline{\omega} \to 0$ as $k \to \infty$. Hence we obtain $\int_B \widetilde{u}_\infty^*\overline{\omega} = 0$, where B is either a disk or an annulus.

Lemma 6.7. *Let $\widetilde{u}_\infty : \mathrm{Int}B \to (-d-2l, -d] \times M$ be a non-constant pseudo-holomorphic map, where B is either a disk or an annulus, and the boundary of this image be contained in $M_{-d} \cup M_{-d-l} \cup (R \times L_1)$. Assume that $\int_B \widetilde{u}_\infty^*\overline{\omega} = 0$. Then there is either a closed orbit of the Reeb vector field or an orbit of the Reeb vector field with the end points in L_1, we denote each by S, such that*

$$\widetilde{u}_\infty(\mathrm{Int}B) = (-d-l, -d) \times S \subset (-d-l, -d) \times M.$$

And we obtain

$$e^{-d}(1-e^{-l})\int_S \gamma = \int_B \widetilde{u}_\infty^*\omega.$$

Proof. From $\overline{\omega} = d\gamma$, $\omega = e^{\theta}(d\theta \wedge \gamma + d\gamma)$ and $\widetilde{u}_{\infty}$ is pseudo-holomorphic, this lemma holds. □

If $\int_{R\times[0,1]} u_n^*\omega < C'$, then we have $\int_{G_k} \widetilde{u}_n^*\omega < C'$ as mentioned after Lemma 6.5. Hence $\int_B \widetilde{u}_{\infty}^*\omega < C'$ and we obtain

$$\int_S \gamma < e^d(1 - e^{-l})^{-1}C'.$$

If we have $C'e^{d+\varepsilon} < C_{\gamma}(L_1)$ for $\varepsilon > 0$, all the images of u_n have to be contained in a compact set of P from Lemma 6.5, Proposition 6.6 and Lemma 6.7.

Thus we finish a proof of Theorem 3.1.

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References

[1] Y. V. Chekanov, Hofer's Symplectic Energy and Lagrangian intersections, in Contact and Symplectic Geometry, Cambridge, 1996, 296–306.

[2] Y. Eliashberg, H. Hofer and D. Salamon, Lagrangian intersectons in contact geometry, Geom. Funct. Anal., **5**-2 (1995), 244–269.

[3] A. Floer, Morse theory for Lagrangian intersections, J. Diff. Geom., **28** (1988), 513–547.

[4] H. Hofer, Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimention three, Invent. Math., **114** (1993), 515–563.