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# Moment and almost sure Lyapunov exponents of mild solutions of stochastic evolution equations with variable delays via approximation approaches

By

Kai Liu and Aubrey Truman

**Abstract**

Several criteria for the asymptotic exponential stability of a class of Hilbert space-valued, non-autonomous stochastic evolution equations with variable delays are presented. This formulation is particularly suitable for the treatment of mild solutions of general stochastic delay partial differential equations. The principal technique of our investigation is to construct a proper approximating strong solution system and carry out a limiting type of argument to obtain the required exponential stability. As a consequence, stability results from A. Ichikawa [8] are generalized to cover a class of non-autonomous stochastic delay evolution equations. In particular, we improve the recent results of T. Taniguchi [14] to remove the time delay interval restriction there.

## 1. Introduction

The purpose of this paper is to investigate exponential stability of the mild solutions for certain Hilbert space-valued stochastic evolution equations. Roughly speaking, we shall consider the following stochastic evolution equation over a certain Hilbert space $H$ with norm $|\cdot|$:

(1.1)
$$\begin{cases} dX_t = (AX_t + A(t, X_t, X_{t-\tau(t)}))dt + B(t, X_t, X_{t-\tau(t)})dW_t, \quad \forall t \in [0, +\infty), \\ X_t = \phi(t), \quad t \in [-h, 0], \end{cases}$$

where $A$ is the infinitesimal generator of a certain $C_0$-semigroup $S(t)$, $t \geq 0$, over $H$ and $A(t, \cdot, \cdot)$ and $B(t, \cdot, \cdot)$ are in general nonlinear mappings from $H \times H$ to $H$ and $H \times H$ to $\mathcal{L}(K, H)$, the family of all bounded linear operators from Hilbert space $K$ into $H$. $W_t$ is some $K$-valued Wiener process. $\phi(t) : [-h, 0] \times$

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$\Omega \to H$, $h > 0$, is a given initial datum such that $\phi(t)$ is $\mathcal{F}_0$-measurable and $\sup_{-h \le r \le 0} E|\phi(r)|^2 < \infty$. $\tau : [0, \infty) \to [0, h]$, $h \ge 0$, is a given continuously differentiable function with $\tau'(t) \le M$, $0 \le M < 1$, which will play the role of variable delays.

Stochastic evolution equations in Hilbert space have been studied by several authors over the last several decades. For instance, G. Da Prato and J. Zabczyk [6], A. Ichikawa [8] and E. Pardoux [15] (amongst others) have established results on the existence and uniqueness of solutions for a certain class of stochastic evolution systems. For variable delay case, the same problems have been studied by J. Real [16] for stochastic linear evolution equations and by T. Caraballo and K. Liu [2] and T. Caraballo, K. Liu and A. Truman [4] for nonlinear cases. On the other hand, under various circumstances there exists an extensive literature on exponential stability of stochastic differential equations in Hilbert space with either null variable delays or not. We should mention here T. Caraballo [1], P. L. Chow [5], G. Da Prato and J. Zabczyk [6], U. G. Haussmann [7], A. Ichikawa [8], R. Khas'minskii and V. Mandrekar [9] and R. Liu and V. Mandrekar [12]. In particular, for null variable delay case U. G. Haussmann [7] obtained the exponential stability in the sense of mean square or almost sure for $A(t, \cdot, \cdot) = 0$ and linear $B(t, \cdot, \cdot)$ in (1.1). For the similar linear equations to [7], T. Caraballo [1] generalized his results to cover the variable delay case. Also, for null variable delay situation A. Ichikawa [8] studied the exponential stability mainly for the mild solutions of the semilinear autonomous stochastic systems (1.1), i.e., for Lipschitz continuous $A(t, x, y) = A(x)$ and $B(t, x, y) = B(x)$ in (1.1). For variable delay one, by using the properties of the stochastic convolution, T. Caraballo and K. Liu [2] considered the exponential stability of the mild solutions for a class of autonomous stochastic evolution equations and T. Taniguchi [14] studied the same problems for the mild solutions of stochastic partial functional differential equations.

In the following sections we shall investigate stability conditions in the sense of mean square and almost sure of the mild solutions for the general nonautonomous stochastic delay evolution equations (1.1). It is worth pointing out that the methods in this paper are quite different from those in [2], [14] and the results derived here are more applicable for practical purposes. Indeed, our results are much stronger than those obtained in [2], [14]. We should also mention that under certain coercivity assumptions, some similar work for strong solution has already been initiated in T. Caraballo and K. Liu [3] and T. Caraballo, K. Liu and A. Truman [4]. In this work, however, we shall remove the coercivity conditions and carry out instead a general Lyapunov function programme for our stability criteria. The main technique is to construct a suitable approximating solution process sequence and carry out a limiting argument to pass on stability of strong solutions to mild solution cases. Lastly, a couple of examples which are hard to treat by using the results mentioned in the above papers, for instance, [8], [14], are studied to illustrate our theory.

## 2. Preliminary results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $P$-null sets). Let $K$ be a real separable Hilbert space, and $\{W_t, t \geq 0\}$ a $K$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with covariance operator $Q$, i.e.,

$$E(W_t, x)_K (W_s, y)_K = (t \wedge s)(Qx, y)_K \quad \text{for all} \quad x, y \in K,$$

where $Q$ is a nonnegative trace class operator from $K$ into itself. In particular, we call $W_t$ a $K$-valued $Q$-Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

Let $H$ be a real Hilbert space and we denote by $\langle \cdot, \cdot \rangle$ its inner product and by $|\cdot|$ its vector and operator norms. Assume $h > 0$ is a given positive constant. In this work, we shall consider the following semilinear stochastic evolution equation over $H$ with variable delays on $\mathcal{I} = [-h, T]$, $\forall T \geq 0$,

$$\begin{cases} dX_t = (AX_t + A(t, X_t, X_{t-\tau(t)}))dt + B(t, X_t, X_{t-\tau(t)})dW_t, & t \in [0, T], \\ X_0 = x, \quad X_t = \phi(t) \in H, & t \in [-h, 0]. \end{cases} \tag{2.1}$$

In particular, throughout this paper we shall impose the following assumptions:

(H1) $A$ is the infinitesimal generator of a $C_0$-semigroup $S(t)$, $t \geq 0$, over $H$ satisfying $\|S(t)\| \leq M \cdot e^{rt}$, $r \in \mathbf{R}^1$, $M \geq 1$. $A(t, \cdot, \cdot)$ and $B(t, \cdot, \cdot)$, $t \geq 0$, are in general measurable nonlinear mappings from $H \times H$ into $H$ and $H \times H$ into $\mathcal{L}(K, H)$ respectively, satisfying the following Lipschitz condition and linear growth condition

$$\begin{aligned} |A(t, y_1, z_1) - A(t, y_2, z_2)| \vee \|B(t, y_1, z_1) - B(t, y_2, z_2)\|_2 \\ \leq k(|y_1 - y_2| + |z_1 - z_2|) \end{aligned} \tag{2.2}$$

and

$$|A(t, y_1, z_1)| \vee \|B(t, y_1, z_1)\|_2 \leq k\,(1 + |y_1| + |z_1|) \tag{2.3}$$

for some constant $k > 0$ and all $y_i$, $z_i \in H$, $i = 1, 2$. Here $\|\cdot\|_2$, or simply $\|\cdot\|$, denotes the Hilbert-Schmidt norm of a nuclear operator, i.e., $\|B(t, y, z)\|_2^2 = tr(B(t, y, z)QB(t, y, z)^*)$, $y$, $z \in H$. $W_t$ is a certain $K$-valued $Q$-Wiener process. $\phi(t) : [-h, 0] \times \Omega \to H$, $h > 0$, is a given initial datum such that $\phi(t)$ is $\mathcal{F}_0$-measurable and $\sup_{-h\leq r\leq 0} E|\phi(r)|^2 < \infty$. $\tau : [0, \infty) \to [0, h]$, $h \geq 0$, is a given differentiable function with $\tau'(t) \leq M$, $0 \leq M < 1$.

We introduce two kinds of solutions of (2.1) as follows similarly to [8]:

**Definition 2.1.** A stochastic process $X_t$, $t \in \mathcal{I}$, is a strong solution of (2.1) if

(i) $X_t$ is adapted to $\mathcal{F}_t$;
(ii) $X_t$ is continuous in $t$ almost sure;

(iii) $X_t \in \mathcal{D}(A)$ on $\mathcal{I} \times \Omega$ with $\int_0^T |AX_t|dt < \infty$ almost surely
and

$$\begin{cases} X_t = x + \int_0^t (AX_s + A(s, X_s, X_{s-\tau(s)}))ds + \int_0^t B(s, X_s, X_{s-\tau(s)})dW_s, \\ X_0 = x, \quad X_t = \phi(t), \quad t \in [-h, 0], \end{cases}$$

for all $t \in \mathcal{I}$ with probability one.

In general this concept is rather strong and a weaker one described below is more appropriate for practical purposes.

**Definition 2.2.** A stochastic process $X_t$, $t \in \mathcal{I}$, is a mild solution of (2.1) if
(i) $X_t$ is adapted to $\mathcal{F}_t$;
(ii) $X_t$ is measurable with $\int_0^T |X_t|^2 dt < \infty$ almost surely
and

$$\begin{cases} X_t = S(t)x + \int_0^t S(t-s)A(s, X_s, X_{s-\tau(s)})ds \\ \qquad + \int_0^t S(t-s)B(s, X_s, X_{s-\tau(s)})dW_s, \\ X_0 = x, \qquad X_t = \phi(t), \quad t \in [-h, 0], \end{cases}$$

for all $t \in \mathcal{I}$ with probability one.

**Remark.** If $X_t$, $t \in \mathcal{I}$, is a strong solution of (2.1) then it is also a mild solution.

The following existence theorem can be obtained similarly by an adapted argument from [2] or [10]. The reader is referred to them for further details on this aspect.

**Theorem 2.1.** *Let $\phi(t)$, $t \in [-h, 0]$, be a given $\mathcal{F}_0$-measurable initial datum with* $\sup_{-h \le r \le 0} E|\phi(r)|^2 < \infty$. *Suppose the hypothesis* (H1) *holds, then* (2.1) *has a unique mild solution $X_t^\phi$, or simply $X_t$, in $C(0, T; L^2(\Omega, \mathcal{F}, P; H))$.*

For our purposes, we can introduce Itô's formula as follows which will play an important role in our stability analysis.

Let $C^2(H)$ denote the space of all real-valued functions $v$ on $H$ with properties:
(i) $v(x) \in C^2(H)$ is twice (Fréchet) differentiable;
(ii) $v'(x)$ and $v''(x)$ are both continuous in $H$ and $\mathcal{L}(H) = \mathcal{L}(H, H)$, respectively.

**Theorem 2.2** (Itô's formula). *Suppose $v \in C^2(H)$ and $\{X_t^\phi,\ t \ge 0\}$ is the strong solution of* (2.1), *then*

$$v(X_t) = v(x) + \int_0^t Lv(s, X_s, X_{s-\tau(s)})ds + \int_0^t \langle v'(X_s), B(s, X_s, X_{s-\tau(s)})dW_s \rangle,$$

*where $Lv(t, x, y) = \langle v'(x), Ax + A(t, x, y)\rangle + 1/2 \cdot tr(v''(x)B(t, x, y)QB^*(t, x, y))$, $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(A)$, $t \ge 0$, is called the infinitesimal generator of Equation* (2.1).

Since Itô's formula is only applicable to the strong solution, we introduce the following approximating systems of (2.1)

$$\begin{cases} dX_t = (AX_t + R(n)A(t, X_t, X_{t-\tau(t)}))dt + R(n)B(t, X_t, X_{t-\tau(t)})dW_t \\ X_0 = R(n)x, \quad X_t = R(n)\phi(t), \quad t \in [-h, 0], \end{cases} \tag{2.4}$$

where $n_0 \le n \in \rho(A)$, the resolvent set of $A$, and $R(n) = nR(n, A)$. The infinitesimal generator $L_n$ corresponding to this equation is $L_n v(t, x, y) = \langle v'(x), Ax + R(n)A(t, x, y)\rangle + 1/2 \cdot tr(v''(x)R(n)B(t, x, y)Q(R(n)B(t, x, y))^*)$, $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(A)$, $t \ge 0$.

**Theorem 2.3** ([10]). *Under the hypotheses of Theorem* 2.1*, Equation* (2.4) *has a unique strong solution* $X_t^\phi(n)$ *in* $C(0, T; L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P; H))$ *for all* $T \ge 0$*. Moreover,* $X_t^\phi(n)$ *converges to the mild solution* $X_t^\phi$ *of* (2.1) *in* $C(0, T; L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P; H))$ *as* $n \to \infty$*, i.e.,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} E\big(|X_t^\phi - X_t^\phi(n)|^2\big) = 0.$$

## 3. The Main Results

In this section, we shall carry out a Lyapunov function programme to study exponential stability of the mild solution of (2.1) in the sense of mean square and pathwise with probability one. Due to mainly paying our attention to stability analysis, throughout this paper we suppose there exists a unique global mild solution of the equation (2.1). In particular, we shall obtain the following consequences whose proof essentially follows those in X. R. Mao [13].

**Theorem 3.1.** *Let* $v(x) : H \to \mathbf{R}^1$ *satisfy*

(i)

$$v(x) \ge c_1 \cdot |x|^2 \quad \textit{for some} \quad c_1 > 0; \tag{3.1}$$

(ii) $v(x)$ *is twice Fréchet differentiable and* $v'(x)$*,* $v''(x)$ *are continuous in* $H$ *and* $\mathcal{L}(H)$ *respectively, and*

$$|v(x)| + |x||v'(x)| + |x|^2|v''(x)| \le c_2|x|^2 \quad \textit{for some} \quad c_2 > 0; \tag{3.2}$$

(iii) *There exist constants* $\alpha > 0$*,* $\mu > 0$*,* $\lambda \in \mathbf{R}_+$ *and a nonnegative function* $\gamma(t)$*,* $t \in \mathbf{R}_+$*, such that*

$$Lv(t, x, y) \le -\alpha v(x) + \lambda v(y) + \gamma(t)e^{-\mu t}, \quad x \in \mathcal{D}(A), \quad y \in \mathcal{D}(A) \tag{3.3}$$

*where* $\gamma(t)$ *satisfies that for any* $\delta > 0$*,* $\gamma(t) = o(e^{\delta t})$*, as* $t \to \infty$*, i.e.,* $\lim_{t\to\infty} \gamma(t)/e^{\delta t} = 0$.

*Assume furthermore the condition* $\alpha > \lambda/(1-M)$ *holds* (*recall* $\tau'(t) \le M$*,* $0 \le M < 1$)*, then there exist constants* $\tau > 0$*,* $C(\phi) > 0$ *such that for the mild solution* $X_t^\phi$ *of* (2.1)*,*

$$E|X_t^\phi|^2 \le C(\phi) \cdot e^{-\tau t}, \qquad \forall t \ge 0. \tag{3.4}$$

*That is, the mild solution is mean square exponentially stable.*

**Remark.** The term $\gamma(t)e^{-\mu t}$ appearing in (3.3) is of the essence for our stability purposes. Indeed, just as the example below shows that any weaker type decay, for instance, polynomial one is not sufficient to ensure exponential stability in either mean square or almost sure sense.

**Example 3.1.** Assume $X_t$ satisfies the following one-dimensional stochastic differential equation

$$dX_t = -pX_t dt + (1+t)^{-q} dB_t, \qquad t \geq 0$$

with initial data $X_0 = 0$, where $p$, $q > 0$ are two positive constants and $B_t$ is a one-dimensional standard Brownian motion.

Choose $v(x) = x^2$, $x \in \mathbf{R}^1$, in Theorem 3.1 and then (3.3) turns out to be

$$Lv(t,x) = 2\langle -px, x\rangle + \left[(1+t)^{-q}\right]^2 = -2pv(x) + (1+t)^{-2q},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^1$. On the other hand, it is easy to obtain the explicit solution

$$X_t = e^{-pt} \int_0^t e^{ps} \cdot (1+s)^{-q} dB_s \equiv e^{-pt} M_t, \qquad t \geq 0$$

which immediately implies that for arbitrarily given $p > 0$, $q > 0$, the Lyapunov exponent

$$\lim_{t\to\infty} \frac{\log E|X_t|^2}{t} = 0.$$

In the meantime, noticing the law of the iterated logarithm

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{2\langle M_t\rangle \log\log\langle M_t\rangle}} = 1 \quad \text{a.s.}$$

and

$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t e^{2ps}(1+s)^{-2q} ds\right)}{t} = 2p,$$

we therefore get for arbitrarily given $p > 0$, $q > 0$, the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{\log|X_t|}{t} = 0 \quad \text{a.s.}$$

*Proof.* Firstly, from (3.3) it is easy to deduce that

$$Lv(t,x,y) \leq -\alpha v(x) + \lambda v(y) + \gamma(t) e^{-(\alpha\wedge\mu)t}, \quad x \in \mathcal{D}(A), \quad y \in \mathcal{D}(A). \tag{3.5}$$

Since $\alpha > \lambda/(1-M)$, we can find a positive constant $\varepsilon \in (0, (\alpha\wedge\mu)/2)$ such that

$$\frac{\lambda e^{\varepsilon h}}{(1-M)(\alpha-\varepsilon)} < 1. \tag{3.6}$$

On the other hand, applying Itô's formula to the function $v(t,x) = e^{\alpha t}v(x)$ and the strong solution $X_t^\phi(n)$ of (2.4) yields

$$
\begin{aligned}
&e^{\alpha t}v(X_t^\phi(n)) - v(X_0^\phi(n)) \\
&\quad = \alpha \int_0^t e^{\alpha s} v(X_s^\phi(n))ds \\
&\qquad + \int_0^t e^{\alpha s}\langle v'(X_s^\phi(n)), \\
&\qquad\qquad AX_s^\phi(n) + R(n)A(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n)), X_{s-\tau(s)}^\phi(n)\rangle ds \\
&\qquad + \int_0^t e^{\alpha s}\langle v'(X_s^\phi(n)), R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))dW_s\rangle \\
&\qquad + 1/2 \cdot \int_0^t e^{\alpha s} tr\{R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n)) \\
&\qquad \cdot Q[R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))]^* v''(X_s^\phi(n))\}ds.
\end{aligned} \tag{3.7}
$$

Therefore, by virtue of (3.5) we can deduce

$$
\begin{aligned}
&e^{\alpha t}Ev(X_t^\phi(n)) \\
&\le Ev(X_0^\phi(n)) + \lambda\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^\phi(n))ds + \int_0^t \gamma(s)e^{[\alpha-(\alpha\wedge\mu)]s}ds \\
&\quad + \int_0^t e^{\alpha s}E\{\langle v'(X_s^\phi(n)), (R(n)-I)A(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))\rangle \\
&\quad + 1/2 \cdot tr[R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n)) \\
&\qquad Q\big(R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))\big)^* \\
&\quad \cdot v''(X_s^\phi(n)) \\
&\quad - B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))QB(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))^* v''(X_s^\phi(n))]\}ds.
\end{aligned} \tag{3.8}
$$

Hence, in view of Conditions (H1), (3.2) and Theorem 2.3, there exists a subsequence of $\{n\}$ in $\rho(A)$ (still denoted by $\{n\}$) such that $X_t^\phi(n) \to X_t^\phi$ in $C(0,T;H)$, as $n \to +\infty$ almost surely. Consequently, letting $n \to \infty$ in (3.8) immediately yields that

$$
\begin{aligned}
e^{\alpha t}Ev(X_t^{\phi}) &\le Ev(X_0^{\phi}) + \lambda\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^{\phi})ds \\
&\quad + \int_0^t \gamma(s)e^{[\alpha-(\alpha\wedge\mu)]s}ds \\
&\le Ev(X_0^{\phi}) + \lambda\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^{\phi})ds \\
&\quad + e^{[\alpha-(\alpha\wedge\mu)]t}\cdot e^{[(\alpha\wedge\mu)-2\varepsilon]t}\cdot\int_0^t \gamma(s)e^{-[(\alpha\wedge\mu)-2\varepsilon]s}ds \\
&\le Ev(X_0^{\phi}) + \lambda\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^{\phi})ds \\
&\quad + e^{(\alpha-2\varepsilon)t}\int_0^t \gamma(s)e^{-[(\alpha\wedge\mu)-2\varepsilon]s}ds
\end{aligned} \tag{3.9}
$$

for arbitrary $t\in[0,T]$, $\forall T\in\mathbf{R}_+$. Consequently,

$$
\begin{aligned}
&Ev(X_t^{\phi}) \\
&\le Ev(\phi(0))\cdot e^{-\alpha t} + \lambda e^{-\alpha t}\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^{\phi})ds + e^{-2\varepsilon t}\int_0^t \gamma(s)e^{-[(\alpha\wedge\mu)-2\varepsilon]s}ds
\end{aligned}
$$

for all $t\ge 0$. Therefore, we have

$$
\begin{aligned}
&\int_0^{\infty} e^{\varepsilon t}Ev(X_t^{\phi})dt \\
&\quad\le Ev(\phi(0))\cdot\int_0^{\infty} e^{-(\alpha-\varepsilon)t}dt \\
&\qquad + \lambda\int_0^{\infty} e^{-(\alpha-\varepsilon)t}\int_0^t e^{\alpha s}Ev(X_{s-\tau(s)}^{\phi})dsdt \\
&\qquad + \int_0^{\infty} e^{-\varepsilon t}\int_0^{\infty}\gamma(s)e^{-[(\alpha\wedge\mu)-2\varepsilon]s}dsdt \\
&\quad\le \frac{1}{\alpha-\varepsilon}Ev(\phi(0)) + \frac{\lambda}{\alpha-\varepsilon}\int_0^{\infty} e^{\varepsilon s}Ev(X_{s-\tau(s)}^{\phi})ds + \frac{M(\varepsilon)}{\varepsilon},
\end{aligned} \tag{3.10}
$$

where $M(\varepsilon)=\int_0^{\infty}\gamma(s)e^{-[(\alpha\wedge\mu)-2\varepsilon]s}ds<\infty$.

However, as the function $\rho(t) = t-\tau(t)$ is strictly increasing with $\lim_{t\to+\infty}\rho(t)=+\infty$, there exists $\delta_1\in(0,h]$ such that $\rho(\delta_1)=0$, $\rho(t)\in[-h,0]$ for all $t\in[0,\delta_1]$ and $\rho(t)>0$ for all $t>\delta_1$. Thus, taking into account the change of variables $u = s - \tau(s)$, it follows

$$\begin{aligned}
&\int_0^\infty e^{\varepsilon s} Ev(X^\phi_{s-\tau(s)})ds \\
&\quad \le \int_0^{\delta_1} e^{\varepsilon s} Ev(X^\phi_{s-\tau(s)})ds + e^{\varepsilon h}\int_{\delta_1}^\infty e^{\varepsilon(s-\tau(s))} Ev(X^\phi_{s-\tau(s)})ds \\
&\quad \le \delta_1 \cdot e^{\varepsilon \delta_1} \sup_{-h\le r\le 0} Ev(\phi(r)) + \frac{e^{\varepsilon h}}{1-M}\int_0^\infty e^{\varepsilon s} Ev(X^\phi_s)ds \\
&\quad \le h \cdot e^{\varepsilon h} \sup_{-h\le r\le 0} Ev(\phi(r)) + \frac{e^{\varepsilon h}}{1-M}\int_0^\infty e^{\varepsilon s} Ev(X^\phi_s)ds
\end{aligned} \tag{3.11}$$

which, together with (3.10), immediately implies that

$$\begin{aligned}
\int_0^\infty e^{\varepsilon t} Ev(X^\phi_t)dt \le \Big(\frac{1}{\alpha-\varepsilon} + \frac{\lambda\cdot h\cdot e^{\varepsilon h}}{\alpha-\varepsilon}\Big) \sup_{-h\le r\le 0} Ev(\phi(r)) \\
+ \frac{M(\varepsilon)}{\varepsilon} + \frac{\lambda e^{\varepsilon h}}{(1-M)(\alpha-\varepsilon)}\int_0^\infty e^{\varepsilon t} Ev(X^\phi_t)dt,
\end{aligned}$$

i.e., noticing (3.6), we have there exists a positive constant $\tilde{C} = \tilde{C}(\alpha,\mu,\lambda,h) < \infty$ such that

$$\int_0^\infty e^{\varepsilon t} Ev(X^\phi_t)dt \le \tilde{C}, \tag{3.12}$$

where

$$\tilde{C} = \frac{1}{1 - \frac{\lambda e^{\varepsilon h}}{(1-M)(\alpha-\varepsilon)}}\Big[\Big(\frac{1}{\alpha-\varepsilon} + \frac{\lambda\cdot h\cdot e^{\varepsilon h}}{\alpha-\varepsilon}\Big) \sup_{-h\le r\le 0} Ev(\phi(r)) + \frac{M(\varepsilon)}{\varepsilon}\Big].$$

Now we are in a position to complete our proof. Firstly, note that by carrying out a similar limiting argument as in (3.8), (3.9) and using the condition (3.2), we can obtain for the above $\varepsilon > 0$,

$$e^{\varepsilon t} Ev(X^\phi_t) \le c_2 E|\phi(0)|^2 + c_2\lambda\int_0^t e^{\varepsilon s} E|X^\phi_{s-\tau(s)}|^2 ds + \int_0^t \gamma(s) e^{[\varepsilon-(\alpha\wedge\mu)]s}ds$$

which, in addition to (3.1), (3.11) and (3.12), immediately yields

$$\begin{aligned}
e^{\varepsilon t} E|X^\phi_t|^2 &\le 1/c_1 \cdot e^{\varepsilon t} Ev(X^\phi_t) \\
&\le \frac{1}{c_1}\Big\{c_2 E|\phi(0)|^2 + c_2\lambda\int_0^t e^{\varepsilon s} E|X^\phi_{s-\tau(s)}|^2 ds \\
&\quad + \int_0^t \gamma(s) e^{[\varepsilon-(\alpha\wedge\mu)]s}ds\Big\} \\
&\le \frac{1}{c_1}\Big\{c_2 E|\phi(0)|^2 + c_2\lambda\Big(he^{\varepsilon h}\sup_{-h\le r\le 0} E|\phi(r)|^2 + \frac{e^{\varepsilon h}\tilde{C}}{1-M}\Big) \\
&\quad + \int_0^\infty \gamma(s) e^{[\varepsilon-(\alpha\wedge\mu)]s}ds\Big\} \\
&:= C(\phi) < \infty,
\end{aligned}$$

i.e.,

$$E|X_t^\phi|^2 \le C(\phi) \cdot e^{-\varepsilon t}$$

for all $t \ge 0$. In other words, the solution is mean square stable and the proof is now complete. $\square$

**Theorem 3.2.** *Assume the hypotheses in Theorem* 3.1 *hold. Then there exist positive constants $K$, $\theta$ and a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$, there exists a positive random number $T(\omega)$ such that*

$$|X_t^\phi|^2 \le K \cdot e^{-\theta t}, \quad \forall t \ge T(\omega). \tag{3.13}$$

*That is, the mild solution is also exponential almost surely stable.*

*Proof.* We shall split our proof into the following several steps.

*Step* 1. We firstly claim that there exists a positive constant $C > 0$, independent of $t \in \mathbf{R}_+$, such that

$$\int_s^t E\|B(s, X_s^\phi, X_{s-\tau(s)}^\phi)\|_2^2 ds \le C < \infty, \quad 0 \le s \le t. \tag{3.14}$$

Indeed, applying Itô's formula as in the proof of Theorem 3.1 to the strong solution $X_t^\phi(n)$ and letting $n \to \infty$, we get for arbitrary $t \in \mathbf{R}_+$

$$\begin{aligned} Ev(X_t^\phi) \le\ & Ev(\phi(0)) + \lambda \int_0^t Ev(X_{s-\tau(s)}^\phi) ds \\ & + \int_0^t \gamma(s) e^{-\mu s} ds - \alpha \int_0^t Ev(X_s^\phi) ds. \end{aligned} \tag{3.15}$$

Evaluating now the delay term in (3.15) by using the change of variable $u = s - \tau(s)$ in the integral and taking into account (3.4), we obtain

$$\begin{aligned} \int_0^t Ev(X_{s-\tau(s)}^\phi) ds &\le \frac{1}{1-M} \int_{-h}^{t-\tau(t)} Ev(X_s^\phi) ds \\ &\le \frac{1}{1-M} \int_{-h}^0 Ev(\phi(s)) ds + \frac{1}{1-M} \int_0^{t-\tau(t)} Ev(X_s^\phi) ds \\ &\le \frac{1}{1-M} \int_{-h}^0 Ev(\phi(s)) ds + \frac{C}{\tau(1-M)}. \end{aligned} \tag{3.16}$$

Consequently, noticing that there exists a positive constant $K_1$ (independent of $t$) such that $\int_0^t \gamma(s) e^{-\mu s} ds \le K_1$, for all $t \ge 0$, then in view of (3.1), (3.2), (3.4), (3.15) and (3.16) it follows

$$\begin{aligned} \int_0^t E|X_s^\phi|^2 ds &\le 1/c_1 \cdot \int_0^t Ev(X_s^\phi) ds \\ &\le \frac{1}{\alpha c_1} \Bigg( c_2 \cdot E|\phi(0)|^2 + K_1 \\ &\quad + \frac{c_2 \lambda h}{1-M} \sup_{-h \le s \le 0} E|\phi(s)|^2 + \frac{\lambda C}{\tau(1-M)} \Bigg) \end{aligned} \tag{3.17}$$

and so

$$\int_0^t E|X_s^\phi|^2 ds \le C_1\,, \quad \forall t \ge 0,$$

where $C_1$ is a positive constant independent on $t$.

Therefore, we can ensure that there exists a positive constant $C_1 > 0$ such that

$$\int_s^t E|X_s^\phi|^2 ds \le C_1 \quad \text{for} \quad 0 \le s \le t. \tag{3.18}$$

Now, taking into account (2.2), (3.2), (3.3) and the above change of variable, (3.18) yields that

$$\begin{aligned}
&\int_s^t E\|B(s, X_s^\phi, X_{s-\tau(s)}^\phi)\|_2^2 ds \\
&\le 2\int_s^t E\|B(s, X_s^\phi, X_{s-\tau(s)}^\phi) - B(s,0,0)\|_2^2 ds \\
&\quad + 2\int_s^t E\|B(s,0,0)\|_2^2 ds \\
&\le k_1 \int_s^t E|X_s^\phi|^2 ds + k_2 \int_s^t E|X_{s-\tau(s)}^\phi|^2 ds + k_3 \int_s^t \gamma(s) e^{-\mu s} ds \\
&\le k_1 \int_s^t E|X_s^\phi|^2 ds + k_4 \int_{-h}^t E|X_s^\phi|^2 ds + k_3 \int_s^t \gamma(s) e^{-\mu s} ds \\
&\le C_2 \quad \text{for} \quad 0 \le s \le t,
\end{aligned} \tag{3.19}$$

where $k_1, k_2, k_3$, $k_4$ and $C_2$ are positive constants (independent of $s$, $t$).

*Step* 2. Next, we claim that for any $T > 0$ there exists a positive constant $M > 0$, independent of $T$, such that

$$E\left(\sup_{0\le t<T} v(X_t^\phi)\right) \le M. \tag{3.20}$$

Indeed, applying Itô's formula to the $v(t, y) = v(y)$ and the strong solution

$X_t^\phi(n)$, we obtain that

$$
\begin{aligned}
&v(X_t^\phi(n))\\
&= v(R(n)X_0^\phi) - \alpha\int_0^t v(X_s^\phi(n))ds + \lambda\int_0^t v(X_{s-\tau(s)}^\phi(n))ds + \int_0^t \gamma(s)e^{-\mu s}ds\\
&+ \int_0^t \langle v'(X_s^\phi(n)), R(n)A(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n)) - A(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))\rangle ds\\
&+ \frac{1}{2}\int_0^t \Big[tr\Big(R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))Q\\
&\quad \cdot B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))^* R(n)^* v''(X_s^\phi(n))\Big)\\
&- tr\left(B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))QB(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))^* v''(X_s^\phi(n))\right)\Big]\,ds\\
&+ \int_0^t \langle v'(X_s^\phi(n)), R(n)B(s, X_s^\phi(n), X_{s-\tau(s)}^\phi(n))dW_s\rangle.
\end{aligned}
$$

Thus, in view of Theorem 2.3 we can pass to the limit in the inequality above, together with (3.2), to obtain for any $T \in \mathbf{R}_+$

$$
\begin{aligned}
E\left(\sup_{0\le t<T} v(X_t^\phi)\right) &\le Ev(X_0^\phi) + \alpha\int_0^T Ev(X_s^\phi)ds\\
&\quad + \lambda\int_0^T Ev(X_{s-\tau(s)}^\phi)ds + \int_0^T \gamma(s)e^{-\mu s}ds\\
&\quad + E\left[\sup_{0\le t\le T}\int_0^t \langle v'(X_s^\phi), B(s, X_s^\phi, X_{s-\tau(s)}^\phi)dW_s\rangle\right].
\end{aligned}
\tag{3.21}
$$

On the other hand, by virtue of Burkholder-Davis-Gundy's inequality and Condition (3.2), we get for arbitrary $T \in \mathbf{R}_+$

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle v'(X_s^\phi), B(s, X_s^\phi, X_{s-\tau(s)}^\phi)dW_s\rangle\right|\right]\\
&\le K_1 E\left[\left(\int_0^T |v'(X_s^\phi)|^2 \|B(s, X_s^\phi, X_{s-\tau(s)}^\phi)\|_2^2 ds\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

which, by using the conditions (3.1), (3.2) and Hölder's inequality, immediately yields

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle v'(X_s^\phi), B(s, X_s^\phi, X_{s-\tau(s)}^\phi)dW_s\rangle\right|\right]\\
&\le \frac{1}{2}E\left[\sup_{0\le s\le T} v(X_s^\phi)\right] + K_2\int_0^T E\|B(s, X_s^\phi, X_{s-\tau(s)}^\phi)\|_2^2 ds
\end{aligned}
\tag{3.22}
$$

where $K_1$, $K_2$ are two positive constants. Therefore, substituting (3.22) into (3.21) immediately yields that

$$\begin{aligned}
&E\Big(\sup_{0\le s\le T} v(X_s^\phi)\Big)\\
&\quad\le Ev(X_0^\phi)+\alpha\int_0^T Ev(X_s^\phi)ds+\lambda\int_0^T Ev(X_{s-\tau(s)}^\phi)ds+\int_0^T\gamma(s)e^{-\mu s}ds\\
&\qquad+\frac{1}{2}E\Big(\sup_{0\le s\le T} v(X_s^\phi)\Big)+K_2\int_0^T E\|B(s,X_s^\phi,X_{s-\tau(s)}^\phi)\|_2^2ds,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&E\left(\sup_{0\le s\le T} v(X_s^\phi)\right)\\
&\quad\le 2Ev(X_0^\phi)+2\alpha\int_0^T Ev(X_s^\phi)ds+2\lambda\int_0^T Ev(X_{s-\tau(s)}^\phi)ds\\
&\qquad+2\int_0^T\gamma(s)e^{-\mu s}ds+2K_2\int_0^T E\|B(s,X_s^\phi,X_{s-\tau(s)}^\phi)\|_2^2ds.
\end{aligned}\tag{3.23}$$

Thus, we can easily obtain our claim by (3.14) and Theorem 3.1.

*Step* 3. Now we are in a position to prove our main result. We only sketch the proof because it is similar to that one in U. G. Haussmann [7].

Firstly, a similar argument to (3.21) implies

$$\begin{aligned}
v(X_T^\phi)\le{}& v(X_N^\phi)+\alpha\int_N^T v(X_s^\phi)ds+\lambda\int_N^T v(X_{s-\tau(s)}^\phi)ds\\
&+\int_N^T\gamma(s)e^{-\mu s}ds\\
&+\left[\sup_{t\in[N,T]}\left|\int_N^t\langle v'(X_s^\phi),B(s,X_s^\phi,X_{s-\tau(s)}^\phi)dW_s\rangle\right|\right],
\end{aligned}\tag{3.24}$$

for arbitrary $T\ge N$, where $N$ is a natural number.

In particular, choosing $N$ large enough it is not difficult to obtain

$$\begin{aligned}
&P\left\{\sup_{t\in[N,N+1]} v(X_t^\phi)\ge\epsilon_N^2\right\}\\
&\quad\le P\left\{v(X_N^\phi)\ge\epsilon_N^2/5\right\}\\
&\qquad+P\left\{\alpha\int_N^{N+1} v(X_s^\phi)ds\ge\epsilon_N^2/5\right\}\\
&\qquad+P\left\{\lambda\int_N^{N+1} v(X_{s-\tau(s)}^\phi)ds\ge\varepsilon_N^2/5\right\}\\
&\qquad+P\left\{\left[\sup_{t\in[N,N+1]}\left|\int_N^t\langle v'(X_s^\phi),B(s,X_s^\phi,X_{s-\tau(s)}^\phi)dW_s\rangle\right|\right]\ge\epsilon_N^2/5\right\},
\end{aligned}\tag{3.25}$$

where $\epsilon_N^2 = Ce^{-\tau N/5}$. Now, we can estimate the terms on the right-hand side of (3.25) using Kolmogorov's inequality and (3.4) for the first two terms. We could also estimate the last one by using Burkholder-Davis-Gundy's lemma, Hölder's lemma and carrying out a similar argument as in Steps 1 and 2. We then get for some $K_3 > 0$,

$$P\Big[\sup_{t\in[N,N+1]} v(X_t^\phi) \geq \epsilon_N^2\Big] \leq K_3 e^{-\tau N/5}, \tag{3.26}$$

and finally a Borel-Cantelli's lemma type argument, together with condition (3.1), completes the proof. □

## 4. Some Corollaries, Examples and Comments

In this section, we shall apply our main results derived above to various circumstances to obtain some useful criteria for practical purposes. As a consequence, we extend the main results from A. Ichikawa [8] to cover general stochastic evolution equations with time non-autonomous type. In the meantime we also improve a result from T. Taniguchi [14] to remove the time delay interval restriction imposed there.

### 4.1. Stochastic evolution equations with null variable delays

Consider the following semilinear stochastic evolution equation over $H$:

$$\begin{cases} dX_t = (AX_t + A(t, X_t))dt + B(t, X_t)dW_t, \quad \forall t \in [0, +\infty), \\ X_0 = x, \end{cases} \tag{4.1}$$

where $A(t,\cdot)$ and $B(t,\cdot)$, $t \geq 0$, are in general measurable nonlinear mappings from $H$ into $H$ and $H$ into $\mathcal{L}(K, H)$ respectively, satisfying the corresponding Lipschitz condition and linear growth condition as in (2.2) and (2.3).

**Corollary 4.1.** *Suppose $\tau(t) \equiv 0$, $t \geq 0$, in Theorems* 3.1 *and* 3.2 *and the corresponding condition* (H1) *holds. Let $v(x) : H \to \mathbf{R}^1$ satisfy*

(i)

$$v(x) \geq c_1 \cdot |x|^2 \quad \textit{for some} \quad c_1 > 0; \tag{4.2}$$

(ii) *$v(x)$ is twice Fréchet differentiable and $v'(x)$, $v''(x)$ are continuous in $H$ and $\mathcal{L}(H)$ respectively, and*

$$|v(x)| + |x||v'(x)| + |x|^2|v''(x)| \leq c_2|x|^2 \quad \textit{for some} \quad c_2 > 0; \tag{4.3}$$

(iii) *There exist constants $\alpha > 0$, $\mu > 0$ and a nonnegative function $\gamma(t)$, $t \in \mathbf{R}_+$, such that*

$$Lv(t, x) \leq -\alpha v(x) + \gamma(t)e^{-\mu t}, \quad tx \in \mathcal{D}(A), \tag{4.4}$$

*where $Lv(t,x) = \langle v'(x), Ax + A(t,x)\rangle + 1/2 \cdot tr(v''(x)B(t,x)QB(t,x)^*)$, $x \in \mathcal{D}(A)$, $t \geq 0$ and $\gamma(t)$ satisfies that for any $\delta > 0$, $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$.*

*Then there exist constants $\tau > 0$, $C > 0$ such that for the mild solution $X_t^x$ of* (4.1),

$$E|X_t^x|^2 \le C \cdot e^{-\tau t}, \quad \forall t \ge 0. \tag{4.5}$$

*That is, the mild solution is mean square exponentially stable. Furthermore, under the same conditions the solution is also exponential almost surely stable.*

**Remark.** In A. Ichikawa [8], stability results (Theorems 3.1 and 5.1 there) are obtained to deal with the mild solutions of the semilinear stochastic evolution equations (4.1). However, as the following example will explain, the results derived there are too restrictive to be applied to some interesting and important examples, especially to the non-autonomous occasions.

**Example 4.1.** Consider the following semilinear stochastic partial differential equation:

$$\begin{cases} dY_t(x) = \frac{\partial^2}{\partial x^2} Y_t(x)dt + e^{-\mu t}\alpha\big(Y_t(x)\big)dW_t, \quad t > 0, \quad x \in (0,1), \\ Y_0(x) = y_0(x), \quad Y_t(0) = Y_t(1) = 0, \quad t \ge 0, \end{cases} \tag{4.6}$$

where $W_t$ is a real standard Wiener process (so, $K = \mathbf{R}^1$) and $\alpha(\cdot) : \mathbf{R}^1 \to \mathbf{R}^1$ is a certain bounded, Lipschitz continuous function and $\mu$ is a positive number. We can set this problem in our formulation by taking $H = L^2[0,1]$ with elements satisfying boundary conditions above, $A = \partial^2/\partial x^2$, $A(t,u) = 0$, $B(t,u) = e^{-\mu t}\alpha(u)$.

Clearly, operator $B(t,\cdot)$ satisfies the corresponding conditions (2.2) and (2.3). On the other hand, let $v(x) = |x|^2$, $x \in H$, and it is easy to deduce (for $u \in \mathcal{D}(A)$)

$$2\langle u, Au + A(t,u)\rangle + \|B(t,u)\|_2^2 \le -2\pi|u|^2 + Ke^{-2\mu t}, \tag{4.7}$$

where $K$ is a certain positive constant.

Since the hypotheses in Corollary 4.1 are fulfilled, we therefore deduce that the mild solution of the equation (4.6) is mean square exponentially stable, that is, there exist positive constants $\tau > 0$, $C > 0$ such that

$$E|X_t|^2 \le C \cdot e^{-\tau t}, \quad \forall t \ge 0,$$

and meanwhile is also exponential almost surely stable.

**Remark.** Observe that Theorems 3.1 and 5.1 in [8] cannot be applied to this occasion since the condition (3.2c) there does not hold.

### 4.2. Stochastic evolution equations with constant variable delays

Assume $h > 0$ and consider the following constant variable delay stochastic evolution equation over $H$ on $\mathcal{I} = [-h, \infty]$,

$$\begin{cases} dX_t = (AX_t + A(t, X_t, X_{t-h}))dt + B(t, X_t, X_{t-h})dW_t, \ \forall t \in [0, +\infty), \\ X_0 = x_0, \quad X_t = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{4.8}$$

where $A(t,\cdot,\cdot)$ and $B(t,\cdot,\cdot)$ are in general nonlinear mappings from $H\times H$ to $H$ and $H\times H$ to $\mathcal{L}(K,H)$, respectively. $\phi(t):[-h,0]\times\Omega\to H$, $h>0$, is a given initial datum such that $\phi(t)$ is $\mathcal{F}_0$-measurable and $\sup_{-h\le r\le 0}E|\phi(r)|^2<\infty$. Also observe that at the present moment $M=0$ in the condition (H1).

**Corollary 4.2.** *Let $v(x):H\to\mathbf{R}^1$ satisfy*

(i)

$$v(x)\ge c_1\cdot|x|^2 \quad \textit{for some} \quad c_1>0; \tag{4.9}$$

(ii) *$v(x)$ is twice Fréchet differentiable and $v'(x)$, $v''(x)$ are continuous in $H$ and $\mathcal{L}(H)$ respectively, and*

$$|v(x)|+|x||v'(x)|+|x|^2|v''(x)|\le c_2|x|^2 \quad \textit{for some} \quad c_2>0; \tag{4.10}$$

(iii) *There exist constants $\alpha>0$, $\mu>0$, $\lambda\in\mathbf{R}_+$ and a nonnegative continuous function $\gamma(t)$, $t\in\mathbf{R}_+$, such that*

$$Lv(t,x,y)\le-\alpha v(x)+\lambda v(y)+\gamma(t)e^{-\mu t}, \quad x\in\mathcal{D}(A), \quad y\in\mathcal{D}(A), \tag{4.11}$$

*where $\gamma(t)$ satisfies that for any $\delta>0$, $\gamma(t)=o(e^{\delta t})$, as $t\to\infty$.*

*Assume furthermore the condition $\alpha>\lambda$ holds, then there exist constants $\tau>0$, $C>0$ such that for the mild solution $X_t^\phi$ of (4.8),*

$$E|X_t^\phi|^2\le C\cdot e^{-\tau t}, \quad \forall t\ge 0. \tag{4.12}$$

*That is, the mild solution is mean square exponentially stable. Furthermore, under the same conditions the solution is also exponential almost surely stable.*

In what follows we shall apply Corollary 4.2 to a stochastic delay system considered by T. Taniguchi in [14] to improve the results derived there.

**Example 4.2** ([14])**.** Consider the semilinear stochastic heat equation with finite variable delays $r_1$ $(r>r_1\ge 0)$

$$\begin{gathered} dZ(t,x)=\delta\frac{\partial^2}{\partial x^2}Z(t,x)dt+\alpha_1 Z(t-r_1,x)d\beta(t), \\ t\ge 0, \quad \delta>0, \quad \alpha_1\ge 0, \\ Z(t,0)=Z(t,1)=0, \quad t\ge 0, \\ Z(s,x)=\phi(s,x), \quad \phi(\cdot,x)\in C([-r,0],\mathbf{R}^1), \quad \phi(s,\cdot)\in L^2(0,1), \\ s\in[-r,0], \quad x\in[0,1], \quad E\|\phi\|_C<\infty, \end{gathered} \tag{4.13}$$

where $\beta(t)$ is a standard Wiener process and $E\|\phi\|_C^2 = E\{\sup_{-r\le s\le 0}\|\phi(s)\|^2_{L^2(0,1)}\}$.

Let $A=\partial^2/\partial x^2$ with the domain

$$\mathcal{D}(A)=\Big\{u\in L^2(0,1),\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\in L^2(0,1), u(0)=u(1)=0\Big\}.$$

Suppose $H = L^2(0,1)$ with the corresponding boundary conditions as above. It is well known that $C_0$-semigroup $S(t)$, $t \geq 0$, generated by the operator $\delta A : L^2(0,1) \to L^2(0,1)$ satisfies $\|S(t)\| \leq e^{-\delta\pi^2 t}$, $t \geq 0$. Hence, by applying Corollary 4.2 to the above equation with $v(x) = \|x\|_H^2$, $x \in L^2(0,1)$, we have (for $u$, $v \in \mathcal{D}(A)$)

$$2\langle u, Au + A(t,u,v)\rangle + \|B(t,u,v)\|_2^2 \leq -2\delta\pi^2\|u\|_H^2 + \alpha_1^2\|v\|_H^2$$

which immediately implies if $\delta\pi^2 > 1/2 \cdot \alpha_1^2$, the solution is mean square exponentially stable and in the meantime is exponentially almost surely stable.

**Remark.** In [14], by using the properties of stochastic convolution integral T. Taniguchi only obtained that if $\delta\pi^2 > 3\alpha_1^2 \cdot e^{\delta\pi^2 r}$, i.e., $r < 1/(\delta\pi^2)$ $\ln(\delta\pi^2/3\alpha_1^2)$, the solution of the equation (4.13) is exponentially stable. In other words, the results derived in [14] involved with a strong restriction to delay interval parameter $r$, i.e., the requirement of the so-called small delay interval.

**Remark.** In [14], a class of more general stochastic partial functional differential equations are considered in addition to Example 4.2. However, it is worth pointing out that the methods employed in this paper can be carried over there in a quite similar manner. As a matter of fact, our results derived in the paper are even much stronger because of the fact that we actually obtain general results in the sense of the special consideration of variable time delay function instead of constant one.

### 4.3. Stochastic evolution equations with fractional power type stability condition.

As the final application, we shall try to investigate the so-called fractional power type stability result to close this paper.

**Corollary 4.3.** *Assume $\tau'(t) \leq M$, $0 \leq M < 1$, $t \in \mathbf{R}_+$. Let $v(x) : H \to \mathbf{R}^1$ satisfy*

(i)

$$v(x) \geq c_1 \cdot |x|^2 \quad \textit{for some} \quad c_1 > 0; \tag{4.14}$$

(ii) *$v(x)$ is twice Fréchet differentiable and $v'(x)$, $v''(x)$ are continuous in $H$ and $\mathcal{L}(H)$ respectively, and*

$$|v(x)| + |x||v'(x)| + |x|^2|v''(x)| \leq c_2|x|^2 \quad \textit{for some} \quad c_2 > 0; \tag{4.15}$$

(iii) *There exist constants $\alpha > 0$, $\nu > 0$, $\mu > 0$, $\theta > 0$, $\lambda \in \mathbf{R}_+$, $0 \leq \sigma \leq 1$ and nonnegative continuous functions $\xi(t)$, $\zeta(t)$ and $\gamma(t)$, $t \in \mathbf{R}_+$, such that*

$$\begin{aligned} Lv(t,x,y) \leq{}& -\alpha v(x) + \lambda v(y) + \xi(t)e^{-\nu t}v(x)^\sigma \\ &+ \zeta(t)e^{-\theta t}v(y)^\sigma + \gamma(t)e^{-\mu t}, \quad x \in \mathcal{D}(A), \quad y \in \mathcal{D}(A), \end{aligned} \tag{4.16}$$

*where $\xi(t)$, $\zeta(t)$ and $\gamma(t)$ satisfy that for any $\delta > 0$, $\xi(t) = o(e^{\delta t})$, $\zeta(t) = o(e^{\delta t})$ and $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$.*

*Assume furthermore the condition $\alpha > \lambda/(1-M)$ holds, then there exist constants $\tau > 0$, $C > 0$ such that for the mild solution $X_t^\phi$ of (2.1),*

$$E|X_t^\phi|^2 \le C \cdot e^{-\tau t}, \quad \forall t \ge 0. \tag{4.17}$$

*That is, the mild solution is mean square exponentially stable. Furthermore, under the same conditions the solution is also exponential almost surely stable.*

**Remark.** Observe that letting $\sigma = 0$ or $\sigma = 1$ in (4.16), we simply obtain Theorem 3.1 once again.

*Proof.* Observe that the case $\sigma = 0$ or $\sigma = 1$ is trivial. For $0 < \sigma < 1$, by virtue of Young's inequality

$$a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for any} \quad a \ge 0,\ b \ge 0,\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

we have for arbitrary $\varepsilon > 0$, the third and fourth terms on the right hand side of (4.16) turn out

$$\xi(t)e^{-\nu t}v(x)^\sigma \le \sigma\varepsilon^{1/\sigma}v(x) + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\xi(t)^{\frac{1}{1-\sigma}} \cdot e^{-\frac{\nu}{1-\sigma}t}$$

and

$$\zeta(t)e^{-\theta t}v(y)^\sigma \le \sigma\varepsilon^{1/\sigma}v(y) + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\zeta(t)^{\frac{1}{1-\sigma}} \cdot e^{-\frac{\theta}{1-\sigma}t}$$

which, together with (4.16), immediately implies that

$$\begin{aligned} &Lv(t,x,y)\\ &\quad \le -\alpha v(x) + \lambda v(y) + \sigma\varepsilon^{1/\sigma}v(x) + \sigma\varepsilon^{1/\sigma}v(y)\\ &\qquad + \Big(\gamma(t) + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\xi(t)^{\frac{1}{1-\sigma}} + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\zeta(t)^{\frac{1}{1-\sigma}}\Big)e^{-(\frac{\nu}{1-\sigma}\wedge\frac{\theta}{1-\sigma}\wedge\mu)t},\\ &\qquad x, y \in \mathcal{D}(A). \end{aligned}$$

Hence, in view of Theorems 3.1 and 3.2, it is easy to deduce that if $\alpha - \sigma\varepsilon^{1/\sigma} > (\lambda + \sigma\varepsilon^{1/\sigma})/(1-M)$, the mild solution is mean square exponentially stable and meanwhile exponential almost surely stable. Observe $\varepsilon > 0$ is an arbitrary constant, the proof of the corollary is therefore complete. $\square$

**Example 4.3.** Consider the following semilinear stochastic partial differential equation:

(4.18)
$$\begin{cases} dY_t(x) = \frac{\partial^2}{\partial x^2}Y_t(x)dt + e^{-t/2}(Y_{t-\tau(t)}(x))^{\frac{1}{3}}dt + \sqrt{\mu}\frac{Y_t(x)}{1+|Y_{t-\tau(t)}(x)|}dW_t,\\ \qquad t > 0, \quad x \in (0,1),\\ Y_0(x) = y_0(x), \quad Y_t(0) = Y_t(1) = 0, \quad t \ge 0, \end{cases}$$

where $\mu \geq 0$ is a nonnegative real number and $\tau(t) : \mathbf{R}^1 \to [0, h]$, is a certain differentiable function with $\tau'(t) \leq 0$. $W_t$ is a real standard Wiener process (so, $K = \mathbf{R}^1$ and $Q = 1$). We can set this problem in our formulation by taking $H = L^2[0,1]$ with the corresponding boundary conditions above, $Au(x) = (d^2/dx^2)u(x)$, $A(t,u,v) = e^{-t/2}v(x)^{1/3}$ and $B(t,u,v) = \sqrt{\mu}u(x)/(1+|v(x)|)$.

Suppose $v(x) = \|x\|_H^2$ and it is easy to deduce that for arbitrary $\delta > 0$ small enough and $u, v \in \mathcal{D}(A)$

$$
\begin{aligned}
(4.19) \qquad & 2\langle Au + A(t,u,v), u\rangle + \|B(t,u,v)\|_2^2 \\
& \leq -2\pi^2 |u|_H^2 + (\delta+\mu)|u|_H^2 + \delta \cdot e^{-t}|v|_H^{2/3}.
\end{aligned}
$$

Therefore, whenever $2\pi^2 > \delta + \mu \geq 0$, or equivalently, $2\pi^2 > \mu \geq 0$ (notice $\delta > 0$ is an arbitrary positive number), we easily deduce from Corollary 4.3 that for arbitrary delay interval $[-h, 0]$, $h > 0$, the mild solution of the equation (4.18) is mean square exponentially stable and also almost surely stable.

**Remark.** Observe once more that Theorems 3.1 and 5.1 in A. Ichikawa [8] cannot be applied to the corresponding null variable delay occasion of Example 4.3 to obtain the required exponential stability.

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Division of Statistics & OR
Department of Mathematical Sciences
University of Liverpool
Peach Street, Liverpool
L69 9ZL, UK

Department of Mathematics
University of Wales Swansea
Singleton Park, Swansea SA2 8PP, UK

## References

[1] T. Caraballo, Asymptotic exponential stability of stochastic partial differential equations with delays, Stochastics, **33** (1990), 27–47.

[2] T. Caraballo and K. Liu, Exponential stability of mild solutions of stochastic partial differential equations with delays, Stochastic Anal. Appl., **17** (1999), 743–763.

[3] T. Caraballo and K. Liu, On exponential stability criteria of stochastic partial differential equations, Stochastic Process Appl., **83** (1999), 289–301.

[4] T. Caraballo, K. Liu and A. Truman, Stability criteria of stochastic partial differential equations with variable delays, Preprint.

[5] P. L. Chow, Stability of nonlinear stochastic evolution equations, J. Math. Anal. Appl., **89** (1982), 400–419.

[6] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, 1992.

[7] U. G. Haussmann, Asymptotic stability of the linear Itô equation in infinite dimensional, J. Math. Anal. Appl., **65** (1978), 219–235.

[8] A. Ichikawa, Stability of semilinear stochastic evolution equations, J. Math. Anal. Appl., **90** (1982), 12–44.

[9] R. Khas'minskii and V. Mandrekar, On the stability of solutions of stochastic evolution equations, The Dynkin Festsch. Prog. Probab. 34, Birkhäuser, Boston, MA, 1994, 185–197.

[10] K. Liu, Lyapunov functionals and asymptotic stability of stochastic delay evolution equations, Stochastics, **63** (1998), 1–26.

[11] K. Liu and X. R. Mao, Exponential stability of non linear stochastic evolution equations, Stochastic Process Appl., **78** (1998), 173–193.

[12] R. Liu and V. Mandrekar, Stochastic semilinear evolution equations: Lyapunov function, stability and ultimate boundedness, J. Math. Anal. Appl., **212** (1997), 537–553.

[13] X. R. Mao, Exponential Stability of Stochastic Differential Equations, Marcel Dekker, New York, 1994.

[14] T. Taniguchi, Almost sure exponential stability for stochastic partial functional differential equations, Stochastic Anal. Appl., **16**-5 (1998), 965–975.

[15] E. Pardoux, Equations aux Dérivées Partielles Stochastiques Nonlinéaires Monotones, Thesis, Université Paris Sud, 1975.

[16] J. Real, Stochastic partial differential equations with delays, Stochastics, **8**-2 (1982–1983), 81–102.