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Polarization change of moduli of vector bundles on surfaces with $p_g > 0$

By

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1. Introduction

Let X be a nonsingular projective surface over the complex number field $\mathbb{C}$, and H an ample line bundle on X. When we fix a line bundle c_1 on X and an integer c_2, there exists a coarse moduli scheme $\mathfrak{M}_H(c_1, c_2)$ parameterizing all rank-two H-μ-stable vector bundles E on X with $\det(E) = c_1$ and $c_2(E) = c_2$.

For two different polarizations H and H', several interesting phenomena are observed about the difference between $\mathfrak{M}_H(c_1, c_2)$ and $\mathfrak{M}_{H'}(c_1, c_2)$ in several papers. For example, if c_2 is sufficiently large with respect to c_1, H, and H', then $\mathfrak{M}_H(c_1, c_2)$ and $\mathfrak{M}_{H'}(c_1, c_2)$ are birationally equivalent ([Q1]). On the other hand, $\mathfrak{M}_H(c_1, c_2)$ is naturally embedded in the moduli space Spl_X of simple sheaves on X. Now we want to know how moduli schemes $\mathfrak{M}_H(c_1, c_2)$ behave in Spl_X, if H runs over the set of *all* ample line bundles on X. Closely related to this problem, we think over the existence problem of trivial polarizations. An irreducible component $\mathcal{M}$ of $\mathfrak{M}_H(c_1, c_2)$ is said to be *trivial* if for any polarization L on X, some vector bundle E contained in $\mathcal{M}$ is also L-μ-stable. A polarization H is said to be *trivial* of type (c_1, c_2) if every irreducible component of $\mathfrak{M}_H(c_1, c_2)$ is trivial. In [Q1], Qin conjectured as follows;

> if c_2 is sufficiently large with respect to X and c_1, then trivial polarizations of type (c_1, c_2) exist.

In case of some ruled surfaces or surfaces with Kodaira dimension zero, this conjecture has an affirmative solution ([Q3] and [Q2]). However in general it is unknown whether this conjecture is valid or not. Now we state the main result in this paper.

Theorem 1.1. *Suppose that the geometric genus $p_g(X)$ of X is positive. Then there exists a constant $C = C(X, c_1)$ depending on X and c_1 such that the following holds.*

If $c_2 \geq C(X, c_1)$, then there exists a polarization $L = L(c_1, c_2)$ depending on c_1 and c_2 such that $\mathfrak{M}_L(c_1, c_2)$ is irreducible and that $\mathfrak{M}_H(c_1, c_2) \cap \mathfrak{M}_L(c_1, c_2)$ is nonempty for any polarization H on X.

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This theorem gives a partial, affirmative solution to Qin's conjecture.

In Section 2 we shall review the wall structure of the ample cone. In Section 3 we shall prove Theorem 1.1. We note that the conclusion of Theorem 1.1 is valid also when X is a relatively minimal elliptic surface with Kodaira dimension one, and $c_1 \cdot f$ is odd, where $f \in \mathrm{NS}(X)$ is the fiber class of elliptic fibration. In this case we will give some comments in Remark 3.7, but we will not give complete proof.

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Notation.

- A scheme is of finite type over $\mathbb{C}$, and a surface is a nonsingular projective surface over $\mathbb{C}$.
- For a surface X, we shall use $\sim$ in order to denote numerically equivalence of divisors on X. $\mathrm{Num}(X)$ is the quotient of $\mathrm{Pic}(X)$ modulo the numerical equivalence. $\mathrm{Amp}(X) \subset \mathrm{Num}(X) \otimes \mathbb{R}$ is the ample cone of X. An ample line bundle on X is often called a *polarization*.
- Two not necessarily integral schemes S_1 and S_2 are *birationally equivalent* if there are dense open subsets $U_1 \subset S_1$ and $U_2 \subset S_2$ such that U_1 is isomorphic to U_2.
- Let X be a surface and H a polarization on X. H-μ-(semi)stability of a torsion free sheaf E on X is slope-(semi)stability. H-(semi)stability is Gieseker-Maruyama (semi)stability. E is strictly H-(μ-)semistable if E is H-(μ-)semistable and not H-(μ-)stable. For a H-semistable sheaf E, $\mathrm{gr}_H(E)$ is $\oplus_{i=1}^t E_i/E_{i-1}$, where $0 = E_0 \subset E_1 \subset \cdots E_t = E$ is a Jordan-Hölder filtration of E.
- Let S be a scheme. For a vector bundle V on S and a line bundle L on S, $\mathrm{Ext}^i_S(V, V \otimes L)^{\,0}$ is the kernel of trace : $\mathrm{Ext}^i_S(V, V \otimes L) \to H^i(S, L)$.

2. Review of walls in the ample cone

The concept of walls in the ample cone often helps us to study how the moduli scheme $\mathfrak{M}_H(c_1, c_2)$ changes as a polarization changes. This concept has been developed in several papers; here we shall mainly refer to [Q4] and [MW].

Definition 2.1. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$ with $4c_2 - c_1^2 > 0$. For $\xi \in \mathrm{Num}(X) \setminus \{0\}$, we define W^ξ to be the set $\{D \in \mathrm{Amp}(X) | D \cdot \xi = 0\}$ in $\mathrm{Amp}(X)$. W^ξ *separates* H and $H' \in \mathrm{Amp}(X)$ if $H \cdot \xi \neq 0 \neq H' \cdot \xi$ and $\mathrm{sgn}(H \cdot \xi) \neq \mathrm{sgn}(H' \cdot \xi)$. We define $W(c_1, c_2)$ to be

$$\left\{ W^\xi \neq \emptyset \;\middle|\; \begin{array}{l} \xi \sim 2D - c_1 \not\sim 0 \text{ for some } D \in \mathrm{Pic}(X) \\ \text{and } -4c_2 + c_1^2 \leq \xi^2 < 0. \end{array} \right\}.$$

A *Wall* of type (c_1, c_2) is an element of $W(c_1, c_2)$.

Proposition 2.2. *The set of walls of type (c_1, c_2) is locally finite in* $\mathrm{Amp}(X)$.

Proof. Refer to [Q4, Proposition 2.1.6] or [MW, Section 1]. □

Proposition 2.3. *Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$ such that $4c_2 - c_1^2 > 0$. Suppose that polarizations H and H' are not contained in any wall of type (c_1, c_2), and that there is* the only one *wall W_0 of type (c_1, c_2) separating H and H' in* $\mathrm{Amp}(X)$.

(a) *If a rank-two vector bundle V with Chern classes (c_1, c_2) is H-μ-stable and not H'-μ-stable, then V is given by a nontrivial extension*

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \longrightarrow 0, \tag{1}$$

where $\mathcal{O}_X(F)$ is a line bundle with $(2F - c_1) \cdot H < 0 < (2F - c_1) \cdot H'$, and Z is a codimension-two subscheme in X with $l(Z) = c_2 + \{(2F - c_1)^2 - c_1^2\}/4$.

(b) *Conversely, suppose that V is a rank-two vector bundle given by a nontrivial extension* (1) *satisfying the same condition as in* (a). *Then V is H-μ-stable and not H'-μ-semistable.*

Proof. (1) is [Q4, p. 400, Proposition 1.2.2]. (2) is [Q4, p. 406, Theorem 1.2.3]. □

Corollary 2.4. *Let $c_1 \in \mathrm{Pic}(X)$, $c_2 \in \mathbb{Z}$, and H be the same as in Proposition* 2.3. *Suppose that an irreducible component $\mathcal{M}$ of $\mathfrak{M}_H(c_1, c_2)$ satisfies that $\mathcal{M} \cap \mathfrak{M}_{H'}(c_1, c_2)$ is empty in Spl_X for some polarization H'. Then $\mathcal{M}_{red}$ is birationally equivalent to $Y \times \mathbb{P}^l$, where l is an integer and Y is a variety with $\dim(Y) < 2c_2 - (c_1^2/2) + q(X)$.*

Proof. If polarizations H and H' are not separated by any wall of type (c_1, c_2), then $\mathfrak{M}_H(c_1, c_2)$ and $\mathfrak{M}_{H'}(c_1, c_2)$ are isomorphic. From this and Proposition 2.2, we can suppose that H and H' in this corollary satisfy the same hypothesis as in Proposition 2.3. Because $\mathcal{M} \cap \mathfrak{M}_{H'}(c_1, c_2)$ is empty, every vector bundle $V \in \mathcal{M}$ is given by a nontrivial extension (1). Now let T be an open, closed, and reduced subscheme in $\coprod_{a \in \mathbb{N}} \mathrm{Pic}(X) \times \mathrm{Hilb}^a(X)$ defined as

$$T = \left\{ \begin{array}{c|c} (\mathcal{O}_X(F), Z) & \begin{array}{c} H \cdot (2F - c_1) < 0 < H' \cdot (2F - c_1) \quad \text{and} \\ l(Z) = c_2 + \{(2F - c_1)^2 - c_1^2\}/4 \end{array} \end{array} \right\}.$$

Here we can prove that T is contained in $\coprod_{a < c_2 - (c_1^2/4)} \mathrm{Pic}(X) \times \mathrm{Hilb}^a(X)$ using the Hodge index theorem, and so

$$\dim T < 2c_2 - (c_1^2/2) + q(X). \tag{2}$$

Next, for $V \in \mathcal{M}$ we define $d(V)$ as $\dim \mathrm{Ext}^1_X(\mathcal{O}_X(c_1 - F) \otimes I_Z, \mathcal{O}_X(F))$, where $\mathcal{O}_X(F)$ and I_Z are sheaves given in (1). Remark that $d(V)$ is well-defined because of the uniqueness of Harder-Narasimhan filtration. We also define $d_{\mathcal{M}}$

as $\min\{d(V)|V \in \mathcal{M}\}$. For the scheme T defined above, we think of a locally closed subset

$$T_{\mathcal{M}} = \{(\mathcal{O}_X(F), Z) \in T|\dim \mathrm{Ext}^1(\mathcal{O}_X(c_1 - F) \otimes I_Z, \mathcal{O}_X(F)) = d_{\mathcal{M}}\}$$

as a reduced subscheme, and denote by $\mathcal{F}$ (resp. $I_{\mathcal{Z}}$) the pull-back of the universal sheaf of $\mathrm{Pic}(X)$ (resp. $\mathrm{Hilb}(X)$) to $T_{\mathcal{M}} \times X$. Then $\mathcal{A} = Ext^1_{T_{\mathcal{M}} \times X/T_{\mathcal{M}}}(\mathcal{O}_X(c_1) \otimes \mathcal{F}^\vee \otimes I_{\mathcal{Z}}, \mathcal{F})$ is a locally free sheaf on $T_{\mathcal{M}}$ because of base change theorem for Ext sheaf. On the other hand one can easily prove that $Hom_{T_{\mathcal{M}} \times X/T_{\mathcal{M}}}(\mathcal{O}_X(c_1) \otimes \mathcal{F}^\vee \otimes I_{\mathcal{Z}}, \mathcal{F}) = 0$ from the definition of T. Hence we have a universal extension

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(c_1) \otimes \mathcal{F}^\vee \otimes I_{\mathcal{Z}} \longrightarrow 0$$

on $\mathbb{P}(\mathcal{A}^\vee) \times X$ (cf. [La, Section 4]). Let $\mathbb{P}(\mathcal{A}^\vee)_0$ be an open subset $\{x \in \mathbb{P}(\mathcal{A}^\vee)|\mathcal{E} \otimes k(x) \text{ is locally free}\}$ in $\mathbb{P}(\mathcal{A}^\vee)$. Then, from Proposition 2.3 (b), $\mathcal{E}$ induces a morphism $F : \mathbb{P}(\mathcal{A}^\vee)_0 \to \mathfrak{M}_H(c_1, c_2)$. From Proposition 2.3 (a) and the definition of $d_{\mathcal{M}}$ one can prove that an open dense subset $\mathcal{M}_0 = \{V \in \mathcal{M}_{red}|d(V) = d_{\mathcal{M}}\}$ in $\mathcal{M}_{red}$ is contained in the image scheme $\mathrm{Im}(F)$. Because $\mathcal{M}$ is an irreducible component of $\mathfrak{M}_H(c_1, c_2)$, there is an irreducible component $\mathbb{P}(\mathcal{A}^\vee)_1$ such that F induces a dominant morphism $F : \mathbb{P}(\mathcal{A}^\vee)_1 \to \mathcal{M}_0$. From the uniqueness of Harder-Narasimhan filtration with respect to H', F is geometrically injective. Because $F : \mathbb{P}(\mathcal{A}^\vee)_1 \to \mathcal{M}_0$ is a dominant, geometrically injective, of finite type morphism between integral schemes over $\mathbb{C}$, F is birational. From this and (2), one can prove this corollary. □

3. Proof of Theorem 1.1

For $(c_1, c_2) \in \mathrm{Pic}(X) \times \mathbb{Z}$, let $M_H(c_1, c_2)$ be the coarse moduli scheme parameterizing S-equivalence classes of H-semistable sheaves with fixed Chern classes $(2, c_1, c_2)$. Similarly, let $M_H(\overline{c_1}, c_2)$ (resp. $\mathfrak{M}_H(\overline{c_1}, c_2)$) be the coarse moduli scheme parameterizing S-equivalence classes of H-semistable sheaves (resp. H-μ-stable vector bundles) $\mathcal{F}$ on X such that

$$\text{(3)} \qquad \mathrm{rank}(\mathcal{F}) = 2,\ c_2(\mathcal{F}) = c_2 \text{ and } c_1(\mathcal{F}) - c_1 \in \mathrm{Pic}^0(X).$$

We shall denote by $\overline{\mathfrak{M}}_H(c_1, c_2)$ (resp. $\overline{\mathfrak{M}}_H(\overline{c_1}, c_2)$) the closure of $\mathfrak{M}_H(c_1, c_2)$ in $M_H(c_1, c_2)$ (resp. the closure of $\mathfrak{M}_H(\overline{c_1}, c_2)$ in $M_H(\overline{c_1}, c_2)$), provided with reduced induced subscheme structure. $\overline{\mathfrak{M}}_H(c_1, c_2)$ and $\overline{\mathfrak{M}}_H(\overline{c_1}, c_2)$ are projective over $\mathbb{C}$. We shall assume that $p_g(X) > 0$ to prove Theorem 1.1. Fix a compact subset $\mathcal{K}$ in $\mathrm{Amp}(X)$ containing some nonempty open set, and a polarization H_0 contained in $\mathcal{K}$.

Now, in order to prepare a constant $C(X, c_1)$ in Theorem 1.1, let us remember how to construct $M_{H_0}(\overline{c_1}, c_2)$, referring to [Gi]. Let $\mathcal{E}$ be the set of all H_0-semistable sheaves satisfying (3). Then $\mathcal{E}$ is a bounded family of coherent sheaves on X. In particular there is an integer $N = N(c_1, c_2)$ such that if n is greater than N, then any member $F \in \mathcal{E}$ satisfies that $h^i(F \otimes H_0^{\otimes n}) = 0$ for $i > 0$, and that $F \otimes H_0^{\otimes n}$ is generated by its global sections. Fix any integer $n > N(c_1, c_2)$. Let $P(m)$ be the Hilbert polynomial $\chi(F \otimes H_0^{\otimes(n+m)})$

of any $F \in \mathcal{E}$, and R be $P(0)$. Because of the choice of n, any $F \in \mathcal{E}$ have a surjection $\mathcal{O}_X^{\oplus R} \twoheadrightarrow F \otimes H_0^{\otimes n}$. Next, we denote the Grothendieck Quot-scheme parameterizing flat families of quotient sheaves of $\mathcal{O}_X^{\otimes R}$ satisfying (3) by $\mathrm{Quot}_{\mathcal{O}_X^{\oplus R}/X}(2, \overline{c_1}, c_2)$. Next, there is an open subscheme $Q_{H_0}^{ss}$ in $\mathrm{Quot}_{\mathcal{O}_X^{\oplus R}/X}(2, \overline{c_1}, c_2)$ such that $\varphi \in \mathrm{Quot}_{\mathcal{O}_X^R/X}(2, c_1, c_2)(T)$ factors through Q^{ss} if and only if its corresponding quotient sheaf $\varphi : \mathcal{O}_{X_T}^R \twoheadrightarrow F_T$ satisfies that, for any closed $t \in T$, $H^0(\varphi \otimes k(t)) : k(t)^{\oplus R} \to H^0(F_T \otimes k(t))$ is an isomorphism and $F_T \otimes k(t)$ is H_1-semistable. $G = PGL(R, \mathbb{C})$ naturally acts on $Q_{H_0}^{ss}$. In [Gi], Gieseker constructed the good quotient $Q_{H_0}^{ss}//G$ of $Q_{H_0}^{ss}$ by G. $M_{H_0}(\overline{c_1}, c_2)$ is this quotient scheme.

Proposition 3.1 ([Li]). *There is a constant $p_1 = p_1(X, H_0, c_1)$ depending on (X, H_0, c_1) such that whenever $c_2 \geq p_1$, $Q_{H_0}^{ss}$ is normal.*

Next, from [GL] and [LQ], there is a constant $p_2 = p_2(X, H_0, c_1)$ depending on (X, H_0, c_1) such that whenever $c_2 \geq p_2$, $\mathfrak{M}_{H_0}(c_1, c_2)$ is irreducible and non-empty. Third, from [Q1, Theorem 2.3], there is a constant $p_3 = p_3(X, \mathcal{K}, c_1)$ such that whenever $c_2 \geq p_3$, then moduli schemes $\mathfrak{M}_H(c_1, c_2)$ are birationally equivalent to each other for all polarizations H such that some rational multiple rH ($r \in \mathbb{Q}$) is contained in $\mathcal{K}$. Last, from [Do], there is a constant $p_4(X, H_0, \mathcal{K}, c_1) \geq \max(p_1, p_2, p_3)$ such that whenever $c_2 \geq p_4$ the open subset

$$M_{H_0}(c_1, c_2) \supset U_{H_0} = \{F | F \text{ is } H_0\text{-}\mu\text{-stable and } \mathrm{Ext}^2(F, F)^0 = 0\} \tag{4}$$

is non-empty. From now on, we shall assume that $c_2 \geq p_4(X, H_0, \mathcal{K}, c_1)$, and take a polarization $H_1 = H_1(c_1, c_2)$ depending on (c_1, c_2) such that some rational multiple rH_1 is contained in any wall of type (c_1, c_2). Because of Proposition 2.2, such a polarization H_1 does exist. Remark that $\overline{\mathfrak{M}}_{H_1}(c_1, c_2)$ is irreducible, non-empty, and the set U_{H_1} defined similarly as U_{H_0} in (4) is non-empty because of the choice of c_2, and H_1. Now we prove Theorem 1.1. The proof is divided in two cases.

Case 3.1. Some point $\overline{\mathfrak{M}}_{H_1}(c_1, c_2)$ represents the S-equivalence classes of a strictly H_1-semistable sheaf $\mathcal{F}$.

Claim 3.2. The sheaf $\mathcal{F}$ in Case 3.1 is L-semistable for any ample line bundle L.

Proof. Because $\mathcal{F}$ is strictly H_1-semistable, $\mathcal{F}$ is an extension of torsion free coherent sheaves with rank one

$$0 \to F \to \mathcal{F} \to G \to 0$$

such that $\chi(F \otimes H_1^{\otimes n}) = \chi(G \otimes H_1^{\otimes n})$ for any $n \in \mathbb{Z}$. From the Riemann-Roch theorem $\chi(F \otimes H_1^{\otimes n}) = n^2 H_1^2 + n(2c_1(F) - K_X) \cdot H_1 + \chi(F)$, and hence we have

$$\chi(F) = \chi(G) \quad \text{and} \quad c_1(F) \cdot H_1 = c_1(G) \cdot H_1. \tag{5}$$

Fix any ample line bundle L. If $c_1(F) \cdot L \neq c_1(G) \cdot L$, then one can show that $W^{c_1(F)-c_1(G)}$ is a wall of type (c_1, c_2) containing H_1 from (5). This contradicts the choice of H_1. Hence $c_1(F) \cdot L = c_1(G) \cdot L$ for any ample line bundle L. Because $\chi(F) = \chi(G)$ from (5), one can show that $\chi(F \otimes L^{\otimes n}) = \chi(G \otimes L^{\otimes n})$ for any n, and so $\mathcal{F}$ is L-semistable. □

Claim 3.3. For any ample line bundle L, $\mathfrak{M}_{H_0}(c_1, c_2) \cap \mathfrak{M}_L(c_1, c_2)$ is non-empty.

Proof. First, we assume that L is not contained in any wall of type (c_1, c_2). Let $\pi : Q^{ss}_{H_0} \to M_{H_0}(\overline{c_1}, c_2)$ be the G.I.T. quotient map, and $\pi : Q^{\mu-s}_{H_0} := \pi^{-1}(\mathfrak{M}_{H_0}(\overline{c_1}, c_2)) \to \mathfrak{M}_{H_0}(\overline{c_1}, c_2)$ be its restriction. $Q^{ss}_{H_0}$ is irreducible by the property of $p_1(X, H_0, c_1)$. $Q^{\mu\text{-}s}_{H_0}$ is its non-empty open subscheme by the property of $p_2(X, H_0, c_1)$, and hence, dense open subscheme. On the other hand, from Claim 3.2, the open subscheme

$$Q^{ss}_{H_0} \supset U^{ss}_L = \{\mathcal{O}_X^{\otimes R} \twoheadrightarrow \mathcal{F} | \mathcal{F} \text{ is } L\text{-semistable}\}$$

is non-empty, and so dense open subscheme. Therefore we have $Q^{\mu\text{-}s}_{H_0} \cap U^{ss}_L \neq \emptyset$ in $Q^{ss}_{H_0}$. From this, there is a sheaf $\mathcal{G} \in \mathfrak{M}_{H_0}(\overline{c_1}, c_2)$ that is L-semistable. If this $\mathcal{G}$ is not L-μ-stable, then $\mathcal{G}$ is H_0-μ-stable and strictly L-μ-semistable, and so $\mathcal{G}$ has a rank-one quotient sheaf G with $(2c_1(G) - c_1(\mathcal{G})) \cdot L = 0$. One can prove $W^{2c_1(G)-c_1(\mathcal{G})}$ is a wall of type (c_1, c_2) containing L. This is contradiction. So $\mathcal{G} \in \mathfrak{M}_{H_0}(\overline{c_1}, c_2)$ is L-μ-stable, and especially $\mathfrak{M}_{H_0}(\overline{c_1}, c_2) \cap \mathfrak{M}_L(\overline{c_1}, c_2) \neq \emptyset$. From this, one can easily prove that

$$\mathfrak{M}_{H_0}(c_1, c_2) \cap \mathfrak{M}_L(c_1, c_2) \neq \emptyset. \tag{6}$$

In case where L belongs to some wall of type (c_1, c_2), one can prove (6) paying attention to [Q4, p. 406, Theorem 1.2.3] and the irreducibility of $\mathfrak{M}_{H_0}(c_1, c_2)$, that is proved from the irreducibility of $Q^{ss}_{H_0}$. □

Now in Case 3.1, Theorem 1.1 holds good provided $L(c_1, c_2) = H_0$ because of Lemma 3.3.

Remark 3.4. Let E and E' be torsion free sheaves having the same S-equivalence class with respect to H_0. Then, for a polarization L, E need not to be L-stable even if E' is L-stable. So we made the proof not in $M_{H_0}(c_1, c_2)$ but in the Quot-scheme. Remark also that in Case 3.1, Theorem 1.1 is valid whether $p_g(X)$ is positive or not.

Case 3.2. Every ($\mathbb{C}$-valued) point in $\overline{\mathfrak{M}}_{H_1}(c_1, c_2)$ represents a H_1-stable sheaf on X.

Now there is a non-zero section $\theta \in H^0(K_X)$ since $p_g \neq 0$. Let S be a scheme over $\mathbb{C}$, and F_S a S-flat coherent sheaf on X_S. Then one can construct a two-form $\Theta_{F_S,\theta} \in H^0(\wedge^2 \Omega_S)$ using θ and F_S from [Mk] or [Ty]. Now we shall think over moduli schemes. As reviewed in Case 3.1, $M_{H_1}(c_1, c_2)$ is the quotient scheme of some subscheme Q^{ss} in $\mathrm{Quot}_{\mathcal{O}_X^R/X}(2,c_1,c_2)$ by $G=\mathrm{PGL}(R,\mathbb{C})$. Let $\pi: Q^{ss}\to M_{H_1}(c_1,c_2)$ be the quotient map. Then, $\pi:\pi^{-1}(\overline{\mathfrak{M}}_{H_1}(c_1,c_2))\to\overline{\mathfrak{M}}_{H_1}(c_1,c_2)$ is a principal G-bundle, from the assumption in Case 3.2 and [Ma, Proposition 6.4]. Let $\mathcal{U}$ be (the restriction of) an universal sheaf of $\pi^{-1}(\overline{\mathfrak{M}}_{H_1}(c_1,c_2))$, and we get a two-form $\Theta_{\mathcal{U},\theta}$ on $\pi^{-1}(\overline{\mathfrak{M}}_{H_1}(c_1,c_2))$. Then we can get a two-form Θ_θ on $\overline{\mathfrak{M}}_{H_1}(c_1,c_2)$ using faithfully-flat quasi-compact descent theory. For a smooth point $x\in U_{H_1}\subset\overline{\mathfrak{M}}_{H_1}$ (U_{H_1} is defined at (4)), Θ_θ induces $\Theta_\theta\otimes k(x):\ T_x\overline{\mathfrak{M}}_{H_1}(c_1,c_2)\to(T_x\overline{\mathfrak{M}}_{H_1}(c_1,c_2))^\vee$. One can see that this homomorphism is equal to $\otimes\,\theta:\mathrm{Ext}^1_X(E_x,E_x)^{\,0}\to\mathrm{Ext}^1_X(E_x,E_x(K_X))^{\,0}$. Next, denote by $D\subset X$ the effective divisor given by $\theta\in H^0(K_X)\setminus\{0\}$. From the exact sequence

$$0\longrightarrow E_x\xrightarrow{\otimes\theta}E_x\otimes K_X\longrightarrow E_x\otimes K_X|_D\longrightarrow 0,$$

we have an exact sequence

$$\mathrm{Hom}_D(E_x|_D,E_x(K_X)|_D)\longrightarrow\mathrm{Ext}^1_X(E_x,E_x)\xrightarrow{\otimes\theta}\mathrm{Ext}^1_X(E_x,E_x(K_X)). \tag{7}$$

Lemma 3.5. *There are constants $p_5(X,H_0,\mathcal{K},c_1,\theta)\geq p_4$ and $l_0(X,\mathcal{K},c_1,\theta)$ depending only on X, $\mathcal{K}$, c_1, and θ as follows. If*

$$c_2\geq p_5(X,H_0,\mathcal{K},c_1,\theta) \tag{8}$$

then, for any polarization H such that some rational multiple rH is contained in $\mathcal{K}$, general point x of $U_H\subset\overline{\mathfrak{M}}_H(c_1,c_2)$ satisfies that $\dim\mathrm{Hom}_D(E_x|D,E_x(K_X)|D)^0\leq l_0$.

Proof. Fix c_1 and H satisfying hypothesis in this lemma. If c_2 is sufficiently large with respect to (X,H,c_1), then there is a rank-two H-μ-stable vector bundle V with Chern classes (c_1,c_2) and $\dim\mathrm{Ext}^2(V,V)^0=0$. For codimension-two subscheme Z such that $Z\cap D=\emptyset$, $V\otimes I_Z$ is H-μ-stable, $c_2(V\otimes I_Z)=c_2(V)+2l(Z)$, and

$$\dim\mathrm{Hom}(V\otimes I_Z|D,V\otimes I_Z(K_X)|D)=\dim\mathrm{Hom}_D(V|D,V(K_X)|D).$$

Though $\dim\mathrm{Ext}^2(V\otimes I_Z,V\otimes I_Z)^0$ might be nonzero, we can use the upper semicontinuity theorem for the function $x\mapsto\dim\mathrm{Hom}_D(E_x|D,E_x(K_X)|D)$ near $V\otimes I_Z$, because $Z\cap D=\emptyset$. In result, we can prove that

$$\{x\in\overline{\mathfrak{M}}_H(c_1,c_2+2n)|\dim\mathrm{Hom}(E_x|D,E_x(K_X)|D)\leq\dim\mathrm{Hom}(V|D,V(K_X)|D)\}$$

contains some non-empty open subset in $\overline{\mathfrak{M}}_H(c_1,c_2+2n)$ ($n\in\mathbb{Z}_{>0}$). Though we omit its proof here, we can prove similar result also for $\overline{\mathfrak{M}}_H(c_1,c_2+2n-1)$ ($n\in\mathbb{Z}_{>0}$). Now, by the choice of p_2, p_3 and p_4, we can prove this lemma if we set $l_0=\dim\mathrm{Hom}_D(V|D,V(K_X)|D)$. □

Now, suppose that c_2 satisfies (8). Let $\tilde{\mathfrak{M}}_{H_1}(c_1,c_2) \to \overline{\mathfrak{M}}_{H_1}(c_1,c_2)$ be a desingularization of the normalization of $\overline{\mathfrak{M}}_{H_1}(c_1,c_2)$. We get a two-form $\tilde{\Theta}_\theta$ on $\tilde{\mathfrak{M}}_{H_1}$, pulling back the two-form Θ_θ on $\overline{\mathfrak{M}}_{H_1}(c_1,c_2)$ mentioned above. (We sometimes abbreviate $\overline{\mathfrak{M}}_{H_1}(c_1,c_2)$ to $\overline{\mathfrak{M}}_{H_1}$ for the sake of simplicity.) From the exact sequence (7) and Lemma 3.5, some $x \in \tilde{\mathfrak{M}}_{H_1}$ satisfies that

$$\dim\ker(\tilde{\Theta}_\theta \otimes k(x) : T_x\tilde{\mathfrak{M}}_{H_1} \to (T_x\tilde{\mathfrak{M}}_{H_1})^\vee) \leq l_0(X,\mathcal{K},c_1,\theta). \tag{9}$$

Because the right side of (9) is independent of c_2, we can easily prove the following.

Lemma 3.6. *There is a constant $l_1 = l_1(X,\mathcal{K},c_1,\theta)$ not depending on c_2 as follows. When the condition (8) is valid, the two-form $\tilde{\Theta}_\theta$ on $\tilde{\mathfrak{M}}_{H_1} = \tilde{\mathfrak{M}}_{H_1}(c_1,c_2)$ constructed just now satisfies that, for $N = [\dim\tilde{\mathfrak{M}}_{H_1}(c_1,c_2)/2] - l_1$, $\wedge^N\tilde{\Theta}_\theta \neq 0$ in $H^0(\wedge^{2N},\Omega_{\tilde{\mathfrak{M}}_{H_1}})$. Here $[\lambda]$ is the largest integer not greater than λ.*

Now we shall prove that $\mathfrak{M}_H(c_1,c_2) \cap \mathfrak{M}_{H_1}(c_1,c_2) \neq \emptyset$ for any polarization H, if c_2 satisfies (8) and the hypothesis (12) mentioned later. Note that if we prove this then we conclude the proof of Theorem 1.1. Indeed, from this and Claim 3.3 in Case 3.1, Theorem 1.1 holds good if

$$C = \max(\, p_5(X,H_0,\mathcal{K},c_1,\theta), \\ [(c_1^2/2) + 3\chi(\mathcal{O}_X) + q(X) + 2l_1(X,\mathcal{K},c_1,\theta)/2]\,).$$

Now we suppose c_2 satisfies (8), and $\mathfrak{M}_H(c_1,c_2) \cap \mathfrak{M}_{H_1}(c_1,c_2)$ is empty for some polarization H. Because $\mathfrak{M}_{H_1}(c_1,c_2)$ is irreducible, $\tilde{\mathfrak{M}}_{H_1}(c_1,c_2)$ is birationally equivalent to $Y \times \mathbb{P}^l$, where Y is a nonsingular variety whose Krull dimension is less than $2c_2 - (c_1^2/2) + q(X)$, from Corollary 2.4. Because $h^0(\wedge^N\Omega_{Y\times\mathbb{P}^l}) = h^0(\wedge^N\Omega_Y)$, and because $\dim H^0(\wedge^N\Omega_Z)$ is birationally invariant for nonsingular complete varieties Z over $\mathbb{C}$, we see that

$$h^0(\wedge^N\Omega_{\tilde{\mathfrak{M}}_{H_1}(c_1,c_2)}) = 0 \qquad \text{if} \quad N \geq 2c_2 - (c_1^2/2) + q(X). \tag{10}$$

From (10) and Lemma 3.6, we see that

$$\dim\tilde{\mathfrak{M}}_{H_1}(c_1,c_2) - 2l_1 \leq 2c_2 - (c_1^2/2) + q(X).$$

From deformation theory, it holds that

$$\dim\tilde{\mathfrak{M}}_{H_1}(c_1,c_2) \geq 4c_2 - c_1^2 - 3\chi(\mathcal{O}_X).$$

Summing up, we see that

$$2c_2 \leq (c_1^2/2) + 3\chi(\mathcal{O}_X) + q(X) + 2l_1, \tag{11}$$

where l_1 is a constant independent of c_2 in Lemma 3.6. Hence, if c_2 satisfies (8) and

$$2c_2 > (c_1^2/2) + 3\chi(\mathcal{O}_X) + q(X) + 2l_1, \tag{12}$$

then $\mathfrak{M}_H(c_1,c_2) \cap \mathfrak{M}_{H_1}(c_1,c_2)$ must be nonempty for any polarization H. Now we conclude the proof of Theorem 1.1.

Remark 3.7. As noted in Introduction, the conclusion of Theorem 1.1 is valid also when X is a relatively minimal elliptic surface with Kodaira dimension one, and $c_1 \cdot f$ is odd, where $f \in \mathrm{NS}(X)$ is the fiber class of elliptic fibration. To prove Theorem 1.1 in this case, we refer to [Br, Theorem 1.1], where the birational structure of the moduli $\mathfrak{M}_{H_f}(\overline{c_1}, c_2)$ of μ-stable vector bundles with respect to $(\overline{c_1}, c_2)$-suitable polarization H_f is studied. Using this result, we can prove that $\kappa(\widetilde{\mathfrak{M}}_{H_f}(\overline{c_1}, c_2))$ is not $-\infty$. In a similar fashion in case where $p_g > 0$, we can show that $\mathfrak{M}_{H_f}(\overline{c_1}, c_2) \cap \mathfrak{M}_H(\overline{c_1}, c_2) \neq \emptyset$ for any polarization H.

Remark 3.8. There are a surface X with $\kappa(X) = 2$ and $p_g(X) > 0$, and $c_1 \in \mathrm{Pic}(X)$, as follows: for any large number N, there is $c_2 \geq N$ such that

$$\sup_{H\colon \text{ample}} \dim \mathfrak{M}_H(c_1, c_2) = +\infty$$

([Ya]). Hence in general, for some polarization H, $\mathfrak{M}_H(c_1, c_2)$ has an irreducible component $\mathcal{M}$ such that $\mathcal{M} \cap \mathfrak{M}_L(c_1, c_2)$ is empty, where L is a trivial polarization in Theorem 1.1.

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