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Inductive limits of topologies, their direct products, and problems related to algebraic structures

By

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Introduction

This paper is a continuation of our previous work [9]. In a half of it, we studied the inductive limit $G = \lim_{\rightarrow} G_n$ of topological groups $G_n, n \geq 1$, and proved that the inductive limit topology $\tau_{ind}^G = \lim_{\rightarrow} \tau_{G_n}$ of topologies τ_{G_n} on G_n does not in general give a group topology on G, contrary to the affirmative statement in [4, Article 210], and then studying τ_{ind}^G in detail, we constructed, under a mild condition (PTA), a group topology $\tau_{BS}^G = \text{BS-}\lim_{\rightarrow} \tau_{G_n}$ on G, called Bamboo-Shoot topology, which is the strongest among those weaker than or equal to τ_{ind}^G. This work provokes us two directions of study.

The one is to study the reason why this kind of pathological phenomena occur rather in general. The other is to construct a good version of inductive limits (like τ_{BS}^G in the category of topological groups) in various categories, such as topological algebras, topological semigroups etc.

Take two inductive systems of topological spaces $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\alpha\}_{\alpha \in A}$ and put $X = \lim_{\rightarrow} X_\alpha, Y = \lim_{\rightarrow} Y_\alpha$. For the direct product $\{X_\alpha \times Y_\alpha\}_{\alpha \in A}$ of these systems, its inductive limit can be identified with $X \times Y$, and on it we have two kinds of topologies, the one is $\tau_{ind}^X \times \tau_{ind}^Y$ with $\tau_{ind}^X := \lim_{\rightarrow} \tau_{X_\alpha}$ and the other is $\tau_{ind}^{X \times Y} := \lim_{\rightarrow}(\tau_{X_\alpha} \times \tau_{Y_\alpha})$. Then, we have in general $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X \times Y}$. We found that a principal reason for pathological phenomena similar to the above one is the mismatch of these two topologies on $X \times Y$. Therefore we propose, in Section 1, Problems A, B and C related to these phenomena, and study them in Sections 4 through 6.

Assume $\{X_\alpha\}_{\alpha \in A}$ be an inductive system in the category of locally convex topological vector spaces (= LCTVSs). Then, the natural inductive limit topology in this category (cf. Definition 2.1) has been given long ago (denoted here as $\tau_{lcv}^X = \text{lcv-}\lim_{\rightarrow} X_\alpha$) and is now used everywhere. As an example, take a space of test functions $\mathcal{D}(M)$ of C^∞-functions with compact supports on a

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differentiable manifold M. Then, the pointwise multiplication in $X = \mathcal{D}(M)$, and the convolution in $X = \mathcal{D}(\mathbf{R}^k)$ are continuous in τ_{lcv}^X, and so X becomes a topological algebra in both cases (Propositions 2.2 and 2.3). Their proofs depend on the special structure of $\mathcal{D}(M)$ and can not be generalized to the general case of inductive systems of topological algebras.

A similar but quite different case is given as follows. Take another $Y = \mathcal{D}(M')$ with M' a differentiable manifold, and put $Z = \mathcal{D}(M \times M')$. Then, the pointwise multiplication $T : X \times Y \to Z$ is not continuous in $\big(\tau_{lcv}^X \times \tau_{lcv}^Y, \tau_{lcv}^Z\big)$, whenever at least one of M and M' is non-compact (Theorem 2.4). Here, $T(\varphi, \psi)(p, p') = \varphi(p)\psi(p')(p \in M, p' \in M')$ for $(\varphi, \psi) \in X \times Y$. Since we have the equivalence of topologies as $\tau_{lcv}^X \times \tau_{lcv}^Y \cong \tau_{lcv}^{X \times Y}$ (Theorem 3.4), the above non-continuity property relates essentially to the difference between linearity and bilinearity, or more exactly, the inductive limit $\tau_{lcv}^{\{*\}}$ in the category of LCTVSs is well fitted to linear structures but not to bilinear maps taken as multiplications.

Inspired by these concrete examples, we propose, in Section 2, Problems D, E and F.

The problem of matching or mismatching of two kinds of topologies on $X \times Y$ is in many respects very important in every category $\mathcal{C}$. It is, in other words, the problem of "commutativity" of two processes: (1) taking an inductive limit of topologies in the category $\mathcal{C}$, and (2) taking a direct product. We take $\tau_{ind}^{\{*\}}$ in general or in the category of topological spaces, $\tau_{BS}^{\{*\}}$ in the category of topological groups, and $\tau_{lcv}^{\{*\}}$ in the category of LCTVSs. Consider them as functors of the corresponding categories consisting of inductive systems. The above "commutativity" for each of these functors is called the condition (DPA) (= *Direct Product is Admitted*).

For the 'Bamboo-Shoot topology functor' $\tau_{BS}^{\{*\}}$ in the category of countable inductive systems of topological groups, and moreover for the extended one in Section 3.4 in the category of general inductive systems of topological groups, the condition (DPA) holds, that is, $\tau_{BS}^G \times \tau_{BS}^H \cong \tau_{BS}^{G \times H}$, where $H = \lim_{\to} H_n$ for an inductive system $\{H_n\}$ (Theorem 3.3).

Following [11] and [3], where countable inductive systems of Banach algebras or of their subgroups are studied, we propose, in Section 3, Problems G and H. For instance, take a countable inductive system of Banach (or topological) algebras $\{X_n\}$, and take subgroups G_n of ${X_n}^\times$ of all the invertible elements in X_n with the restricted topology $\tau_{G_n} := \tau_{X_n}|_{G_n}$. On the limit group $G = \lim_{\to} G_n$, we want to compare two topologies $\tau_{BS}^G = \text{BS-}\lim_{\to} \tau_{G_n}$ and $\tau_{lcv}^X|_G$ with $X = \lim_{\to} X_n$.

For the functor $\tau_{lcv}^{\{*\}}$, the condition (DPA) holds in general in the category of inductive systems of LCTVSs (Theorem 3.4).

For the functor $\tau_{ind}^{\{*\}}$ in the category of inductive systems of topological spaces, the condition (DPA) holds only in a certain restricted subcategory. So we look for a better version of inductive limit for which the condition (DPA) holds in much wider subcategory. In Section 7 of [6], we take the category of uniform spaces and propose a variant τ_{wBS}^X of τ_{ind}^X, under a condition called (wPTA), but an essential progress has not yet been achieved.

This paper is organized as follows.

In Section 1, we discuss some generalities and show several examples about the condition (DPA) for inductive systems of topological spaces. Among other things, the product topology $\tau_{ind}^X \times \tau_{ind}^Y$ on $X \times Y$ is characterized as the strongest product topology such that $\preceq \tau_{ind}^{X\times Y}$ (Theorem 1.3). We propose Problems A, B and C for inductive systems of topological spaces.

In Section 2, inductive limits topologies in various categories are discussed, especially about locally convex vector topology $\tau_{lcv}^{\{*\}}$. Taking spaces of test functions, we give examples of inductive systems of topological algebras or similar ones for which multiplications in the limit algebras or so are continuous or not continuous. Problems D, E and F are proposed in relation to these subjects.

In Section 3, we discuss about relations between the Bamboo-Shoot topology $\tau_{BS}^{\{*\}}$ and locally convex topology $\tau_{lcv}^{\{*\}}$. Further the condition (DPA) is proved to hold, for the extended Bamboo-Shoot topology $\tau_{BS}^{\{*\}}$ for inductive systems of topological groups, and also for locally convex vector topology $\tau_{lcv}^{\{*\}}$ for inductive systems of LCTVSs. Relatedly, Problems G and H are proposed.

In Section 4, sufficient conditions for (DPA) are discussed for inductive systems of topological spaces. This is a half of Problem A. Here the local compactness and the sequential local compactness play a decisive role.

In Section 5, the case where X is an inductive limit space $\lim_{\rightarrow} X_n$ and Y is a fixed topological space, is treated. These are discussions about Problem B.

In Section 6, necessary conditions to have the condition (DPA) in the case of topological spaces are discussed. This is the other half of Problem A.

A previous version of this paper has appeared in [6], and a summalized version in [7].

1. Inductive limits and direct products

1.1. Preliminaries

Let us consider an inductive system in a certain category $\mathcal{C}$, of topological spaces, of topological groups, of topological vector spaces, or of topological algebras, etc., as

$$\{(X_\alpha, \tau_{X_\alpha}), \alpha \in A; \phi_{\beta,\alpha}, \alpha \preceq \beta, \alpha, \beta \in A\},$$

where the index set A is a directed set, each X_α is an object in $\mathcal{C}$ with topology τ_{X_α}, and $\phi_{\beta,\alpha}$ is a (continuous) homomorphism $X_\alpha \to X_\beta$ in $\mathcal{C}$ satisfying the consistency condition: $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$ for any $\alpha \preceq \beta \preceq \gamma$.

Then, on an inductive limit space $X := \lim_{\rightarrow} X_\alpha$, we define the corresponding algebraic structure. On the other hand, we have also an inductive limit topology, denoted as $\lim_{\rightarrow} \tau_{X_\alpha}$ or simply as τ_{ind}^X, in which a subset D of X is open, by definition, if and only if ${\phi_\alpha}^{-1}(D) \subset X_\alpha$ is open in τ_{X_α} for each $\alpha \in A$. Here, ϕ_α denotes the canonical homomorphism from X_α to X.

In this paper, we study about the harmonicity of the limit topology τ_{ind}^X with the algebraic structure on X. Furthermore, we consider an appropriate variant of τ_{ind}^X in each category $\mathcal{C}$ (denote it by $\tau_{\mathcal{C}}^X$ provisionally here) and study various kinds of harmonicity.

Meantime, as is explained in Introduction, we find that one of the important points of discussions is the problem of commutativity of (1) taking the inductive limit $\tau_{\mathcal{C}}^X$ and (2) taking direct products. This commutativity is expressed symbolically as $\tau_{\mathcal{C}}^X \times \tau_{\mathcal{C}}^Y \cong \tau_{\mathcal{C}}^{X\times Y}$, for two inductive systems $\{(X_\alpha, \tau_{X_\alpha}), \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}), \alpha \in A\}$ with $Y = \lim_{\rightarrow} Y_\alpha$.

More in detail, let us explain our problems in the following.

1.2. Inductive limits of topological groups

Let $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$ be an inductive system of topological groups with a directed set A as index set. Here τ_{G_α} denotes the group topology on G_α and we are given an inductive system of continuous group homomorphisms $\phi_{\alpha_2,\alpha_1}; G_{\alpha_1} \to G_{\alpha_2} (\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2)$ satisfying $\phi_{\alpha_3,\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi_{\alpha_3,\alpha_1}$ for $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$. Put $G := \lim_{\rightarrow} G_\alpha$ and $\tau_{ind}^G := \lim_{\rightarrow} \tau_{G_\alpha}$ the inductive limit of groups and that of topologies respectively. Then, as seen in [9], the multiplication $G \times G \ni (g, h) \mapsto gh \in G$ is not necessarily continuous with respect to the inductive limit topology τ_{ind}^G, or more exactly, with respect to $(\tau_{ind}^G \times \tau_{ind}^G, \tau_{ind}^G)$.

Inspired by this rather critical phenomenon, we start in this paper to study the inductive limit topologies in detail in more general setting.

1.3. A continuity criterion

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of topological spaces, and take another inductive system $\{(Z_\alpha, \tau_{Z_\alpha}); \alpha \in A\}$ of topological spaces with the same index set A and with an inductive system of continuous maps $\phi'_{\alpha_2,\alpha_1}$: $Z_{\alpha_1} \to Z_{\alpha_2}$. Then, assume that we are given a system of maps F_α of X_α to Z_α for $\alpha \in A$ which is *consistent* in the sense that $F_{\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi'_{\alpha_2,\alpha_1} \circ F_{\alpha_1}$ for $\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$. Then this system induces a map $F : X \to Z := \lim_{\rightarrow} Z_\alpha$ such that $F \circ \phi_\alpha = \phi'_\alpha \circ F_\alpha (\alpha \in A)$, where ϕ_α (resp. ϕ'_α) denotes the natural map from X_α to X (resp. Z_α to Z), continuous with respect to $(\tau_{X_\alpha}, \tau_{ind}^X)$ (resp. to $(\tau_{Z_\alpha}, \tau_{ind}^Z)$). Furthermore the following fact is easy to prove.

Lemma 1.1. *If every map $F_\alpha : X_\alpha \to Z_\alpha$ is continuous in $(\tau_{X_\alpha}, \tau_{Z_\alpha})$ for $\alpha \in A$, then the induced map $F : X \to Z$ is continuous in $(\tau_{ind}^X, \tau_{ind}^Z)$.*

Let us apply this lemma to the above case of inductive limits of topological groups, by setting

$$(X_\alpha, \tau_{X_\alpha}) = (G_\alpha \times G_\alpha, \tau_{G_\alpha} \times \tau_{G_\alpha}), \quad (Z_\alpha, \tau_{Z_\alpha}) = (G_\alpha, \tau_{G_\alpha}),$$

and $F_\alpha : X_\alpha \to Z_\alpha$ as $F_\alpha(g_\alpha, h_\alpha) = g_\alpha h_\alpha$. Then, since τ_{G_α} is a group topology on G_α, the map F_α is continuous for each $\alpha \in A$, and so, as their natural limit, the multiplication map $F(g,h) = gh$ of $X = G \times G$ to $Z = G$ is continuous, by Lemma 1.1, with respect to the topologies $\tau_{ind}^{G\times G} := \lim_{\rightarrow} (\tau_{G_\alpha} \times \tau_{G_\alpha})$ on $G \times G = X$ and $\tau_{ind}^{G} := \lim_{\rightarrow} \tau_{G_\alpha}$ on $G = Z$.

Remark 1.1. For an inductive system $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$, we may assume without loss of generality that $\phi_\alpha : X_\alpha \to X$ is an injection, or $X_\alpha \subset X$ for $\alpha \in A$, replacing X_α by its image $X'_\alpha := \phi_\alpha(X_\alpha)$ if necessary. In that case, the quotient topology $\tau_{X'_\alpha}$ of τ_{X_α} is attributed to X'_α. For some problems, it is also possible to take the relative topology $\tau_X|_{X'_\alpha}$ for $X'_\alpha \subset X$.

Note that when a directed set A is countable, there exists a sub-directed-set isomorphic to $\mathbf{N}$ or to $\{1, 2, \ldots, k\} \subset \mathbf{N}$ which is cofinal to A.

1.4. Direct products of inductive limits of topologies

On the other hand, it is easy to see the following fact for the direct product of inductive limits of topologies. Take two inductive limits of topological spaces $(X, \tau_{ind}^X) = (\lim_{\rightarrow} X_\alpha, \lim_{\rightarrow} \tau_{X_\alpha})$ and $(Y, \tau_{ind}^Y) = (\lim_{\rightarrow} Y_\alpha, \lim_{\rightarrow} \tau_{Y_\alpha})$, and consider their direct products.

Proposition 1.2. *The product space $X \times Y$ is naturally identified with the inductive limit space $\lim_{\rightarrow} (X_\alpha \times Y_\alpha)$. On this space the direct product of inductive limit topologies $\tau_{ind}^X \times \tau_{ind}^Y = (\lim_{\rightarrow} \tau_{X_\alpha}) \times (\lim_{\rightarrow} \tau_{Y_\alpha})$ is weaker than or equal to the inductive limit of product topologies $\tau_{ind}^{X\times Y} := \lim_{\rightarrow} (\tau_{X_\alpha} \times \tau_{Y_\alpha})$, or in a symbolic notation, $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X\times Y}$. In particular, for a subset of product type $D \times E \subset X \times Y$, it is open in the former topology if and only if so is in the latter.*

For an inductive limit of topological groups $G := \lim_{\rightarrow} G_\alpha$, taking into account the above result cited in Section 1.2, we see from Lemma 1.1 that, in the case where the multiplication $G \times G \ni (g,h) \mapsto gh \in G$ is not continuous with respect to τ_{ind}^G, the product topology $\tau_{ind}^G \times \tau_{ind}^G$ should be strictly weaker than the inductive limit topology $\tau_{ind}^{G\times G} := \lim_{\rightarrow} (\tau_{G_\alpha} \times \tau_{G_\alpha})$. Thus we come naturally to the following problem.

Problem A. *Let the notations be as above. Then, give a necessary and sufficient condition for the equivalence of two topologies $\tau_{ind}^X \times \tau_{ind}^Y$ and $\tau_{ind}^{X\times Y} := \lim_{\rightarrow} (\tau_{X_\alpha} \times \tau_{Y_\alpha})$ on $X \times Y$, where $(X, \tau_{ind}^X) = (\lim_{\rightarrow} X_\alpha, \lim_{\rightarrow} \tau_{X_\alpha})$ and $(Y, \tau_{ind}^Y) = (\lim_{\rightarrow} Y_\alpha, \lim_{\rightarrow} \tau_{Y_\alpha})$.*

This is, in a sense, the problem of commutativity of two processes: (1) taking inductive limits and (2) taking direct products, for two inductive systems of topological spaces. This is the problem on the condition (DPA) for $\tau_{ind}^{\{*\}}$.

Remark 1.2. It is sufficient in general to treat the case where the index sets for two inductive systems are the same. In fact, in case where the second inductive system has another directed set B as its index set as $\{(Y_\beta, \tau_\beta); \beta \in B\}$, then consider the direct product $\Gamma = A \times B$ with the order $(\alpha, \beta) \preceq (\alpha', \beta')$ defined by $\alpha \preceq \alpha'$ and $\beta \preceq \beta'$. Take a sub-directed-set A' cofinal to Γ, and put for $\gamma = (\alpha, \beta) \in A'$, $X_\gamma = X_\alpha, Y_\gamma = Y_\beta$. Then we come to the case where two inductive systems have the same index set A'.

1.5. Examples and further problems

Let us examine the simple example, Example 1.2 in [9], from the stand point of general topology.

Example 1.1. Let $G_n = F^n \times \mathbf{Q}, F = \mathbf{R}$, $\mathbf{Q}$ or $\mathbf{T}$ with the usual non-discrete topology τ_n for $n \in \mathbf{N}$. Then, $G = \lim_{\to} G_n = (\prod' F) \times \mathbf{Q}$, where $\prod' F$ denotes the restricted direct product of countable number of F's. The multiplication on G is not continuous with respect to $\tau_{ind}^G = \lim_{\to} \tau_{G_n}$. Hence, $\tau_{ind}^G \times \tau_{ind}^G \prec \tau_{ind}^{G \times G}$.

Furthermore, considering G_n as a topological space and express it as a direct product of two spaces as $X_n \times Y$, with $X_n = F^n, Y = \mathbf{Q}$. Then, $X := \lim_{\to} X_n = \lim_{\to} F^n = \prod' F$, and we see that the direct product topology $\tau_{ind}^X \times \tau_Y$ is strictly weaker than $\tau_{ind}^{X \times Y} = \lim_{\to} (\tau_{X_n} \times \tau_Y)$ at every point of $X \times Y$, by reexamining the proof in Example 1.2 in [9] for non-continuity of the multiplication on G.

In the above case, the topological space Y is fixed, and so the following problem is also important to study.

Problem B. *Let $(X, \tau_{ind}^X) = (\lim_{\to} X_\alpha, \lim_{\to} \tau_{X_\alpha})$ be an inductive limit of topological spaces and (Y, τ_Y) a fixed topological space. Then, give a necessary and sufficient condition for the equivalence of two topologies $\tau_{ind}^X \times \tau_Y$ and $\tau_{ind}^{X \times Y} := \lim_{\to} (\tau_{X_\alpha} \times \tau_Y)$ on $X \times Y$.*

The former Problem A contains this Problem B, but it is worth to study Problem B by itself. We may expect that a solution to Problem B helps to solve Problem A. However the situation is not so simple that Problem A is reduced to Problem B, because, for instance, the topology τ_Y cannot be in general recovered from the system $\tau_{Y_n} = \tau_Y|_{Y_n}$, as shown in the next example.

Example 1.2. Let $(Y, \tau_Y) = (\mathbf{R}, \tau_{\mathbf{R}})$, where $\tau_{\mathbf{R}}$ denotes the usual topology on $\mathbf{R}$. Let number all the elements of $\mathbf{Q}$ as $\{q_1, q_2, q_3, \dots\}$, and put $Y_n = (Y \setminus \mathbf{Q}) \sqcup \{q_1, q_2, \dots, q_n\}$ for $n = 1, 2, \dots$. We give a topology τ_{Y_n} as the restriction of τ_Y onto Y_n. Then, we recover the original space Y as $\lim_{\to} Y_n$, but how about the topology τ_Y on Y? Can we recover it as the inductive limit $\tau_{ind}^Y = \lim_{\to} \tau_{Y_n}$? The answer is no: $\tau_Y \prec \tau_{ind}^Y$. More exactly we know the following.

(i) For any subset $D \subset \mathbf{Q}$, the set $Y \setminus D = (Y \setminus \mathbf{Q}) \sqcup (\mathbf{Q} \setminus D)$ is τ_{ind}^Y-open.

(ii) A fundamental system of τ_{ind}^Y-neighborhood of $y \in Y$ is given by a family $\{y\} \cup ((y - \epsilon, y + \epsilon) \cap (Y \setminus \mathbf{Q})), \epsilon > 0$. Here (a, b) denotes an $\tau_{\mathbf{R}}$-open interval of $\mathbf{R}$ determined by a and b.

(iii) τ_{ind}^{Y} induces on $\mathbf{Q} \subset Y$ the discrete topology. The τ_{ind}^{Y}-closure of $Y \setminus \mathbf{Q}$ is Y, and that of $\mathbf{Q}$ is $\mathbf{Q}$ itself.

As a peculiar fact about the topology τ_{Y_n}, we note that no points of Y_n have compact neighborhoods.

This kind of phenomenon is interesting to study and we propose the following problem.

Problem C. *Let (Y, τ_Y) be a topological space and $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ be an inductive system of topological spaces such that $Y_\alpha \subset Y$ and $Y = \lim_{\rightarrow} Y_\alpha$ as sets. Assume that the restriction $\tau_Y|_{Y_\alpha}$ of the topology τ_Y onto Y_α is equal to τ_{Y_α}. Then, $\tau_Y \preceq \tau_{ind}^{Y} := \lim_{\rightarrow} \tau_{Y_\alpha}$. Look for a necessary and sufficient condition for the equivalence of these two topologies on Y.*

1.6. A characterization of the product topology $\tau_{ind}^{X} \times \tau_{ind}^{Y}$

For the product $X \times Y$ of two inductive limits of topological spaces $(X, \tau_{ind}^{X}) = (\lim_{\rightarrow} X_\alpha, \lim_{\rightarrow} \tau_{X_\alpha})$ and $(Y, \tau_{ind}^{Y}) = (\lim_{\rightarrow} Y_\alpha, \lim_{\rightarrow} \tau_{Y_\alpha})$, we have by Proposition 1.2, the relation $\tau_{ind}^{X} \times \tau_{ind}^{Y} \preceq \tau_{ind}^{X \times Y} := \lim_{\rightarrow} (\tau_{X_\alpha} \times \tau_{Y_\alpha})$.

Further we can characterize the product topology as the strongest topology on $X \times Y$ among direct product topologies weaker than $\tau_{ind}^{X \times Y}$. More exactly, we have the following.

Theorem 1.3. *Let τ'_X and τ'_Y be topologies on X and Y respectively such that $\tau'_X \times \tau'_Y \preceq \tau_{ind}^{X \times Y}$. Then, $\tau'_X \preceq \tau_{ind}^{X}$, $\tau'_Y \preceq \tau_{ind}^{Y}$, and so $\tau'_X \times \tau'_Y \preceq \tau_{ind}^{X} \times \tau_{ind}^{Y}$.*

Proof. Let $D \subset X$ be an open subset in τ'_X. Then, $D \times Y$ is open in $\tau_{ind}^{X \times Y}$ by assumption. So, for any α, $(D \times Y) \cap (X_\alpha \times Y_\alpha) = (D \cap X_\alpha) \times Y_\alpha$ is open in $\tau_{X_\alpha} \times \tau_{Y_\alpha}$, whence, $D \cap X_\alpha$ is open in τ_{X_α}. Therefore D is open in τ_{ind}^{X}. □

The above facts evoke studies on inductive limit topologies in various kinds of categories, such as the Bamboo-Shoot topology τ_{BS}^{G} in the category of topological groups in [9] and its generalization, the locally convex vector topology τ_{lcv}^{X} in the category of locally convex topological vector spaces, and so on.

We will discuss on them in the succeeding sections.

2. Inductive limit topologies in various categories

As noticed in Sectoin 1.2, for an inductive limit $G = \lim_{\rightarrow} G_n$ of topological groups $G_n, n \geq 1$, the multiplication map is not necessarily continuous with respect to the inductive limit topology $\tau_{ind}^{G} = \lim_{\rightarrow} \tau_{G_n}$. So we have introduced in [9] a so-called Bamboo-Shoot topology τ_{BS}^{G} on G as the strongest group topology $\preceq \tau_{ind}^{G}$, under the condition (PTA) on the inductive system $\{G_n\}$. Concerning these subject we will discuss in the next section.

In these respects, it is also natural to ask the similar question for other topological algebraic objects, such as topological vector spaces (= TVSs), topological semigroups, topological rings, and topological algebras etc.

2.1. Case of locally convex topological vector spaces

A good category of TVSs is the category of locally convex topological vector spaces (= LCTVSs) over a field $F = \mathbf{R}$ or $\mathbf{C}$. In that category, we know well how to define an inductive limit of topologies.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of LCTVSs with $\phi_{\alpha_2,\alpha_1} : X_{\alpha_1} \to X_{\alpha_2}, \alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$, a homomorphism in the category of LCTVSs, that is, a continuous linear map. On the vector space $X = \lim_{\rightarrow} X_\alpha$, we consider a locally convex vector topology as in the following definition.

Definition 2.1. On the limit space $X = \lim_{\rightarrow} X_\alpha$ of an inductive system $\{X_\alpha\}$ of LCTVSs, a locally convex vector topology, denoted by lcv-$\lim_{\rightarrow} \tau_{X_\alpha}$ or τ_{lcv}^X, is defined as the one for which a fundamental system of neighborhood of the null element 0 is given as $\{U \subset X; \tau_{ind}^X$-open, convex, balanced (i.e., $\lambda x \in U$ for $x \in U, \lambda \in F, |\lambda| \leq 1$), and absorbing$\}$ (cf. [12, I.1, Definition 6, p. 27]).

For discussions in the following, it is better to introduce a simple characterization of neighborhoods of $0 \in X$, which is taken as the definition of the neighborhood system of $0 \in X$ in [10, Section 13, p. 126].

Lemma 2.1. *Let $X = \lim_{\rightarrow} X_\alpha$ be the inductive limit of an inductive system $\{X_\alpha\}_{\alpha\in A}$ of LCTVSs. In each X_α, a neighborhood of $0 \in X_\alpha$ contains by definition a convex, balanced, absorbing, open set. Then, in the lcv-limit topology τ_{lcv}^X, a subset $V \subset X$ is a neighborhood of $0 \in X$ if and only if each ${\phi_\alpha}^{-1}(V) \subset X_\alpha$ contains a τ_{X_α}-neighborhood of $0 \in X_\alpha$, where ϕ_α denotes the cannonical homomorphism of X_α to X.*

A proof of this lemma can be found for instance in [6].
We propose the following problem.

Problem D. *Assume that every space X_α in an inductive system of LCTVSs has an additional structure or operation of the same kind, which induces as its inductive limit such a structure or an operation on the limit space $X := \lim_{\rightarrow} X_\alpha$. Is this structure or operation consistent with the lcv-limit topology τ_{lcv}^X?*

2.2. Multiplication or product in an inductive system

Let us first consider two concrete cases to show what kind of things we want to study.

Let M be a non-compact differentiable manifold, and $M_n \nearrow M, n \geq 1$, be an increasing sequence of relatively compact, open submanifolds such that the closure $\overline{M_n}$ is contained in M_{n+1}. The space of complex-valued test functions (C^∞-functions with compact supports) on M, denoted by $\mathcal{D}(M)$, is a LCTVS obtaind as an inductive limit of the inductive system $X_n = \mathcal{D}(\overline{M_n}) := \{\varphi \in C^\infty(M); \mathrm{supp}(\varphi) \subset \overline{M_n}\}, n \in \mathbf{N}$. Here $\mathcal{D}(\overline{M_n})$ is topologized in a usual manner by means of a countable number of seminorms.

Let us consider two kinds of operations in $X = \mathcal{D}(M)$. First one is the pointwise multiplication $T : X \times X \to X$, given as $T(\varphi_1, \varphi_2)(p) = \varphi_1(p)\varphi_2(p)$ $(p \in M)$, and the second one is the convolution $T(\varphi_1, \varphi_2) = \varphi_1 * \varphi_2$ in the case of $M = \mathbf{R}^k$. We ask if they are continuous or not in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$.

Note that, for the first T, $\mathrm{supp}(\varphi_1\varphi_2) \subset \mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_2)$, and so it maps $X_n \times X_n$ into X_n. On the other hand, for the second T, $\mathrm{supp}(\varphi_1 * \varphi_2)$ becomes bigger and is in general comparable to $\mathrm{supp}(\varphi_1) + \mathrm{supp}(\varphi_2)$, and so T maps $X_n \times X_n$ into $X_{\beta(n)}$ with a $\beta(n) > n$.

2.2.1. Continuity of the multiplication in $\mathcal{D}(M)$

Proposition 2.2. *In the space of test functions $X = \mathcal{D}(M)$, the multiplication map $T(\varphi_1, \varphi_2) = \varphi_1\varphi_2$ is continuous in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$.*

Proof. First choose a sequence of open, relatively compact submanifolds $M_n \nearrow M$ as above, and put $X_n = \mathcal{D}(\overline{M_n})$. We consider $X = \mathcal{D}(M)$ as the inductive limit of the system $\{X_{2n-1}\}_{n\geq 1}$, that is, $X = \lim_{\to} X_{2n-1}$, and use $\{X_{2n}\}_{n\geq 1}$ as auxiliary assistants.

For each n, choose a function $\omega_n \in X_{2n}$ such that $\omega_n = 1$ on M_{2n-1} and $\mathrm{supp}(\omega_n) \subset M_{2n}$. Then, $\omega_n\varphi = \varphi$ for $\varphi \in X_{2n-1}$, and $\omega_n\psi \in X_{2n}$ for any $\psi \in X$.

Now take a convex τ^X_{lcv}-neighborhood U of $0 \in X$. Then, define a $\tau_{X_{2n}}$-neighborhood V_{2n} of $0 \in X_{2n}$ by induction on n in such a way that $T(V_{2n}, V_{2n}) \subset U \cap X_{2n}$ by the continuity of $T|_{X_{2n}\times X_{2n}}$, and that $\omega_j V_{2n} \subset V_{2j}$ for $1 \leq j < n$. Put $V_{2n-1} := V_{2n} \cap X_{2n-1}$ and $V = \mathrm{Conv}(\cup_{n\geq 1} V_{2n-1})$. Then, since $V \cap X_{2n-1} \supset V_{2n-1}$ for $n \geq 1$, V is a neighborhood of $0 \in X$ in $\tau^X_{lcv} := \lim_{\to} \tau_{X_{2n-1}}$ by Lemma 2.1.

We prove $T(V, V) \subset U$, which shows the continuity of T in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$. Take $\varphi, \psi \in V$. Then,

$$\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_m\varphi_m, \qquad \psi = \beta_1\psi_1 + \beta_2\psi_2 + \cdots + \beta_m\psi_m,$$

with $\alpha_j \geq 0, \sum_{1\leq j\leq m} \alpha_j = 1, \varphi_j \in V_{2j-1}$ and $\beta_k \geq 0, \sum_{1\leq k\leq m} \beta_k = 1, \psi_k \in V_{2k-1}$. Since

$$\varphi\psi = \sum_{j,k} \alpha_j\beta_k \cdot \varphi_j\psi_k, \qquad \sum_{j,k} \alpha_j\beta_k = 1,$$

it is enough for us to prove $\varphi_j\psi_k \in U$ for each j, k.

(a) In case $j = k$, $\varphi_j\psi_j = T(\varphi_j, \psi_j) \in T(V_{2j-1}, V_{2j-1}) \subset U \cap X_{2j} \subset U$.

(b) In case $j < k$, $\varphi_j\psi_k = \varphi_j(\omega_j\psi_k) \in V_{2j-1}V_{2j} \subset V_{2j}V_{2j} \subset U \cap X_{2j} \subset U$. □

2.2.2. Continuity of the convolution in $\mathcal{D}(\mathbf{R}^k)$

Proposition 2.3. *In the space of test functions $X = \mathcal{D}(\mathbf{R}^k)$, the convolution map $T(\varphi, \psi) = \varphi * \psi$ is continuous in $(\tau_{lcv}^X \times \tau_{lcv}^X, \tau_{lcv}^X)$.*

Proof. Let $M = \mathbf{R}^k$. For $n \geq 1$, put

$$M_n = \{x = (x_1, x_2, \ldots, x_k) \in M; |x| < n\} \quad \text{with} \quad |x| = \max_{1 \leq i \leq k} |x_i|,$$

and $X_n = \mathcal{D}(\overline{M_n})$. The convolution is given by

$$T(\varphi, \psi)(x) = \varphi * \psi(x) = \int_{\mathbf{R}^k} \varphi(x-y)\psi(y)dy$$

and maps $X_n \times X_m$ to X_{n+m}.

Take a convex, balanced, closed neighborhood W in τ_{lcv}^X of $0 \in X$. Then, for any $n \geq 1$, there exists a $\tau_{X_{2n}}$-neighborhood $U_{2n} \subset W \cap X_{2n}$ given as

$$U_{2n} = \{\varphi \in X_{2n}; \sup_{x \in M_{2n}} |D^s \varphi(x)| \leq \varepsilon, s = (s_1, s_2, \ldots, s_k), |s| \leq k_n\},$$

$$\text{with} \quad D^s = D_1^{s_1} D_2^{s_2} \cdots D_k^{s_k}, D_i = \frac{\partial}{\partial x_i}, |s| = s_1 + s_2 + \cdots + s_k.$$

Put, for $y \in M$, $\varphi_y(x) := \varphi(x-y)$. Then, for any φ in $U_{2n} \cap X_n$, a τ_{X_n}-neighborhood of $0 \in X_n$, we have

$$\varphi_y \in U_{2n} \subset W \cap X_{2n} \quad \text{for} \quad y \in M_n.$$

On the other hand, put

$$V_n := U_{2n} \cap X_n \cap \left\{\varphi \in X, \int_{\mathbf{R}^k} |\varphi(x)|dx < 1\right\}.$$

Then, V_n is a τ_{X_n}-neighborhood of $0 \in X_n$. By Lemma 2.1, $V = \mathrm{Conv}\left(\bigcup_{n \geq 1} V_n\right)$ is a τ_{lcv}^X-neighborhood of $0 \in X$. We assert that $T(V, V) \subset W$.

To prove this, take $\varphi, \psi \in V$. Then,

$$\varphi = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_m \varphi_m, \qquad \psi = \beta_1 \psi_1 + \beta_2 \psi_2 + \cdots + \beta_m \psi_m,$$

with $\alpha_j \geq 0, \sum_{1 \leq j \leq m} \alpha_j = 1, \varphi_j \in V_j$ and $\beta_k \geq 0, \sum_{1 \leq k \leq m} \beta_k = 1, \psi_k \in V_k$. Since

$$T(\varphi, \psi) = \varphi * \psi = \sum_{j,k} \alpha_j \beta_k \cdot \varphi_j * \psi_k, \qquad \sum_{j,k} \alpha_j \beta_k = 1,$$

and since W is convex, it is enough for us to prove $\varphi_j * \psi_k \in W$ for each j, k.

So, take $\varphi \in V_j, \psi \in V_k$ and assume $j \geq k$. Then,

$$T(\varphi, \psi) = \int_{|y| \leq k} \varphi_y(x)\psi(y)dy, \quad \int_{|y| \leq k} |\psi(y)|dy < 1,$$

and $\varphi_y \in W \cap X_{2j}$. The element $T(\varphi, \psi) \in X_{2j}$ can be approximated in the topology $\tau_{X_{2j}}$ by Riemann sums for the integral in y. Since the integral of $|\psi(y)|$ is less than 1, and since $W \cap X_{2j}$ is convex, balanced and closed, every Riemann sums belong to $W \cap X_{2j}$ and accordingly their limit function (in x) $T(\varphi, \psi)$ belongs also to $W \cap X_{2j}$.

This proves that $T(V, V) \subset W$ and so the continuity of the map T. □

2.2.3. General case

In the above two cases, multiplications T are both commutative, but in the above proofs the commutativity is not important but the special structure of the space $\mathcal{D}(M)$ is fully used. So, the proofs can not be generalized directly in the following general situation.

Problem E. *Assume that an inductive system $\{X_\alpha; \alpha \in A\}$ of LCTVSs has multiplications, consistent in the sense that, for any α, there exists a $\beta(\alpha)$ such that $T_\alpha : X_\alpha \times X_\alpha \to X_{\beta(\alpha)}$ is a continuous bilinear map, and that, for any $\alpha_1, \alpha_2 \in A$, there exists a $\gamma \in A$ such that $\gamma \succeq \alpha_j, \beta(\gamma) \succeq \beta(\alpha_j), j = 1, 2$, and T_{α_j}'s are naturally induced from T_γ. Then the system $\{T_\alpha\}$ induces as its inductive limit a multiplication T on $X = \lim_{\to} X_\alpha$.*

Is the limit map T continuous with respect to $\tau_{lcv}^X = \text{lcv-}\lim_{\to} \tau_{X_\alpha}$?

2.3. Multiplication map between two spaces of test functions

Let M and M' be two differentiable manifolds. We assume that at least one of them, say M', is non-compact.

The space of testing functions $X = \mathcal{D}(M)$ is equipped with a locally convex vector topology τ'_X, where $\tau'_X = \tau_X$ the usual C^∞-topology in the case M is compact, and $\tau'_X = \tau_{lcv}^X := \text{lcv-lim}_{\to} \tau_{X_n}$ with $X_n = \mathcal{D}(M_n)$ as above in the case M is non-compact. The space $Y = \mathcal{D}(M')$ is equipped with the lcv-limit topology $\tau_{lcv}^Y := \text{lcv-lim}_{\to} \tau_{Y_n}$ with $Y_n = \mathcal{D}(M'_n)$, where $\{M'_n, n = 1, 2, \ldots\}$ is a sequence of relatively compact open submanifolds such that $\overline{M'_n} \subset M'_{n+1}$ and $M' = \cup_{n \geq 1} M'_n$. We can give to the product space $X \times Y = \mathcal{D}(M) \times \mathcal{D}(M')$ the lcv-limit topology $\tau_{lcv}^{X \times Y}$ which is equal to $\text{lcv-lim}_{\to}(\tau_X \times \tau_{Y_n})$ if M is compact, and to $\text{lcv-lim}_{\to} (\tau_{X_n} \times \tau_{Y_n})$ if M is non-compact.

Now put $Z := \mathcal{D}(M \times M')$. Then, we ask if the multiplication (or product) map $T : X \times Y \to Z$, given as $T(\varphi, \psi)(p, p') = \varphi(p) \cdot \psi(p')$, $p \in M$, $p' \in M'$, for $\varphi \in X$, $\psi \in Y$, is continuous with respect to $(\tau'_X \times \tau_{lcv}^Y, \tau_{lcv}^Z)$.

This time, the answer is definitely no, as is seen from the following.

Theorem 2.4. *Let M and M' be two differentiable manifolds. Assume that one of them, say M', is non-compact. Then, the multiplication map T : $\mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto \varphi \cdot \psi \in \mathcal{D}(M \times M')$ is not continuous in $(\tau'_X \times \tau_{lcv}^Y, \tau_{lcv}^Z)$, where $X = \mathcal{D}(M)$, $Y = \mathcal{D}(M')$, $Z = \mathcal{D}(M \times M')$, and $\tau'_X = \tau_X$ or $\tau'_X = \tau_{lcv}^X$ according as M is compact or not.*

Proof. Let us prove the discontinuity in $(\tau'_X \times \tau_{lcv}^Y, \tau_{lcv}^Z)$. To give a common proof irrespective of whether M is compact or not, we consider even when

M is compact a sequence of submanifolds $M_1 \subset M_2 \subset \cdots$ such that $M_n = M$ for $n \geq 2$.

Let $k = \dim M$ and consider ${I_1}^k$ with $I_1 = (-1, 1) \subset \mathbf{R}$, open interval. We choose an open submanifold $M_1 \subset M$ contained in a co-ordinate neighborhood in such a way that, through an appropriate co-ordinate map, M_1 is mapped onto ${I_1}^k$ and its closure $\overline{M_1}$ is mapped onto ${J_1}^k$, where $J_1 = [-1, 1] \subset \mathbf{R}$, the closure of $I_1 = (-1, 1)$. For simplicity, we identify M_1 with ${I_1}^k$ and $\overline{M_1}$ with ${J_1}^k$. Take a sequence of open submanifolds $M_n, n \geq 2$, $M_n \nearrow M$, appropriately in case M is non-compact, and put $(M \times M')_n = M_n \times M'_n$. Then, we are now given $X_n = \mathcal{D}(\overline{M_n}), Y_n = \mathcal{D}(\overline{M'_n})$ and $Z_n = \mathcal{D}(\overline{M_n} \times \overline{M'_n})$, noting $\overline{(M \times M')_n} = \overline{M_n} \times \overline{M'_n}$. The topologies τ^X_{lcv}, τ^Y_{lcv} and τ^Z_{lcv} are respectively defined by these inductive sequences. Note that when M is compact, the inductive sequence $X_n, n = 1, 2, \ldots$, is superfluous and $X_n = X$ and $\tau_{X_n} = \tau_X$ for $n \geq 2$ and τ^X_{lcv} is nothing but τ_X.

Put $E_j = (\overline{M_1} \times \overline{M'_{j+1}}) \setminus (\overline{M_1} \times \overline{M'_j}) \subset M \times M'$, and consider a subset W of $Z = \mathcal{D}(M \times M')$ given by

$$W := \{\omega \in Z; \sup_{(x,y)\in E_j} |D_1^{j+1}\omega(x, y)| < 1 \quad (j = 1, 2, 3, \ldots)\} \quad \text{with} \quad D_1 = \frac{\partial}{\partial x_1},$$

where $x = (x_1, x_2, \ldots , x_k) \in \overline{M_1} = {J_1}^k, y \in M'$. Then, W is convex, and for any n, $W \cap Z_n$ is τ_{Z_n}-open. Hence, W is a τ^Z_{lcv}-open neighborhood of $0 \in Z$.

Assume that the map $T : X \times Y \to Z$ is $(\tau^X_{lcv} \times \tau^Y_{lcv}, \tau^Z_{lcv})$-continuous. Then there exists a τ^X_{lcv}-neighborhood U of $0 \in X$ and a τ^Y_{lcv}-neighborhood V of $0 \in Y$ such that $T(U, V) \subset W$. For $U \cap X_1$ with $X_1 = \mathcal{D}(J_1^k)$, there exist an $m \in \mathbf{N}$ and an $\epsilon > 0$ such that, for an element $\phi \in X_1 = \mathcal{D}(J_1^k)$, the condition

$$\sup_{x\in J_1^k} |{D_1}^{s_1} D_2^{s_2} \cdots D_k^{s_k} \phi(x)| < \epsilon \quad (0 \leq s_i \leq m\,(1 \leq i \leq k)) \quad \text{with} \quad D_j = \frac{\partial}{\partial x_j}$$

implies that $\phi \in U \cap X_1$. For this m, consider $V \cap Y_{m+1}$ with $Y_{m+1} = \mathcal{D}(\overline{M'_{m+1}})$, then for some $\psi \in V \cap Y_{m+1}$,

$$\eta := \sup\left\{|\psi(y)|; y \in \overline{M'_{m+1}} \setminus \overline{M'_m}\right\} > 0.$$

Thus we get, for any $\phi \in X_1$ satisfying the above condition, the evaluation

$$\eta \cdot \sup_{-1\leq x_1\leq 1} |{D_1}^{m+1}\phi(x)| \leq \sup_{(x,y)\in E_m} |{D_1}^{m+1} \{\phi(x)\psi(y)\}| < 1.$$

This implies that, for any $\phi \in X_1 = \mathcal{D}(J_1^k)$, there holds the following inequality

$$\sup_{-1\leq x_1\leq 1} |{D_1}^{m+1}\phi(x)| \leq \frac{1}{\epsilon\eta} \cdot \max_{0\leq s_i\leq m(1\leq i\leq k)} \sup_{x\in J_1^k} |{D_1}^{s_1} {D_2}^{s_2} \cdots {D_k}^{s_k} \phi(x)|,$$

for any $x \in \mathrm{supp}(\phi) \subset {J_1}^k$. This is clearly not true.

This means that T is not continuous in $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$. □

Taking into account Propositions 2.2, 2.3 and Theorem 2.4, we propose the following problem.

Problem F. *Take three inductive systems of LCTVSs* $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$, $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$, *and* $\{(Z_\alpha, \tau_{Z_\alpha}); \alpha \in A\}$, *and let their inductive limits be* (X, τ^X_{lcv}), (Y, τ^Y_{lcv}) *and* (Z, τ^Z_{lcv}). *Assume that, for every* $\alpha \in A$, *there exists a continuous multiplication (bilinear map)* $T_\alpha : X_\alpha \times Y_\alpha \to Z_{\beta(\alpha)}$ *with a* $\beta(\alpha) \succeq \alpha$, *which are consistent with these inductive systems so that there exists a multiplication* $T : X \times Y \to Z$ *as their inductive limit. Then, under what conditions,* T *is continuous in* $(\tau^X_{lcv} \times \tau^Y_{lcv}, \tau^Z_{lcv})$*?*

The bilinear map $T : X \times Y \to Z$ is factored through the natural map $X \times Y \to X \otimes Y$, and similarly for T_α's. So we can ask also about relations with topologies on the tensored spaces $X \otimes Y$ and $X_\alpha \otimes Y_\alpha$.

Remark 2.1. In comparison to the so-called kernel theorem for distributions (cf. [10, Theorem 51.7]), we give some remarks. In the situation in Theorem 2.4 with M' non-compact, take a distribution S on $M \times M'$ or $S \in \mathcal{D}'(M \times M')$. Then the bilinear functional $\mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto S(T(\varphi, \psi))$ is not necessarily continuous in the product topology, because so is not the bilinear map $T : \mathcal{D}(M) \times \mathcal{D}(M') \to \mathcal{D}(M \times M')$.

By the same reason, the natural imbedding map from $\mathcal{D}(M) \bigotimes_\pi \mathcal{D}(M')$ ($\cong \mathcal{D}(M) \bigotimes_\varepsilon \mathcal{D}(M')$ thanks to the nuclearity) to $\mathcal{D}(M \times M')$ is not continuous. However, the ϵ-topology on this imbedded subspace, which coincides with π-topology due to the nuclearity, is strictly weaker than the restricted topology τ from $\mathcal{D}(M \times M')$, as is easily seen. Nevertheless its completion $\mathcal{D}(M) \widehat{\bigotimes}_\epsilon \mathcal{D}(M')$ $= \mathcal{D}(M) \widehat{\bigotimes}_\pi \mathcal{D}(M')$ is just equal to $\mathcal{D}(M \times M')$ as a space (cf. [4, Article 125]).

2.4. Spaces of finitely many times differentiable functions

Let r be a non-negative integer and M' is a non-compact $C^{(r)}$-class differentiable manifold. Let us consider the space $Y = C^{(r)}_c(M')$ of $C^{(r)}$-class functions with compact supports. For $r = 0$, Y is nothing but the space of continuous functions with compact supports.

To topologize Y, we take a locally finite covering $\{U_j; 1 \le j < \infty\}$ of Y such that the closure $\overline{U_j}$ of every U_j is contained in a coordinate neighborhood. For each U_j, identifying it with the corresponding domain of coordinates $y = (y_1, y_2, \ldots, y_\ell), \ell = \dim M'$, we define a seminorm $\rho_{U_j}(h)$ for $h \in Y$ by

$$\rho_{U_j}(h) := \sum_{|t| \le r} \sup_{y \in U_j} |D^t h(y)|,$$

where $t = (t_1, t_2, \ldots, t_\ell), |t| = t_1 + t_2 + \cdots + t_\ell$, and

$$D^t = {D_1}^{t_1} {D_2}^{t_2} \cdots {D_\ell}^{t_\ell} \qquad \text{with} \quad D_i = \frac{\partial}{\partial y_i}.$$

Moreover put for $h \in Y$,

$$\|h\| := \sum_{1 \le j < \infty} \rho_{U_j}(h).$$

Then, the summation on the right hand side is actually finite, and $\|h\|$ gives a norm on the space Y. The topology on Y defined by this norm is denoted by $\tau^{Y}_{\|\cdot\|}$.

Now take a sequence of open, relatively compact submanifolds $M'_1 \subset M'_2 \subset \cdots$ as above. For the subspace $Y_n = C^{(r)}(\overline{M'_n})$, we restrict the norm $\|\cdot\|$ on it, then Y_n becomes a Banach space. Denote by τ_{Y_n} its topology and consider the lcv-limit $\tau^{Y}_{lcv} :=$ lcv-$\lim_{\to} \tau_{Y_n}$ on Y. Then τ^{Y}_{lcv} does not depend on the choices of $\{U_j\}$ and $\{M'_n\}$, and it is strictly stronger than the norm topology $\tau^{Y}_{\|\cdot\|}$ because M' is non-compact.

Let $Z = C^{(\infty,r)}_c(M \times M')$ be the space of functions $f(x, y)$ in $(x, y) \in M \times M'$, which is simultaneously of class $C^{(\infty)}$ in $x \in M$ and of class $C^{(r)}$ in $y \in M'$, and compactly supported. We can topologize it in two ways.

The first way is to utilise sequences $(M \times M')_n = M_n \times M'_n$ and $Z_n = C^{(\infty,r)}(\overline{M_n} \times \overline{M'_n})$ to get the lcv-limit $\tau^{Z}_{lcv} := \lim_{\to} \tau_{Z_n}$.

The second way is to utilize sequences $M_n \times M'$, non-compact, and $Z'_n = C^{(\infty,r)}_c(\overline{M_n} \times M')$. We equip Z'_n the topology $\tau_{Z'_n}$ given by the usual way in the variable $x \in M_n$ and by the norm $\|\cdot\|$ in the variable $y \in M'$ using as above a locally finite covering $U_j, 1 \le j < \infty$, of M' and evaluating derivatives in y up to the degree r. Then we get another lcv-limit $\tau^{Z}_{lcv,r} :=$ lcv-$\lim_{\to} \tau_{Z'_n}$.

On the space Z, the first topology is strictly stronger than the second one: $\tau^{Z}_{lcv,r} \prec \tau^{Z}_{lcv}$.

For the continuity of the multiplication map $T : X \times Y \to Z$, we can choose two kinds of topologies both on Y and on Z. However, for any choice of topologies, the map T is not continuous, as stated in the following theorem. A proof of it can be given by word for word interpretation of the proof of Theorem 2.4.

Theorem 2.5. *Let M be a differentiable manifold and M' be a non-compact $C^{(r)}$-class manifold for some r, $0 \le r < \infty$. Put $X = \mathcal{D}(M), Y = C^{(r)}_c(M')$ and $Z = C^{(\infty,r)}_c(M \times M')$. Then, the multiplication map $T : X \times Y \ni (\varphi, \psi) \mapsto \varphi \cdot \phi \in Z$ is not continuous in $(\tau'_X \times \tau^{Y}_{lcv}, \tau^{Z}_{lcv,r})$, where $\tau'_X = \tau_X$ if M is compact, and $\tau'_X = \tau^{X}_{lcv}$ if M is non-compact.*

Remark 2.2. In general, take two inductive systems of LCTVSs $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ and put $X = \lim_{\to} X_\alpha, Y = \lim_{\to} Y_\alpha$. The direct product of these systems is defined as $\{(X_\alpha \times Y_\alpha, \tau_{X_\alpha \times Y_\alpha}), \alpha \in A\}$ with $\tau_{X_\alpha \times Y_\alpha} = \tau_{X_\alpha} \times \tau_{Y_\alpha}$. Then its inductive limit is isomorphic to the direct product $X \times Y$ as vector spaces, and as topologies on this space, $\tau^{X}_{lcv} \times \tau^{Y}_{lcv} \preceq \tau^{X \times Y}_{lcv} :=$ lcv-$\lim_{\to} \tau_{X_\alpha \times Y_\alpha}$.

Actually these two locally convex vector topologies on $X \times Y$ are mutually equivalent as will be proved in Theorem 3.4.

3. Bamboo-Shoot topology τ_{BS}^{G} and locally convex topology τ_{lcv}^{X}

3.1. Bamboo-Shoot topology for PTA-groups

For an inductive system of topological groups $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$, assume that the index set A is cofinal to a sub-directed-set isomorphic to $\mathbf{N}$. Then we introduced in [9, Section 2] a condition called (PTA), and under this condition, we defined the so-called Bamboo-Shoot topology τ_{BS}^{G} on $G = \lim_{\rightarrow} G_\alpha$, and proved that it is the strongest one among group topologies weaker than or equal to the inductive limit topology τ_{ind}^{G} on G.

For later use, we refer these things here. Assume that $A = \mathbf{N}$. For an inductive sytem $\{(G_n, \tau_{G_n}); n \in \mathbf{N}\}$, let $\phi_n : G_n \to G$ be the canonical homomorphisms. Then, we define

Condition (PTA). *Fix an $n \in \mathbf{N}$. For any τ_{G_n}-neighborhood U of the identity element e_n of G_n, there exists a τ_{G_n}-neighborhood $V \subset U$ of $e_n \in G_n$, symmetric (i.e., $V^{-1} = V$), and satisfying that, for any $m > n$, and for any τ_{G_m}-neighborhood W_m of $e_m \in G_m$, there exists a τ_{G_m}-neighborhood W'_m of $e_m \in G_m$ such that $\phi_m(W'_m)\phi_n(V) \subset \phi_n(V)\phi_m(W_m)$.*

In the case where the condition (PTA) holds, the Bamboo-Shoot topology τ_{BS}^{G} on $G = \lim_{\rightarrow} G_n$ is defined as the group topology for which a fundamental system of neighborhoods of the identity element $e \in G$ is given by the family of subsets of the following form: for a system $\{U_j; j \in \mathbf{N}\}$ of τ_{G_j}-neighborhood of $e_j \in G_j$, put for $k \in \mathbf{N}$,

$$U[k] := \bigcup_{n \geq k} U(n,k) \qquad \text{with}$$

$$U(n,k) := \phi_n(U_n)\phi_{n-1}(U_{n-1}) \cdots \phi_k(U_k)\phi_k(U_k)\phi_{k+1}(U_{k+1}) \cdots \phi_n(U_n) \ (n \geq k).$$

3.2. Bamboo-Shoot topology and locally convex topology

The group topology τ_{BS}^{G} has an intimate relation to the locally convex vector topology τ_{lcv}^{X} as in the following problem.

Problem G. *Let $\{(X_n, \|\cdot\|_n); n \in \mathbf{N}\}$ be an inductive system of Banach algebras. Then $X = \lim_{\rightarrow} X_n$ has naturally a structure of algebra. Take an inductive system of topological subgroups G_n of $(X_n^{\times}, \tau_{X_n^{\times}})$ the group of all invertible elements in X_n, with the restriction $\tau_{X_n^{\times}}$ of $\|\cdot\|_n$-topology on $X_n^{\times}$. In the case where the condition* (PTA) *holds, what is the relation between the Bamboo-Shoot topology τ_{BS}^{G} on $G = \lim_{\rightarrow} G_n$ and the restriction $\tau_{lcv}^{X}|_G$ onto G of the locally convex vector topology τ_{lcv}^{X}?*

In [11], A. Yamasaki studied the following two cases.

Case 1. $X_n = M(n, \mathbf{C})$, the algebra of all $n \times n$ matrices over $F = \mathbf{C}$, and $G_n = GL(n, \mathbf{C})$. Their limit is $G = GL(\infty, \mathbf{C})$.

Case 2. $X_n = M(n,\Lambda)$ with $\Lambda = C(D,\mathbf{C})$, the algebra of all $\mathbf{C}$-valued continuous functions on a compact space D, and $G_n = GL(n,\Lambda)$. Then, their limit is $G = GL(\infty,\Lambda)$.

In both cases, the algebras X_n are Banach algebras and all the homomorphism $\phi_{m,n}$ are norm-preserving isomorphisms. According to his results,

in *Case* 1, $\tau_{BS}^G = \tau_{ind}^G$ and it coincides with the restriction $\tau_{lcv}^X|_G$, and

in *Case* 2, $\tau_{BS}^G \prec \tau_{ind}^G$ and $\tau_{BS}^G = \tau_{lcv}^X|_G$. Further it is proved that (G,τ_{BS}^G) is isomorphic to $C(D, GL(\infty,\mathbf{C}))$ as topological groups.

Recently, T. Edamatsu [3] studied the general case where all the $\phi_{m,n}$ are norm-preserving isomorphisms, and proved that these two kinds of topologies coincide with each other: $\tau_{BS}^G = \tau_{lcv}^X|_G$, for $G_n = {X_n}^\times, n \geq 1$.

Slightly generalizing the above problem, we can propose the following one. Let $\{(X_n,\tau_{X_n}); n \in \mathbf{N}\}$ be an inductive system of topological algebras. Then $X = \lim_{\rightarrow} X_n$ has naturally an algebra structure.

Problem H. *Assume that every (X_n,τ_{X_n}) is locally convex as a TVS. Then, with the locally convex limit topology τ_{lcv}^X, does the algebra X become a topological algebra?*

Furthermore, let $G_n := {X_n}^\times$ be the set of all invertible elements in X_n. Then, G_n is a topological group with the relative topology $\tau_{G_n} := \tau_{X_n}|_{G_n}$, and they form an inductive system of topological groups. Then, under the condition (PTA)*, what is the relation between the Bamboo-Shoot topology τ_{BS}^G on G and the restriction $\tau_{lcv}^X|_G$ onto G of the locally convex limit topology τ_{lcv}^X on X?*

Here, since the scalar multiplication for X is continuous with respect to τ_{lcv}^X, the problem is reduced to the continuity of the product map: $X \times X \ni (x_1,x_2) \mapsto x_1x_2 \in X$.

We also remark here that studies in different directions on inifinite dimensional Lie groups, containing the theory of their representations, are continued for example in [2] and in [8].

3.3. Coincidense of topologies $\tau_{BS}^{(X,+)}$ and τ_{lcv}^X

For an inductive limit space $X = \lim_{\rightarrow} X_\alpha$, in the category of LCTVSs, assume that the index set A has a countable cofinal subset, then A contains a cofinal subset isomorphic to $\mathbf{N}$, and replacing A by the latter one, we may assume from the beginning $A = \mathbf{N}$ (cf. Remark 4.1).

In that case, we have another limit topology $\tau_{BS}^{(X,+)}$ other than τ_{lcv}^X, when every $X_n, n \in \mathbf{N}$, is considered as an additive group forgetting scalar multiplication. In fact, since X_n's are abelian groups, the condition (PTA) holds automatically, and so the Bamboo-Shoot topology $\tau_{BS}^{(X,+)}$ can be defined for this additive group structure.

Proposition 3.1. *For a countable inductive system of LCTVSs, there holds* $\tau_{BS}^{(X,+)} = \tau_{lcv}^{X}$ *on the limit space* $X = \lim_{\rightarrow} X_n$.

Proof. Note that $\tau_{BS}^{(X,+)}$ is characterized as the strongest group topology among those $\preceq \tau_{ind}^{X}$, and that the addition is continuous with respect to τ_{lcv}^{X}. Then, we have $\tau_{lcv}^{X} \preceq \tau_{BS}^{(X,+)}$.

On the other hand, we see from the definition that a base of $\tau_{BS}^{(X,+)}$-neighborhood of $0 \in X$ is given by the family of sets of the form $\bigcup_{n\geq 1}(U_1 + U_2 + \cdots + U_n)$, where every U_n runs over open convex τ_{X_n}-neighborhoods of $0 \in X_n$. So they are all τ_{ind}^{X}-open and convex, and accordingly τ_{lcv}^{X}-open, whence $\tau_{lcv}^{X} \succeq \tau_{BS}^{(X,+)}$. □

3.4. Extension of Bamboo-Shoot topologies and their products

In the category of topological groups, we can extend in an abstract way the notion of Bamboo-Shoot topology on an inductive limit group $G = \lim_{\rightarrow} G_\alpha$ for any (not necessarily countable) inductive system $\{(G_\alpha, \tau_{G_\alpha}),\ \alpha \in A; \phi_{\beta,\alpha}, \alpha \preceq \beta\}$.

In fact, we see easily from axioms of neighborhood system of the unit element for a topological group (e.g., (GT1) $\sim$ (GT5) in [9, Section 1.3]) that there exists, on an inductive limit group $G = \lim_{\rightarrow} G_\alpha$, the strongest group topology under the condition that every canonical homomorphism $\phi_\alpha : G_\alpha \to G$ is continuous. We call it the *extended Bamboo-Shoot topology* and denote it again by τ_{BS}^{G}.

In the case where the inductive system is countable and the condition (PTA) holds for it, this topology coincides with the Bamboo-Shoot topology τ_{BS}^{G} constructed explicitly in [9], and reviewed in Section 3.1.

Lemma 3.2. *Let* (K, τ_K) *be a topological group, and take a system of homomorphisms* $\Psi_\alpha : G_\alpha \to K$*, consistent in the sense that* $\Psi_\beta \circ \phi_{\beta,\alpha} = \Psi_\alpha$ *for* $\alpha \preceq \beta$*. Then, a homomorphism* $\Psi : G \to K$ *is canonically induced. If every* Ψ_α *is continuous in* $(\tau_{G_\alpha}, \tau_K)$*, then,* Ψ *is continuous in* (τ_{BS}^{G}, τ_K)*.*

Furthermore the extended Bamboo-Shoot topology τ_{BS}^{G} *is the strongest group topology* $\preceq \tau_{ind}^{G}$ *on* $G = \lim_{\rightarrow} G_\alpha$ *having this property.*

In the category of topological groups, the problem similar to Problem A is affirmatively solved as follows.

Let $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$ and $\{(H_\alpha, \tau_{H_\alpha}); \alpha \in A\}$ be inductive systems of topological groups. Let $G = \lim_{\rightarrow} G_\alpha$ and $H = \lim_{\rightarrow} H_\alpha$ be their inductive limit groups, and the canonical homomorphisms be $\phi_\alpha : G_\alpha \to G$ and $\psi_\alpha : H_\alpha \to H$.

Then, we have the direct product of inductive systems as $\{(G_\alpha \times H_\alpha, \tau_{G_\alpha \times H_\alpha}); \alpha \in A\}$ with $\tau_{G_\alpha \times H_\alpha} = \tau_{G_\alpha} \times \tau_{H_\alpha}$. Its inductive limit is canonically identified with the direct product $G \times H$.

Theorem 3.3. (i) *Let* $G = \lim_{\to} G_\alpha$, $H = \lim_{\to} H_\alpha$, *and* $G \times H = \lim_{\to}(G_\alpha \times H_\alpha)$ *be as above. Then the extended Bamboo-Shoot topologies* τ_{BS}^G, τ_{BS}^H, *and* $\tau_{BS}^{G\times H}$ *on* G, H, *and* $G \times H$ *respectively satisfy*

$$\tau_{BS}^G \times \tau_{BS}^H \cong \tau_{BS}^{G\times H} \qquad \text{on} \quad G \times H.$$

(ii) *In the case of countable inductive systems, if* $\{(G_n, \tau_{G_n}); n \in \mathbf{N}\}$ *and* $\{(H_n, \tau_{H_n}); n \in \mathbf{N}\}$ *satisfy the condition* (PTA)*, then so does their direct product* $\{(G_n \times H_n, \tau_{G_n\times H_n});\ n \in \mathbf{N}\}$.

Proof. (i) The relation $\tau_{BS}^G \times \tau_{BS}^H \preceq \tau_{BS}^{G\times H}$ follows from the facts that $\tau_{BS}^G \times \tau_{BS}^H$ gives a group topology on $G \times H$ and that $\tau_{BS}^G \times \tau_{BS}^H \preceq \tau_{ind}^G \times \tau_{ind}^H \preceq \tau_{ind}^{G\times H}$, because the extended Bamboo-Shoot topology $\tau_{BS}^{G\times H}$ is the strongest group topology which is weaker than or equal to the inductive limit topology $\tau_{ind}^{G\times H}$.

Let us prove the converse relation. We assert that the homomorphism $G \ni g \mapsto (g, e_H) \in G \times H$ is continuous in $(\tau_{BS}^G, \tau_{BS}^{G\times H})$, where e_H denote the unit element of H.

To prove this, we apply Lemma 3.2 for $(K, \tau_K) = (G\times H, \tau_{BS}^{G\times H})$. Consider, for each $\alpha \in A$, the homomorphism $G_\alpha \ni g_\alpha \mapsto (\phi_\alpha(g_\alpha), e_H) \in K = G \times H$. Then, it is continuous in τ_{G_α} and $\tau_K = \tau_{BS}^{G\times H}$, because, by definition of the extended Bamboo-Shoot topology $\tau_{BS}^{G\times H}$, the canonical map $G_\alpha \times H_\alpha \ni (g_\alpha, h_\alpha) \mapsto (\phi_\alpha(g_\alpha), \psi_\alpha(h_\alpha)) \in G \times H$ is continuous in $(\tau_{G_\alpha\times H_\alpha}, \tau_{BS}^{G\times H})$, and the imbedding map $G_\alpha \ni g_\alpha \mapsto (g_\alpha, e_{H_\alpha}) \in G_\alpha \times H_\alpha$ is of course continuous. Therefore, by Lemma 3.2, we get the asserted continuity.

Similarly the homomorphism $H \ni h \mapsto (e_G, h) \in G \times H$ is continuous in $(\tau_{BS}^H, \tau_{BS}^{G\times H})$. Therefore, the map

$$\Phi : G \times H \ni (g, h) \longmapsto ((g, e_H), (e_G, h)) \in (G \times H) \times (G \times H)$$

is continuous in $\tau_{BS}^G \times \tau_{BS}^H$ and $\tau_{BS}^{G\times H} \times \tau_{BS}^{G\times H}$. Since the extended Bamboo-Shoot topology $\tau_{BS}^{G\times H}$ on $G \times H$ is a group topology, the product map

$$\Psi : (G \times H) \times (G \times H) \ni ((g, h), (g', h')) \longmapsto (gg', hh') \in (G \times H)$$

is continuous in $\tau_{BS}^{G\times H} \times \tau_{BS}^{G\times H}$ and $\tau_{BS}^{G\times H}$. Thus the product of maps $\Psi \cdot \Phi : G \times H \ni (g, h) \mapsto (g, h) \in G \times H$ is continuous in $\tau_{BS}^G \times \tau_{BS}^H$ and $\tau_{BS}^{G\times H}$.

This means that the former topology is stronger or equal to the latter one.

(ii) Fix $n \geq 1$, and take a $\tau_{G_n\times H_n}$-neighborhood W of the unit element $(e_{G_n}, e_{H_n}) \in G_n \times H_n$. Then there exist symmetric τ_{G_n}-neighborhood U of e_{G_n} and τ_{H_n}-neighborhood V of e_{H_n} such that $U \times V \subset W$ and that they satisfy the following condition (by assumption). For any $m > n$, and for any τ_{G_m}-neighborhood U' of e_{G_m} and any τ_{H_m}-neighborhood V' of e_{H_m}, there exist such ones U'' and V'' for which there hold

$$\phi_m(U'')\phi_n(U) \subset \phi_n(U)\phi_m(U') \quad \text{and} \quad \psi_m(V'')\psi_n(V) \subset \psi_n(V)\psi_m(V').$$

Hence,

$$(\phi_m \times \psi_m)(U'' \times V'') \cdot (\phi_n \times \psi_n)(U \times V) \\ \subset (\phi_n \times \psi_n)(U \times V) \cdot (\phi_m \times \psi_m)(U' \times V').$$

Since the family of neighborhoods of $G_m \times H_m$ of the form $U' \times V'$ forms a fundamental basis of neighborhoods of unit element, the above relation proves that the condition (PTA) holds for $G \times H$ or more exactly for the direct product of inductive systems. □

3.5. Direct product of locally convex vector topology

Let $\{(X_\alpha, \tau_{X_\alpha});\ \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ be inductive systems of LCTVSs, and put $X = \lim_\rightarrow X_\alpha, Y = \lim_\rightarrow Y_\alpha$. The direct product of these systems is defined as $\{(X_\alpha \times Y_\alpha, \tau_{X_\alpha \times Y_\alpha}); \alpha \in A\}$ with $\tau_{X_\alpha \times Y_\alpha} := \tau_{X_\alpha} \times \tau_{Y_\alpha}$. Then its inductive limit is isomorphic to the direct product $X \times Y$ as vector spaces. For topologies on this space, we already know that $\tau_{lcv}^X \times \tau_{lcv}^Y \preceq \tau_{lcv}^{X \times Y} :=$ lcv-$\lim_\rightarrow \tau_{X_\alpha \times Y_\alpha}$.

On the other hand, we have a variant of Lemma 3.2, in the category of LCTVSs, and applying it similarly as Lemma 3.2 to the proof of Theorem 3.3, we see that the condition (DPA) holds in general for the 'lcv-limit functor' $\tau_{lcv}^{\{*\}}$ as follows.

Theorem 3.4. *Let $X = \lim_\rightarrow X_\alpha, Y = \lim_\rightarrow Y_\alpha$ be inductive limits in the category of LCTVSs. The direct product space $X \times Y$ is identified with the inductive limit of the direct product of inductive systems. Then, as locally convex vector topologies on $X \times Y$, there holds the equivalence*

$$\tau_{lcv}^X \times \tau_{lcv}^Y \cong \tau_{lcv}^{X \times Y} := \text{lcv-}\lim_{\rightarrow} \tau_{X_\alpha \times Y_\alpha}.$$

Proof. It is enough to prove the converse relation $\tau_{lcv}^X \times \tau_{lcv}^Y \succeq \tau_{lcv}^{X \times Y}$.

The linear map $\Phi^X : X \ni x \mapsto (x, 0) \in X \times Y$ is continuous in τ_{lcv}^X and $\tau_{lcv}^{X \times Y}$. This can be shown by applying a variant of Lemma 3.2 in the category of LCTVSs to $(K, \tau_K) = (X \times Y, \tau_{lcv}^{X \times Y})$.

In fact, discussing as in the proof of Theorem 3.3, we see that, for each $\alpha \in A$, the corresponding map: $X_\alpha \ni x_\alpha \mapsto (\phi_\alpha(x_\alpha), 0) \in K = X \times Y$ is continuous in τ_{X_α} and $\tau_K = \tau_{lcv}^{X \times Y}$. Then the desired continuity follows from the variant of Lemma 3.2.

Similar for the linear map $\Phi^Y : Y \ni y \mapsto (0, y) \in X \times Y$.

So the map $\Phi : X \times Y \ni (x, y) \longmapsto ((x, 0), (0, y)) \in (X \times Y) \times (X \times Y)$ is continuous in $\tau_{lcv}^X \times \tau_{lcv}^Y$ and $\tau_{lcv}^{X \times Y} \times \tau_{lcv}^{X \times Y}$. On the other hand, the addition $\Psi : (X \times Y) \times (X \times Y) \ni ((x, y), (x', y')) \longmapsto (x + x', y + y') \in (X \times Y)$ is naturally continuous in $\tau_{lcv}^{X \times Y} \times \tau_{lcv}^{X \times Y}$ and $\tau_{lcv}^{X \times Y}$.

Thus, we see that the product of maps $\Psi \cdot \Phi : (x, y) \mapsto (x, y)$ is continuous in $\tau_{lcv}^X \times \tau_{lcv}^Y$ and $\tau_{lcv}^{X \times Y}$. This means that the former topology is stronger or equal to the latter one, on $X \times Y$. This is to be proved. □

4. Sufficient conditions for Problem A

For sufficient conditions for Problem A or B, the local compactness and the local sequential compactness play important roles. Here we study them for Problem A.

4.1. A sufficient condition for $\tau_{ind}^{X} \times \tau_{ind}^{Y} \simeq \tau_{ind}^{X\times Y}$

As in Section 1.4, let

$$\begin{aligned} &\{(X_\alpha, \tau_{X_\alpha}), \alpha \in A; \phi_{\beta,\alpha}, \alpha \preceq \beta\} \quad \text{and} \\ &\{(Y_\alpha, \tau_{Y_\alpha}), \alpha \in A; \psi_{\beta,\alpha}, \alpha \preceq \beta\} \end{aligned} \tag{4.1}$$

be inductive systems of topological spaces and put $X = \lim_{\to} X_\alpha, Y = \lim_{\to} Y_\alpha$. First let us give a simple sufficient condition for the 'commutativity' of (1) taking inductive limits and (2) taking direct products, for inductive limits of topologies. When this commutativity holds, we say that the condition (DPA) (= *Direct Product is Admitted*) holds in this case.

Theorem 4.1. *Assume that A has a cofinal sub-directed-set isomorphic to* $\mathbf{N}$. *For two inductive systems of topological spaces in* (4.1), *assume that every X_α and Y_α are locally compact Hausdorff spaces. Then, as topologies on $X \times Y$ with $X = \lim_{\to} X_\alpha, Y = \lim_{\to} Y_\alpha$, identified with $\lim_{\to}(X_\alpha \times Y_\alpha)$, the product topology $\tau_{ind}^{X} \times \tau_{ind}^{Y}$ and the inductive limit topology $\tau_{ind}^{X\times Y} := \lim_{\to}(\tau_{X_\alpha} \times \tau_{Y_\alpha})$ are mutually equivalent*: $\tau_{ind}^{X} \times \tau_{ind}^{Y} \cong \tau_{ind}^{X\times Y}$, *that is, the condition* (DPA) *holds.*

Proof. By assumption, we may assume that $A = \mathbf{N}$ as directed set. Since $\tau_{ind}^{X} \times \tau_{ind}^{Y} \preceq \tau_{ind}^{X\times Y}$ in general, it is sufficient for us to prove the converse relation.

Take a point $(x, y) \in X \times Y$ and its $\tau_{ind}^{X\times Y}$-open neighborhood O. We may assume, for simplicity that the canonical maps $\phi_n : X_n \to X$ and $\psi_n : Y_n \to Y$ are injective, and consider X_n as a subset of X through ϕ_n, and similarly for $Y_n \subset Y$. Starting from a certain $n = n_0$, we have $x \in X_n$ and $y \in Y_n$. Put $O_n = (X_n \times Y_n) \cap O$, then it is a $(\tau_{X_n} \times \tau_{Y_n})$-open set containing (x, y). Therefore there exist a τ_{X_n}-open, relatively compact $U_n \subset X_n$ and a τ_{Y_n}-open, relatively compact $V_n \subset Y_n$ such that $x \in U_n, y \in V_n$ and $U_n \times V_n \subset O_n$. Denote by $\overline{U_n}$ the τ_{X_n}-closure of U_n in X_n, then it is equal to the closure in $(X_m, \tau_{X_m}), m > n$, and also in (X, τ_{ind}^{X}), because of its compactness. Similar for the τ_{Y_n}-closure $\overline{V_n}$. We assert that the sequences $\{U_n\}$ and $\{V_n\}$ can be taken as $\overline{U_n} \subset U_{n+1}, \overline{V_n} \subset V_{n+1}$. To see this, we construct them by induction on n applying repeatedly the following elementary lemma. Then, putting $U = \bigcup_{n\geq n_0} U_n$ and $V = \bigcup_{n\geq n_0} V_n$, we get $(\tau_{ind}^{X} \times \tau_{ind}^{Y})$-open neighborhood $U \times V$ of (x, y) contained in O. This proves that $\tau_{ind}^{X} \times \tau_{ind}^{Y} \succeq \tau_{ind}^{X\times Y}$. □

Lemma 4.2. *Let (Z, τ) and (Z', τ') be two locally compact Hausdorff topological spaces with topologies τ and τ'. Let $O \subset Z \times Z'$ be a $(\tau \times \tau')$-open* *subset and $C \subset Z$ and $C' \subset Z'$ be respectively τ-compact and τ'-compact such that $C \times C' \subset O$. Then, there exist a τ-open $D \supset C$ and a τ'-open $D' \supset C'$ such that they are relatively compact and $\overline{D} \times \overline{D'} \subset O$, where $\overline{D}$ denotes the closure of D.*

Remark 4.1. Let A be a directed set. A subset B of A is said to be *cofinal* to A if for any $\alpha \in A$ there exists a $\beta \in B$ such that $\alpha \preceq \beta$, and A is called in [9] of *fish-bone type* if it contains a cofinal totally ordered subset. On the other hand, any totally ordered set contains a well ordered subset cofinal to it.

In the sequel, we usually treat the case where the index set A is of fish-bone type, and accordingly we may assume from the biginning that A is well ordered. In the set of ordinals corresponding to cofinal subsets of A, there exists a minimum which is called the *cofinality* (or "*caractère final*" in [1, III, p. 89, Exercise 16]) of A and is denoted by $\mathrm{cf}(A)$. A well ordered set A contains a cofinal set isomorphic to $\mathbf{N}$ if and only if $\mathrm{cf}(A) = \omega_0$, the first infinite ordinal.

4.2. Other sufficient conditions

We give other sufficient conditions assuming on X_n and Y_n a stronger condition (SC) than the local sequential compactness.

Definition 4.1. For a subset D of a topological space Z, its *sequential closure*, denoted by $\mathrm{scl}(D)$, is defined as

$$\mathrm{scl}(D) := \{z \in Z; \exists z_n \in D \quad \text{suchthat} \quad \lim_{n\to\infty} z_n = z\},$$

and D is called *sequentially compact* if every sequence in it has a subsequence converging to a point in D, and further Z is called *locally sequentially compact* if every point in it has an open neighborhood U for which $\mathrm{scl}(U)$ is sequentially compact.

Our condition (SC) on Z is defined as follows.

(SC) *For every sequentially compact subset K and an open set O containing it, there exists an open set G such that $K \subset G \subset \mathrm{scl}(G) \subset O$ and that $\mathrm{scl}(G)$ is sequentially compact.*

Under this condition (SC), we can give two kinds of sufficient conditions for Problem A as follows. For the inductive system (4.1), assume that the directed set A has a cofinal sub-directed-set isomorphic to $\mathbf{N}$. Then we may put $A = \mathbf{N}$, and assume that $X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X$ by the identification through the canonical maps ϕ_n.

Theorem 4.3. *Let $A = \mathbf{N}$ for an inductive system* (4.1) *of topological spaces, and assume that every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}) satisfies the condition* (SC). *Then, in the case where they all satisfy the first countability axiom, the condition* (DPA) *holds, i.e., for $X = \lim_{\rightarrow} X_n$ and $Y = \lim_{\rightarrow} Y_n$, there holds the equivalence $\tau^X_{ind} \times \tau^Y_{ind} \cong \tau^{X\times Y}_{ind} := \lim_{\rightarrow}(\tau_{X_n} \times \tau_{Y_n})$ on $X \times Y$.*

Theorem 4.4. *Let $A = \mathbf{N}$ for* (4.1) *and assume the condition* (SC) *for every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}). Then, in the case where the system* (4.1) *satisfies $\tau_{X_{n+1}}|_{X_n} = \tau_{X_n}$, $\tau_{Y_{n+1}}|_{Y_n} = \tau_{Y_n}$ for $n \geq 1$, and the condition*

(Gδ) X_n *is a G_δ-set of X_{n+1}, and Y_n is a G_δ-set of Y_{n+1}, for $n \geq 1$,*

there holds for $X \times Y$ the equivalence $\tau^X_{ind} \times \tau^Y_{ind} \cong \tau^{X\times Y}_{ind} := \lim_{\rightarrow}(\tau_{X_n} \times \tau_{Y_n})$.

Remark 4.2. If a sequentially compact normal space satisfies the first countability axiom, then it satisfies the condition (SC). However it is not necessarily locally compact as the following example shows. Let $X = (0, \omega)^{\mathbf{N}}$ with the usual product topology, where ω is the first uncountable ordinal, $(0, \omega)$ denotes the interval consisting of all ordinal numbers $0 < \alpha < \omega$, and the topology on $(0, \omega)$ is defined by open intervals (β, γ) consisting of α such that $\beta < \alpha < \gamma$.

A proof is given in [6] for that X is normal.

4.3. Proofs of Theorems 4.3 and 4.4

First we prepare the following lemmas.

Lemma 4.5. *In a topological space satisfying the condition* (SC), *for a sequentially compact subset K and an open set O containing it, there exists a sequentially compact G_δ-set P such that $K \subset P \subset O$.*

Proof. Take G in the condition (SC) as G_1, and define open sets G_n inductively on n in such a way that $K \subset G_n \subset \mathrm{scl}(G_n) \subset G_{n-1}$ and that $\mathrm{scl}(G_n)$ is sequentially compact. Then, put $P = \bigcap_{n\geq 1} G_n = \bigcap_{n\geq 1} \mathrm{scl}(G_n)$. □

Lemma 4.6. *In a topological space, let $K_k, k = 1, 2, \ldots$, be a decreasing sequence of sequentially compact subsets, and O an open set. Assume $\bigcap_{k\geq 1} K_k \subset O$, then there exists a k such that $K_k \subset O$.*

Proof of Theorem 4.3. It is sufficient to prove $\tau^X_{ind} \times \tau^Y_{ind} \succeq \tau^{X\times Y}_{ind}$. Take a point $(x, y) \in X \times Y$ and its $\tau^{X\times Y}_{ind}$-open neighborhood O. we may assume $(x, y) \in X_1 \times Y_1$. Then there exist a τ_{X_1}-open neighborhood P_1 of x, and a τ_{Y_1}-open neighborhood Q_1 of y such that $\mathrm{scl}(P_1) \times \mathrm{scl}(Q_1) \subset O$ with sequentially compact $\mathrm{scl}(P_1)$ and $\mathrm{scl}(Q_1)$. Starting from these P_1 and Q_1, we construct inductively on n, $P_1 \subset P_2 \subset \cdots \subset P_n, Q_1 \subset Q_2 \subset \cdots \subset Q_n$, satisfying for $1 \leq i \leq n$,

(1) P_i, Q_i are open neighborhoods of x, y in $(X_i, \tau_{X_i}), (Y_i, \tau_{Y_i})$ respectively,
(2) $\mathrm{scl}(P_i)$ and $\mathrm{scl}(Q_i)$ are respectively sequentially compact,
(3) $\mathrm{scl}(P_i) \times \mathrm{scl}(Q_i) \subset O$.

To construct P_{n+1}, Q_{n+1}, we can view $C_n = \mathrm{scl}(P_n)$ and $D_n = \mathrm{scl}(Q_n)$ as sequentially compact subsets of $(X_{n+1}, \tau_{X_{n+1}})$ and $(Y_{n+1}, \tau_{Y_{n+1}})$ respectively since they are images of continuous maps of such subsets in X_n and Y_n. From $C_n \times D_n \subset O$, we see thanks to the first countability axiom on $(X_{n+1}, \tau_{X_{n+1}})$ that, for every $\xi \in C_n$, there exists a $\tau_{X_{n+1}}$-open neighborhood $V(\xi)$ such that $V(\xi) \times D_n \subset O$. Put $G = \bigcup_{\xi \in C_n} V(\xi)$, a $\tau_{X_{n+1}}$-open. Then, $C_n \subset G, G \times D_n \subset O$. Now apply the condition (SC) for X_{n+1} and $C_n \subset G$, then there exists a $\tau_{X_{n+1}}$-open P_{n+1} such that $C_n \subset P_{n+1} \subset \mathrm{scl}(P_{n+1}) \subset G$ and that $C_{n+1} = \mathrm{scl}(P_{n+1})$ is sequentially compact in X_{n+1}.

Similar arguments for the second component of $C_{n+1} \times D_n \subset O$, proves that there exists a $\tau_{Y_{n+1}}$-open $Q_{n+1} \subset Y_{n+1}$ such that $C_{n+1} \times D_{n+1} \subset O$ with $D_{n+1} = \mathrm{scl}(Q_{n+1})$ sequentially compact in $(Y_{n+1}, \tau_{Y_{n+1}})$.

Now put $P = \bigcup_{n \geq 1} P_n$, $Q = \bigcup_{n \geq 1} Q_n$, then, P is τ^X_{ind}-open, Q is τ^Y_{ind}-open, and $(x, y) \in P \times Q \subset O$. □

Proof of Theorem 4.4. Take a point $(x, y) \in X \times Y$ and its $\tau^{X \times Y}_{ind}$-open neighborhood O. As in the proof of Theorem 4.3, we may assume $(x, y) \in X_1 \times Y_1$. This time, we construct series of sets P_n, Q_n, C_n, D_n, inductively on n, in such a way that

(1) $P_n \subset X_n$ is τ_{X_n}-open, $Q_n \subset Y_n$ is τ_{Y_n}-open, and $x \in P_1 \subset P_2 \subset \cdots \subset P_n$, $y \in Q_1 \subset Q_2 \subset \cdots \subset Q_n$,

(2) $C_n \subset X_n$ and $D_n \subset Y_n$ are sequentially compact G_δ-sets such that $P_n \subset C_n$, $Q_n \subset D_n$, $C_n \times D_n \subset O$.

First take $V(x) \subset X_1$, a τ_{X_1}-open, and $V(y) \subset Y_1$, a τ_{Y_1}-open such that $V(x) \times V(y) \subset O$. Then, applying (SC) for $\{x\} \subset V(x)$, we have a τ_{X_1}-open P_1 such that $x \in P_1 \subset \mathrm{scl}(P_1) \subset V(x)$ and that $\mathrm{scl}(P_1)$ is sequentially compact. Applying Lemma 4.5 to $\mathrm{scl}(P_1) \subset V(x)$, we get a sequentially compact G_δ-set C_1 such that $\mathrm{scl}(P_1) \subset C_1 \subset V(x)$. Similarly we have two subsets $Q_1 \subset D_1$ of Y_1.

Now assume that desired subsets have been constructed until n. Then, from the assumptions, we see that C_n (resp. D_n) is a G_δ-set of X_{n+1} (resp. Y_{n+1}). So, there exists an expression $C_n = \bigcap_{k \geq 1} W_{n,k}$ with monotone-decreasing open sets $W_{n,k}$ of X_{n+1}. Applying the condition (SC), let us choose monotone-decreasing $\tau_{X_{n+1}}$-open $V_{n,k} \subset X_{n+1}$ $(k = 1, 2, \cdots)$ such that $C_n \subset V_{n,k} \subset \mathrm{scl}(V_{n,k}) \subset W_{n,k}$ and that $\mathrm{scl}(V_{n,k})$ is sequentially compact. Suppose that they have been chosen until $k = m - 1$. Then, apply (SC) to $C_n \subset V_{n,m-1} \cap W_{n,m}$, and we get $V_{n,m}$ such that $C_n \subset V_{n,m} \subset \mathrm{scl}(V_{n,m}) \subset V_{n,m-1} \cap W_{n,m}$. Thus, we have an expression of C_n as $C_n = \bigcap_{k \geq 1} V_{n,k} = \bigcap_{k \geq 1} \mathrm{scl}(V_{n,k})$. Similarly, we obtain such an expression of D_n as $D_n = \bigcap_{k \geq 1} V'_{n,k} = \bigcap_{k \geq 1} \mathrm{scl}(V'_{n,k})$. Finally,

$$\bigcap_{k \geq 1} \big(\mathrm{scl}(V_{n,k}) \times \mathrm{scl}(V'_{n,k})\big) = C_n \times D_n \subset O.$$

Then, by Lemma 4.6, there exists a k such that $\mathrm{scl}(V_{n,k}) \times \mathrm{scl}(V'_{n,k}) \subset O$. Applying Lemma 4.5 to $\mathrm{scl}(V_{n,k+1}) \subset V_{n,k}$, we get a sequentially compact G_δ-set C_{n+1} such that $\mathrm{scl}(V_{n,k+1}) \subset C_{n+1} \subset V_{n,k}$.

Similarly we have such a subset D_{n+1} as $\mathrm{scl}(V'_{n,k+1}) \subset D_{n+1} \subset V'_{n,k}$.

Now put $P_{n+1} = V_{n,k+1}, Q_{n+1} = V'_{n,k+1}$. Then, $P_n \subset C_n \subset P_{n+1} \subset C_{n+1}$, $Q_n \subset D_n \subset Q_{n+1} \subset D_{n+1}$, and $C_{n+1} \times D_{n+1} \subset O$. Thus, finishing the construction of P_n's and Q_n's, we put $P = \bigcup_{n \geq 1} P_n$, $Q = \bigcup_{n \geq 1} Q_n$. Then P is τ^X_{ind}-open, Q is τ^Y_{ind}-open, and $(x, y) \in P \times Q \subset O$ as desired. □

5. The case of a fixed Y and Problem B

In the following, we study in detail Problems A and B, especially necessary conditions for converses of theorems in Section 4. In this section, we study the case where Y is fixed, or the case where $(Y_n, \tau_{Y_n}) = (Y, \tau_Y)$ for any $n \geq 1$. This is our Problem B.

5.1. Comments to converses of Theorems 4.1, 4.3 and 4.4

Statements for direct converses of these theorems contain necessarily a global characterization such as "*X_n is a locally compact space*". However, this kind of global characterization of spaces X_n and Y_n are not possible in its nature of inductive sequences of topological spaces, and so, possible converses should be at first stated in languages of local characterizations of these spaces. This can be seen from the following examples.

Example 5.1. Let $X = \mathbf{R}$ and $X_n = (-n, n) \cup \mathbf{Q}$ with an open interval $(-n, n)$, where X is equipped with a usual topology $\tau_\mathbf{R}$ of $\mathbf{R}$, and X_n with its relative topology $\tau_{X_n} = \tau_\mathbf{R}|_{X_n}$. Then, no X_n is locally compact, whereas so is the inductive limit space X (cf. Theorems 5.3 and 5.5). Note that the space $(\mathbf{Q}, \tau_\mathbf{Q} = \tau_\mathbf{R}|_\mathbf{Q})$ is totally disconnected and normal.

Example 5.2. Let $Y = \prod'_{k\geq 1} \mathbf{R}_k$ with $\mathbf{R}_k = \mathbf{R}$ be the restricted direct product of $\mathbf{R}$. Put $Y_n = \prod_{k=1}^n \mathbf{R}_k = \mathbf{R}^n$, $Y'_n = \left(\prod_{k=1}^{n-1} \mathbf{R}_k\right) \times \mathbf{Q} \subset Y_n$, and imbed Y_n into Y_{n+1} as $Y_n \ni y \mapsto (y, 0) \in Y_{n+1}$. The space Y_n is equipped with the usual Euclidean metric, and the space Y'_n with its relative topology. Then, Y_n is locally compact, whereas no point of Y'_n has a compact neighborhood. However the topological space Y considered as the inductive limit of $(Y_n, \tau_{Y_n}), n \geq 1$, is also equal to the inductive limit of $(Y'_n, \tau_{Y'_n}), n \geq 1$, since there is a mixed inductive system given by $Y''_{2n+1} := Y_n$, $Y''_{2n} := Y'_n$, $(n \geq 1)$, which converges to (Y, τ^Y_{ind}).

Now let $\{X_n; n \in \mathbf{N}\}$ be an inductive system of separable locally compact spaces and put $X = \lim_{\rightarrow} X_n$. Consider two inductive systems of direct product type as $\{X_n \times Y_m; (n, m) \in \mathbf{N} \times \mathbf{N}\}$, and $\{X_n \times Y'_m; (n, m) \in \mathbf{N} \times \mathbf{N}\}$, where $(n, m) \preceq (n', m')$ in $\mathbf{N} \times \mathbf{N}$ if and only if $n \leq n'$, $m \leq m'$. Then we get as their inductive limits the same space $X \times Y$. Denote by $\tau^{X\times Y}_{ind,1}$ and $\tau^{X\times Y}_{ind,2}$ the inductive limit topologies on $X \times Y$ corresponding to the first and the second system respectively. We assert that $\tau^{X\times Y}_{ind,1} \cong \tau^{X\times Y}_{ind,2} \cong \tau^X_{ind} \times \tau^Y_{ind}$.

In fact, the first equivalence is affirmed by considering a mixed inductive system (Z_n, τ_{Z_n}), $n > 1$, with $(Z_{2n+1}, \tau_{Z_{2n+1}}) := (X_n \times Y_n, \tau_{X_n} \times \tau_{Y_n})$, $(Z_{2n}, \tau_{Z_{2n}}) := (X_n \times Y'_n, \tau_{X_n} \times \tau_{Y'_n})$. Another equivalence $\tau^{X\times Y}_{ind,1} \cong \tau^X_{ind} \times \tau^Y_{ind}$ is guaranteed by Theorem 4.1 thanks to the local compactness of X_n's and Y_n's.

Furthermore, in the case the index m is fixed, as for the topologies on $\lim_{n\to\infty}(X_n \times Y_m) = X \times Y_m$ and on $\lim_{n\to\infty}(X_n \times Y'_m) = X \times Y'_m$, we get the equivalence $\tau^X_{ind} \times \tau_{Y_m} = \tau^{X\times Y_m}_{ind}$ by Theorem 4.1, but the inequivalence

$\tau_{ind}^{X} \times \tau_{Y'_m} \prec \tau_{ind}^{X\times Y'_m}$ by Theorem 5.3 below. In more significant notation,

$$\left(\lim_{\to} \tau_{X_n}\right) \times \tau_{Y'_m} \prec \lim_{n\to\infty} \left(\tau_{X_n} \times \tau_{Y'_m}\right) \quad (m \geq 1),$$

$$\left(\lim_{\to} \tau_{X_n}\right) \times \left(\lim_{\to} \tau_{Y'_m}\right) \cong \lim_{\to}{}_{n,m} \left(\tau_{X_n} \times \tau_{Y'_m}\right) = \lim_{\to} \left(\tau_{X_n} \times \tau_{Y'_n}\right).$$

Note also that the above order in $\mathbf{N} \times \mathbf{N}$ is not the lexicographic one.

The last statement in the above example shows that Problem A cannot be reduced simply to Problem B in general, and the relation between them is rather delicate.

5.2. A sufficient condition for $\tau_{ind}^{X} \times \tau_Y \simeq \tau_{ind}^{X\times Y}$

Let us now begin to treat Problem B. Fix a topological space (Y, τ_Y). Put $Z_n = X_n \times Y$, $\tau_{Z_n} = \tau_{X_n} \times \tau_Y$, and $Z = \lim_{\to} Z_n, \tau_{ind}^{Z} = \lim_{\to} \tau_{Z_n}$. We identify Z with $X \times Y$ and τ_{ind}^{Z} with $\tau_{ind}^{X\times Y}$. We know in general $\tau_{ind}^{X} \times \tau_Y \preceq \tau_{ind}^{X\times Y}$, and the problem is to guarantee the converse relation. A simple sufficient condition is given as follows.

Proposition 5.1. *Assume for the inductive system $\{(X_n, \tau_{X_n})\}$ that X_n is imbedded homeomorphically into X_{n+1} for $n \geq 1$, and for the counter part (Y, τ_Y) that Y is locally compact Hausdorff. Then there holds the equivalence $\tau_{ind}^{X} \times \tau_Y \cong \tau_{ind}^{X\times Y}$.*

Proof. For a point $z = (x, y) \in Z = X \times Y$, take a τ_{ind}^{Z}-neighborhood W of z. It is enough for us to prove that there exist a τ_{ind}^{X}-neighborhood U of $x \in X$ and a neighborhood V of $y \in Y$ such that $U \times V \subset W$.

We may assume that $x \in X_1$. Then there exist a τ_{X_1}-open neighborhood $U_1 \subset X_1$ and a relatively compact, open neighborhood Q of $y \in Y$ such that $U_1 \times V \subset W \cap Z_1$ with $V = \mathrm{Cl}(Q)$. Starting from U_1, we construct a τ_{X_n}-open neighborhood U_n of $x \in X_n$ in such a way that $U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots$, $U_n \times V \subset W \cap Z_n$. If this is done, then $U = \bigcup_{n\geq 1} U_n$ is a τ_{ind}^{X}-open neighborhood of $x \in X$ and $U \times V \subset W$ as demanded.

Now assume U_k's have been chosen for $1 \leq k \leq n$. Then, since $U_n \times V \subset W \cap Z_n \subset W \cap Z_{n+1}$, and V is compact, there exists for any $\xi \in U_n$ a $\tau_{X_{n+1}}$-open neighborhood $U(\xi)$ of ξ such that $U(\xi) \times V \subset W \cap Z_{n+1}$. Put $U_{n+1} = \bigcup_{\xi \in U_n} U(\xi)$, then U_{n+1} is $\tau_{X_{n+1}}$-open, and we have done. □

5.3. Normalization of situations

To simplify the situations we put some natural assumptions from the beginning.

First we assume for simplicity that the index set A contains a cofinal subset isomorphic to $\mathbf{N}$ as directed set, and so we take $A = \mathbf{N}$ later on except when the contrary is announced.

It may be assumed without essential loss of generality that

(00-X) *each canonical map* $\phi_{n+1,n} : X_n \to X_{n+1} \quad (n \geq 1)$ *is injective*,

and so considering as $X_n \subset X_{n+1}$ and $X = \bigcup_{n \geq 1} X_n$, we can omit the notations $\phi_{m,n}$ and ϕ_n rather freely, and then,

(01-X) *each* $\phi_{n+1,n}$ *is a homeomorphism, or* $\tau_{X_{n+1}}|_{X_n} \cong \tau_{X_n}$.

For (01-X), we remark that the topologies τ_{X_n} can be replaced by $\tau_{ind}^X|_{X_n}$ to get the same inductive limit topology τ_{ind}^X, and then (01-X) holds for new topologies on X_n's. From now on, we assume (00-X) and (01-X) for $\{X_n\}$.

Let us remark the following fact.

Lemma 5.2. *Assume that at a point* $x_0 \in X$ *there exists a* τ_{X_j}*-open neighborhood* $U_0 \subset X_j$ *for a certain* j *such that the image* $\phi_{n,j}(U_0) = U_0$ *is* τ_{X_n}*-open in* X_n*, for any* $n > j$*. Then, the topologies* $\tau_{ind}^X \times \tau_Y$ *and* $\tau_{ind}^{X \times Y}$ *are mutually equivalent at a point* $(x_0, y) \in X \times Y$*, that is, the neighborhood systems of* (x_0, y) *in both topologies are mutually equivalent. More exactly, on a subset* $U_0 \times Y \subset X \times Y$*, we have the equivalence* $\left(\tau_{ind}^X|_{U_0}\right) \times \tau_Y \cong \tau_{ind}^{X \times Y}|_{U_0 \times Y}$.

Proof. It is enough to remark that $U_0 \subset X = \lim_{\to} X_n$ is τ_{ind}^X-open from the assumption. □

Taking into account the above fact, to study necessary conditions for $\tau_{ind}^X \times \tau_Y \cong \tau_{ind}^{X \times Y}$, we can put an assumption to deny the above simple sufficient condition. Thus, taking an appropriate cofinal sequence if necessary, we may put the following assumption for $\{X_n\}$ from the beginning:

(1-X) *for any* n*,* X_n *as a subset of* X_{n+1} *has no* $\tau_{X_{n+1}}$*-inner point of* X_{n+1}.

5.4. Necessary conditions for $\tau_{ind}^X \times \tau_Y \simeq \tau_{ind}^{X \times Y}$

We follow the discussion of A. Yamasaki in [11] to get the following necessary condition.

Theorem 5.3. *Let* $A = \mathbf{N}$ *and* Y *be fixed. Assume the condition* (1-X) *and the following*:

(2-x_0) *for* $n \gg 1$*,* $x_0 \in X_n$ *has a countable fundamental system of* τ_{X_n}*-neighborhoods*;

(3-y_0)*,* $y_0 \in Y$ *has a countable fundamental system of neighborhoods consisting of closed ones*;

(4-y_0)*,* $y_0 \in Y$ *does not have a sequentially compact neighborhood.*

Then, $\tau_{ind}^X \times \tau_Y \prec \tau_{ind}^{X \times Y} := \lim_{\to} (\tau_{X_n} \times \tau_Y)$ *at* $(x_0, y_0) \in X \times Y$.

Our method of proof is to apply the following lemma.

Lemma 5.4. *Assume that $y_0 \in Y$ has a countable fundamental system of neighborhoods. For $x_0 \in X_1$, assume that there exists, for each n, a τ_{Z_n}-open neighborhood W_n of $z_0 = (x_0, y_0)$ in $Z_n = X_n \times Y$ such that*

(a) $W_{n+1} \cap Z_n = W_n$, *and*

(b) *for a fundamental system $V_j (j \geq 1)$ of neighborhoods of $y_0 \in Y$, there holds, for each fixed n, $U_{X_n} \times V_n \not\subset W_n$ for any τ_{X_n}-neighborhood of $x_0 \in X_n$.*

Then, $W = \bigcup_{n \geq 1} W_n$ is a τ^Z_{ind}-open neighborhood of $z_0 \in Z$, and $U_X \times V_Y \not\subset W$ for any τ^X_{ind}-neighborhood U_X of $x_0 \in X$ and any τ_Y-neighborhood V_Y of $y_0 \in Y$. In other words, $\tau^X_{ind} \times \tau_Y \prec \tau^Z_{ind}$ at the point $z_0 \in Z$.

Proof of Theorem 5.3. According to Lemma 5.4, let us determine W_n by induction on n. We may start from $n = 1$ and put $W_1 = Z_1$. Assume that W_k have been determined for $1 \leq k \leq n-1$. Then, by (1-X) and (2-x_0), there exists a sequence $x_{n,k} \in X_n \setminus X_{n-1} (k \geq 1)$ converging in τ_{X_n} to x_0. By (3-y_0) and (4-y_0), we have a fundamental system of neighborhoods $V_{Y,j} (j \geq 1)$ of $y_0 \in Y$, closed but not sequentially compact. Then by (4-y_0), there exists for every j a sequence $y_{j,k} \in V_{Y,j} (k \geq 1)$ with no accumulation point in Y. Put $D_n := \{(x_{n,k}, y_{n,k}); k \geq 1\} \subset Z_n$. Then, D_n is τ_{Z_n}-closed and $D_n \cap Z_{n-1} = \emptyset$. By (01-X), we have a τ_{Z_n}-open $W'_n \subset Z_n$ such that $W_{n-1} = Z_{n-1} \cap W'_n$. Put $W_n = W'_n \cap (Z_n \setminus D_n)$, then W_n is τ_{Z_n}-open and $W_n \cap Z_n = W_{n-1}$, whence (a) holds.

Furthermore, since $D_n \cap (U_{X_n} \times V_{Y,n}) \neq \emptyset$ and $D_n \cap W_n = \emptyset$, we have $U_{X_n} \times V_{Y,n} \not\subset W_n$. Hence (b) holds, as desired. □

Reformulating the above result in a global form, we get a kind of converse, in the case of a fixed Y, of affirmative assertions in theorems in Section 4 as follows.

Theorem 5.5. *Assume* (1-X) *and the following*:

(2-X), *each (X_n, τ_{X_n}) satisfies the first countability axiom*;

(3-Y), *Y is regular and satisfies the first countability axiom.*

Then, $\tau^X_{ind} \times \tau_Y \prec \tau^{X \times Y}_{ind}$ at any point $(x, y) \in X \times Y$ for which $y \in Y$ has no sequentially compact neighborhood.

6. Necessary conditions for $\tau^X_{ind} \times \tau^Y_{ind} \simeq \tau^{X \times Y}_{ind}$ and Problem A

Let $A = \mathbf{N}$. Let us consider two inductive systems $\{X_n\}$ and $\{Y_n\}$, and put $Z_n = X_n \times Y_n$ and identify $Z = \lim_{\rightarrow} Z_n$ with $X \times Y$, then $\tau^Z_{ind} = \tau^{X \times Y}_{ind}$. Assume (00-X) and (01-X) for $\{X_n\}$ and similarly (00-Y) and (01-Y) for $\{Y_n\}$, for simplicity.

6.1. Conditions for $\tau^X_{ind} \times \tau^Y_{ind} \prec \tau^{X \times Y}_{ind}$ at a point

We study when the above two inductive limit topologies on $Z = X \times Y$ are different with each other at a point $z_0 = (x_0, y_0) \in Z$.

Theorem 6.1. *Assume the following*:
(1-X) X_n *has no* $\tau_{X_{n+1}}$*-inner point of* X_{n+1} *for* $n \geq 1$;
(2-X) X_n *satisfies the first countability axiom for* $n \geq 1$;
(3-Y_{n_0}) Y_{n_0} *is regular and satisfies the first countability axiom*;
(4-Y_{n_0}-y_0) $y_0 \in Y_{n_0}$ *has no sequentially compact neighborhood*;
(5-Y_{n_0}) Y_{n_0} *is* τ_{Y_n}*-closed in* Y_n *for all* $n > n_0$.
Then, $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^{X \times Y}$ *at* $(x_0, y_0) \in X \times Y$ *for any* $x_0 \in X_{n_0}$.

Proof. We may assume $x_0 \in X_1$, and $n_0 = 1$ whence $y_0 \in Y_1$. Let us apply an appropriate version of Lemma 5.4. To do so, we determine τ_{Z_n}-open neighborhood $W_n \subset Z_n$ of $z_0 = (x_0, y_0)$ by induction on n.

For $n = 1$, put $W_1 = Z_1$ and suppose that W_k has been chosen for $1 \leq k \leq n-1$. Then, by (1-X) and (2-X), there exists a sequence $x_{n,k} \in X_n \setminus X_{n-1} (k \geq 1)$, τ_{X_n}-convergent to x_0. On the other hand, by (3-Y_{n_0}) with $n_0 = 1$, there exists a fundamental system of neighborhoods of $y_0 \in Y_1$ consisting of τ_{Y_1}-closed $V_{Y_1,j} (j \geq 1)$. Then, because of the condition (5-Y_{n_0}) with $n_0 = 1$, each $V_{Y_1,j}$ is τ_{Y_n}-closed in Y_n for $n \geq 1$, and therefore τ_Y-closed in Y. By (4-Y_{n_0}-y_0), there exists, for each j, a sequence $y_{j,k} \in V_{Y_1,j} (k \geq 1)$ with no τ_{Y_1}-accumulation point in Y_1, and by (5-Y_{n_0}), no such one in each (Y_n, τ_{Y_n}). Put $D_n = \{(x_{n,k}, y_{n,k}); k \geq 1\}$, then $D_n \subset X_n \times Y_1 \subset X_n \times Y_n = Z_n$. Further, $D_n \cap W_{n-1} = \emptyset$, and D_n is $\tau_{X_n} \times \tau_{Y_1}$-closed, and so, closed also in $\tau_{X_n} \times \tau_{Y_n} = \tau_{Z_n}$. By (01-X) and (01-Y), we have a τ_{Z_n}-open $W'_n \subset Z_n$ such that $W_{n-1} = Z_{n-1} \cap W'_n$. Put $W_n = W'_n \cap (Z_n \setminus D_n)$, then $W_{n-1} = Z_{n-1} \cap W_n$. So, $W = \bigcup_{n \geq 1} W_n$ is a τ_{ind}^Z-open neighborhood of $z_0 \in Z$.

Note that $W_n \cap D_n = \emptyset$ and $(U_{X_n} \times V_{Y_1,n}) \cap D_n \neq \emptyset$ for any τ_{X_n}-open neighborhood U_{X_n} of $x_0 \in X_n$, then we have $U_{X_n} \times V_{Y_1,n} \not\subset W_n$. Now suppose $U_X \times V_Y \subset W$ for some τ_X-open neighborhood U_X of $x_0 \in X$ and τ_Y-open one V_Y of $y_0 \in Y$. Then, $V_Y \supset V_{Y_1,j}$ for $j \gg 1$. Take such a one $j = n$, then taking intersections with $X_n \times Y_n = Z_n$, we have $U_{X_n} \times V_{Y_1,n} \subset W_n$ with $U_{X_n} = U_X \cap X_n$, a contradiction. Thus the τ_{ind}^Z-open neighborhood W is not a neighborhood of z_0 in $\tau_{ind}^X \times \tau_{ind}^Y$. This proves the assertion of the theorem. □

Reformulating the above result in a global form, we get a converse of Theorem 4.1 as follows.

Theorem 6.2. *Assume* (1-X) *and* (2-X)*and further assume the following*:
(3′-Y) *each* (Y_n, τ_{Y_n}) *is regular and satisfies the first countability axiom*;
(5′-Y) Y_n *is closed in* $(Y_{n+1}, \tau_{Y_{n+1}})$, *for* $n \geq 1$.
Then, if $y_0 \in Y$ *has no sequentially compact neighborhood in any* (Y_n, τ_{Y_n}), *there holds* $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^Z$ *at* $(x_0, y_0) \in Z$ *for any* $x_0 \in X$.

Remark 6.1. The additional conditions (5-Y_{n_0}) in Theorem 6.1 and (5′-Y) in Theorem 6.2 are asked to avoid situations similar to that in Example 5.2.

To get much faithful converses to Theorems 4.1, 4.3 and 4.4, we should get rid of the first countability axiom. We will discuss this point in the future.

Theorem 6.3. *Let X_n and Y_n be all regular Hausdorff spaces satisfying the first countability axiom. Assume the conditions* (1-X) *and* (5′-X) *for* $\{X_n\}$ *and similarly* (1-Y) *and* (5′-Y) *for* $\{Y_n\}$. *Then* $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ *if and only if X_n and Y_n are all locally sequentially-compact.*

6.2. Case of metrizable spaces

In the case of metrizable spaces, they are automatically regular and satisfy the first countability axiom, and furthermore sequential compactness is equivalent to compactness. Therefore, in that case, we get from Theorems 4.1 and 6.2 the following simple necessary and sufficient condition for the commutativity of "inductive limit" and "direct product": $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y} := \lim_{\rightarrow}(\tau_{X_n} \times \tau_{Y_n})$.

Theorem 6.4. *Assume the conditions* (00-X), (01-X), (1-X) *and* (5′-X) *for* $\{X_n\}$, *and similarly* (00-Y), (01-Y), (1-Y) *and* (5′-Y) *for* $\{Y_n\}$. *Let X_n and Y_n be all metrizable spaces. Then,* $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ *if and only if X_n and Y_n are locally compact.*

Remark 6.2. In the case of topological groups $G = \lim_{\rightarrow} G_n$ as in [9] or [11], the first countability axiom is equivalent to the metrizability. So, the result in [11] for the necessity for that τ_{ind}^G gives a group topology on G, can be understood as for a metrizable case.

6.3. Local compactness of inductive limit spaces

The local compactness of each X_n and Y_n plays important roles in our discussions until now. Concerning a relationship between local compactness of spaces (X_n, τ_{X_n}) and that of the inductive limit space $X = \lim_{\rightarrow} X_n$ with $\tau_{ind}^X = \lim_{\rightarrow} \tau_{X_n}$, we have the following result.

Proposition 6.5. *Assume that, for* $\{(X_n, \tau_{X_n})\}$, *every X_n is T_1-space, that is, each of two points has a neighborhood not containing the other one.*

(i) *For a τ_{ind}^X-compact set C in X, there exists an n such that $C \subset X_n$.*

(ii) *For an $x \in X$, it has a τ_{ind}^X-compact neighborhood in X if and only if there exists an n such that X_n contains a τ_{ind}^X-compact neighborhood of x.*

(iii) *In the case of topological groups, where (X_n, τ_{X_n}) are topological groups, the inductive limit (X, τ_{ind}^X) is a locally compact group if and only if there exists an n such that (X_n, τ_{X_n}) is an open subgroup of X which itself is locally compact.*

Proof. Enough to prove (i). Assume the contrary. Put $C_n = X_n \cap C$. Then, since $C_1 \subset C_2 \subset \cdots C_n \subset \cdots, \cup_{n\geq 1} C_n = C$, there exists an infinite sequence $n(1) < n(2) < \cdots$ such that $C_{n(j)} \setminus C_{n(j-1)} \neq \emptyset$. Transfering to a cofinal sub-directed-set if necessary, it can be assumed that $n(j) = j (j \in \mathbf{N})$. Thus, we can fix a sequence of points $x_j \in C_j \setminus C_{j-1}, j \in \mathbf{N}$. Put $R_n = \{x_n, x_{n+1}, \cdots\} \subset C$. Then, for any j, the intersection $R_n \cap X_j$ is finite, and so R_n is closed in (X, τ_{ind}^X). On the other hand, $\bigcap_{n\geq 1} R_n = \emptyset$. This contradicts the finite intersection property for C. □

6.4. A comment to Problem C

From Proposition 6.5 above, we get a simple necessary and sufficient condition for the equivalence $\lim_{\rightarrow}(\tau_Y|_{Y_n}) \cong \tau_Y$ in Problem C, in the case of a locally compact space (Y, τ_Y).

Proposition 6.6. *Let (Y, τ_Y) be a locally compact Hausdorff space, and $Y_n, n \geq 1$, be an increasing sequence of subsets of Y such that $\bigcup_{n\geq 1} Y_n = Y$. Put $\tau_{Y_n} = \tau_Y|_{Y_n}$. Then, $\lim_{\rightarrow} \tau_{Y_n} \cong \tau_Y$ if and only if, for any $y \in Y$, there exists a k such that $y \in Y_k$ and it has a relatively compact, open neighborhood in τ_{Y_k} which is also τ_{Y_n}-open for $n \geq k$.*

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