J. Math. Kyoto Univ. (JMKYAZ)
42-3 (2002), 599–606

Some nontrivial homology classes on the space of symplectic forms

By

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Abstract

We construct a family of examples of non-zero relative homolgy classes on some infinite dimensional spaces of symplectic forms.

1. Introduction

In this note, we construct some family of examples of non-zero relative homology classes on the infinite dimensional space of symplectic forms on smooth manifolds of any dimension at least four. The main tool for investigation is the relative version of Gromov invariants, which is by now well established by the works of Fukaya-Ono [2] and Liu-Tian [4], [5]. More accurately, let (M, ω) be a compact symplectic manifold of dimension at least four. Associated to M, we define an infinite dimensional space $\mathfrak{V}$ consisting of pairs of symplectic forms and compatible almost complex structures. In $\mathfrak{V}$, there are subspaces $\mathfrak{V}_{[\omega]}$ of such pairs whose symplectic forms have a fixed cohomology class $[\omega]$. We construct geometrically a map

$$r : S^{n-1} \to \mathfrak{V}_{[\omega]}$$

from a sphere to one of those subspaces. The aim is to show that this map can be extended to a map from an n-ball to $\mathfrak{V}$, but, by a homological reason, cannot be extended if we require the image to be contained in $\mathfrak{V}_{[\omega]}$. We demonstrate this task by defining a relative version of Gromov-like invariant, using the notion of virtual moduli cycles, which is roughly speaking the intersection of the image of the extended map

$$\tilde{r} : B^n \to \mathfrak{V}$$

with some locus $\mathfrak{N} \subset \mathfrak{V} - \mathfrak{V}_{[\omega]}$, and the nontriviality of this invariant assures us that the relative homology class of $(\mathfrak{V}, \mathfrak{V} - \mathfrak{N})$ represented by the image of r is essential.

This paper begins with an explanation, by a typical example, following Kronheimer [3], of the situation we are mainly interested in. Next we set

Received November 20, 2001

up the infinite dimensional spaces of symplectic forms and almost complex structures, and explain the construction of the invariants mentioned above. Also, we describe a lemma concerning the virtual moduli cycles which is in need for our construction, and show the main result. Finally, we exhibit simple examples containing higher dimensional cases.

Note. After an earlier version of this article has been finished, an e-print written by O. Buse [1] has appeared. It makes use of the same invariant to investigate the homology of $Symp_0(S^2 \times S^2 \times M)$.

2. Kronheimer's example

In this section, we recall the example given by Kronheimer [3]. Let Y_0 be the quotient $\mathbb{C}^2/C_m$, where C_m is the cyclic subgroup of order m inside the group of scalars. Let $\tilde{Y_0}$ be the total space of the line bundle $\mathcal{O}(-m)$ over $\mathbb{C}P^1$. This space $\tilde{Y_0}$ forms a resolution of the space Y_0,

$$\sigma_0 : \tilde{Y_0} \to Y_0$$

by blowing up the origin of Y_0. The space $\tilde{Y_0}$ has a natural $(m-1)$- parameter family of deformations $\tilde{Y_u}, u \in \mathbb{C}^{m-1}$ such that the total space of the family forms a complex manifold

$$\tilde{\mathbf{Y}} = \mathcal{O}(-1)^m.$$

Here, $\tilde{Y_u}$ is the fiber $\tilde{q}^{-1}(u)$ of the map

$$\tilde{q} : \tilde{\mathbf{Y}} \to \mathbb{C}^{m-1}$$

defined by evaluating any generic (in particular, fiberwise linearly independent) collection of $m-1$ sections of the dual bundle $\mathcal{O}(1)^m$. An important remark is that the map $\tilde{q}$ describes the space $\tilde{\mathbf{Y}}$ as a C^∞ product

$$\tilde{\mathbf{Y}} \simeq \tilde{Y_0} \times \mathbb{C}^{m-1}$$

but only the fiber over $0 \in \mathbb{C}^{m-1}$ contains a *holomorphic* sphere. Moreover, the existence of the holomorphic sphere in $\tilde{Y_0}$ is a $transverse\,phenomenon$ in a sense that the natural map (restriction of the Kodaira-Spencer map)

$$T_0\mathbb{C}^{m-1} \to H^1(C, \nu(C))$$

(see [3]) is an isomorphism, here C is the holomorphic sphere and $\nu(C)$ is the normal bundle of C in $\tilde{Y_0}$. This transversality is needed for the calculation of the virtual moduli cycles below (see Lemma 3.3).

3. The main construction

Let M be a smooth, oriented manifold of dimension at least four, and $\Omega^2(M)$ be the space of two forms. We consider the space $\mathfrak{Y} \subset \Omega^2(M) \times End(TM)$ which is defined as follows.

$$\mathfrak{Y} = \left\{ (\omega, J) \,\middle|\, \begin{array}{l} \omega \text{ is a symplectic form on } M, \\ \text{and } J \text{ runs over the space of} \\ \omega\text{-compatible almost complex structures.} \end{array} \right\}$$

Suppose now we are given a family of Kählerian manifolds $\tilde{X}_u$ over an open ball U in $\mathbb{C}^n$,

$$\tilde{\pi} : \tilde{\mathbf{X}} \to U,$$

which has a curve C on the fiber of the origin, $\tilde{X}_0$. We put the following assumptions.

Assumption. (1) The occurrence of the curve C is a *transverse phenomenon* in the sense of the previous section. Namely,

$$H^0(C, \nu(C)) = 0$$

and the map

$$T_0\mathbb{C}^n \to H^1(C, \nu(C))$$

is isomorphic.

(2) We can contract the curve C to obtain a new (singular) family of projective varieties

$$\pi : \mathbf{X} \to U.$$

(3) The natural identification of $\tilde{\pi}|_{\partial U}$ and $\pi|_{\partial U}$ gives an isomorphism as families of projective varieties.

Without losing generality, we can assume the family $\pi : \mathbf{X} \to U$ is embedded in $\mathbb{CP}^N \times U$ for some N, so each fiber has a natural Kähler form, and that all the fibers over U except X_0 are differentially isomorphic (these are isomorphic to the desingularized fiber $\tilde{X}_0$). Especially, we have a natural family of pairs of symplectic forms and compatible (almost) complex structures on the C^∞-manifold $\tilde{X}_0$, parameterized by the sphere S^{2n-1} which links the origin of U. In the same way. the original family $\tilde{\pi} : \tilde{\mathbf{X}} \to U$ gives us a map from U to $\mathfrak{Y}$.

Remark. The cohomology classes of the family of symplectic forms described above is the same under natural isomorphisms of X_u, $u \in S^{2n-1}$. Let us denote the class $[\omega]$. On the otherhand, the homology class of the sphere C is zero, when evaluated by the cohomology class of the symplectic form, because it is contracted in the family $\mathbf{X}$ embedded in $\mathbb{C}P^N \times U$. It follows that the natural symplectic structure ω_0 of the desingularized fiber $\tilde{X_0}$ is never cohomologous to $[\omega]$, since the class of ω_0 evaluates the curve C nontrivially.

In particular, it follows that the original family $\tilde{\pi} : \tilde{\mathbf{X}} \to U$ can not be embedded in $\mathbb{C}P^N \times U$ for any N.

We want to associate such pairs an invariant to detect the homological non-trivialiness of that family on some subspace of the space of pairs $\mathfrak{V}$. Let $\tilde{r} : B^{2n} \to \mathfrak{V}$ be the map our family $\tilde{\pi} : \tilde{\mathbf{X}} \to U$ gives. We define

$$\mathfrak{N} = \text{ The locus in } \mathfrak{V} \text{ on which there is a } J\text{-holomorphic curve of class}[C].$$

Note $\mathfrak{N}$ is nonempty (because of the very existence of C) and virtually a submanifold of codimension $2n$. In general, the locus $\mathfrak{N}$ will have singularities and ill dimensional components, but suppose now that the image of the map $\tilde{r}$ intersects $\mathfrak{N}$ only at $\tilde{r}(0) = (\omega_{\tilde{r}(0)}, J_{\tilde{r}(0)})$ transversally. We want to define an invariant by associating to the map $\tilde{r}|_{\partial B^{2n}}$ a number of $J_{\tilde{r}(0)}$ holomorphic curve on $(M,\ \omega_{\tilde{r}(0)})$. This naive attempt is now possible by the existence of virtual moduli cycles so that the number is well-defined in that it depends only on the relative homology class of the pair of maps $(\tilde{r},\ \tilde{r}|_{\partial B^{2n}})$. We formulate this claim as follows.

Lemma 3.1. *Let $\mathcal{M}_{g,\beta}(M,\omega,J)$ be the moduli space of stable J-holomorphic maps of degree $\beta \in H_2(M;\mathbb{Z})$ and genus g, and $\mathfrak{N}$ be the locus in $\mathfrak{V}$ on which $\mathcal{M}_{g,\beta}(M,\omega,J)$ is nonempty. Let $(B,\ \partial B)$ be a d-dimensional manifold with boundary. Take a map*

$$a : [0,1] \times \partial B \to \mathfrak{V}$$

such that

$$Image(a) \cap \mathfrak{N} = \emptyset.$$

Furthermore, suppose we can extend the map a as follows:

$$\tilde{a} : [0,1] \times B \to \mathfrak{V}, \quad \tilde{a}|_{[0,1]\times\partial B} = a.$$

Then, the following claims are valid.

(1) *For each $s \in [0,1]$, we can associate a 0-dimensional rational cycle $\mathcal{M}^{virt}_{s,g,\beta}$.*

(2) $\mathcal{M}^{virt}_{I,g,\beta} = \bigcup_{s\in[0,1]} \mathcal{M}^{virt}_{s,g,\beta}$ *is a 1-dimensional rational chain, and satisfies*

$$\partial\mathcal{M}^{virt}_{I,g,\beta} = \mathcal{M}^{virt}_{1,g,\beta} - \mathcal{M}^{virt}_{0,g,\beta}.$$

This can be proved by the methods used in Fukaya and Ono [3] or Liu and Tian [4], [5]. The construction of the virtual cycle is done locally, so the condition that $Image(a) \cap \mathfrak{N} = \emptyset$ assures the bordism in claim 2.

From this, it is easy to deduce the next result.

Lemma 3.2. *In the lemma above, the sum of the coefficients of the* 0*-dimensional rational cycle* $\mathcal{M}^{virt}_{s,g,\beta}$ *(any parameter* s *gives the same number, because of the second assertion of the above lemma) is an invariant of the relative homology class of the map* a: $(X, \partial X) \to (\mathfrak{V}, \mathfrak{V} - \mathfrak{N})$. *In particular, if this number, denoted* $n_{g,\beta}$, *is nonzero, then the map* a *represents a nontrivial relative homology class of the space* $(\mathfrak{V}, \mathfrak{V} - \mathfrak{N})$.

We want to apply this lemma to our situation

$$(\pi : \mathbf{X} \to B^{2n}\, , \tilde{\pi} : \tilde{\mathbf{X}} \to B^{2n}\, , C)$$

and compute the number $n_{g,\beta}$, or at least show it is nonzero. Recall that we assumed that $\mathbf{X}$ is embedded in some product space $\mathbb{CP}^N \times U$, preserving the fiber structure. The fibers of $\pi : \mathbf{X} \to B^{2n}$ except X_0 are diffeomorphic and have canonical symplectic forms which are cohomologous under this diffeomorphism.

There is an induced map

$$r|_{\partial B^{2n}} : S^{2n-1} \to \mathfrak{V}$$

whose image is contained in the subspace $\mathfrak{V}_{[\omega]}$ of $\mathfrak{V}$ which consists of symplectic forms with cohomology class $[\omega]$ and their compatible almost complex structures and it has an extension

$$\tilde{r} : B^{2n} \to \mathfrak{V}$$

defined by the family $\tilde{\pi} : \tilde{X} \to U$. If we extend r to a 1-parameter family

$$\tilde{r}|_{\partial B^{2n} \times [0,1]} : S^{2n-1} \times [0,1] \to \mathfrak{V}, \quad \tilde{r}|_{\partial B^{2n} \times 0} = r|_{\partial B^{2n}},$$

it can also be extended to

$$\tilde{r}|_{B^{2n} \times [0,1]} : B^{2n} \times [0,1] \to \mathfrak{V}, \quad \tilde{r}|_{B^{2n} \times 0} = r|_{B^{2n}}$$

within a fixed homotopy class.

Now, our family $\tilde{\pi} : \tilde{\mathbf{X}} \to B^{2n}$ has a J-holomorphic curve C on the fiber X_0, whose homology class is evaluated to zero by the cohomology class $[\omega]$ (see the remark above). Our computation is based on the following claim.

Lemma 3.3. *Under the assumption* 1 *above, the number* $n_{g(C),[C]}$ *is* 1.

Proof. This follows from the fact that the obstruction bundle for the curve C (more precisely, the stable map $i : C \to C$) is exactly the space $H^1(C, \nu(C))$, and so the surjectivity of the map $T_0\mathbb{C}^n \to H^1(C, \nu(C))$ means our naive moduli (the 'point' i) equals to the virtual moduli. Note that we can determine the sign of the invariant because of the complex orientation. $\square$

Let $\mathcal{S}$ be the space of all symplectic forms on X and $\mathfrak{N}_0$ be the image of $\mathfrak{N}$ in $\mathcal{S}$ under the projection $p : \mathfrak{V} \to \mathcal{S}$. Combining the above lemmas, we have

Theorem 3.4. *Let $\pi : \mathbf{X} \to U$, $\tilde{\pi} : \tilde{\mathbf{X}} \to U$ and C as above. Then, if the assumptions* 1, 2 *and* 3 *are satisfied, the composition map $p \circ (\tilde{r}, r) : (B^{2n}, S^{2n-1}) \to (\mathcal{S}, \mathcal{S} - \mathfrak{N}_0)$ gives a nontrivial relative homology class of the pair $(\mathcal{S}, \mathcal{S} - \mathfrak{N}_0)$ of spaces of symplectic forms on $\tilde{X}_0$.*

Proof. This should be almost clear. We have constructed a relative homology class in the space consists of symplectic forms and compatible almost complex structures, but the space of compatible almost complex structures is contractible for any symplectic form. So, this class will give a relative homology class for the space of symplectic forms. □

4. Examples

4.1. Hirzebruch-Jung singularities on surfaces

We described these examples in the second section for some detail. As mentioned there, these are originally studied by Kronheimer [3]. First, we describe the relevant situation precisely.

Let $\mathbf{X} \to U$ be a family of projective surfaces X_u, $u \in U$, which is embedded in $\mathbb{C}P^N \times U$, U a open neighbourhood of $0 \in \mathbb{C}^m$. Suppose the only singularity of the family is the Hirzebruch-Jung singular point p on the fiber X_0. Then, the restriction of the Kähler form on $\mathbb{C}P^N|_{S^{2m-1}}$ gives a map $r : S^{2m-1} \to \mathfrak{V} - \mathfrak{N}$. Let $\mathcal{S}_{[\omega]}$ be the space of all symplectic forms on X having the fixed cohomology class $[\omega]$. Then we have the following result.

Theorem 4.1. *The map r above can be extended to a pair of maps $(\tilde{r}, r) : (B^{2m}, S^{2m-1}) \to (\mathfrak{V}, \mathfrak{V} - \mathfrak{N})$, whose composition with the projection p represents a relative homology class of the pair $(\mathcal{S}_{[\omega_{B^{2m}}]}, \mathcal{S}_{[\omega]})$, where $\mathcal{S}_{[\omega_{B^{2m}}]} = \cup_{t \in B^{2m}} \mathcal{S}_{[\omega_t]}$ is a suitable family of spaces of symplectic forms with fixed cohomology parameterized by B^{2m} and $[\omega_{S^{2m-1}}] = [\omega]$ is fixed.*

Proof. We only need to construct a suitable family of Kähler forms on the desingularized family $\tilde{\mathbf{X}} \to U$. This can be performed by gluing Kähler forms coming from the Fubini-Study metric on one hand (near the boundary of U), and any smooth family of Kähler metrics on the other (near the origin of U. Recall that small deformations of the complex structure of a Kähler manifold gives Kähler manifolds.), using a cut-off function on the base. For more details, see Kronheimer [3]. □

Remark. As Kronheimer noted, we don't have to restrict ourselves to Hirzebruch-Jung singularities, and any isolated singularities will suffice if it has a simultaneous resolution satisfying moderate assumptions, the exact form of which should be clear from the above arguments.

4.2. A higher dimensional example

As another example, we slightly modify the above example so that the relevant symplectic manifold could be arbitrary high dimension. Namely, take the same total space $\tilde{Y}$ as in the above example, but this time take the base to be some linear hyperplane of $\mathbb{C}^m$. That is, denoting by

$$p_{m,k} : \mathbb{C}^m \to \mathbb{C}^k, \quad 0 < k < m$$

any projection, our families are

$$p_{m,k} \circ \tilde{\pi} : \tilde{\mathbf{X}} \to \mathbb{C}^k$$

A simple calculation will show these examples satisfy the required assumptions, so we have constructed nontrivial relative homology classes in the space of symplectic forms for arbitrary dimension.

Remark. It is natural to ask for the absolute homology classes of the space $\mathfrak{V}_{[\omega]}$ (In dimension four, this was accomplished by Kronheimer's original work [3], using a relative version of Seiberg-Witten invariant). The connected component of the space of symplectic forms with a constant cohomology is, due to Moser, identified with $Diff_0(M)/Symp_0(M,\omega)$, where $Diff_0(M)$ is the subgroup of $Diff(M)$ consisted of elements isotropic to the identity, and $Symp_0(M,\omega)$ is the intersection of $Diff_0(M)$ with $Symp(M,\omega)$. So, Noting that our families are trivial as C^∞ families, and taking the homotopy exact sequence of the Serre fibration

$$Symp_0(M,\omega) \to Diff_0(M) \to Diff_0(M)/Symp_0(M,\omega)$$

we see that these absolute homology classes in $Diff_0(M)/Symp_0(M,\omega)$ will give nontrivial even dimensional homotopy classes of $Symp_0(M,\omega)$. As mentioned in the introduction, Buse [1] investigated one direction of this problem.

Acknowledgements. The author would like to thank Professor Kenji Fukaya for encouragement and advise.

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