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Structure of group C^*-algebras of the generalized Mautner groups

By

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Abstract

We construct finite composition series of group C^*-algebras of the generalized Mautner groups whose subquotients are tensor products of commutative C^*-algebras, noncommutative tori and the C^*-algebra of compact operators. As an application, we estimate the stable rank and connected stable rank of the C^*-algebras of generalized real Mautner groups.

Introduction

We first define the generalized Mautner groups $M_{n,m}$ by the semi-direct products $\mathbb{C}^n \rtimes_\alpha \mathbb{R}^m$ where the actions α of $\mathbb{R}^m$ on $\mathbb{C}^n$ are defined by the multiplication of the matrices

$$\alpha_t = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}, \quad t = (t_k) \in \mathbb{R}^m, z_j \in \mathbb{T}(1 \leq j \leq n)$$

with $z_j = e^{2\pi i \sum_{k=1}^m c_{jk} t_k}$ with $c_{jk} \in \mathbb{R}$. Then they are simply connected solvable Lie groups. The Mautner group is a special case of $M_{2,1}$ such as $c_{11} = 1$ and c_{21} an irrational number. Note that this definition is different from that of the extended Mautner groups of [AM, p. 138].

In this paper we will investigate the algebraic structure of the C^*-algebras of $M_{n,m}$. In the cases of either $n \geq 2$, $m = 1$, or $n = 1$, they are studied in [Sd2], [Sd4]. In what follows we assume that $n, m \geq 2$. We emphasize that it would be the first step for the cases with $m \geq 2$.

Notation. Let $C^*(G)$ be the (full) group C^*-algebra of a Lie group G (cf. [Dx, Part II]). For a locally compact Hausdorff space X, we denote by $C_0(X)$ the C^*-algebra of all complex-valued continuous functions on X vanishing at

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infinity, and let $C(X) = C_0(X)$ when X is compact. Let $\mathbb{K} = \mathbb{K}(H)$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space H.

1. Structure of the group C^*-algebra of $M_{n,m}$

Let $C^*(M_{n,m})$ be the group C^*-algebra of $M_{n,m} = \mathbb{C}^n \rtimes_\alpha \mathbb{R}^m$. Then via Fourier transform we have that

$$C^*(M_{n,m}) \cong C^*(\mathbb{C}^n) \rtimes_\alpha \mathbb{R}^m \cong C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{R}^m,$$

where the right hand side means the C^*-crossed product of $C_0(\mathbb{C}^n)$ by $\mathbb{R}^m$ with the action $\hat{\alpha}$ defined by the complex conjugate action $\hat{\alpha}_t = \alpha_t^*$. If $z_j = 1$ for some j, then $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{R}^m \cong C_0(\mathbb{C}) \otimes (C_0(\mathbb{C}^{n-1}) \rtimes_{\hat{\alpha}} \mathbb{R}^m)$. So we may assume that $z_j \neq 1$ for all j in the following. If $c_{jk} = 0$ $(1 \leq j \leq n)$ for some $1 \leq k \leq m$, then $C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{R}^m \cong C_0(\mathbb{R}) \otimes (C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{R}^{m-1})$. So we may also assume that for any k, $c_{jk} \neq 0$ for some j in our setting.

Since the origin 0_n of $\mathbb{C}^n$ is fixed under $\hat{\alpha}$, we have the following exact sequence:

$$0 \to C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes_{\hat{\alpha}} \mathbb{R}^m \to C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{R}^m \to C_0(\mathbb{R}^m) \to 0.$$

By the assumption of α in the introduction, the above ideal have a finite composition series $\{\mathfrak{L}_k\}_{k=1}^n$ whose subquotients are given by

$$\mathfrak{L}_{n-k+1}/\mathfrak{L}_{n-k} \cong \oplus^{\binom{n}{k}} C_0((\mathbb{C} \setminus \{0_1\})^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$$

for $1 \leq k \leq n$, where $\oplus^{\binom{n}{k}}$ is the combination $\binom{n}{k}$ times direct sum, and $\mathfrak{L}_k = C_0(X_k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$ with $X_k \setminus X_{k-1}$ the disjoint union $\sqcup_{\binom{n}{n+1-k}} (\mathbb{C} \setminus \{0_1\})^{n+1-k}$ obtained from considering $\hat{\alpha}$-invariant direct factors of the form $(\mathbb{C} \setminus \{0_1\})^{n+1-k}$ in X_k. Moreover, since $\hat{\alpha}$ is trivial in the radius direction of each direct factor $\mathbb{C} \setminus \{0_1\}$ of $(\mathbb{C} \setminus \{0_1\})^k$, we have the following decomposition:

$$C_0((\mathbb{C} \setminus \{0_1\})^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m \cong C_0(\mathbb{R}^k) \otimes (C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m).$$

By the above argument we obtain that

Proposition 1.1. *Let $M_{n,m}$ $(n, m \geq 2)$ be the generalized Mautner groups. Then $C^*(M_{n,m})$ has no nontrivial projections.*

Proof. The statement follows from that the above subquotients of $C^*(M_{n,m})$ have no nontrivial projections. □

Remark. The above proposition is also true in the cases of either $m = 1$ or $n = 1$. The first case is implicitly obtained in [Sd2], and the second one in [Sd4]. However, it is not true in general cases for $n = 1$ if actions are not multi-rotations. See [Sd1], [Sd3] for other results on the projection problem.

By the above reduction, it suffices to analyze the crossed product $C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$.

We now suppose that the action $\hat{\alpha}$ is transitive on $\mathbb{T}^k$. In this case we have $k \leq m$. Then we obtain by [Gr, Corollary 2.10] that

$$C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m \cong C(\mathbb{R}^m/\mathbb{R}^m_{1_k}) \rtimes \mathbb{R}^m \cong C^*(\mathbb{R}^m_{1_k}) \otimes \mathbb{K}(L^2(\mathbb{T}^k)),$$

where $\mathbb{R}^m_{1_k}$ is the stabilizer of $1_k = (1, \dots, 1) \in \mathbb{T}^k$. Since $\mathbb{R}^m_{1_k}$ is a closed subgroup of $\mathbb{R}^m$ and $\mathbb{R}^m/\mathbb{R}^m_{1_k} \cong \mathbb{T}^k$, it is isomorphic to $\mathbb{R}^{m-k} \times \mathbb{Z}^k$. In fact, by Pontryagin duality we have an exact sequence $1 \to \mathbb{Z}^k \to \mathbb{R}^m \to (\mathbb{R}^m_{1_k})^\wedge \to 1$ of abelian groups of characters so that $(\mathbb{R}^m_{1_k})^\wedge \cong \mathbb{R}^{m-k} \times \mathbb{T}^k$. Hence

$$C^*(\mathbb{R}^m_{1_k}) \cong C^*(\mathbb{R}^{m-k} \times \mathbb{Z}^k) \cong C_0(\mathbb{R}^{m-k} \times \mathbb{T}^k).$$

Next suppose that the action $\hat{\alpha}$ is not transitive on $\mathbb{T}^k$, which occurs for either any $k > m$ or some $k \leq m$. We note that the crossed product $C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$ can be decomposed into the successive crossed products

$$(\cdots((C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \cdots) \rtimes_{\hat{\alpha}^m} \mathbb{R},$$

where $\hat{\alpha}_t = (\hat{\alpha}^1_{t_1}, \dots, \hat{\alpha}^m_{t_m})$ for $t = (t_j) \in \mathbb{R}^m$. Then we deduce that

$$C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong \begin{cases} C(\mathbb{T}^{k-1}) \otimes (C(\mathbb{T}) \rtimes_{\hat{\alpha}^1} \mathbb{R}), & \text{or} \\ C(\mathbb{T}^{k-k_1}) \otimes (C(\mathbb{T}^{k_1}) \rtimes_{\hat{\alpha}^1} \mathbb{R}), & 2 \leq k_1 \leq k, \end{cases}$$

where the stabilizers of $\hat{\alpha}_1$ on either $\mathbb{T}$ or each direct factor of $\mathbb{T}^{k_1}$ are not $\mathbb{R}$. Moreover, in the first case, we obtain by [Gr, Corollary 2.10] that

$$C(\mathbb{T}) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong C(\mathbb{R}/\mathbb{Z}) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong C(\mathbb{T}_1) \otimes \mathbb{K},$$

where $C(\mathbb{T}_1) \cong C^*(\mathbb{Z})$, $\mathbb{T}_1 = \mathbb{T}$ and $\mathbb{Z}$ is the stabilizer of $\hat{\alpha}_1$. In the second case we note that the stabilizers of $\hat{\alpha}^1$ on $\mathbb{T}^{k_1}$ are $\{0\}$ or $\mathbb{Z}$, hence discrete, so that $C(\mathbb{T}^{k_1}) \rtimes_{\hat{\alpha}^1} \mathbb{R}$ is regarded as a foliation C^*-algebra $C^*(\mathbb{T}^{k_1}, \mathfrak{F})$ with the foliation $\mathfrak{F}$ consisting of all orbits in $\mathbb{T}^{k_1}$ by $\hat{\alpha}_1$. Since $(\mathbb{T}^{k_1}, \mathfrak{F})$ is a foliated $\mathbb{T}^{k_1-1}$-bundle over $\mathbb{T}$ with $\mathfrak{F}$ transversal to each fibers $\mathbb{T}^{k_1-1}$, we obtain that (cf. [Cn, II.8], [MS])

$$C(\mathbb{T}^{k_1}) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong C^*(\mathbb{T}^{k_1}, \mathfrak{F}) \cong (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}$$

where the action $\hat{\alpha}^1$ of $\mathbb{Z}$ is a suitable restriction to the transversal $\mathbb{T}^{k_1-1}$, and the crossed product $C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}$ is a special case of noncommutative tori, say $\mathbb{T}^{k_1}_{\Theta_1}$ with $\Theta_1 = (c_{j_1 1}, \dots, c_{j_{k_1-1} 1})$ and $1 \leq j_1 < \cdots < j_{k_1-1} \leq n$. (Note that the similar argument as above was first shown in [Sd2].)

Therefore, we have that

Lemma 1.2. *Let $C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}$ be the subalgebra of $C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$ as above. Then*

$$\begin{aligned} C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R} &\cong \begin{cases} C(\mathbb{T}^{k-1} \times \mathbb{T}_1) \otimes \mathbb{K}, & \textit{or} \\ C(\mathbb{T}^{k-k_1}) \otimes (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K} \end{cases} \\ &\cong \begin{cases} C(\mathbb{T}^k) \otimes \mathbb{K}, & \textit{or} \\ C(\mathbb{T}^{k-k_1}) \otimes \mathbb{T}^{k_1}_{\Theta_1} \otimes \mathbb{K}, & \textit{for } 2 \leq k_1 \leq k. \end{cases} \end{aligned}$$

Remark. If the stabilizers of $\hat{\alpha}_1$ on each direct factor of $\mathbb{T}^k$ is not $\mathbb{R}$, we deduce that

$$C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R} \cong \begin{cases} C(\mathbb{T}) \otimes \mathbb{K} & \text{if } k = 1, \\ \mathbb{T}^k_{\Theta_1} \otimes \mathbb{K} & \text{if } k \geq 2. \end{cases}$$

Next we consider the crossed product $(C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}$. By using Lemma 1.2 we obtain that

$$(C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \cong \begin{cases} (C(\mathbb{T}^{k-1} \times \mathbb{T}_1) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, & \text{(I)} \quad \text{or} \\ (C(\mathbb{T}^{k-k_1}) \otimes (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R}, & \text{(II)}. \end{cases}$$

In the case (I) we have that

$$(C(\mathbb{T}^{k-1} \times \mathbb{T}_1) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \cong \begin{cases} (C(\mathbb{T}^{k-1}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K} \otimes C_0(\mathbb{R}_2), & (\mathrm{I}_1) \quad \text{or} \\ (C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K} & (\mathrm{I}_2) \end{cases}$$

since $\hat{\alpha}^2$ is trivial on $C(\mathbb{T}_1) \otimes \mathbb{K}$ in the case (I_2), and for the case (I_1) we note that

$$C(\mathbb{T}_1) \otimes \mathbb{K} \otimes C_0(\mathbb{R}_2) \cong C(\mathbb{T}) \rtimes_{(\hat{\alpha}_1, \hat{\alpha}_2)} \mathbb{R}^2,$$

where $\mathbb{R}_2 = \mathbb{R}$. By the same observation with the case for $\hat{\alpha}^1$, and since $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, we deduce that $(C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}$ is isomorphic to one of the following:

$$\begin{cases} C(\mathbb{T}^{k-2} \times \mathbb{T}_2) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}, \quad \text{or} \\ C(\mathbb{T}^{k-1-k_2}) \otimes (C(\mathbb{T}^{k_2-1}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}, \quad \text{or} \\ C(\mathbb{T}^{k-k_2}) \otimes (C(\mathbb{T}^{k_2-2}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}. \end{cases}$$

for $2 \leq k_2 \leq k-1$, where $\mathbb{T}_2 = \mathbb{T}$ as $\mathbb{T}_1$, and the stabilizers of $\hat{\alpha}^2$ on each direct factor of $\mathbb{T}^{k_2}$ are not $\mathbb{R}$, and for the third case, we obtain that

$$\begin{aligned} (C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K} &\cong (C(\mathbb{T}^{k-k_2} \times \mathbb{T}^{k_2-1} \times \mathbb{T}) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \\ &\cong (C(\mathbb{T}^{k-k_2} \times \mathbb{T}^{k_2-1} \times \mathbb{T}_1) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \\ &\cong C(\mathbb{T}^{k-k_2}) \otimes (C(\mathbb{T}^{k_2-2}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K} \end{aligned}$$

because the stabilizers of $\hat{\alpha}_2$ on each direct factor of $\mathbb{T}^{k_2-1}$ are discrete.

In the case (II) we conclude that

$$\begin{aligned} &(C(\mathbb{T}^{k-k_1}) \otimes (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \\ &\quad \cong \begin{cases} (C(\mathbb{T}^{k-k_1} \times \mathbb{T}^{k_1-2}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K} \otimes C(\mathbb{T}_2), & (\mathrm{II}_1) \quad \text{or} \\ ((C(\mathbb{T}^{k-k_1} \times \mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K} & (\mathrm{II}_2) \end{cases} \end{aligned}$$

since the actions $\hat{\alpha}^1$ and $\hat{\alpha}^2$ commute in the case (II_2), and for the case (II_1),

$$\begin{aligned} (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^2} \mathbb{R} &\cong (C(\mathbb{T}^{k_1}) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \\ &\cong (C(\mathbb{T}^{k_1-1} \times \mathbb{T}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \rtimes_{\hat{\alpha}^1} \mathbb{R} \\ &\cong (C(\mathbb{T}^{k_1-1}) \otimes C(\mathbb{T}_2) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^1} \mathbb{R} \\ &\cong (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \otimes C(\mathbb{T}_2) \otimes \mathbb{K} \end{aligned}$$

where the stabilizers of $\hat{\alpha}_2$ on $\mathbb{T}$ of $\mathbb{T}^{k_1-1} \times \mathbb{T} = \mathbb{T}^{k_1}$ are $\mathbb{Z}$ and $\mathbb{R}$ on elsewhere. Moreover, using Lemma 1.2 for $\hat{\alpha}_2$ we have that

$$
\begin{aligned}
&((C(\mathbb{T}^{k-k_1} \times \mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^2} \mathbb{R}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K} \\
&\quad \cong \begin{cases} ((C(\mathbb{T}^{k-k_1-1} \times \mathbb{T}^{k_1-1}) \otimes C(\mathbb{T}_2) \otimes \mathbb{K}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, \quad \text{or} \\ ([C(\mathbb{T}^{k-k_1-l_{21}} \times \mathbb{T}^{k_1-1-l_{22}}) \otimes (C(\mathbb{T}^{k_2-1}) \rtimes_{\hat{\alpha}^2} \mathbb{Z})] \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}, \end{cases}
\end{aligned}
$$

where $2 \leq k_2 = l_{21} + l_{22} \leq k-1$. Summing up the above argument we obtain that

Lemma 1.3. *Let $(C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R}$ be the subalgebra of $C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{R}^m$ as above. Then it is isomorphic to one of the following types:*

$$
\begin{cases}
C(\mathbb{T}^{k-1}) \otimes C_0(\mathbb{T}_1 \times \mathbb{R}_2) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-2}) \otimes C(\mathbb{T}_1 \times \mathbb{T}_2) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-1-k_2}) \otimes (C(\mathbb{T}^{k_2-1}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_2}) \otimes (C(\mathbb{T}^{k_2-2}) \rtimes_{\hat{\alpha}^2} \mathbb{Z}) \otimes C(\mathbb{T}_1) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1}) \otimes (C(\mathbb{T}^{k_1-2}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes C_0(\mathbb{T}_2) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1-1}) \otimes (C(\mathbb{T}^{k_1-1}) \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes C(\mathbb{T}_2) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1-l_{21}}) \otimes ([C(\mathbb{T}^{k_1-1-l_{22}}) \otimes (C(\mathbb{T}^{k_2-1}) \rtimes_{\hat{\alpha}^2} \mathbb{Z})] \rtimes_{\hat{\alpha}^1} \mathbb{Z}) \otimes \mathbb{K}
\end{cases}
$$

for $2 \leq k_2 = l_{21} + l_{22} \leq k-1$. These cases can be rewritten by

$$
(C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \cong \begin{cases}
C_0(\mathbb{T}^k \times \mathbb{R}) \otimes \mathbb{K}, \\
C(\mathbb{T}^k) \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_2}) \otimes \mathbb{T}^{k_2}_{\Theta_2} \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_2+1}) \otimes \mathbb{T}^{k_2-1}_{\Theta_2} \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1+1}) \otimes \mathbb{T}^{k_1-1}_{\Theta_1} \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1}) \otimes \mathbb{T}^{k_1}_{\Theta_1} \otimes \mathbb{K}, \\
C(\mathbb{T}^{k-k_1-l_{21}}) \otimes \mathbb{T}^{k_1+l_{21}}_{(\Theta_1,\Theta_2)} \otimes \mathbb{K},
\end{cases}
$$

where $\mathbb{T}^{k_1+l_{21}}_{(\Theta_1,\Theta_2)} = C(\mathbb{T}^{k_1+l_{21}-2}) \rtimes_{(\hat{\alpha}^1,\hat{\alpha}^2)} \mathbb{Z}^2$ is a noncommutative torus.

Remark. If the stabilizers of $\hat{\alpha} = (\hat{\alpha}^1, \hat{\alpha}^2)$ on each direct factor of $\mathbb{T}^k$ are not $\mathbb{R}^2$, and $k_j \geq 1$ for $(j = 1, 2)$, then we have that

$$
(C(\mathbb{T}^k) \rtimes_{\hat{\alpha}^1} \mathbb{R}) \rtimes_{\hat{\alpha}^2} \mathbb{R} \cong \begin{cases}
C_0(\mathbb{T} \times \mathbb{R}) \otimes \mathbb{K}, & k = 1, \\
C(\mathbb{T}_1 \times \mathbb{T}_2) \otimes \mathbb{K}, & k = 2, \\
\mathbb{T}^{k-1}_{\Theta_2} \otimes C(\mathbb{T}_1) \otimes \mathbb{K}, \\
\mathbb{T}^{k-1}_{\Theta_1} \otimes C(\mathbb{T}_2) \otimes \mathbb{K}, \\
\mathbb{T}^{k}_{(\Theta_1,\Theta_2)} \otimes \mathbb{K},
\end{cases}
$$

where $\hat{\alpha}$ is transitive in the above cases for $k = 1, 2$.

In the general case, we obtain by the similar computation that

Theorem 1.4. *Let $C(\mathbb{T}^k)\rtimes_{\hat\alpha}\mathbb{R}^m$ be the crossed product cited above with the stabilizers of $\hat\alpha$ on each direct factor of $\mathbb{T}^k$ not equal to $\mathbb{R}^m$, and $k_j \geq 1$ for any $1 \leq j \leq m$ as in the above remark. If $\hat\alpha$ is transitive on $\mathbb{T}^k$, then we deduce that*

$$C(\mathbb{T}^k)\rtimes_{\hat\alpha}\mathbb{R}^m \cong C_0(\mathbb{R}^{m-k}\times\mathbb{T}^k)\otimes\mathbb{K}, \quad k\leq m,$$

and in other cases, we obtain that

$$C(\mathbb{T}^k)\rtimes_{\hat\alpha}\mathbb{R}^m \cong \begin{cases} C_0(\mathbb{T}^k\times\mathbb{R}^{h_1})\otimes\mathbb{K}, \\ C_0(\mathbb{T}^{k-l_j}\times\mathbb{R}^{h_2})\otimes\mathbb{T}^{l_j}_{\Theta_j}\otimes\mathbb{K}, \\ C_0(\mathbb{T}^{k-l_{j12}}\times\mathbb{R}^{h_3})\otimes\mathbb{T}^{l_{j12}}_{(\Theta_{j_1},\Theta_{j_2})}\otimes\mathbb{K}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_0(\mathbb{T}^{k-l_{j_1\cdots(m-2)}}\times\mathbb{R}^{h_{m-1}})\otimes\mathbb{T}^{l_{j_1\cdots(m-2)}}_{(\Theta_1,\cdots,\Theta_{m-2})}\otimes\mathbb{K}, \\ C(\mathbb{T})\otimes\mathbb{T}^{k-1}_{(\Theta_1,\cdots,\Theta_{m-1})}\otimes\mathbb{K}, \\ \mathbb{T}^k_{(\Theta_1,\cdots,\Theta_m)}\otimes\mathbb{K}, \end{cases}$$

where $\mathbb{R}^{h_j}$ $(0\leq h_j\leq m-j)$ are subspaces of $\hat{\mathbb{R}}^m$, and

$$\mathbb{T}^{l_j}_{\Theta_j},\cdots,\mathbb{T}^k_{(\Theta_1,\cdots,\Theta_m)} = C(\mathbb{T}^{k-m})\rtimes_{(\hat\alpha_1,\cdots,\hat\alpha_m)}\mathbb{Z}^m$$

are noncommutative tori, and if $1\leq s\leq m-1$, then $1\leq j_1<j_2<\cdots<j_s\leq m$ for $\mathbb{T}^l_{(\Theta_{j_1},\Theta_{j_2},\cdots,\Theta_{j_s})}$, where $\Theta_{j_p}=(c_{q_1p},\cdots,c_{q_{l-s}p})$ with $1\leq q_1<\cdots<q_{l-s}\leq n$ for $1\leq p\leq s$.

Proof. We have shown the transitive case after Proposition 1.1. In other cases, by induction on m we deduce that

$$C(\mathbb{T}^k)\rtimes_{\hat\alpha}\mathbb{R}^m \cong C_0(\mathbb{T}^{k-l}\times\mathbb{R}^h)\otimes\mathbb{T}^l_{(\Theta_{j_1},\cdots,\Theta_{j_s})}\otimes\mathbb{K}$$

for some $0\leq l\leq k$ and $0\leq h\leq m-s-1$. Then $(C(\mathbb{T}^k)\rtimes_{\hat\alpha}\mathbb{R}^m)\rtimes_{\hat\alpha^{m+1}}\mathbb{R}$ is isomorphic to one of the following types:

$$\begin{cases} ((C(\mathbb{T}^{k-l-k'})\otimes\mathbb{T}^l_{(\Theta_{j_1},\cdots,\Theta_{j_s})})\otimes\mathbb{K}\otimes C_0(\mathbb{R}^{h-h'}) \\ \qquad\qquad\qquad\otimes(C(\mathbb{T}^{k'})\rtimes_{(\hat\alpha,\hat\alpha^{m+1})}\mathbb{R}^{h'+k'}\times\mathbb{R}), \\ ((C(\mathbb{T}^{k-l})\otimes\mathbb{T}^l_{(\Theta_{j_1},\cdots,\Theta_{j_s})})\rtimes_{\hat\alpha^{m+1}}\mathbb{R})\otimes\mathbb{K}\otimes C_0(\mathbb{R}^h), \end{cases}$$

where $C(\mathbb{T}^{k'})\rtimes_{\hat\alpha}\mathbb{R}^{h'+k'}\cong C_0(\mathbb{T}^{k'}\times\mathbb{R}^{h'})\otimes\mathbb{K}$. For the first case, we obtain that

$$C(\mathbb{T}^{k'})\rtimes_{(\hat\alpha,\hat\alpha^{m+1})}\mathbb{R}^{h'+k'}\times\mathbb{R}\cong C_0(\mathbb{T}^{k'}\times\mathbb{R}^{h'+1})\otimes\mathbb{K}.$$

For the second case, we note that

$$\begin{aligned}\mathbb{T}^l_{(\Theta_{j_1},\cdots,\Theta_{j_s})}\rtimes_{\hat\alpha^{m+1}}\mathbb{R} &\cong (C(\mathbb{T}^{l-s})\rtimes_{\hat\alpha}\mathbb{Z}^s)\rtimes_{\hat\alpha^{m+1}}\mathbb{R} \\ &\cong (C(\mathbb{T}^{l-s})\rtimes_{\hat\alpha^{m+1}}\mathbb{R})\rtimes_{\hat\alpha}\mathbb{Z}^s.\end{aligned}$$

Therefore we have that

$$(C(\mathbb{T}^{k-l}) \otimes \mathbb{T}^{l}_{(\Theta_{j_1},\dots,\Theta_{j_s})}) \rtimes_{\hat{\alpha}^{m+1}} \mathbb{R} \cong (C(\mathbb{T}^{k-l} \times \mathbb{T}^{l-s}) \rtimes_{\hat{\alpha}^{m+1}} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{Z}^s.$$

By assumption of $\hat{\alpha}$, if $k-l \geq 1$, then the stabilizers of $\hat{\alpha}^{m+1}$ on each direct factor of $\mathbb{T}^{k-l}$ are not $\mathbb{R}$. Then

$$C(\mathbb{T}^{k-s}) \rtimes_{\hat{\alpha}^{m+1}} \mathbb{R} \cong \begin{cases} C(\mathbb{T}_{m+1}) \otimes \mathbb{K} & \text{if } k-s=1, \\ (C(\mathbb{T}^{k-s-1}) \rtimes_{\hat{\alpha}^{m+1}} \mathbb{Z}) \otimes \mathbb{K} & \text{if } k-s \geq 2, \end{cases}$$

where $\mathbb{T}_{m+1} = \mathbb{T}$. In particular, if $k-l=1$ and $\hat{\alpha}^{m+1}$ trivial on $\mathbb{T}^{l-s}$, then the crossed product in the left hand side is isomorphic to $C(\mathbb{T} \times \mathbb{T}^{l-s}) \otimes \mathbb{K}$. Finally, note that

$$(C(\mathbb{T}^{k-s-1}) \rtimes_{\hat{\alpha}^{m+1}} \mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z}^s \cong \mathbb{T}^{k}_{(\Theta_{j_1},\dots,\Theta_{j_s},\Theta_{j_{s+1}})},$$

where $\Theta_{j_{s+1}}$ corresponds to $\hat{\alpha}^{m+1}$ on $\mathbb{T}^{k-s-1}$. □

Summing up the arrangement discussed above we obtain that

Theorem 1.5. *Let $M_{n,m} = \mathbb{C}^n \rtimes_\alpha \mathbb{R}^m$ be the generalized Mautner group. Then there exists a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ of $C^*(M_{n,m})$ whose subquotients are given by*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{C}^l \times \mathbb{R}^m) & j=K, \\ C_0(\mathbb{C}^l \times \mathbb{R}^{k_j}) \otimes (C(\mathbb{T}^{k_j}) \rtimes_{\hat{\alpha}} \mathbb{R}^m) & 1 \leq j \leq K-1, \end{cases}$$

for some $0 \leq l \leq n$ and $1 \leq k_j \leq k_{j-1} \leq n$, and the crossed product $C(\mathbb{T}^{k_j}) \rtimes_{\hat{\alpha}} \mathbb{R}^m$ is isomorphic to one of the C^*-tensor products appeared in Theorem 1.4.

Remark. We note that the product space $\mathbb{C}^l \times \mathbb{R}^m$ is homeomorphic to the space of all 1-dimensional representations of $M_{n,m}$. The composition series $\{\mathfrak{I}_j\}_{j=1}^K$ is a refinement of $\{\mathfrak{L}_k\}_{k=1}^n$ constructed before Proposition 1.1. The algebraic structure of group C^*-algebras of simply connected, solvable Lie groups in more general cases would be examined in another paper elsewhere.

2. Application

As an application of Theorem 1.5, we obtain that

Theorem 2.1. *Let $H = \mathbb{R}^n \rtimes_\beta \mathbb{R}^m$ be the subgroup of the generalized Mautner group $G = \mathbb{C}^n \rtimes_\alpha \mathbb{R}^m$ with $\alpha = \beta + i\beta$. Then $C^*(H)$ has a finite composition series $\{\mathfrak{D}_j\}_{j=1}^L$ whose subquotients are given by*

$$\mathfrak{D}_j/\mathfrak{D}_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{d+m}) & j=L, \\ \begin{cases} C_0(X_j) \otimes \mathbb{K}, \quad \textit{or} \\ C_0(Y_j) \otimes \mathfrak{B}_j \otimes \mathbb{K} \end{cases} & 1 \leq j \leq L-1 \end{cases}$$

for some $0 \le d \le n$, *where* $\mathfrak{B}_j$ *is isomorphic to either a suitable* $\mathbb{T}^{l_j}_{(\Theta_{j_1},\dots,\Theta_{j_s})}$ *or its quotient, and* X_j, Y_j *are suitable quotient spaces of the product spaces* $\mathbb{C}^l \times \mathbb{R}^{k_j+h_j} \times \mathbb{T}^{k_j-l_j}$ *as in Theorems* 1.4 *and* 1.5.

Proof. Note that $C^*(H)$ is a quotient of $C^*(G)$ by assumption. Then we can construct a finite composition series of $C^*(H)$ by taking quotients of the composition series of $C^*(G)$ in Theorem 1.5. Also note that the noncommutative tori $\mathbb{T}^{l_j}_{(\Theta_{j_1},\dots,\Theta_{j_s})}$ are simple or not according to whether the action on the corresponding restriction is free. □

We now denote by $\hat{G}_1$ the space of all 1-dimensional representations of a Lie group G. As a corollary of Theorems 1.5 and 2.1, using some formulas of the stable rank and connected stable rank for C^*-algebras we obtain that

Corollary 2.2. *Let* G *be the generalized Mautner group. Then we have that*

$$\begin{cases} \begin{cases} \mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \hat{G}_1, & \text{if } \dim \hat{G}_1 \text{ is even,} \\ 2 \vee \dim_{\mathbb{C}} \hat{G}_1 \le \mathrm{sr}(C^*(G)) \le \dim_{\mathbb{C}} \hat{G}_1 + 1, & \text{if } \dim \hat{G}_1 \text{ is odd,} \end{cases} \\ \mathrm{csr}(C^*(G)) \le 2 \vee \mathrm{csr}(C_0(\hat{G}_1)) = [(\dim \hat{G}_1 + 1)/2] + 1, \end{cases}$$

where $\mathrm{sr}(\cdot)$, $\mathrm{csr}(\cdot)$ *respectively mean the stable rank and connected stable rank of* C^**-algebras,* $\dim_{\mathbb{C}} = [\dim(\cdot)/2] + 1$, *and* $\vee$ *is the maximum.*

Proof. We apply the following formulas by [Rf, Theorems 3.6, 4.3, 4.4, 4.11 and 6.4] and [Sh, Theorems 3.9 and 3.10] to the composition series obtained in Theorem 1.5 (or Theorem 2.1 for $C^*(H)$) inductively:

$$\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \le \mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \quad \mathrm{csr}(\mathfrak{A}) \le \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I})$$

for an exact sequence $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ of C^*-algebras, and

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) \le 2, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \le 2.$$

By [Rf, Proposition 1.7] and [Sh, p. 381] (cf. [Ns]), we know that

$$\begin{cases} \mathrm{sr}(C_0(X)) = \dim_{\mathbb{C}} X^+, \quad \mathrm{csr}(C_0(\mathbb{R})) = 2, \\ \mathrm{csr}(C_0(\mathbb{R}^2)) = 1, \quad \mathrm{csr}(C_0(\mathbb{R}^d)) = [(d+1)/2] + 1 \quad \text{for } d \ge 3, \end{cases}$$

where X^+ means the one-point compactification of a locally compact T^2-space X. Moreover, we note that $\mathrm{sr}(C^*(G)) \ge 2$ by [ST2, Lemma 3.7]. These imply the conclusion. □

Remark. The above corollary partially answers Rieffel's question [Rf, p. 313] describing the stable rank of group C^*-algebras in terms of groups. See [Sd2], [Sd3], [ST1], [ST2] for some results related to this question. If K_1-group of $C^*(G)$ is nontrivial, in other words, if $\dim G$ is odd (cf. [Cn, II.C]), then we obtain that $\mathrm{csr}(C^*(G)) \ge 2$ by [Eh, Corollary 1.6].

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