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Weyl-Schur duality for Cartan-type Lie superalgebra $W(n)$

By

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Introduction

Cartan-type Lie superalgebras is a subclass in the classification of the finite dimensional simple Lie superalgebras over $\mathbb{C}$. Lie superalgebra $W(n)$, which consists of all the superderivations on Grassmann algebra $\Lambda(n)$, is one of Cartan-type Lie superalgebras. The irreducible representations of $W(n)$ are described in [5]. In this article, we want to develop an analogue of the classical Weyl-Schur duality for $W(n)$. In other words, we try to decompose the m-fold tensor product $\otimes^m \Lambda(n)$ of the defining representation $\Lambda(n)$ of $W(n)$ under the diagonal actions of $W(n)$ and its commutant algebra. In [7], K. Nishiyama has considered the decomposition of the m-fold tensor product of the defining representation of Cartan-type Lie algebra W_n under the condition $m \leq n$.

In the present article, we give the decomposition of $\otimes^m \Lambda(n)$ under the condition $m \leq n$ in Section 2. As an attempt toward the general cases, in Section 3, we consider the case $n = 2$, where m is arbitrary. Let us explain this more explicitly.

Let End$[m]$ be a semigroup of all the mappings from a finite collection of integers $[m] := \{1, 2, \cdots, m\}$ into itself. In [8] and [9], using the semigroup End$[m]$, Nishiyama and the author determined the commutant algebra of $W(n)$ in the m-fold tensor product of the defining representation under the condition $m \leq n$. More precisely, let ψ be the defining representation of $W(n)$ on $\Lambda(n)$ and φ be a representation of End$[m]$ on $\otimes^m \Lambda(n)$ (the definition is given in Section 1). If $m \leq n$, then the commutant algebra of $\psi^{\otimes m}(W(n))$ in $\mathrm{End}(\otimes^m \Lambda(n))$ is the algebra generated by $\varphi(\mathrm{End}[m])$ in $\mathrm{End}(\otimes^m \Lambda(n))$ ([9]).

Therefore, along the idea of Schur and Weyl, we want to decompose the space $\otimes^m \Lambda(n)$ as a $W(n) \times \mathrm{End}[m]$-module. However, in our case, the representation of $W(n)$ on $\otimes^m \Lambda(n)$ is not semisimple, hence we are forced to consider the quotient representations. Nishiyama suggested the following conjectures.

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Conjecture 1. Let $\rho \otimes \sigma$ be a finite-dimensional irreducible representation of $W(n) \times \mathrm{End}[m]$. Then we have

$$\dim \mathrm{Hom}_{W(n)\times \mathrm{End}[m]}(\otimes^m \Lambda(n), \rho \otimes \sigma) \leq 1,$$

where ρ (resp. σ) is an irreducible representation of $W(n)$ (resp. $\mathrm{End}[m]$).

Conjecture 2. For a $W(n) \times \mathrm{End}[m]$-module U, we put

$$\mathfrak{R}_{W(n)}(U) := \{\rho \in W(n)^\wedge \mid \mathrm{Hom}_{W(n)}(U, \rho) \neq (0)\}$$

and

$$\mathfrak{R}_{\mathrm{End}[m]}(U) := \{\sigma \in \mathrm{End}[m]^\wedge \mid \mathrm{Hom}_{\mathrm{End}[m]}(U, \sigma) \neq (0)\},$$

where $W(n)^\wedge$ (resp. $\mathrm{End}[m]^\wedge$) is the set of the equivalence classes of all the irreducible modules of $W(n)$ (resp. $\mathrm{End}[m]$). Then, for any $\rho \in \mathfrak{R}_{W(n)}(\otimes^m \Lambda(n))$, there is one and only one $\sigma \in \mathfrak{R}_{\mathrm{End}[m]}(\otimes^m \Lambda(n))$ such that

$$\dim \mathrm{Hom}_{W(n)\times \mathrm{End}[m]}(\otimes^m \Lambda(n), \rho \otimes \sigma) \neq 0.$$

Furthermore, the following mapping is a bijection:

$$W(n)^\wedge \supseteq \mathfrak{R}_{W(n)}(\otimes^m \Lambda(n)) \ni \rho \leftrightarrow \sigma \in \mathfrak{R}_{\mathrm{End}[m]}(\otimes^m \Lambda(n))/\mathbf{1}_{\mathrm{End}[m]} \subseteq \mathrm{End}[m]^\wedge,$$

where $\mathbf{1}_{\mathrm{End}[m]}$ denotes the trivial representation of $\mathrm{End}[m]$.

In the simplest case $n = 1$, we have an affirmative result (see [11]). Unfortunately, along the method of [11], it seems difficult to prove conjectures above in general. In this article, we apply Nishiyama's method in [7] to the cases $m \leq n$ and get the affirmative answers to the above conjectures. Also we consider the case $n = 2$, where m is arbitrary. Through the detailed calculation, we obtain more concrete results.

In [10], Sergeev has given the decomposition of the k-fold tensor space of the natural representation of Lie superalgebra $\mathfrak{gl}(n, m)$. Because $W(2)$ is isomorphic to the classical Lie superalgebra $A(1, 0) \cong \mathfrak{sl}(2, 1)$ (see [5]), it is possible to deduce the decomposition of $W(2) \times \mathfrak{S}_m$-module $\otimes^m(\Lambda(2)/\mathbb{C})$ from Sergeev's results. In the third section of this article, we give the decomposition of $W(2) \times \mathfrak{S}_m$-module $\otimes^m(\Lambda(2)/\mathbb{C})$ by a method completely different from Sergeev's one. Note that our proof is constructive, and the information of the explicit decomposition will be helpful when we consider the decomposition of $W(n) \times \mathfrak{S}_m$-module $\otimes^m(\Lambda(n)/\mathbb{C})$ for arbitrary $n < m$. The latter is our goal, and to consider the case $n = 2$ is a preliminary step for this aim.

Let us describe the contents of each section briefly. In the first section, we give the basic notations and preliminary results. In the second section, we give the decomposition under the condition $m \leq n$. In this section, firstly, we decompose the top level $\otimes^m(\Lambda(n)/\mathbb{C})$ as a $W(n) \times \mathfrak{S}_m$-module using the traditional method, that is, first to calculate the commutant algebra, then to get the decomposition. With this result, along the way appeared in [7], we obtain the decomposition of the whole space (Theorem 2.6). In the third section, we consider the case $n = 2$. Along the similar arguments in Section 2, we first decompose the top level of the representation $\otimes^m \Lambda(2)$. However, since the commutant algebra of $W(2)$ is not available at hand, we cannot follow the arguments in Section 2 literally. Instead, we use the classical Weyl-Schur duality in the lowest degree space. For a space of general degree, we can decompose it into two parts. One is the space consisting of images of the operators of $W_1(2)$ coming from the lower degree spaces. The other is new. We decompose these new spaces as $\mathfrak{S}_m$-modules (Lemma 3.2). In this way, we obtain all the representations which actually appear in the top level (Theorem 3.3). We believe that this method will be helpful to consider the case $m > n$. For the decomposition of the whole space, we again use the method in [7] and obtain Theorem 3.12.

Convention. The ground field is always $\mathbb{C}$. In the following, we will omit it if it does not cause a confusion.

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1. Notations and preliminaries

1.1. Cartan type Lie superalgebra $W(n)$ and its defining representation

Let $\Lambda(n)$ be the Grassmann algebra generated by $\{\xi_i \mid i = 1, 2, \ldots, n\}$ and $\Lambda_j(n)$ its j-th homogeneous piece. Naturally, $\Lambda(n) = \sum_{j:\ \text{even}} \Lambda_j(n) \oplus \sum_{j:\ \text{odd}} \Lambda_j(n)$ is a $\mathbb{Z}_2$-graded space. Let $W(n)$ be the set of all superderivations over $\Lambda(n)$, or explicitly,

$$W(n) = \left\{ \sum_{i=1}^{m} f_i \partial_i \mid f_i \in \Lambda(n) \right\},$$

where ∂_i is a superderivation determined by $\partial_i \xi_j = \delta_{ij}$. Put

$$W_k(n) = \left\{ \sum_i f_i \partial_i \mid f_i \in \Lambda_{k+1}(n) \right\}.$$

Then $W(n) = \sum_{k=-1}^{n-1} W_k(n)$ gives a $\mathbb{Z}$-gradation and $W_0(n) \cong \mathfrak{gl}(n)$ canonically. In the following, we identify $W_0(n)$ with $\mathfrak{gl}(n)$. Furthermore, $W_k(n)$ has a natural structure of $\mathfrak{gl}(n)$-module.

Let ψ be the defining representation of $W(n)$ on $\Lambda(n)$. We consider the m-fold tensor product of ψ, or, the representation $\psi^{\otimes m}$ on $\otimes^m \Lambda(n)$.

Let $\Lambda(n, m)$ be the Grassmann algebra generated by $\{\xi_{ij} \mid i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, m\}$, then

$$\otimes^m \Lambda(n) \cong \Lambda(n, m).$$

Through this isomorphism, $W(n)$ has a representation on $\Lambda(n,m)$ denoted by the same symbol $\psi^{\otimes m}$, and so $\Lambda(n,m)$ is also a $\mathfrak{gl}(n)$-module. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n)$ as $\mathfrak{h} = \langle \xi_i \partial_i \mid i = 1, 2, \ldots, n\rangle$, then we have the following weight space decomposition:

$$\Lambda(n,m) = \sum_{\mu}^{\oplus} \Lambda_\mu,$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a weight with respect to $\mathfrak{h}$ and Λ_μ is the weight space of weight μ. We put $|\mu| := \mu_1 + \mu_2 + \cdots + \mu_n$.

1.2. The semigroup End$[m]$ and its representations

Denote by $[m]$ the set $\{1, 2, \ldots, m\}$ of integers, and put $\mathrm{End}[m] = \{\varphi : [m] \to [m]\}$. Then End$[m]$ becomes a semigroup and the subset of its group elements coincides with the symmetric group $\mathfrak{S}_m$. Let $\mathcal{R}(m)$ be the semigroup ring generated by End$[m]$, and put

$$\mathcal{R}_k(m) := \langle \varphi \in \mathrm{End}[m] \mid \sharp\varphi([m]) \le k\rangle,$$

where $\sharp N$ means the cardinality of a finite set N and $\langle A\rangle$ denotes the vector space spanned by A. We write $\mathcal{R}_k$ instead of $\mathcal{R}_k(m)$ if it does not cause a confusion.

For the semigroup End$[m]$, Hewitt and Zuckerman [1] gave the classification of irreducible representations and Nishiyama [7] simplified their results. Here we follow Nishiyama's formulation. Put

$$\mathcal{P}_{k,m-k} := \{\varphi \in \mathrm{End}[m] \mid \varphi([k]) \subseteq [k]\} \subseteq \mathrm{End}[m].$$

Clearly $\mathcal{P}_{k,m-k}$ is a sub-semigroup of End$[m]$ and call it a *maximal parabolic subsemigroup*. It projects naturally onto End$[k]$, where $[k] = \{1, 2, \ldots, k\}$, and this projection, denoted by P, is a semigroup morphism. Let (σ, U_σ) be a representation of End$[k]$. Then the above projection P naturally induces a representation $P^\star\sigma$ of $\mathcal{P}_{k,m-k}$ by $P^\star\sigma(\varphi) = \sigma(P(\varphi))$. Denote this representation $(P^\star\sigma, U_\sigma)$ of $\mathcal{P}_{k,m-k}$ by the same letter σ, then the induced representation $\mathrm{Ind}\,\sigma = \mathrm{Ind}_{\mathcal{P}_{k,m-k}}^{\mathrm{End}[m]}\sigma$ has the following property according to [7].

Proposition 1.1 ([7], Proposition 3.1). *Let (σ, U_σ) be an irreducible representation of* End$[k]$ *on which $\mathcal{R}_{k-1}$ acts trivially. Then the* End$[m]$*-module* $\mathrm{Ind}_{\mathcal{P}_{k,m-k}}^{\mathrm{End}[m]}\sigma$ *has the unique irreducible quotient.*

By this proposition, the following definition makes sense.

Definition. Let σ_λ be an irreducible module of $\mathfrak{S}_k$ corresponding to the partition λ. We extend it to the representation of End$[k]$ on which $\mathcal{R}_{k-1}$ acts trivially. The unique irreducible quotient of $\mathrm{Ind}_{\mathcal{P}_{k,m-k}}^{\mathrm{End}[m]}\sigma_\lambda$ is denoted by Σ_λ and call it a *standard representation* of End$[m]$. If D is the Young diagram of shape λ, we also denote Σ_λ as Σ_D.

Let $\mathsf{P}(k)$ be the set of all the partitions of k. In [7], the following theorem is proved.

Theorem 1.2 ([7], Theorem 3.5)**.** *The standard representations*

$$\{\Sigma_\lambda \mid \lambda \in \mathsf{P}(k) \ \ (1 \le k \le m)\}$$

of End$[m]$ *are mutually inequivalent, and they give a complete set of representatives of equivalence classes of irreducible representations of* End$[m]$.

1.3. End$[m]$-module $\Lambda(n,m)$ and skew $GL_n \times GL_m$-duality

For any $\varphi \in \mathrm{End}[m]$ and for any homogenous element $\xi_{i_1 j_1} \wedge \cdots \wedge \xi_{i_r j_r} \in \Lambda(n,m)$, we define

$$\varphi(\xi_{i_1 j_1} \wedge \cdots \wedge \xi_{i_r j_r}) := \xi_{i_1 \varphi(j_1)} \wedge \cdots \wedge \xi_{i_r \varphi(j_r)},$$

and extend it linearly. Then $\Lambda(n,m)$ becomes an End$[m]$-module, and the actions of End$[m]$ and $W(n)$ commute with each other (see [8]). Put

$$\mathcal{V}_k = \mathcal{R}_k \Lambda(n,m)$$

for $k \ge 1$ and $\mathcal{V}_0 = \mathbb{C}$. Then $(\mathcal{V}_k)_{0 \le k \le m}$ is a natural filtration of $W(n) \times \mathrm{End}[m]$-module $\Lambda(n,m)$. Consider the graded module

$$\mathrm{gr}\mathcal{V} = \oplus_{k \ge 0} \mathcal{V}_k / \mathcal{V}_{k-1} =: \oplus_{k \ge 0} \mathcal{V}(k) \quad (\mathcal{V}_{-1} = (0)).$$

Clearly $\mathcal{V}(k) = \mathcal{V}_k / \mathcal{V}_{k-1}$ has a $\mathbb{Z}$-graded structure which inherits from $\Lambda(n,m)$, and is a $W(n) \times \mathrm{End}[m]$-module. In particular, $\mathcal{V}(m) \cong \otimes^m (\Lambda(n)/\mathbb{C} \cdot \mathbf{1})$, and $\mathcal{R}_k$ acts trivially on $\mathcal{V}(m)$ for $k < m$. This means $\mathcal{V}(m)$ is essentially an $\mathfrak{S}_m$-module.

Let ρ_μ be an irreducible module of $\mathfrak{gl}(n)$ with highest weight μ. In the article [2], Howe has established the following decomposition.

Theorem 1.3. *As a $\mathfrak{gl}(n) \times \mathfrak{gl}(m)$-module, $\Lambda = \otimes^m \Lambda(n)$ is decomposed as*

$$\otimes^m \Lambda(n) \cong \sum_\lambda{}^{\oplus} \rho_\lambda \otimes \rho_{\lambda^t}.$$

In the above direct sum, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ ranges over all partitions with $\lambda_1 \le m$. The diagram λ^t denotes the transpose of λ. Let $\Lambda^N := \sum_{|\mu|=N} \Lambda_\mu$, then Λ^N is stable under the action of $\mathfrak{gl}(n) \times \mathfrak{gl}(m)$ and we have a decomposition:

$$\Lambda^N \cong \sum_{|\lambda|=N} \rho_\lambda \otimes \rho_{\lambda^t}.$$

Furthermore, $x_\lambda := \xi_{11} \cdots \xi_{1\lambda_1} \xi_{21} \cdots \xi_{2\lambda_2} \cdots \xi_{n1} \cdots \xi_{n\lambda_n}$ is a joint highest weight vector of $\mathfrak{gl}(n) \times \mathfrak{gl}(m)$-module $\rho_\lambda \otimes \rho_{\lambda^t}$.

2. Howe correspondence for $W(n) \times \mathrm{End}[m]$-module $\Lambda(n, m)\,(m \leq n)$

In this section, we consider the decomposition of $W(n) \times \mathrm{End}[m]$-module $\Lambda(n, m)$ under the condition $m \leq n$. It seems difficult to do it directly. Having considered the fact that in the $W(n)$-module $\Lambda(n)$ there exists an invariant space $\mathbb{C} = \mathbb{C} \cdot \mathbf{1}$ under $W(n)$, we first decompose $W(n) \times \mathrm{End}[m]$-module $\otimes^m(\Lambda(n)/\mathbb{C}) \cong \mathcal{V}(m)$, just done as in [7].

2.1. Decomposition of the top level $\mathcal{V}(m)$ (the case $m \leq n$)

To decompose the $W(n) \times \mathrm{End}[m]$-module $\mathcal{V}(m)$, it is enough to decompose $\mathcal{V}(m)$ as a $W(n) \times \mathfrak{S}_m$-module because $\mathcal{R}_k (k < m)$ acts on the top level trivially. Under the condition $m \leq n$, we can determine the commutant algebra of $W(n)$ in the space $\mathrm{End}(\mathcal{V}(m))$, so we can follow the traditional method used in Weyl-Schur duality to get the decomposition.

First, let us determine the commutant algebra.

Lemma 2.1. *Assume that $m \leq n$. Then, the commutant algebra of $W(n)$ in the space $\mathrm{End}(\mathcal{V}(m))$ is isomorphic to $\mathbb{C}[\mathfrak{S}_m]$, the group ring of the symmetric group of degree m.*

Proof. We can prove the lemma by a similar method as in Theorem 2.3 of [9]. So we omit the proof. ☐

Since the algebra $\mathbb{C}[\mathfrak{S}_m]$ is semisimple, we can decompose $\mathcal{V}(m)$ as follows:

$$\mathcal{V}(m) \cong \sum\nolimits_{\lambda \in \mathsf{P}(m)}^{\oplus} \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m)) \otimes \sigma_\lambda,$$

where $\mathsf{P}(m)$ is the set of partitions of size m, and σ_λ is the irreducible representation of $\mathfrak{S}_m$ corresponding to a partition λ. By the above lemma, we know that $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m))$ is a $W(n)$-module and it is indecomposable. Moreover, by the following proposition, it is a lowest weight module of $W(n)$.

Proposition 2.2. *Assume that $m \leq n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition of m. Then the $W(n)$-module $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m))$ is a lowest weight module with lowest weight $(0, \ldots, 0, \mu_m, \ldots, \mu_1)$, where $\mu = (\mu_1, \ldots, \mu_m)$ is the transpose of the partition λ.*

Proof. Let e_λ be a standard Young symmetrizer of the partition λ, then $e_\lambda^2 = e_\lambda$, and $\mathbb{C}[\mathfrak{S}_m] \cdot e_\lambda$ is a realization of σ_λ. First, we want to show that

$$\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m)) \cong e_\lambda(\mathcal{V}(m)) \quad (\text{as } W(n)\text{-module}).$$

In fact, by the explicit realization $\sigma_\lambda = \mathbb{C}[\mathfrak{S}_m] \cdot e_\lambda$, it is easy to check that the mapping

$$F : \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m)) \longrightarrow e_\lambda(\mathcal{V}(m))$$

defined by

$$F(f) := f(e_\lambda) \qquad (f \in \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m)))$$

is a $W(n)$-module isomorphism. So it is enough to prove that $e_\lambda(\mathcal{V}(m))$ is a lowest weight module with lowest weight $(0, \ldots, 0, \mu_m, \ldots, \mu_1)$.

Let $\mathcal{V}^k(m)$ denote the subspace of $\mathcal{V}(m)$ of degree k, then $\mathcal{V}(m) = \sum_{k=m}^{nm} \mathcal{V}^k(m)$ and $\mathcal{V}^k(m)$ is a $\mathfrak{gl}(n) \times \mathfrak{S}_m$-module. Let $V = \Lambda_1(n) \cong \mathbb{C}^n$, then by the classical Weyl-Schur duality,

$$\mathcal{V}^m(m) \cong \sum_{\lambda \in \mathsf{P}(m)}^{\oplus} \rho_{\lambda^t} \otimes \sigma_\lambda$$

as a $\mathfrak{gl}(n) \times \mathfrak{S}_m$-module, where $\lambda^t := (\mu_1, \cdots, \mu_m)$ denotes the transpose of λ and ρ_{λ^t} denotes the representation of $\mathfrak{gl}(n)$ corresponding to λ^t. Moreover, we have

$$\rho_{\lambda^t} \cong e_\lambda(\otimes^m \Lambda_1(n)) \cong e_\lambda(\mathcal{V}^m(m)).$$

Hence, $e_\lambda(\mathcal{V}^m(m))$ is an irreducible $\mathfrak{gl}(n)$-module with the lowest weight $(0, \ldots, 0, \mu_m, \ldots, \mu_1)$. Note that

$$W_{-1}(e_\lambda(\mathcal{V}^m(m))) = e_\lambda(W_{-1}\mathcal{V}^m(m)) \equiv (0) \pmod{\mathcal{V}(m-1)},$$

because $\mathcal{V}^m(m)$ has the smallest possible degree.

Now it is enough to prove that $e_\lambda(\mathcal{V}(m))$ is generated by $e_\lambda(\mathcal{V}^m(m))$ as $W(n)$-module. But this is easy if we notice that the element $\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm} \in \mathcal{V}^m(m)$ is a cyclic vector for $W(n)$-module $\mathcal{V}(m)$ and $e_\lambda(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm}) \neq 0$. □

Let $L_\lambda := e_\lambda(\mathcal{V}(m)) \cong \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m))$. By the above proposition, L_λ is a lowest weight module of $W(n)$ with lowest weight $(0, \ldots, 0, \mu_m, \ldots, \mu_1)$, where $(\mu_1, \mu_2, \ldots, \mu_m) = \lambda^t$. So it has a unique irreducible quotient, denoted by π_λ. Then the following theorem holds.

Theorem 2.3. *Assume that $m \leq n$. Then, $W(n) \times \mathfrak{S}_m$-module $\mathcal{V}(m)$ decomposes as*

$$\mathcal{V}(m) \cong \sum\nolimits_{\lambda \in \mathsf{P}(m)}^{\oplus} L_\lambda \otimes \sigma_\lambda. \tag{2.1}$$

Moreover, $\mathcal{V}(m)$ is quotient multiplicity-free with irreducible quotients $\{\pi_\lambda \otimes \sigma_\lambda \mid \lambda \in \mathsf{P}(m)\}$, that is,

$$\dim \mathrm{Hom}_{W(n) \times \mathfrak{S}_m}(\mathcal{V}(m), \pi_\lambda \otimes \sigma_\lambda) = 1 \quad (\forall \lambda \in \mathsf{P}(m)).$$

Proof. From Lemma 2.1, the formula of the decomposition is obvious. Noting that L_λ is a lowest weight module, we have

$$\dim \mathrm{Hom}_{W(n)}(L_\lambda, \pi_\lambda) = 1.$$

Combining with the formula (2.1), we complete the proof of the theorem. □

2.2. Duality for general level

To decompose the module $\Lambda(n,m)$ of $W(n)\times\mathrm{End}[m]$, first we study the structures of the general level $\mathcal{V}(k)=\mathcal{V}_k/\mathcal{V}_{k-1}$.

Let $\Lambda^+(n):=\Lambda(n)/\mathbb{C}$. Then there is a natural injection

$$\otimes^k\Lambda^+(n)\hookrightarrow\otimes^k\Lambda^+(n)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\subseteq\mathcal{V}(k).$$

By the results in Section 2.1, as a $W(n)\times\mathfrak{S}_k$-module, $\otimes^k\Lambda^+(n)$ is decomposed as

$$\otimes^k\Lambda^+(n)\cong\sum\nolimits^{\oplus}_{\lambda\in\mathsf{P}(k)}\mathrm{Hom}_{\mathfrak{S}_k}(\sigma_\lambda,\otimes^k\Lambda^+(n))\otimes\sigma_\lambda,$$

where $\mathrm{Hom}_{\mathfrak{S}_k}(\sigma_\lambda,\otimes^k\Lambda^+(n))$ is a lowest weight module of $W(n)$.

Lemma 2.4.

$$\mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{k,m-k}}\otimes^k\Lambda^+(n)\cong\mathcal{V}(k).$$

Proof. The proof is the same as that of Lemma 4.1 of [7]. □

From this lemma, we have

$$\begin{aligned}\mathcal{V}(k)&\cong\mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{k,m-k}}(\otimes^k\Lambda^+(n))\\&\cong\mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{k,m-k}}\sum\nolimits^{\oplus}_{\lambda\in\mathsf{P}(k)}\mathrm{Hom}_{\mathfrak{S}_k}(\sigma_k,\otimes^k\Lambda^+(n))\otimes\sigma_\lambda\\&\cong\sum\nolimits^{\oplus}_{\lambda\in\mathsf{P}(k)}\mathrm{Hom}_{\mathfrak{S}_k}(\sigma_k,\otimes^k\Lambda^+(n))\otimes\mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{k,m-k}}\sigma_\lambda.\end{aligned}$$

Therefore it is easy to see that the irreducible representation $\pi_\lambda\otimes\Sigma_\lambda$ ($\lambda\in\mathsf{P}(k)$) is an irreducible quotient of $\mathcal{V}(k)$.

Theorem 2.5. (1) *Any irreducible quotient of* $W(n)\times\mathrm{End}[m]$*-module* $\mathcal{V}(k)$ *is of the form* $\pi_\lambda\otimes\Sigma_\lambda$ ($\lambda\in\mathsf{P}(k)$).

(2) *For any* $\lambda\in\mathsf{P}(k)$,

$$\dim\mathrm{Hom}_{W(n)\times\mathrm{End}[m]}(\mathcal{V}(k),\pi_\lambda\otimes\Sigma_\lambda)=1.$$

Proof. The assertion (2) is immediate from the remark before the theorem. The proof of (1) is essentially the same as that of Theorem 4.2 of [7]. □

2.3. Howe correspondence for $W(n)\times$ End$[m]$-module $\Lambda(n,m)$ ($m\le n$)

In this subsection, we consider the whole $W(n)\times\mathrm{End}[m]$-module $\Lambda(n,m)$ and give an affirmative answer to Conjectures 1 and 2 posed in Introduction under the condition $m\le n$. We say that an irreducible representation $\pi\otimes\Sigma$ has quotient multiplicity k in a representation U if

$$\dim\mathrm{Hom}_{W(n)\times\mathrm{End}[m]}(U,\pi\otimes\Sigma)=k.$$

Theorem 2.6. (1) *The tensor product $\otimes^m \Lambda(n)$ is quotient multiplicity free, that is, for an irreducible representation $\pi \otimes \Sigma$ of $W(n) \times \mathrm{End}[m]$,*

$$\dim \mathrm{Hom}_{W(n)\times \mathrm{End}[m]}(\otimes^m \Lambda(n), \pi \otimes \Sigma) \leq 1.$$

(2) *The quotient multiplicity is one if and only if $\pi \otimes \Sigma$ is isomorphic to $\pi_\lambda \otimes \Sigma_\lambda$ for some partition λ of size k $(1 \leq k \leq m)$.*

Since the proof of this theorem is essentially the same as that of [7, Theorem 5.1], we omit it.

3. The decompositions of $W(2) \times \mathrm{End}[m]$-module $\Lambda(2, m)$

In the above sections, we have discussed decompositions of $\Lambda(n, m)$ as a $W(n) \times \mathrm{End}[m]$-module under the condition $m \leq n$. For the case $m > n$, we do not have a satisfactory result yet. But for $n = 2$, using the representation theory of $\mathfrak{sl}(2)$, we can make an explicit calculation and get a detailed decomposition of the space $\Lambda(2, m)$ as a $W(2) \times \mathrm{End}[m]$-module.

We recall some notations used in the above sections and fix some new notations. For convenience, we denote $\psi^{\otimes m}(X)$ by the same symbol X for any $X \in W(2)$. In the case of $n = 2$, we have $W_1(2) = \langle C_1, C_2\rangle$, where $C_i = \xi_i C$ with $C = \xi_1\partial_1 + \xi_2\partial_2$. Let

$$e := \xi_1\partial_2,\ f := \xi_2\partial_1,\ h_i := \xi_i\partial_i\ (i = 1, 2)\ \text{ and }\ h := \xi_1\partial_1 - \xi_2\partial_2;$$

then $\mathcal{H} = \langle h_1, h_2\rangle/\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{gl}(2)$. Let $\{\alpha_1, \alpha_2\}$ be the dual base of $\{h_1, h_2\}$ and $\alpha = \alpha_1 - \alpha_2$. In the coordinate system, we denote $\alpha_1 = (1, 0), \alpha_2 = (0, 1)$. Note that $\{\alpha_1, \alpha_2\}$ is the set of weights of $\mathfrak{gl}(2)$-module $W_1(2)$, and α_1 is the highest weight.

Let $\Phi_2(l)$ be the subset of the dominant integral weights of $\mathfrak{gl}(2)$ defined by

$$\Phi_2(l) := \{\mu \mid |\mu| = l, \mu_1 \leq l\} \cup \{\mu \mid l + 1 \leq |\mu| \leq 2(l - 1), \mu_1 \leq l - 1\}$$

and $\Phi_2 := \cup_{l=1}^{2m-2} \Phi_2(l)$. Let

$$\mathsf{P}_2(l) := \{\lambda = (\lambda_1, \dots, \lambda_l) \mid |\lambda| = l, \lambda_2 \leq 2\}$$

be a subset of the set of all the partitions of l, and $\mathsf{P}_2 := \cup_{l=1}^{m} \mathsf{P}_2(l)$. Define a mapping Θ_l from $\mathsf{P}_2(l)$ to $\Phi_2(l)$ as follows:

$$\Theta_l(\lambda) := (r + k, l - r) \qquad \text{for } \lambda = (\lambda_1, \dots, \lambda_r, 0, \dots, 0) \in \mathsf{P}_2(l) \quad \text{with } \lambda_r \geq 1,$$

where $k = \max\{0, \lambda_1 - 2\}$. Then Θ_l is well-defined and gives a bijection.

Denote the inverse map of Θ_l by Γ_l. For any $\mu \in \Phi_2(l)$, put $k = |\mu| - l$, then $\Gamma_l(\mu)$ can be expressed as $\Gamma_l(\mu) = (\mu_1 - k, \mu_2 - k, \underbrace{1, \dots, 1}_{k})^t$. For convenience, define $\Gamma_l(\mu) = 0$ if $\mu \notin \Phi_2(l)$.

Let $\mathcal{V}_\mu(l)$ be the weight space of weight μ in the $\mathfrak{gl}(2)$-module $\mathcal{V}(l)$ (see Section 1.3) and $V_\mu^+(l)$ be the subspace of $\mathcal{V}_\mu(l)$ consisting of the highest weight vectors of $\mathfrak{gl}(2)$. Let V_μ^+ be the set consisting of the highest weight vectors of $\mathfrak{gl}(2)$-module $\Lambda(2,m)$ of weight μ. Then $V_\mu^+(l) = \mathcal{R}_l(V_\mu^+)/\mathcal{R}_{l-1}(V_\mu^+)$ and $V_\mu^+ \cong \sum_{l=0}^m V_\mu^+(l)$ as $\mathfrak{S}_m$-module.

3.1. The top level as $\mathfrak{gl}(2) \times \mathfrak{S}_m$-module

In this subsection, we consider the decomposition of $\mathfrak{gl}(2) \times \mathfrak{S}_m$-module $\mathcal{V}(m)$, the top level of $\otimes^m \Lambda(2)$.

Lemma 3.1. *Let μ be a weight of $\mathfrak{gl}(2)$-module $\mathcal{V}(m)$, then $\mathcal{V}_\mu(m)$ is decomposed as*

$$\mathcal{V}_\mu(m) = f(\mathcal{V}_{\mu+\alpha}(m)) \oplus V_\mu^+(m).$$

By this lemma, the decomposition of $\mathcal{V}(m)$ as a $\mathfrak{gl}(2) \times \mathfrak{S}_m$-module follows from the decomposition of $\mathfrak{S}_m$-module $V_\mu^+(m)$.

Let us consider the lowest degree space $\mathcal{V}^m(m)$ of $\mathcal{V}(m)$. We have $\mathcal{V}^m(m) \cong \otimes^m V$ as $\mathfrak{gl}(2)$-module and $\mathcal{V}^m(m) \otimes \mathrm{sgn} \cong \otimes^m V$ as $\mathfrak{S}_m$-module, where $V \cong \mathbb{C}^2$ is the defining representation space of $\mathfrak{gl}(2)$. By the Weyl-Schur duality, $V_\mu^+(m) \cong \sigma_{\mu^t} \cong \sigma_{\Gamma_m(\mu)}$ if $|\mu| = m$. For the space $\mathcal{V}^{m+r}(m)(r \geq 1)$, we have

Lemma 3.2. *Take $\mu \in \Phi_2(m)$ and put $r = |\mu| - m$. As a module of $\mathfrak{S}_m$, $V_\mu^+(m)$ is decomposed as*

$$\begin{aligned} V_\mu^+(m) &\cong \mathrm{Ind}_{\mathfrak{S}_{m-r}\times\mathfrak{S}_r}^{\mathfrak{S}_m}((\otimes^{m-r}\Lambda^+(2))^+_{\mu-(r,r)} \otimes \mathbf{1}_{\mathfrak{S}_r}) \\ &\cong \sigma_{\Gamma_m(\mu)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1)} \oplus \sigma_{\Gamma_m(\mu-\alpha_2)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}, \end{aligned}$$

where $\mathbf{1}_{\mathfrak{S}_r}$ denotes the identity representation, and $\Lambda^+(2) = \Lambda(2)/\mathbb{C}\cdot\mathbf{1}$.

Proof. Let $U_{\mu-(r,r)} = (\otimes^{m-r}\Lambda^+(2))_{\mu-(r,r)}$ and $U^+_{\mu-(r,r)} = (\otimes^{m-r}\Lambda^+(2))^+_{\mu-(r,r)}$. We abbreviate $\mathrm{Ind}_{\mathfrak{S}_{m-r}\times\mathfrak{S}_r}^{\mathfrak{S}_m}$ as Ind in this proof.

At first, we prove the following isomophism:

$$\mathcal{V}_\mu(m) \cong \mathrm{Ind}(U_{\mu-(r,r)} \otimes \mathbf{1}_{\mathfrak{S}_r}). \tag{3.1}$$

Define the mapping F as follows:

$$F(\varphi \otimes x \otimes 1) = \varphi(x \cdot \xi_{1,m-r+1}\xi_{2,m-r+1}\cdots\xi_{1,m}\xi_{1,m}) \quad (\varphi \in \mathfrak{S}_m,\ x \in U_{\mu-(r,r)}),$$

and extend it linearly to the space $\mathbb{C}[\mathfrak{S}_m] \otimes U_{\mu-(r,r)} \otimes \mathbf{1}_{\mathfrak{S}_r}$. Obviously, F is well-defined $\mathfrak{S}_m$-mapping and is a surjection. Therefore, from the following equality of dimensions

$$\dim \mathcal{V}_\mu(m) = \frac{m!}{r!(m-\mu_1)!(\mu_1-r)!} = \dim \mathrm{Ind}\,(U_{\mu-(r,r)} \otimes \mathbf{1}),$$

F is an isomorphism and (3.1) is deduced.

Note that $U^+_{\mu-(r,r)}$ is the subspace of $U_{\mu-(r,r)}$ consisting of the highest weight vectors of $\mathfrak{gl}(2)$ of weight $\mu-(r,r)$, so $\mathrm{Ind}(U^+_{\mu-(r,r)}\otimes \mathbf{1}_{\mathfrak{S}_r})$ is a subspace of $\mathrm{Ind}(U_{\mu-(r,r)}\otimes \mathbf{1}_{\mathfrak{S}_r})$. Furthermore, the mapping F sends $\mathrm{Ind}(U^+_{\mu-(r,r)}\otimes \mathbf{1}_{\mathfrak{S}_r})$ to $V^+_\mu(m)$, because

$$e\cdot F(\varphi\otimes x\otimes 1)=\varphi((e\cdot x)\cdot \xi_{1,m-r+1}\xi_{2,m-r+1}\cdots\xi_{1,m}\xi_{1,m})=0 \quad (x\in U^+).$$

On the other hand,

$$\begin{aligned}\dim V^+_\mu(m)&=\dim\mathcal{V}_\mu(m)-\dim\mathcal{V}_{\mu+\alpha}(m)=\frac{m!(\mu_1-\mu_2+1)}{r!(m-\mu_1)!(\mu_1-r+1)!}\\&=\dim\mathrm{Ind}(U^+_{\mu-(r,r)}\otimes\mathbf{1}).\end{aligned}$$

Therefore, the first isomorphism in this lemma is established by the restriction of F on the space $\mathrm{Ind}(U^+_{\mu-(r,r)}\otimes \mathbf{1}_{\mathfrak{S}_r})$.

By the Weyl-Schur duality, $U^+_{\mu-(r,r)}\cong\sigma_{\Gamma_{m-r}(\mu-(r,r))}\cong\sigma_{(\mu-(r,r))^t}$. By Young's rule (see [4]), we complete the proof. □

Combining Lemmas 3.1 and 3.2, we obtain

Theorem 3.3. *As a $\mathfrak{gl}(2)\times\mathfrak{S}_m$-module, $\mathcal{V}(m)$ decomposes as follows:*

$$\begin{aligned}\mathcal{V}(m)\cong&\sum_{\substack{|\mu|=m\\ \mu_1\ge\mu_2\ge0}}\rho_\mu\otimes\sigma_{\Gamma_m(\mu)}\oplus\\&\sum_{\substack{m+1\le|\mu|\le 2m\\ m\ge\mu_1\ge\mu_2\ge1}}\rho_\mu\otimes\left[\sigma_{\Gamma_m(\mu)}\oplus\sigma_{\Gamma_m(\mu-\alpha_1)}\oplus\sigma_{\Gamma_m(\mu-\alpha_2)}\oplus\sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}\right]\\\cong&\sum_{\mu\in\Phi_2(m)}\Big(\rho_\mu\oplus\rho_{\mu+\alpha_1}\oplus\rho_{\mu+\alpha_2}\oplus\rho_{\mu+\alpha_1+\alpha_2}\Big)\otimes\sigma_{\Gamma_m(\mu)}.\end{aligned}$$

Here, if $\mu+\alpha_2$ is not a dominant weight (resp. $\mu_1+1>m$), we interpret $\rho_{\mu+\alpha_2}=0$ (resp. $\rho_{\mu+\alpha_1}=0$).

3.2. The top level as $W(2)\times\mathfrak{S}_m$-module

From Theorem 3.3, we obtain the decomposition of $\mathcal{V}(m)$ as a $\mathfrak{gl}(2)\times\mathfrak{S}_m$-module. In this subsection, we consider it as a $W(2)\times\mathfrak{S}_m$-module.

Because $\mathfrak{S}_m$ is contained in the commutant algebra of $W(2)$, it is obvious that

$$\mathcal{V}(m)\cong{\sum_\lambda}^{\oplus}\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda,\mathcal{V}(m))\otimes\sigma_\lambda,$$

where λ runs over all the partitions of m. By Theorem 3.3, if λ is not in $\mathsf{P}_2(m)$, $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda,\mathcal{V}(m))=(0)$, and if $\lambda\in\mathsf{P}_2(m)$,

$$\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda,\mathcal{V}(m))\cong\rho_{\Theta_m(\lambda)}\oplus\rho_{\Theta_m(\lambda)+\alpha_1}\oplus\rho_{\Theta_m(\lambda)+\alpha_2}\oplus\rho_{\Theta_m(\lambda)+\alpha_1+\alpha_2}.$$

Using the same method as Proposition 2.2, we have

$$\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_\lambda, \mathcal{V}(m)) \cong e_B(\mathcal{V}(m)) \qquad \text{(as } W(2)\text{-module)},$$

where B is a Young tableau of shape λ, and e_B is the standard Young symmetrizer of B. Put $\pi_\mu := e_B(\mathcal{V}(m))$, where $\mu = \Theta_m(\lambda)$. Then, by the above formula, π_μ is independent of the choice of B as a $W(2)$-module, and as $\mathfrak{gl}(2)$-modules,

$$\pi_\mu \cong \rho_\mu \oplus \rho_{\mu+\alpha_1} \oplus \rho_{\mu+\alpha_2} \oplus \rho_{\mu+\alpha_1+\alpha_2}.$$

Let us show that the $W(2)$-module π_μ is irreducible. First, we consider actions of the operators in $W_1(2)$ and $W_2(2)$.

Let $k_\mu = (\mu_1 - \mu_2 + 1)^{-1}$, and put

$$\tilde{C}_2 \stackrel{\text{def}}{=} C_2 - k_\mu f C_1, \quad \tilde{\partial}_1 \stackrel{\text{def}}{=} \partial_1 + k_\mu f \partial_2.$$

From the relations $[e, C_1] = 0, [e, \tilde{C}_2] = C_1 - k_\mu h C_1$, we know that the mappings $C_1 : V^+_\mu(m) \to V^+_{\mu+\alpha_1}(m)$, $\tilde{C}_2 : V^+_\mu(m) \to V^+_{\mu+\alpha_2}(m)$ are well-defined. Similarly, from the relations $[e, \partial_2] = 0, [e, \tilde{\partial}_1] = -\partial_2 + k_\mu h \partial_2$, we deduce that the mappings $\tilde{\partial}_1 : V^+_\mu(m) \to V^+_{\mu-\alpha_1}(m)$, $\partial_2 : V^+_\mu(m) \to V^+_{\mu-\alpha_2}(m)$ are well-defined.

Lemma 3.4. *For the mappings C_1, and $\tilde{C}_2$, the following sequences are exact*:

$$0 \to V^+_{(m-\mu_2,\mu_2)} \xrightarrow{C_1} V^+_{(m-\mu_2+1,\mu_2)}(m) \xrightarrow{C_1} \cdots \xrightarrow{C_1} V^+_{(m,\mu_2)}(m) \quad \textit{for } 1 \le \mu_2 \le [\tfrac{m}{2}], \tag{3.2}$$

$$V^+_{(\mu_2,\mu_2)} \xrightarrow{C_1} V^+_{(\mu_2+1,\mu_2)}(m) \xrightarrow{C_1} \cdots \xrightarrow{C_1} V^+_{(m,\mu_2)}(m) \quad \textit{for } m \ge \mu_2 > [\tfrac{m}{2}], \tag{3.3}$$

$$0 \to V^+_{(\mu_1,m-\mu_1)}(m) \xrightarrow{\tilde{C}_2} V^+_{(\mu_1,m-\mu_1+1)}(m) \xrightarrow{\tilde{C}_2} \cdots \xrightarrow{\tilde{C}_2} V^+_{(\mu_1,\mu_1)}(m) \quad \textit{for } m > \mu_1 \ge [\tfrac{m}{2}]. \tag{3.4}$$

Proof. Because $2C_1^2 = [C_1, C_1] = 0$, we have $\mathrm{Im}\, C_1 \subseteq \ker C_1$. For any $x \in \ker C_1 \cap V^+_\mu(m)$, put $x_1 = C_2\partial_2 x$ and $x_2 = \partial_2 C_2 x$. Then $x_1 + x_2 = [\partial_2, C_2]x = \mu_1 x$. Furthermore, $x_1, x_2 \in \ker C_1 \cap V^+_\mu(m)$ by the relations $[e, C_2\partial_2] = -e - \partial_2 C_1$ and $[C_1, C_2\partial_2] = -C_2 e$. Therefore,

$$C_1(\partial_1 + k_\mu f\, \partial_2)x_2 = C_1\partial_1 x_2 + k_\mu f \cdot (\partial_2)^2 C_2 x_2 = \mu_2 x_2.$$

That means $x_2 \in \mathrm{Im}(C_1|_{V^+_{\mu-\alpha_1}(m)})$. On the other hand,

$$\begin{aligned} C_1(\partial_1 + k_\mu f\, \partial_2)x_1 &= \mu_2 x_1 - k_\mu C_2\partial_2 x_1 \\ &= (\mu_2 - \mu_1 k_\mu)x_1 + k_\mu \partial_2 C_2 x_1 = (\mu_1 - \mu_2)(\mu_2 - 1)k_\mu x_1, \end{aligned}$$

so if $\mu_1 > \mu_2 > 1$, then $x_1 \in \mathrm{Im}(C_1|_{V^+_{\mu-\alpha_1}(m)})$. If $\mu_2 = 1$, then (3.2) is obvious. Hence, (3.2) and (3.3) is obtained.

(3.4) is obtained by the same method as above. □

Combining Lemmas 3.2 and 3.4, we obtain

Lemma 3.5. *We have the following isomorphisms as $\mathfrak{S}_m$-modules.*

(1) $\ker(\tilde{\partial}_1 |_{V_\mu^+(m)}) \cap \ker(\partial_2 |_{V_\mu^+(m)}) \cong \sigma_{\Gamma_m(\mu)}$.

(2) $\ker(C_1|_{V_\mu^+(m)}) = \mathrm{Im}(C_1|_{V_{\mu-\alpha_1}^+(m)}) \cong \sigma_{\Gamma_m(\mu-\alpha_1)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ *if* $m > \mu_1 > \mu_2$,
$\ker(C_1|_{V_\mu^+(m)}) \cong \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ *if* $\mu_1 = \mu_2$.

(3) $\ker(\tilde{C}_2|_{V_\mu^+(m)}) = \mathrm{Im}(\tilde{C}_2|_{V_{\mu-\alpha_2}^+(m)}) \cong \sigma_{\Gamma_m(\mu-\alpha_2)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ *if* $m > \mu_1 > \mu_2$,
$\mathrm{Im}(\tilde{C}_2|_{V_{\mu-\alpha_2}^+(m)}) \cong \sigma_{\Gamma_m(\mu-\alpha_2)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ *if* $\mu_1 = \mu_2$.

(4) $\ker(C_1|_{V_\mu^+(m)}) \cap \ker(\tilde{C}_2|_{V_\mu^+(m)}) \cong \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$.

Proof. By Lemma 3.2,

$$V_\mu^+(m) \cong \sigma_{\Gamma_m(\mu)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1)} \oplus \sigma_{\Gamma_m(\mu-\alpha_2)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}.$$

(1) Because $\sigma_{\Gamma_m(\mu)}$ does not appear in the decompositions of $V_{\mu-\alpha_1}^+(m)$ and $V_{\mu-\alpha_2}^+(m)$, we get $\sigma_{\Gamma_m(\mu)} \subseteq \ker(\tilde{\partial}_1 |_{V_\mu^+(m)}) \cap \ker(\partial_2 |_{V_\mu^+(m)})$.

For any $x \in \ker(\tilde{\partial}_1 |_{V_\mu^+(m)}) \cap \ker(\partial_2 |_{V_\mu^+(m)})$, we have $\partial_1\partial_2 C_1 C_2 x = -\mu_2(\mu_1+1)x$. But there is only $\sigma_{\Gamma_m(\mu)}$ which appears in the decompositions of both $V_\mu^+(m)$ and $V_{\mu+\alpha_1+\alpha_2}^+(m)$. So $x \in \sigma_{\Gamma_m(\mu)}$. This establishes the isomorphism of (1).

(2) Assume that $m > \mu_1 > \mu_2$. By Lemma 3.4, $\ker(C_1|_{V_\mu^+(m)}) = \mathrm{Im}(C_1|_{V_{\mu-\alpha_1}^+(m)})$. Because $\sigma_{\Gamma_m(\mu-\alpha_1)}$ and $\sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ do not appear the decomposition formula of $V_{\mu+\alpha_1}^+(m)$, we have $\sigma_{\Gamma_m(\mu-\alpha_1)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)} \subseteq \ker(C_1|_{V_\mu^+(m)})$. Because there are only $\sigma_{\Gamma_m(\mu-\alpha_1)}$ and $\sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ appeared in the decompositions of both $V_{\mu-\alpha_1}^+(m)$ and $V_\mu^+(m)$, therefore, $\sigma_{\Gamma_m(\mu-\alpha_1)} \oplus \sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)}$ can only appear as irreducible components of $\ker(C_1|_{V_\mu^+(m)})$. Thus, we get the first formula in (2).

If $\mu_1 = \mu_2$, then $V_{\mu-\alpha_1}^+(m) = 0$ and so $\mathrm{Im}(C_1|_{V_{\mu-\alpha_1}^+(m)}) = 0$. For the space $\ker(\tilde{\partial}_1 |_{V_\mu^+(m)})$, it is obvious that $\sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)} \subseteq \ker(C_1|_{V_\mu^+(m)})$. On the other hand,

$$\mathrm{Im}(C_1|_{V_\mu^+(m)}) = \ker(C_1|_{V_{\mu+\alpha_1}^+(m)}) = \sigma_{\Gamma_m(\mu)} \oplus \sigma_{\Gamma_m}(\mu-\alpha_2).$$

Again, using the decomposition of $V_{(\mu_2,\mu_2)}^+(m)$, we obtain that $\sigma_{\Gamma_m(\mu-\alpha_1-\alpha_2)} \supseteq \ker(C_1 |_{V_\mu^+(m)})$. Therefore, the second formula in (2) is established.

(3) It is similar to (2).

(4) It is easy to see that (2) and (3) imply (4). □

Now we return to the $W(2)$-module $\pi_\mu = e_B(\mathcal{V}(m))$.

Lemma 3.6. *The $W(2)$-module π_μ is irreducible.*

Proof. From the definition of π_μ and Theorem 3.3 and Lemma 3.5, the following formulas are deduced easily:

$$\pi_\mu = e_B(\mathcal{V}^{|\mu|}(m)) \oplus e_B(\mathcal{V}^{|\mu|+1}(m)) \oplus e_B(\mathcal{V}^{|\mu|+2}(m)),$$

and

$$\begin{aligned} e_B(\mathcal{V}^{|\mu|}(m)) &\cong \rho_\mu, \\ e_B(\mathcal{V}^{|\mu|+1}(m)) &= C_1(e_B(\mathcal{V}^{|\mu|}(m))) + \tilde{C}_2(e_B(\mathcal{V}^{|\mu|}(m))) \cong \rho_{\mu+\alpha_1} + \rho_{\mu+\alpha_2}, \\ e_B(\mathcal{V}^{|\mu|+2}(m)) &= C_1\tilde{C}_2(e_B(\mathcal{V}^{|\mu|}(m))) = C_1C_2(e_B(\mathcal{V}^{|\mu|}(m))) \cong \rho_{\mu+\alpha_1+\alpha_2}. \end{aligned}$$

Therefore, it is enough to prove that

$$\{x \in \pi_\mu \mid \partial_1(x) = 0, \partial_2(x) = 0\} = e_B(\mathcal{V}^{|\mu|}(m)).$$

Let L_μ denote the set in the left hand side. It is obvious that $e_B(\mathcal{V}^{|\mu|}(m)) \subseteq L_\mu$. Assume that $x \in L_\mu \cap \mathcal{V}^{|\mu|+1}_{\mu'}(m)$ and $x = C_1(y_1) + C_2(y_2)$, where $\mu' = (\mu_1', \mu_2')$ and $y_1 \in \mathcal{V}^{|\mu|}_{\mu'-\alpha_1}(m), y_2 \in \mathcal{V}^{|\mu|}_{\mu'-\alpha_2}(m)$. By $\partial_1 x = \partial_2 x = 0$, we have

$$0 = \partial_1 x = \partial_1 C_1 y_1 + \partial_1 C_2 y_2 = \mu_2' y_1 - f y_2, \tag{3.5}$$

$$0 = \partial_2 x = \partial_2 C_1 y_1 + \partial_2 C_2 y_2 = -e y_1 + \mu_1' y_2. \tag{3.6}$$

Assume that $y_2 \neq 0$, then $y_1 \neq 0$ and by the isomorphism $e_B(\mathcal{V}^{|\mu|}(m)) \cong \rho_\mu$, there is an integer $N > 0$ such that $e^N y_2 = 0$ and $e^{N-1} y_2 \neq 0$. From (3.6), $e^{N+1} y_1 = 0$ and $e^N y_1 \neq 0$. Therefore,

$$\begin{aligned} \mu_1' \mu_2' e^N y_1 &= \mu_1' e^N f y_2 = \mu_1' N e^{N-1}(h + N - 1) y_2 \\ &= N(\mu_1' - \mu_2' + N) e^{N-1} y_2 = N(\mu_1' - \mu_2' + N) e^N y_1. \end{aligned}$$

This forces $N = \mu_2'$ and contradicts the fact $y_2 \in \mathcal{V}^{|\mu|}_{\mu'-\alpha_2}$. Hence $y_1 = y_2 = 0$ and $L_\mu \cap e_B(\mathcal{V}^{|\mu|+1}(m)) = \{0\}$.

Using the same method as above, we get $L_\mu \cap e_B(\mathcal{V}^{|\mu|+2}(m)) = \{0\}$. Thus the proof is completed. □

Summarizing the above results, we obtain the main theorem in this subsection.

Theorem 3.7. *As a $W(2) \times \mathfrak{S}_m$-module, the top level $\mathcal{V}(m)$ is decomposed as follows*:

$$\begin{aligned} \mathcal{V}(m) &\cong \sum_{\substack{|\mu|=m \\ m \geq \mu_1 \geq \mu_2 \geq 0}} \pi_\mu \otimes \sigma_{\Gamma_m(\mu)} \oplus \sum_{\substack{m+1 \leq |\mu| \leq 2m-2 \\ m \geq \mu_1 \geq \mu_2 \geq 1}} \pi_\mu \otimes \sigma_{\Gamma_m(\mu)} \\ &\cong \sum_{\mu \in \Phi_2(m)} \pi_\mu \otimes \sigma_{\Gamma_m(\mu)}. \end{aligned}$$

Compare this with Theorem 2.3 in the case $m \leq n$. Then we find that the first summand in the above formula is similar to that of Theorem 2.3, for which the lowest weight vectors come from the space of the lowest degree m. But in the case $m > n$ (at least in the case $n = 2$), when the degree increases, new $W(n)$-modules appear, such as in the second summand in the above formula. This is an evident difference from the case $m \leq n$.

3.3. The decompositions for general levels

In this subsection, we consider the decompositions of general levels $\mathcal{V}(l)$. The method of proofs is similar to those appeared in Section 2, so we omit the proofs and give only the results.

Recall $\mathcal{V}(l) = \mathcal{R}_l(\Lambda(2,m))/\mathcal{R}_{l-1}(\Lambda(2,m))$. We know that $\mathcal{V}(l)$ is a $\mathbb{Z}$-graded $W(2)$-module with the lowest degree l and the highest degree $2l$. Using the results of the above subsections, we have the following decomposition.

Theorem 3.8. (1) *As a* $W(2) \times \mathrm{End}[m]$*-module,*

$$\begin{aligned}\mathcal{V}(l) &\cong \sum_{\substack{|\mu|=l \\ l\geq\mu_1\geq\mu_2\geq 0}} \pi_\mu \otimes \mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{l,m-l}} \sigma_{\Gamma_l(\mu)} \oplus \sum_{\substack{l+1\leq|\mu|\leq 2l-2 \\ l\geq\mu_1\geq\mu_2\geq 1}} \pi_\mu \otimes \mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{l,m-l}} \sigma_{\Gamma_l(\mu)} \\ &\cong \sum_{\mu\in\Phi_2(l)} \pi_\mu \otimes \mathrm{Ind}^{\mathrm{End}[m]}_{\mathcal{P}_{l,m-l}} \sigma_{\Gamma_l(\mu)}.\end{aligned}$$

(2) *Any irreducible quotient of* $W(2) \times \mathrm{End}[m]$*-module* $\mathcal{V}(l)$ *is of the form* $\pi_\mu \otimes \Sigma_{\Gamma_l(\mu)}$*, where* $\mu = (\mu_1, \mu_2)$ *runs over the set* $\Phi_2(l)$*.*

(3) *For any* $\mu \in \Phi_2(l)$*,*

$$\dim \mathrm{Hom}_{W(2)\times\mathrm{End}[m]}(\mathcal{V}(l), \pi_\mu \otimes \Sigma_{\Gamma_l(\mu)}) = 1.$$

3.4. End[m]-module $\Lambda(2,m)$

The aim of this subsection is to decompose End[m]-module $\Lambda(2,m)$. By Lemma 3.2, it is enough to consider End[m]-module V_μ^+. Note that, as vector spaces, $V_\mu^+ = \oplus_{l=0}^m V_\mu^+(l)$.

Let $U_\mu(l) := \ker(\tilde{\partial}_1 |_{V_\mu^+(l)}) \cap \ker(\partial_2 |_{V_\mu^+(l)})$, then $U_\mu(l)$ is a subspace of V_μ^+, and as an $\mathfrak{S}_m$-module, $U_\mu(l) \cong \mathrm{Ind}^{\mathfrak{S}_m}_{\mathfrak{S}_l\times\mathfrak{S}_{m-l}} (\sigma_{\Gamma_l(\mu)} \otimes \mathbf{1})$.

Lemma 3.9.

$$V_\mu^+(l-1) \subseteq \mathcal{R}_{l-1} \cdot U_\mu(l).$$

Proof. Let $r = |\mu| - l$ and put $K^+ := (\otimes^{l-r-2}\Lambda^+(2))^+_{\mu-(r+1,r+1)}$ where $\Lambda^+(2) = \Lambda(2)/\mathbb{C}\cdot\mathbf{1}$. Then $K^+ \cong \sigma_{\Gamma_{l-r-2}(\mu-(r+1,r+1))}$ is an irreducible module of $\mathfrak{S}_{l-r-2}$. From Theorem 3.8 and Lemma 3.2,

$$V_\mu^+(l-1) = \langle \varphi(x \cdot \xi_{1,l-r-1}\xi_{2,l-r-1} \cdots \xi_{1,l-1}\xi_{2,l-1}) \mid \varphi \in \mathfrak{S}_m, x \in K^+ \rangle.$$

Hence it is enough to show that there is a nonzero element $x \in K^+$ such that

$$x \cdot \xi_{1,l-r-1}\xi_{2,l-r-1} \cdots \xi_{1,l-1}\xi_{2,l-1} \in \mathcal{R}_{l-1} \cdot U_\mu(l).$$

Put

$$x_0 = e_{B_1}(\xi_{1,1}\xi_{1,2} \cdots \xi_{1,\mu_1-r-1}\xi_{2,\mu_1-r+1} \cdots \xi_{2,1-r-2}),$$

where B_1 is the Young tableau on the left side of Figure 1. Then x_0 is a highest weight vector of $\mathfrak{gl}(2)$. Because B_1 is a Young tableau of shape $\Gamma_{l-r-2}(\mu-(r+1, r+1))$, so $x_0 \in K^+$. Put

$$y_0 = e_{B_2}(\xi_{1,l-1}\xi_{1,1} \cdots \xi_{1,\mu_1-r-1}\,\xi_{2,l}\xi_{2,\mu_1-r} \cdots \xi_{2,l-r-2}\,\xi_{1,l-r-1}\xi_{2,l-r-1} \cdots \xi_{1,l-2}\xi_{2,l-2}),$$

where B_2 is the Young tableau on the right side of Figure 1. Then y_0 is a highest weight vector of $\mathfrak{gl}(2)$ and $\partial_1 y_0 = 0$ (in $\mathcal{V}(l)$), $\partial_2 y_0 = 0$ (in $\mathcal{V}(l)$), that is,

$$y_0 \in V_\mu^+(l) \cap \ker(\partial_1 \mid_{V_\mu^+(l)}) \cap \ker(\partial_2 \mid_{V_\mu^+(l)}) = U_\mu(l).$$

Let φ be an element of $\mathcal{R}_{l-1}$ defined as follows,

$$\varphi_0 = \begin{pmatrix} 1 & 2 & \cdots & l-1 & l & l+1 & \cdots & m \\ 1 & 2 & \cdots & l-1 & l-1 & l-1 & \cdots & l-1 \end{pmatrix},$$

Then $\varphi_0(y_0) = c \cdot x_0 \cdot \xi_{1,l-r-1}\xi_{2,l-r-1} \cdots \xi_{1,l-1}\xi_{2,l-1}$, where c is a non-zero constant. Therefore, the lemma is proved. □

1	μ_1-r
2	μ_1-r+1
$\vdots$	$\vdots$
μ_2-r-1	$l-r-2$
$\vdots$	
μ_1-r-1	

$l-1$	l	$l-r-1$	$\cdots$	$l-2$
1	μ_1-r			
2	μ_1-r+1			
$\vdots$	$\vdots$			
μ_2-r-1	$l-r-2$			
$\vdots$				
μ_1-r-1				

Figure 1

From the above lemma, we know that the End$[m]$-module V_μ^+ is generated by the subspace $V_\mu^+(l_0)$, where $l_0 = \min\{m, |\mu|\}$. Combining the results of Section 3.1 with Theorem 3.5 of [7], we get

Corollary 3.10. *If* $0 < |\mu| \leq m$, *then*

$$\mathfrak{R}_{\mathrm{End}[m]}(V_\mu^+) = \{\Sigma_D \mid D = \Gamma_{|\mu|}(\mu)\},$$

and if $|\mu| > m$ *and* $\mu_1 \leq m-1$, *then*

$$\mathfrak{R}_{\mathrm{End}[m]}(V_\mu^+) = \{\Sigma_D \mid D = \Gamma_m(\mu), \Gamma_m(\mu-\alpha_1), \Gamma_m(\mu-\alpha_2), \Gamma_m(\mu-\alpha_1-\alpha_2)\},$$

where the definition of $\mathfrak{R}_{\mathrm{End}[m]}(V_\mu^+)$ *is given in Introduction.*

Summarizing the above results, we deduce

Proposition 3.11. *The set of irreducible quotients of* End$[m]$*-module* $\Lambda(2,m)$ *is as follows.*

$$\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m)) = \{\Sigma_D \mid D = \Gamma_{l_0}(\mu), l_0 = \min\{m, |\mu|\}\} \cup \{\mathbf{1}_{\mathrm{End}[m]}\},$$

where $\mathbf{1}_{\mathrm{End}[m]}$ *denotes the trivial representation of* End$[m]$.

Proof. It is direct from Lemma 3.1 and Corollary 3.10. □

3.5. Howe's correspondence for $W(2)\times$End$[m]$-module $\Lambda(2,m)$

In this subsection, we give a decomposition of $W(2)\times$End$[m]$-module $\Lambda(2,m)$ and establish the main results of Section 3, which is Howe's correspondence for $n=2$.

Theorem 3.12. (1) *The tensor product* $\Lambda(2,m)$ *is quotient multiplicity free, that is, if* $\pi\otimes\Sigma$ *is an irreducible representation of* $W(2)\times$ End$[m]$, *then*

$$\dim \mathrm{Hom}_{W(2)\times \mathrm{End}[m]}(\Lambda(2,m), \pi\otimes\Sigma) \leq 1.$$

(2) *The sets of irreducible quotients of* $W(2)$*-module* $\Lambda(2,m)$ (*denoted by* $\mathfrak{R}_{W(n)}(\Lambda(2,m))$) *and* End$[m]$*-module* $\Lambda(2,m)$ (*denoted by* $\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m))$) *are as follows,*

$$\mathfrak{R}_{W(n)}(\Lambda(2,m)) = \{\pi_\mu \in W(2)^\wedge \mid \mu \in \Phi_2\},$$
$$\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m)) = \{\Sigma_{\Gamma_{l_0}(\mu)} \mid \mu\in\Phi_2, l_0 = \min\{m,|\mu|\}\} \cup \{\mathbf{1}_{\mathrm{End}[m]}\}.$$

(3) *The equality in* (1) *is valid if and only if* $\pi = \pi_\mu \in \mathfrak{R}_{W(n)}(\Lambda(2,m))$ *and* $\Sigma = \Sigma_D \in \mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m))/\mathbf{1}_{\mathrm{End}[m]}$ *and* $D = \Gamma_{l_0}(\mu)$ *with* $l_0 = \min\{m,|\mu|\}$.
Furthermore, the above correspondence

$$\mathfrak{R}_{W(n)}(\Lambda(2,m)) \ni \mu \longrightarrow \Gamma_{l_0}(\mu) \in \mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m))/\mathbf{1}_{\mathrm{End}[m]}$$

is a bijection from $\mathfrak{R}_{W(n)}(\Lambda(2,m))$ *to* $\mathfrak{R}_{\mathrm{End}[m]}(\Lambda(2,m))/\mathbf{1}_{\mathrm{End}[m]}$.

Proof. From the exact sequence:

$$0 \longrightarrow \mathcal{V}_{k-1} \longrightarrow \mathcal{V}_k \longrightarrow \mathcal{V}(k) \longrightarrow 0,$$

the following inequality is deduced,

(3.7)

$$\dim \mathrm{Hom}_{W(2)\times \mathrm{End}[m]}(\mathcal{V}_m, \pi\otimes\Sigma) \le \sum_{k=0}^{m} \dim \mathrm{Hom}_{W(2)\times \mathrm{End}[m]}(\mathcal{V}(k), \pi\otimes\Sigma)$$

for any irreducible module π of $W(2)$ and an irreducible module Σ of End$[m]$. Therefore, by the decomposition of Theorem 3.8 of the general level spaces $\mathcal{V}(k)$, we obtain (1).

If the equality in (1) is valid, then Σ is an irreducible quotient of End$[m]$-module. By Proposition 3.11, there is a $\mu \in \Phi_2$ such that $\Sigma = \Sigma_{\Gamma_{l_0}(\mu)}$ or Σ is the trivial representation, where $l_0 = \min\{m, |\mu|\}$. From (3.7), we know that if $\Sigma = \Sigma_{\Gamma_{l_0}(\mu)}$, then $\pi = \pi_\mu$, and if Σ is the trivial representation, then π is also the trivial representation of $W(2)$.

Let us show that the quotient multiplicity of the trivial representation is zero. Take $f \in \mathrm{Hom}_{W(2)\times\mathrm{End}[m]}(\Lambda(2,m),\mathbb{C})$, then $\xi_{i,j} - f(\xi_{i,j})$ is in the kernel of f. By the action of ∂_i on $\xi_{i,j} - f(\xi_{i,j})$, we obtain $f(\mathbb{C}) = 0$. By the similar argument as above, we conclude that $f = 0$. Thus we complete the proof of the "only if "-part of (3).

For the proof of "if "-part of (3), using the same method as that of [7] ([7], Theorem 5.1), we can get it. For the rest of this theorem, it is simple and direct, we omit it. □

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