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On the existence of weak solutions to the steady compressible Navier-Stokes equations when the density is not square integrable

By

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Abstract

We consider the steady compressible Navier-Stokes equations in the isentropic regime in a bounded domain of $\mathbb{R}^3$. We show that the renormalized continuity equation holds even if the density is not square integrable. We use this result to prove existence of weak solutions under the sole hypothesis $\gamma > 3/2$ for the adiabatic constant.

1. Introduction

In this paper, we investigate the existence of renormalized finite energy weak solutions to the Navier-Stokes system of equations describing the flow of an isentropic compressible fluid in a bounded domain $\Omega \subset \mathbb{R}^3$. This system reads

$$-\mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2)\nabla \operatorname{div} \boldsymbol{u} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \rho^\gamma = \rho \boldsymbol{f} + \boldsymbol{g} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div}(\rho \boldsymbol{u}) = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

Here the unknown functions are $\rho(x)$, $\boldsymbol{u}(x) = (u^1(x), u^2(x), u^3(x))$, $x \in \Omega$ and they respectively represent the density and the velocity of the fluid. The term $\rho \boldsymbol{f} + \boldsymbol{g}$, with $\boldsymbol{f}(x) = (f^1(x), f^2(x), f^3(x))$ and $\boldsymbol{g}(x) = (g^1(x), g^2(x), g^3(x))$ two given vectors fields on Ω, corresponds to the external forces. The viscosity coefficients μ_1 and μ_2 are constants, such that

$$\mu_1 > 0, \quad \frac{2}{3}\mu_1 + \mu_2 \geq 0 \tag{1.3}$$

and the adiabatic constant γ satisfies

$$\gamma > \frac{3}{2} \quad \text{if } \operatorname{curl} \boldsymbol{f} = 0 \qquad \text{or} \qquad \gamma > \frac{5}{3} \quad \text{if } \operatorname{curl} \boldsymbol{f} \neq 0. \tag{1.4}$$

The equations (1.1), (1.2) are complemented by the no-slip boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \partial\Omega \tag{1.5}$$

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and the additional condition

$$\int_\Omega \rho\, dx = M \tag{1.6}$$

where M is a given positive constant which represents the total mass of the fluid in the volume Ω.

We also observe that for any sufficiently smooth solution $(\rho, \boldsymbol{u})$,

$$\operatorname{div}(b(\rho)\boldsymbol{u}) + \{\rho b'(\rho) - b(\rho)\} \operatorname{div} \boldsymbol{u} = 0 \tag{1.7}$$

holds with any function $b \in C^1([0, +\infty))$.

First, we explain what we mean by renormalized finite energy weak solution to the problem (1.1), (1.2), (1.5), (1.6).

Consider functions

$$b \in C^0([0, +\infty)) \cap C^1((0, +\infty)), \quad \exists \lambda_0 < 1, \ \ |b'(t)| \le ct^{-\lambda_0}, \ \ \forall t \in (0, 1] \tag{1.8}$$

with growth conditions at infinity

$$|b'(t)| \le ct^{\lambda_1}, \quad |tb'(t) - b(t)| \le ct^{\lambda_2}, \quad \forall t \ge 1 \quad \text{where } \lambda_1 \lambda_2 \in \mathbb{R}, \tag{1.9}$$

c denoting a positive constant. Let $3/2 \le p < +\infty$. A couple $(\rho, \boldsymbol{u})$ is called a renormalized finite energy weak solution to the problem (1.1), (1.2), (1.5), (1.6) if and only if

(i) $\rho \in L^p(\Omega)$, $\rho \ge 0$ a.e. in Ω and satisfies (1.6), $\boldsymbol{u} \in [W_0^{1,2}(\Omega)]^3$;

(ii) equation (1.1) holds in $[\mathcal{D}'(\Omega)]^3$;

(iii) equation (1.2) holds in $\mathcal{D}'(\mathbb{R}^3)$ provided $(\rho, \boldsymbol{u})$ is extended by zero outside Ω;

(iv) equation (1.7) is satisfied in $\mathcal{D}'(\mathbb{R}^3)$ provided $(\rho, \boldsymbol{u})$ is extended by zero outside Ω, for any function b belonging to the class (1.8), (1.9) with

$$-1 < \lambda_1 < \frac{p}{2} - 1, \quad 0 < \lambda_2 \leqslant \frac{p}{2} \tag{1.10}$$

(v) it satisfies

$$\int_\Omega \{\mu_1 |\nabla \boldsymbol{u}|^2 + (\mu_1 + \mu_2)(\operatorname{div} \boldsymbol{u})^2\}\, dx \le \int_\Omega (\rho \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{g} \cdot \boldsymbol{u})\, dx. \tag{1.11}$$

Now, we can state our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$ and $M > 0$. Suppose that $\boldsymbol{f} \in [L^\infty(\Omega)]^3$, $\boldsymbol{g} \in [L^\infty(\Omega)]^3$, μ_1 and μ_2 satisfy* (1.3) *and γ satisfies* (1.4)*. Then there exists a renormalized finite energy weak solution $(\rho, \boldsymbol{u})$ to the problem* (1.1), (1.2), (1.5), (1.6) *satisfying $\rho \in L^{s(\gamma)}(\Omega)$ where*

$$s(t) = \begin{cases} 3(t-1) & \text{if} \quad t < 3, \\ 2t & \text{if} \quad t \ge 3. \end{cases} \tag{1.12}$$

Theorem 1.1 generalizes a similar result of Lions [5] where the adiabatic coefficient satisfies the restriction

$$\gamma > \frac{5}{3}. \tag{1.13}$$

In virtue of (1.13) and similar estimates to those presented in Section 4, ρ is bounded in $L^2(\Omega)$. This in turn implies that $(\rho, \boldsymbol{u})$ satisfies the renormalized continuity equation (1.7) in the sense of Di-Perna and Lions [2] with suitable b. This is one of the main arguments of the Lions proof. It is not available under the sole hypothesis $\gamma > 3/2$.

In [3], E. Feireisl proved compactness of solutions to the nonsteady isentropic Navier-Stokes equations under the hypothesis $\gamma > 3/2$. By using cut-off operators T_k similar to those introduced in Section 5, he showed that

$$\sup_{k>0} \limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{0,\gamma+1} \le C. \tag{1.14}$$

Here, $\{\rho_n\}_{n\in\mathbb{N}^*}$ is a sequence of densities corresponding to solutions of nonsteady isentropic Navier-Stokes equations an ρ its weak limit. This estimate is crucial in his proof.

The main goal of this paper is to show that the estimate (1.14) holds also in the steady case. Once it is established, we prove that, with suitable b, the renormalized continuity equation is verified. We also show how to use this fact to prove the existence result.

Let us conclude this section by recalling some notations used throughout the paper. By a domain $\Omega \subset \mathbb{R}^3$, we mean a connected open set of $\mathbb{R}^3$. As usual, $\mathcal{D}(\Omega)$ denotes the space of indefinitely differentiable functions with compact support in Ω and $\mathcal{D}'(\Omega)$ its dual, the space of distributions on Ω; $L^p(\Omega)$ resp. $L^p_{loc}(\Omega)$, $1 \le p \le +\infty$, the Lebesgue spaces of L^p-integrable functions resp. the space of locally L^p-integrable functions; $W^{1,p}(\Omega)$, $1 \le p \le +\infty$, are the Sobolev spaces; $W^{1,p}_0(\Omega)$, $1 \le p \le +\infty$, is the subspace of functions of $W^{1,p}(\Omega)$ with zero traces on $\partial\Omega$ and $W^{-1,p'}(\Omega)$, $1 \le p < +\infty$, its dual. Finally, the characteristic function of a set A will always be denoted by 1_A.

2. Some results about the continuity equation

Continuity equation enjoys some properties which will be important in the sequel. We recall them in the present section.

First statement deals with the extension outside Ω of the continuity equation.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\rho \in L^p(\Omega)$, $p \ge 2$, $\boldsymbol{u} \in [W^{1,2}_0(\Omega)]^3$ and $f \in L^1(\Omega)$. Assume that*

$$\operatorname{div}(\rho\boldsymbol{u}) = f \qquad in \quad \mathcal{D}'(\Omega). \tag{2.1}$$

Then, extending ρ, $\boldsymbol{u}$ and f by zero outside Ω and denoting again by ρ, $\boldsymbol{u}$ and f the new functions, we have

$$\operatorname{div}(\rho\boldsymbol{u}) = f \qquad in \quad \mathcal{D}'(\mathbb{R}^3). \tag{2.2}$$

Proof. We have to show

$$-\int_{\mathbb{R}^3} \rho \boldsymbol{u} \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^3} f\varphi \, dx, \ \ \forall \varphi \in \mathcal{D}(\mathbb{R}^3) \tag{2.3}$$

provided ρ, $\boldsymbol{u}$ and f are extended by zero outside Ω. To this end, consider the sequence of functions

$$\begin{gathered} \Phi_n \in \mathcal{D}(\Omega), \quad n \in \mathbb{N}^*, \quad 0 \le \Phi_n \le 1, \\ \Phi_n(x) = 1, \quad x \in \left\{ y \in \Omega, \quad \operatorname{dist}(y, \partial\Omega) \ge \frac{2}{n} \right\}, \\ \Phi_n(x) = 0, \quad x \in \left\{ y \in \Omega, \quad \operatorname{dist}(y, \partial\Omega) \le \frac{1}{n} \right\}, \\ |\nabla \Phi_n(x)| \le 2n, \quad \forall x \in \Omega. \end{gathered} \tag{2.4}$$

Clearly

$$\Phi_n \to 1 \quad \text{pointwise in } \Omega \text{ as } n \to \infty, \tag{2.5}$$

$$\operatorname{supp}\nabla\Phi_n \subset \left\{ x \in \Omega, \quad \frac{1}{n} \le \operatorname{dist}(x, \partial\Omega) \le \frac{2}{n} \right\}, \quad |\operatorname{supp}\nabla\Phi_n| \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

In virtue of (2.1),

$$-\int_{\mathbb{R}^3} \rho \boldsymbol{u} \cdot \nabla\varphi \Phi_n \, dx = \int_{\mathbb{R}^3} f\varphi\Phi_n \, dx + \int_{\mathbb{R}^3} \rho \boldsymbol{u} \cdot \nabla \Phi_n \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3). \tag{2.7}$$

Due to (2.4), (2.5) and the Lebesgue theorem, the first two integrals of this identity tend respectively to $-\int_{\mathbb{R}^3} \rho\boldsymbol{u} \cdot \nabla\varphi \, dx$ and $\int_{\mathbb{R}^3} f\varphi \, dx$. The third integral is bounded by

$$C \sup_{x \in \mathbb{R}^3} |\varphi(x)| \, \|\rho\|_{0,2,\operatorname{supp}\nabla\Phi_n} \left\| \frac{\boldsymbol{u}}{\operatorname{dist}(x, \partial\Omega)} \right\|_{0,2,\Omega}.$$

In accordance with (2.6) and due to the summability of ρ, $\|\rho\|_{0,2,\operatorname{supp}\nabla\Phi_n} \to 0$ as $n \to \infty$ and since $\boldsymbol{u} \in [W_0^{1,2}(\Omega)]^3$, by Hardy's inequality, $\boldsymbol{u}/\operatorname{dist}(x, \partial\Omega) \in [L^2(\Omega)]^3$. Hence (2.7) as $n \to \infty$ yields (2.3). Proof of Lemma 2.1 is thus complete. □

Next result is a consequence of the theory of renormalized solutions to the transport equation by Di-Perna and Lions [2].

Lemma 2.2. *Let $p \ge 2$, let λ_1, λ_2 such that*

$$-1 < \lambda_1 \le \frac{p}{2} - 1 \quad \textit{and} \quad 0 < \lambda_2 \le \frac{p}{2}. \tag{2.8}$$

Assume that $\rho \in L^p_{loc}(\mathbb{R}^3)$, $\rho \ge 0$ a.e. in $\mathbb{R}^3$, $\boldsymbol{u} \in [W^{1,2}_{loc}(\mathbb{R}^3)]^3$, and $f \in L^{q'}_{loc}(\mathbb{R}^3)$, $1 \le q \le p/\lambda_1$ if $\lambda_1 > 0$, $1 < q < +\infty$ if $\lambda_1 \le 0$, satisfy eqation (2.2). Then

$$\operatorname{div}(b(\rho)\boldsymbol{u}) + \{\rho b'(\rho) - b(\rho)\} \operatorname{div} \boldsymbol{u} = f b'(\rho) \qquad \textit{in} \quad \mathcal{D}'(\mathbb{R}^3), \tag{2.9}$$

for any function $b \in C^1([0,+\infty))$ *satisfying* (1.9) *with* (2.8). *Moreover, if* $f \equiv 0$, *the assumptions on* b *can be relaxed to* (1.8), (1.9) *and* (2.8).

Proof. First, we deal with the case $b \in C^1([0,+\infty))$ and $\lambda_1 > 0$. Regularizing (2.2) by usual mollifier S_ε, $0 < \varepsilon < 1$, we get

$$\operatorname{div}(S_\varepsilon(\rho)\boldsymbol{u}) = r_\varepsilon + S_\varepsilon(f) \qquad \text{a.e. in} \quad \mathbb{R}^3, \tag{2.10}$$

where $r_\varepsilon = \operatorname{div}(S_\varepsilon(\rho)\boldsymbol{u}) - \operatorname{div}(S_\varepsilon(\rho\boldsymbol{u}))$. Due to the generalized Friedrichs lemma (see [2, Lemma II.1]), $r_\varepsilon \to 0$ in $L^r_{loc}(\mathbb{R}^3)$, $1/r = 1/p + 1/2$ as $\varepsilon \to 0^+$.

We multiply equation (2.10) by $b'(S_\varepsilon(\rho))$ to obtain

$$\begin{aligned} &\operatorname{div}(b(S_\varepsilon(\rho))\boldsymbol{u}) + \{S_\varepsilon(\rho)b'(S_\varepsilon(\rho)) - b(S_\varepsilon(\rho))\}\operatorname{div}\boldsymbol{u} \\ &\quad = b'(S_\varepsilon(\rho))r_\varepsilon + b'(S_\varepsilon(\rho))S_\varepsilon(f) \qquad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{2.11}$$

Now, we pass to the limit $\varepsilon \to 0^+$. Clearly, $S_\varepsilon(\rho) \to \rho$ in $L^p_{loc}(\mathbb{R}^3)$ and $S_\varepsilon(f) \to f$ in $L^{q'}_{loc}(\mathbb{R}^3)$ and as a consequence a.e. in $\mathbb{R}^3$. Thanks to the growth conditions (1.9), (2.9), Vitali's theorem gives $b(S_\varepsilon(\rho)) \to b(\rho)$, $\{S_\varepsilon(\rho)b'(S_\varepsilon(\rho)) - b(S_\varepsilon(\rho))\} \to \{\rho b'(\rho) - b(\rho)\}$ and $b'(S_\varepsilon(\rho)) \rightharpoonup b'(\rho)$ respectively in $L^{6/5}_{loc}(\mathbb{R}^3)$, $L^2_{loc}(\mathbb{R}^3)$ and $L^q_{loc}(\mathbb{R}^3)$. For any bounded measurable set $\omega \subset \mathbb{R}^3$, $\int_\omega b'(S_\varepsilon(\rho))r_\varepsilon\, dx$ is bounded by $C\|b'(S_\varepsilon(\rho))\|_{0,r',\omega}\|r_\varepsilon\|_{0,r,\omega}$ and therefore tends to zero. Equation (2.11) thus yields (2.8).

Next, we consider the case $f \equiv 0$, b satisfying (1.8), (1.9), (2.9) with $\lambda_1 > 0$. For $0 < h < 1$, we put $b_h(\,\cdot\,) = b(\,\cdot + h)$ and apply to it the first part of the proof to get

$$\operatorname{div}(b_h(\rho)\boldsymbol{u}) + \{\rho b_h'(\rho) - b_h(\rho)\}\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{2.12}$$

It is clear that, as $h \to 0^+$, $b_h(t) \to b(t)$, $\forall t \ge 0$, and that, due to (1.8), $tb_h'(t) - b_h(t) \to tb'(t) - b(t)$, $\forall t \ge 0$. Moreover, for any fixed $R \ge 1$

$$\max_{[0,R]} |b_h(t)| \le \max_{[0,2R]} |b(t)|$$

and thanks again to (1.8)

$$\max_{[0,R]} |tb_h'(t) - b_h(t)| \le \max_{[0,2R]} |tb'(t) - b(t)| + ch^{1-\lambda_0} + h \max_{[1,2R]} |b'(t)|.$$

Finally, for any bounded measurable $\omega \subset \mathbb{R}^3$, we have

$$\begin{aligned} \|b_h(\rho)1_{\{\rho \ge R\}}\|_{0,\frac{6}{5},\omega} &\le cR^{(\lambda_1+1)-\frac{5}{6}p}\|\rho\|_{0,p,\omega}^{\frac{5p}{6}}, \\ \|\{\rho b_h'(\rho) - b_h(\rho)\}1_{\{\rho \ge R\}}\|_{0,2,\omega} &\le c(R^{\lambda_1-\frac{p}{2}} + R^{\lambda_2-\frac{p}{2}})\|\rho\|_{0,p,\omega}^{\frac{p}{2}}. \end{aligned}$$

So, we can write

$$\begin{aligned} \int_{\mathbb{R}^3} b_h(\rho)\boldsymbol{u}\cdot\nabla\varphi\, dx = &\int_{\mathbb{R}^3} b_h(\rho)1_{\{\rho \le R\}}\boldsymbol{u}\cdot\nabla\varphi\, dx \\ &+ \int_{\mathbb{R}^3} b_h(\rho)1_{\{\rho > R\}}\boldsymbol{u}\cdot\nabla\varphi\, dx, \ \varphi \in \mathcal{D}(\mathbb{R}^3). \end{aligned}$$

Due to the Lebesgue theorem, the first integral tends to $\int_{\mathbb{R}^3} b(\rho)1_{\{\rho\le R\}}\boldsymbol{u}\cdot\nabla\varphi\,dx$ as $h\to 0^+$. The second integral tends to zero as $R\to+\infty$ because it is bounded by $cR^{(\lambda_1+1)-5p/6}\|\rho\|_{0,p}^{5p/6}\|\boldsymbol{u}\|_{0,6}\|\nabla\varphi\|_{0,\infty}$. The analysis of the convergence of the other term is similar. Lemma 2.2 is thus proved. ☐

Now, for $k>0$, we put

$$b_k(t)=\begin{cases} b(t) & \text{if} \quad t\in[0,k],\\ b(k) & \text{if} \quad t\in(k,+\infty)\end{cases} \tag{2.13}$$

where $b\in C^1([0,+\infty))$. Then $b_k\in C^1([0,k)\cup(k,+\infty))$, $\lim_{t\to k^-}b_k'(t)=b'(k)$ and $\lim_{t\to k^+}b_k'(t)=0$. If $b'(k)=0$, then $b_k\in C^1([0,+\infty))$ and the composite mapping $b_k'\circ\rho$ is well defined a.e. in $\mathbb{R}^3$. If $b'(k)\neq 0$, we have to define $b_k'\circ\rho$ and we do it e.g. as follows

$$b_k'\circ\rho(x)=\begin{cases} b_k'(\rho(x)) & \text{if} \quad x\in\{\rho\neq k\},\\ 0 & \text{if} \quad x\in\{\rho=k\}.\end{cases} \tag{2.14}$$

With this definition in mind, Lemma 2.2 yields the following statement which we need in the sequel.

Corollary 2.1. *Let* $\rho\in L^p_{loc}(\mathbb{R}^3)$, $p\ge 2$, $\rho\ge 0$ a.e. *in* $\mathbb{R}^3$, $\boldsymbol{u}\in[W^{1,2}_{loc}(\mathbb{R}^3)]^3$ *and* $f\in L^1_{loc}(\mathbb{R}^3)$ *satisfy equation* (2.2)*. Then*

(2.15)
$$\operatorname{div}(b_k(\rho)\boldsymbol{u})+\{\rho b_k'(\rho)-b_k(\rho)\}\operatorname{div}\boldsymbol{u}=fb_k'(\rho) \qquad in \quad \mathcal{D}'(\mathbb{R}^3),\ \ \forall k>0$$

with any function b_k *satisfying* (2.13) *where* $b\in C^1([0,+\infty))$*. Moreover, if* $f\equiv 0$*, the assumptions on* b *can be relaxed to* (1.8).

Proof. First, we claim that

$$k\operatorname{div}\boldsymbol{u}=f \qquad \text{a.e. in} \quad \{\rho=k\},\quad \forall k>0. \tag{2.16}$$

Indeed, take $b\in\mathcal{D}(\mathbb{R})$ such that supp$b\subset\mathbb{R}^+$, $b(t)=t$ in $((3/4)k,(5/4)k)$ and put $b^+_{k,\varepsilon}=S_{\varepsilon/2}(b_{k+\varepsilon})$, $b^-_{k,\varepsilon}=S_{\varepsilon/2}(b_{k-\varepsilon})$ where $S_{\varepsilon/2}$ is one dimensional regularizing operator. We have as $\varepsilon\to 0^+$

$$b^+_{k,\varepsilon}(t)\to b_k(t),\ \ \forall t\in\mathbb{R}^+,\ \ (b^+_{k,\varepsilon})'(t)\to b_k'(t),\ \ \forall t\neq k,\ \ (b^+_{k,\varepsilon})'(k)\to 1,$$
$$b^-_{k,\varepsilon}(t)\to b_k(t),\ \ \forall t\in\mathbb{R}^+,\ \ (b^-_{k,\varepsilon})'(t)\to b_k'(t),\ \ \forall t\neq k,\ \ (b^-_{k,\varepsilon})'(k)\to 0.$$

Lemma 2.2 applied to equation (2.2) yields

$$\operatorname{div}(b^\pm_{k,\varepsilon}(\rho)\boldsymbol{u})+\{\rho(b^\pm_{k,\varepsilon})'(\rho)-b^\pm_{k,\varepsilon}(\rho)\}\operatorname{div}\boldsymbol{u}=f(b^\pm_{k,\varepsilon})'(\rho) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

Therefore, as $\varepsilon\to 0^+$,

$$\begin{aligned}\operatorname{div}(b_k(\rho)\boldsymbol{u})+\{\rho b_k'(\rho)1_{\{\rho\neq k\}}+k1_{\{\rho=k\}}-b_k(\rho)\}\operatorname{div}\boldsymbol{u}\\ =\{b_k'(\rho)1_{\{\rho\neq k\}}+1_{\{\rho=k\}}\}f \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3)\end{aligned}$$

and

$$\operatorname{div}(b_k(\rho)\boldsymbol{u}) + \{\rho b_k'(\rho)1_{\{\rho\neq k\}} - b_k(\rho)\}\operatorname{div}\boldsymbol{u} = b_k'(\rho)1_{\{\rho\neq k\}} f \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

Subtracting the last two equations yields (2.16).

Now, let us consider the case $b \in C^1([0,+\infty))$. We denote the extension of b by $b(0)$ in $(-\infty, 0)$ again by b. Lemma 2.2 applied to $S_\varepsilon(b_k)$ where S_ε, $0<\varepsilon<1$, is one dimensional regularization, yields

(2.17)
$$\operatorname{div}(S_\varepsilon(b_k)(\rho)\boldsymbol{u}) + \{\rho S_\varepsilon(b_k)'(\rho) - S_\varepsilon(b_k)(\rho)\}\operatorname{div}\boldsymbol{u} = S_\varepsilon(b_k)'(\rho) f \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

As $\varepsilon \to 0^+$,

$$S_\varepsilon(b_k)(t) \to b_k(t), \ \ \forall t \in [0,+\infty), \ \ S_\varepsilon(b_k)'(t) \to b_k'(t), \ \ \forall t \in [0,k) \cup (k,+\infty).$$

In accordance with (2.16) therefore

$$\begin{aligned} S_\varepsilon(b_k)(\rho) &\to b_k(\rho) \qquad \text{a.e. in} \quad \mathbb{R}^3, \\ S_\varepsilon(b_k)'(\rho) &\to b_k'(\rho) \qquad \text{a.e. in} \quad \{\rho \neq k\}, \\ \rho S_\varepsilon(b_k)'(\rho)\operatorname{div}\boldsymbol{u} &= S_\varepsilon(b_k)'(\rho) f \qquad \text{a.e. in} \quad \{\rho = k\}. \end{aligned}$$

Moreover, the particular form of functions b_k implies that $S_\varepsilon(b_k)$ and $S_\varepsilon(b_k)'$ are uniformly bounded with respect to ε. We pass to the limit $\varepsilon \to 0^+$ in (2.17) by using the Lebesgue theorem and we get (2.15).

The case $f \equiv 0$ and b satisfying (1.8) can be treated by following the lines of proof of Lemma 2.2 starting by (2.12). □

3. Approximation

The proof of Theorem 1.1 will be carried out by means of the following approximation.

$$\begin{aligned} &-\mu_1\Delta\boldsymbol{u}_\delta - (\mu_1+\mu_2)\nabla\operatorname{div}\boldsymbol{u}_\delta + \operatorname{div}(\rho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta) \\ &\qquad\qquad + \nabla\{\rho_\delta^\gamma + \delta\rho_\delta^\beta\} = \rho_\delta\boldsymbol{f} + \boldsymbol{g} \qquad \text{in} \quad \Omega, \end{aligned} \tag{3.1}$$

$$\operatorname{div}(\rho_\delta\boldsymbol{u}_\delta) = 0 \qquad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}_\delta = \boldsymbol{0} \qquad \text{on} \quad \partial\Omega, \tag{3.3}$$

$$\int_\Omega \rho_\delta \, dx = M \tag{3.4}$$

where

$$\beta = \max(\gamma, 3), \quad \ \delta \in (0,1). \tag{3.5}$$

Similar approximation was used in the nonsteady case in paper [4].

Existence of finite energy weak solutions to the system (3.1) through (3.4) is guaranteed by the following theorem of P. L. Lions, see [5], Theorem 6.7, p. 114 and Section 6.10, p. 158–162.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$, $\boldsymbol{f} \in [L^\infty(\Omega)]^3$, $\boldsymbol{g} \in [L^\infty(\Omega)]^3$, μ_1 and μ_2 satisfy* (1.3), *$\beta > 5/3$ and $M > 0$. Assume that $p \in C^1([0,+\infty))$ is a strictly increasing function such that*

$$\exists C_1, C_2 > 0,\ \ C_1 t^\beta \le p(t) \le C_2 t^\beta,\ \ \forall t \in [1,+\infty). \tag{3.6}$$

Then there exists a couple $(\rho, \boldsymbol{u})$, $\rho \in L^{s(\beta)}(\Omega)$, where $s(\cdot)$ is defined in (1.12), *$\rho \ge 0$* a.e. *in Ω, $\int_\Omega \rho\, dx = M$, $\boldsymbol{u} \in [W_0^{1,2}(\Omega)]^3$ satisfying*

(3.7)
$$-\mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2)\nabla \operatorname{div} \boldsymbol{u} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = \rho \boldsymbol{f} + \boldsymbol{g} \qquad in \quad [\mathcal{D}'(\Omega)]^3,$$

$$\operatorname{div}(\rho \boldsymbol{u}) = 0 \qquad in \quad \mathcal{D}'(\Omega). \tag{3.8}$$

Moreover $(\rho, \boldsymbol{u})$ verifies the inequality

$$\int_\Omega \{\mu_1 |\nabla \boldsymbol{u}|^2 + (\mu_1 + \mu_2)(\operatorname{div} \boldsymbol{u})^2\}\, dx \le \int_\Omega (\rho \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{g} \cdot \boldsymbol{u})\, dx. \tag{3.9}$$

Such a couple $(\rho, \boldsymbol{u})$ is called a finite energy weak solution to the system (3.7) *and* (3.8).

Remark 1. The reader easily verifies by density argument that equation (3.7) holds with any test function $\boldsymbol{\varphi} \in [W_0^{1,2}(\Omega)]^3$.

Due to Theorem 3.1 complemented by Lemmas 2.1 and 2.2, for any $\delta \in (0,1)$, there exists a finite energy weak solution $(\rho_\delta, \boldsymbol{u}_\delta)$ to the system (3.1) through (3.5) such that $\rho_\delta \in L^{2\beta}(\Omega)$ and, provided ρ_δ and $\boldsymbol{u}_\delta$ are extended by zero outside Ω, it holds

$$\operatorname{div}(\rho_\delta \boldsymbol{u}_\delta) = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3) \tag{3.10}$$

and

$$\operatorname{div}(b(\rho_\delta)\boldsymbol{u}_\delta) + \{\rho_\delta b'(\rho_\delta) - b(\rho_\delta)\} \operatorname{div} \boldsymbol{u}_\delta = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3) \tag{3.11}$$

with any b satisfying (1.8), (1.9) and (2.9) with $p = 2\beta$.

In the following sections, we shall prove that there exists a weakly convergent subsequence of $\{(\rho_\delta, \boldsymbol{u}_\delta)\}_{\delta \in (0,1)}$ such that its weak limit is a renormalized finite energy weak solution to the original problem (1.1), (1.2), (1.5), (1.6). From now, we shall suppose without loss of generality that the sequence $(\rho_\delta, \boldsymbol{u}_\delta)$ is defined on the whole space $\mathbb{R}^3$.

4. Estimates independent of δ

We shall start by the problem

$$\operatorname{div} \boldsymbol{v} = f,\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}. \tag{4.1}$$

The following result is due to Bogovskiĭ [1].

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a linear operator $\boldsymbol{\mathcal{B}} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3) : \{f \in L^p(\Omega), \int_\Omega f\,dx = 0\} \to [W_0^{1,p}(\Omega)]^3$, for any $1 < p < \infty$, satisfying*

$$\operatorname{div} \boldsymbol{\mathcal{B}}(f) = f \quad \text{a.e. } in\ \Omega, \qquad \|\nabla \boldsymbol{\mathcal{B}}(f)\|_{0,p} \le C(\Omega,p)\|f\|_{0,p}. \tag{4.2}$$

From (3.9), it follows

$$\|\nabla \boldsymbol{u}_\delta\|_{0,2} \le C\{1 + \|\rho_\delta\|_{0,\frac{6}{5}} 1_{\{\gamma > \frac{5}{3}\}}\}. \tag{4.3}$$

Put $p_\delta = \rho_\delta^\gamma + \delta\rho_\delta^\beta$ and consider $\boldsymbol{\varphi} = \boldsymbol{\mathcal{B}}(p_\delta^\theta - (1/|\Omega|)\int_\Omega p_\delta^\theta\,dy)$ as test function in equation (3.1). Due to Remark 1, it is an admissible test function provided $2 \le 2/\theta$, i.e. provided $\theta \le 1$. We get

$$\begin{aligned} \int_\Omega p_\delta^{1+\theta}\,dx = \frac{1}{|\Omega|}\int_\Omega p_\delta^\theta\,dy \int_\Omega p_\delta\,dx - \int_\Omega (\rho_\delta \boldsymbol{f}\cdot\boldsymbol{\varphi} + \boldsymbol{g}\cdot\boldsymbol{\varphi})\,dx \\ + \int_\Omega \{\mu_1 \nabla \boldsymbol{u}_\delta : \nabla\boldsymbol{\varphi} + (\mu_1+\mu_2)\operatorname{div}\boldsymbol{u}_\delta \operatorname{div}\boldsymbol{\varphi}\}\,dx \\ - \int_\Omega \rho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta : \nabla\boldsymbol{\varphi}\,dx. \end{aligned} \tag{4.4}$$

For the first term at the right side of (4.4) it holds by Hölder's inequality and interpolation

$$\begin{aligned} \frac{1}{|\Omega|}\int_\Omega p_\delta^\theta\,dy \int_\Omega p_\delta\,dx &\le |\Omega|^{-\frac{\theta}{1+\theta}}\{\|\rho_\delta\|_{0,1}^{\eta\gamma}\|\rho_\delta\|_{0,(1+\theta)\gamma}^{(1-\eta)\gamma} \\ &\quad + \delta\|\rho_\delta\|_{0,1}^{\widetilde{\eta}\beta}\|\rho_\delta\|_{0,(1+\theta)\beta}^{(1-\widetilde{\eta})\beta}\}\|p_\delta\|_{0,(1+\theta)}^\theta \\ &\le C\{\|\rho_\delta\|_{0,(1+\theta)\gamma}^{(1-\eta)\gamma} + \delta\|\rho_\delta\|_{0,(1+\theta)\beta}^{(1-\widetilde{\eta})\beta}\}\|p_\delta\|_{0,(1+\theta)}^\theta \end{aligned}$$

where

$$0 < \eta = \frac{\theta}{(1+\theta)\gamma - 1} < 1 \quad \text{and} \quad 0 < \widetilde{\eta} = \frac{\theta}{(1+\theta)\beta - 1} < 1.$$

By Hölder's and Sobolev's inequalities and then by Lemma 4.1, the second term can be estimated as follows

$$\begin{aligned} \int_\Omega \rho_\delta \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx &\le \|\rho_\delta\|_{0,(1+\theta)\gamma}\|\boldsymbol{f}\|_{0,\frac{6\gamma(1+\theta)}{5\gamma(1+\theta)-6}}\|\boldsymbol{\varphi}\|_{0,6} \qquad \text{provided} \quad (1+\theta) > \frac{6}{5\gamma} \\ &\le C\|\rho_\delta\|_{0,(1+\theta)\gamma}\|\nabla\boldsymbol{\varphi}\|_{0,2} \\ &\le C\|\rho_\delta\|_{0,(1+\theta)\gamma}\|p_\delta\|_{0,2\theta}^\theta \qquad \text{provided} \quad \theta \le 1 \\ &\le C\|\rho_\delta\|_{0,(1+\theta)\gamma}\|p_\delta\|_{0,(1+\theta)}^\theta \qquad \text{provided} \quad 2\theta \le (1+\theta) \text{ i.e. } \theta \le 1 \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \boldsymbol{g}\cdot\boldsymbol{\varphi}\,dx &\le \|\boldsymbol{g}\|_{0,\frac{6}{5}}\|\boldsymbol{\varphi}\|_{0,6} \le C\|\nabla\boldsymbol{\varphi}\|_{0,2} \\ &\le C\|p_\delta\|_{0,2\theta}^\theta \le C\|p_\delta\|_{0,(1+\theta)}^\theta \qquad \text{provided} \quad \theta \le 1. \end{aligned}$$

In virtue of the Schwarz inequality, (4.3), Lemma 4.1 and interpolation, the third integral is estimated by

$$\begin{aligned}
&\int_\Omega \{\mu_1 \nabla \boldsymbol{u}_\delta : \nabla \boldsymbol{\varphi} + (\mu_1+\mu_2)\operatorname{div}\boldsymbol{u}_\delta \operatorname{div}\boldsymbol{\varphi}\}\,dx \\
&\qquad \le C\|\nabla \boldsymbol{u}_\delta\|_{0,2}\|\nabla\varphi\|_{0,2} \\
&\qquad \le C\{1+\|\rho_\delta\|_{0,\frac{6}{5}} 1_{\{\gamma>\frac{5}{3}\}}\}\|p_\delta\|^\theta_{0,(1+\theta)} \qquad \text{provided} \quad \theta\le 1 \\
&\qquad \le C\{1+\|\rho_\delta\|^{\overline{\eta}}_{0,1}\|\rho_\delta\|^{(1-\overline{\eta})}_{0,(1+\theta)\gamma} 1_{\{\gamma>\frac{5}{3}\}}\}\|p_\delta\|^\theta_{0,(1+\theta)} \\
&\qquad \le C\{1+\|\rho_\delta\|^{(1-\overline{\eta})}_{0,(1+\theta)\gamma} 1_{\{\gamma>\frac{5}{3}\}}\}\|p_\delta\|^\theta_{0,(1+\theta)}
\end{aligned}$$

where

$$0<\overline{\eta}=\frac{5\gamma(1+\theta)-6}{6\gamma(1+\theta)-6}<1.$$

Last, but not least, again by Hölder's inequality, (4.3), Lemma 4.1 and interpolation

$$\begin{aligned}
&\int_\Omega \rho_\delta \boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta : \nabla\boldsymbol{\varphi}\,dx \\
&\quad \le \|\rho_\delta\|_{0,(1+\theta)\gamma}\|\boldsymbol{u}_\delta\|^2_{0,6}\|\nabla\boldsymbol{\varphi}\|_{0,\frac{3\gamma(1+\theta)}{2\gamma(1+\theta)-3}} \qquad \text{provided} \quad (1+\theta)>\frac{3}{2\gamma} \\
&\quad \le C\|\rho_\delta\|_{0,(1+\theta)\gamma}\{1+\|\rho_\delta\|^{2\overline{\eta}}_{0,1}\|\rho_\delta\|^{2(1-\overline{\eta})}_{0,(1+\theta)\gamma}1_{\{\gamma>\frac{5}{3}\}}\}\|p_\delta\|^\theta_{0,\frac{3\gamma\theta(1+\theta)}{2\gamma(1+\theta)-3}} \\
&\quad \le C\|\rho_\delta\|_{0,(1+\theta)\gamma}\{1+\|\rho_\delta\|^{2(1-\overline{\eta})}_{0,(1+\theta)\gamma}1_{\{\gamma>\frac{5}{3}\}}\}\|p_\delta\|^\theta_{0,(1+\theta)}
\end{aligned}$$

provided

$$\frac{3\gamma\theta(1+\theta)}{2\gamma(1+\theta)-3}\le(1+\theta)\Leftrightarrow\theta\le\frac{2\gamma-3}{\gamma}.$$

Putting these estimates into (4.4) we obtain

$$\begin{aligned}
\|p_\delta\|_{0,(1+\theta)} \le C\{&(1+\delta\|\rho_\delta\|^{(1-\widetilde{\eta})\beta}_{0,(1+\theta)\beta}+\|\rho_\delta\|^{(1-\eta)\gamma}_{0,(1+\theta)\gamma}+\|\rho_\delta\|_{0,(1+\theta)\gamma}) \\
&+(\|\rho_\delta\|^{(1-\overline{\eta})}_{0,(1+\theta)\gamma}+\|\rho_\delta\|^{2(1-\overline{\eta})+1}_{0,(1+\theta)\gamma})1_{\{\gamma>\frac{5}{3}\}}\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\|\rho_\delta\|^\gamma_{0,(1+\theta)\gamma}+\delta\|\rho_\delta\|^\beta_{0,(1+\theta)\beta} \\
&\qquad \le C\{(1+\delta\|\rho_\delta\|^{(1-\widetilde{\eta})\beta}_{0,(1+\theta)\beta}+\|\rho_\delta\|^{(1-\eta)\gamma}_{0,(1+\theta)\gamma}+\|\rho_\delta\|_{0,(1+\theta)\gamma}) \\
&\qquad\quad +(\|\rho_\delta\|^{(1-\overline{\eta})}_{0,(1+\theta)\gamma}+\|\rho_\delta\|^{2(1-\overline{\eta})+1}_{0,(1+\theta)\gamma})1_{\{\gamma>\frac{5}{3}\}}\}.
\end{aligned}$$

From the last estimate, by means of the Young inequality, we get

$$\|\rho_\delta\|^\gamma_{0,(1+\theta)\gamma}+\delta\|\rho_\delta\|^\beta_{0,(1+\theta)\beta}\le C$$

where C is a constant independent of δ. Summing up the conditions on θ, it is not difficult to check that the optimal $\theta = (2\gamma - 3)/\gamma$ if $3/2 < \gamma < 3$ and $\theta = 1$ if $\gamma \geq 3$.

Let us conclude this part by summarizing the properties of the sequence $(\rho_\delta, \boldsymbol{u}_\delta)$.

Lemma 4.2. *Let* $(\rho_\delta, \boldsymbol{u}_\delta)$ *be a finite energy weak solution to the problem* (3.1) *through* (3.4) *from Theorem* 3.1. *Then*

$$\|\rho_\delta\|_{0,s(\gamma)} \leq C, \tag{4.5}$$

$$\delta^{\frac{1}{\beta}} \|\rho_\delta\|_{0,\frac{s(\gamma)}{\gamma}\beta} \leq C, \tag{4.6}$$

where $s(\cdot)$ *is defined in* (1.12) *and*

$$\|\nabla \boldsymbol{u}_\delta\|_{0,2} \leq C. \tag{4.7}$$

Lemma 4.2 implies directly the following statement.

Lemma 4.3. *There exist functions* ρ, $\overline{\rho^\gamma}$, $\boldsymbol{u}$ *and a subsequence of* $\{(\rho_\delta, \boldsymbol{u}_\delta)\}_{\delta\in(0,1)}$ *such that*

$$\rho_\delta \rightharpoonup \rho \quad \textit{in } L^{s(\gamma)}(\mathbb{R}^3), \quad \rho_\delta^\gamma \rightharpoonup \overline{\rho^\gamma} \quad \textit{in } L^{\frac{s(\gamma)}{\gamma}}(\mathbb{R}^3), \quad \rho = 0 \quad \textit{in } \mathbb{R}^3 \setminus \Omega, \tag{4.8}$$

$$\delta \rho_\delta^\beta \to 0 \qquad \textit{in} \quad L^1(\mathbb{R}^3), \tag{4.9}$$

$$\boldsymbol{u}_\delta \rightharpoonup \boldsymbol{u} \quad \textit{in } [W^{1,2}(\mathbb{R}^3)]^3, \quad \boldsymbol{u}_\delta \to \boldsymbol{u} \qquad \textit{in} \quad [L^p(\mathbb{R}^3)]^3, \quad 2 \leq p < 6,$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \textit{in} \quad \mathbb{R}^3 \setminus \Omega, \tag{4.10}$$

(4.11)
$$\rho_\delta \boldsymbol{u}_\delta \rightharpoonup \rho \boldsymbol{u} \quad \textit{in } [L^{\frac{6s(\gamma)}{s(\gamma)+6}}(\mathbb{R}^3)]^3, \quad \rho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} \quad \textit{in } [L^{\frac{3s(\gamma)}{s(\gamma)+3}}(\mathbb{R}^3)]^{3\times 3}.$$

It holds

(4.12)
$$-\mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2)\nabla \operatorname{div} \boldsymbol{u} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \overline{\rho^\gamma} = \rho \boldsymbol{f} + \boldsymbol{g} \qquad \textit{in} \quad [\mathcal{D}'(\Omega)]^3,$$

$$\operatorname{div}(\rho \boldsymbol{u}) = 0 \qquad \textit{in} \quad \mathcal{D}'(\mathbb{R}^3) \tag{4.13}$$

and the inequality (3.9) *is verified.*

Remark 2. By density argument one can easy seen that equation (4.12) holds for any test function $\boldsymbol{\varphi} \in [W_0^{1,(s(\gamma)/\gamma)'}(\Omega)]^3$.

5. Functions T_k and their basic properties

For $k > 0$, we define

$$T_k(t) = \begin{cases} t & \text{if} \quad t \in [0, k], \\ k & \text{if} \quad t > k. \end{cases} \tag{5.1}$$

It is easy to check that

$$\operatorname*{esssup}_{t\in[0,s]}|tT_k'(t)-T_k(t)|\le T_k(s)1_{\{s\ge k\}},\quad \forall s\ge 0. \tag{5.2}$$

Next, due to the inequalities $a^\gamma - b^\gamma \ge (a-b)^\gamma$, $|T_k(a)-T_k(b)|\le |a-b|$, $\forall 0\le b\le a<+\infty$,

$$|T_k(t)-T_k(s)|^{\gamma+1}\le (t^\gamma - s^\gamma)(T_k(t)-T_k(s)),\qquad \forall s,t\ge 0. \tag{5.3}$$

Thanks to the inequality $|\{\rho_\delta\ge k\}|\le (1/k)\int_\Omega \rho_\delta 1_{\{\rho_\delta\ge k\}}\,dx\le (1/k)\int_\Omega\rho_\delta\,dx\le M/k$, we get

$$\|\rho_\delta 1_{\{\rho_\delta\ge k\}}\|_{0,p}\le\left(\frac{M}{k}\right)^{\frac{1}{p}-\frac{1}{s(\gamma)}}\|\rho_\delta\|_{0,s(\gamma)},\qquad \forall 1\le p<s(\gamma). \tag{5.4}$$

Denote $\overline{T_k(\rho)}\in L^\infty(\Omega)$ the weak-star limit of the sequence $\{T_k(\rho_\delta)\}_{\delta\in(0,1)}$. In accordance with the estimates

$$\begin{aligned}\|\overline{T_k(\rho)}-\rho\|_{0,p}&\le\liminf_{\delta\to 0^+}\|T_k(\rho_\delta)-\rho_\delta\|_{0,p}\\ &\le 2\liminf_{\delta\to 0^+}\|\rho_\delta 1_{\{\rho_\delta\ge k\}}\|_{0,p}\le Ck^{\frac{1}{s(\gamma)}-\frac{1}{p}},\qquad \forall 1\le p<s(\gamma),\end{aligned}$$

it holds

$$\overline{T_k(\rho)}\to\rho\quad\text{in } L^p(\Omega),\qquad \forall 1\le p<s(\gamma)\quad \text{as } k\to+\infty. \tag{5.5}$$

Similarly,

$$T_k(\rho)\to\rho\quad\text{in } L^p(\Omega),\qquad \forall 1\le p<s(\gamma)\quad \text{as } k\to+\infty. \tag{5.6}$$

6. Effective pressure

We begin by recalling some facts useful in the sequel. In this section, we are using Einstein summation convention over repeated indexes.

We introduce the operators

$$\mathcal{A}_i:\mathcal{S}(\mathbb{R}^3)\to\mathcal{S}(\mathbb{R}^3),\ \ i=1,2,3,\ \ \mathcal{A}_i(f)(x)=\mathcal{F}^{-1}\left[\frac{\mathrm{i}\xi_i}{|\xi|^2}\mathcal{F}(f)(\xi)\right](x) \tag{6.1}$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. The Marcinkiewicz theorem about multipliers yields

$$\|\nabla\mathcal{A}_i(f)\|_{0,p,\mathbb{R}^3}\le C(p)\|f\|_{0,p,\mathbb{R}^3},\ \ 1<p<\infty,\ \ i=1,2,3 \tag{6.2}$$

and, moreover, due to the Sobolev embeddings

$$\|\mathcal{A}_i(f)\|_{0,p^*,\mathbb{R}^3}\le C(p)\|f\|_{0,p,\mathbb{R}^3},\ \ p^*=\frac{3p}{3-p},\ \ 1<p<3,\ \ i=1,2,3. \tag{6.3}$$

Next, we denote

$$\mathcal{R}_{ij} = \partial_i \mathcal{A}_j, \qquad i, j = 1, 2, 3 \tag{6.4}$$

and observe that

$$\mathcal{R}_{ij} = \mathcal{R}_{ji}, \;\; \forall i, j = 1, 2, 3, \;\; \mathcal{R}_{ii}(f) = f \tag{6.5}$$

and, due to Parseval equality,

(6.6)
$$\int_{\mathbb{R}^3} \mathcal{R}_{ij}(f) g \, dx = \int_{\mathbb{R}^3} f \mathcal{R}_{ij}(g) \, dx, \;\; \forall f \in L^p(\mathbb{R}^3), \;\; \forall g \in L^{p'}(\mathbb{R}^3), \;\; \forall i, j = 1, 2, 3.$$

Let us emphasize that, in virtue of (6.2), the closure of the operator $\mathcal{R}_{ij}$, $\forall i, j = 1, 2, 3$, denoted again $\mathcal{R}_{ij}$, is a strongly continuous linear operator from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $1 < p < \infty$.

Next, we recall two results related to the div-curl lemma. The proof of the first one can be found in Yi [6] and that of the second one in Feireisl [3].

Lemma 6.1. *Let $1 < p_1, p_2, q_1, q_2 \le \infty$. Suppose that*

$$\boldsymbol{f}_n \rightharpoonup \boldsymbol{f} \quad \text{in } [L^{p_1}(\Omega)]^3, \qquad \boldsymbol{g}_n \rightharpoonup \boldsymbol{g} \quad \text{in } [L^{p_2}(\Omega)]^3$$

and

$$\operatorname{div} \boldsymbol{f}_n \to \operatorname{div} \boldsymbol{f} \quad \text{in } W^{-1,q_1}(\Omega), \qquad \operatorname{curl} \boldsymbol{g}_n \to \operatorname{curl} \boldsymbol{g} \quad \text{in } [W^{-1,q_2}(\Omega)]^3.$$

Then

$$\boldsymbol{f}_n \cdot \boldsymbol{g}_n \rightharpoonup \boldsymbol{f} \cdot \boldsymbol{g} \qquad \text{in } L^r(\Omega), \qquad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} < 1.$$

Lemma 6.2. *Let $1 < p < \infty$. Suppose that*

$$f_n \rightharpoonup f \quad \text{in } L^p(\mathbb{R}^3), \;\; g_n \rightharpoonup g \quad \text{in } L^q(\mathbb{R}^3), \;\; \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Then

$$f_n \mathcal{R}_{ij}(g_n) - g_n \mathcal{R}_{ij}(f_n) \rightharpoonup f \mathcal{R}_{ij}(g) - g \mathcal{R}_{ij}(f) \qquad \text{in } L^r(\mathbb{R}^3), \;\; \forall i, j = 1, 2, 3.$$

Now, we are in position to prove the following lemma.

Lemma 6.3. *For any $k > 0$, it holds*

$$\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\,\overline{T_k(\rho)} = (2\mu_1 + \mu_2)\{\overline{T_k(\rho) \operatorname{div} \boldsymbol{u}} - \overline{T_k(\rho)} \operatorname{div} \boldsymbol{u}\} \quad \text{a.e. in } \Omega \tag{6.7}$$

where $\overline{\rho^\gamma T_k(\rho)}$ and $\overline{T_k(\rho) \operatorname{div} \boldsymbol{u}}$ are the weak limits of $\{\rho_\delta^\gamma T_k(\rho_\delta)\}_\delta$ and $\{T_k(\rho_\delta) \operatorname{div} \boldsymbol{u}_\delta\}_\delta$ respectively in $L^{s(\gamma)/\gamma}(\Omega)$ and $L^2(\Omega)$.

Proof. Thanks to the properties of functions T_k and of the operators $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{A}}(T_k(\rho_\delta)) \in [W^{1,p}(\mathbb{R}^3)]^3$, $\forall 1 < p < \infty$. According to Remark 1, the function $\boldsymbol{\varphi} = \boldsymbol{\mathcal{A}}(T_k(\rho_\delta))\eta$, where $\eta \in \mathcal{D}'(\Omega)$, is an admissible test function for equation (3.1). Using conveniently (6.5), (6.6), after a long but straightforward computation, we arrive at

(6.8)
$$\begin{aligned}
&\int_\Omega \{\rho_\delta^\gamma - (2\mu_1+\mu_2)\operatorname{div}\boldsymbol{u}_\delta\}T_k(\rho_\delta)\eta\,dx + \delta\int_\Omega \rho_\delta^\beta T_k(\rho_\delta)\eta\,dx \\
&\quad = -\int_\Omega (\rho_\delta^\gamma + \delta\rho_\delta^\beta)\mathcal{A}_i(T_k(\rho_\delta))\partial_i\eta\,dx + (\mu_1+\mu_2)\int_\Omega \mathcal{A}_i(T_k(\rho_\delta))\partial_i\eta\operatorname{div}\boldsymbol{u}_\delta\,dx \\
&\qquad + \mu_1\int_\Omega \partial_j u_\delta^i \mathcal{A}_i(T_k(\rho_\delta))\partial_j\eta\,dx - \mu_1\int_\Omega u_\delta^i \mathcal{R}_{ij}(T_k(\rho_\delta))\partial_j\eta\,dx \\
&\qquad + \mu_1\int_\Omega u_\delta^i T_k(\rho_\delta)\partial_i\eta\,dx - \int_\Omega \rho_\delta u_\delta^i u_\delta^j \mathcal{A}_i(T_k(\rho_\delta))\partial_j\eta\,dx \\
&\qquad - \int_\Omega (\rho_\delta f^i + g^i)\mathcal{A}_i(T_k(\rho_\delta))\eta\,dx + \int_\Omega u_\delta^j\{T_k(\rho_\delta)\mathcal{R}_{ij}(\rho_\delta u_\delta^i\eta) \\
&\qquad - \mathcal{R}_{ij}(T_k(\rho_\delta))\rho_\delta u_\delta^i\eta\}\,dx - \int_\Omega \rho_\delta u_\delta^i \mathcal{R}_{ij}(u_\delta^j T_k(\rho_\delta))\eta\,dx.
\end{aligned}$$

Now, the aim is to pass to the limit $\delta \to 0^+$. Firstly, the following convergences are direct consequences of Lemma 4.2 and properties of functions T_k and operators $\boldsymbol{\mathcal{A}}$:

(6.9)
$$\begin{aligned}
\{\rho_\delta^\gamma - (2\mu_1+\mu_2)\operatorname{div}\boldsymbol{u}_\delta\}T_k(\rho_\delta) &\rightharpoonup \overline{\rho^\gamma T_k(\rho)} \\
&\quad - (2\mu_1+\mu_2)\overline{T_k(\rho)\operatorname{div}\boldsymbol{u}} \qquad \text{in} \quad L^{\frac{s(\gamma)}{\gamma}}(\mathbb{R}^3),
\end{aligned}$$

$$\delta\int_\Omega \rho_\delta^\beta T_k(\rho_\delta)\eta\,dx \to 0, \qquad \delta\int_\Omega \rho_\delta^\beta \mathcal{A}_i(T_k(\rho_\delta))\partial_i\eta\,dx \to 0, \tag{6.10}$$

$$\boldsymbol{u}_\delta T_k(\rho_\delta) \rightharpoonup \boldsymbol{u}\overline{T_k(\rho)} \qquad \text{in} \quad [L^6(\mathbb{R}^3)]^3. \tag{6.11}$$

Next, in virtue of (4.11), (6.11) and of the weak convergence of $T_k(\rho_\delta)$ to $\overline{T_k(\rho)}$ in $L^p(\mathbb{R}^3)$, $\forall 1 < p < \infty$, we have

$$\mathcal{R}_{ij}(T_k(\rho_\delta)) \rightharpoonup \mathcal{R}_{ij}(\overline{T_k(\rho)}) \qquad \text{in} \quad L^p(\mathbb{R}^3),\ \forall 1 < p < \infty, \tag{6.12}$$

$$\mathcal{R}_{ij}(\eta\rho_\delta u_\delta^i) \rightharpoonup \mathcal{R}_{ij}(\eta\rho u^i) \qquad \text{in} \quad L^{\frac{6s(\gamma)}{s(\gamma)+6}}(\mathbb{R}^3) \tag{6.13}$$

and

$$\mathcal{R}_{ij}(u_\delta^j T_k(\rho_\delta)) \rightharpoonup \mathcal{R}_{ij}(u^j\overline{T_k(\rho)}) \qquad \text{in} \quad L^6(\mathbb{R}^3). \tag{6.14}$$

From (6.12), we easily deduce that

$$\mathcal{A}_i(T_k(\rho_\delta)) \rightharpoonup \mathcal{A}_i(\overline{T_k(\rho)}) \qquad \text{in} \quad W^{1,p}(\mathbb{R}^3),\ \forall\frac{3}{2} < p < +\infty, \tag{6.15}$$

which, thanks to the compact embedding $W^{1,3}(\Omega) \subset\subset L^q(\Omega)$, $1 \le q < \infty$, yields

$$\mathcal{A}_i(T_k(\rho_\delta)) \to \mathcal{A}_i(\overline{T_k(\rho)}) \quad \text{in} \quad L^q(\Omega),\ \forall 1 \le q < \infty. \tag{6.16}$$

Using Lemma 6.1 with $\boldsymbol{f}_\delta = \rho_\delta \boldsymbol{u}_\delta$ and $\boldsymbol{g}_\delta = \nabla \mathcal{A}_j(\boldsymbol{u}_\delta^j T_k(\rho_\delta))$, on condition $6s(\gamma)/(5s(\gamma)-6) < 6$ i.e. $\gamma > 3/2$, we get

$$\int_\Omega \rho_\delta u_\delta^i \mathcal{R}_{ij}(u_\delta^j T_k(\rho_\delta))\eta\, dx \to \int_\Omega \rho u^i \mathcal{R}_{ij}(u^j \overline{T_k(\rho)})\eta\, dx. \tag{6.17}$$

From Lemma 6.2 with $f_\delta = T_k(\rho_\delta)$ and $g_\delta = \rho_\delta u_\delta^i \eta$, we deduce that

$$\begin{gathered} T_k(\rho_\delta)\mathcal{R}_{ij}(\rho_\delta u_\delta^i \eta) - \rho_\delta u_\delta^i \eta \mathcal{R}_{ij}(T_k(\rho_\delta)) \rightharpoonup \overline{T_k(\rho)}\mathcal{R}_{ij}(\rho u^i \eta) - \rho u^i \eta \mathcal{R}_{ij}(\overline{T_k(\rho)}) \\ \text{in} \quad L^r(\mathbb{R}^3),\ 1 < r < \frac{6s(\gamma)}{s(\gamma)+6}. \end{gathered} \tag{6.18}$$

On condition $6s(\gamma)/(s(\gamma)+6) > 6/5$ i.e. $\gamma > 3/2$ and in accordance with (4.10)

$$\begin{aligned} &\int_\Omega u_\delta^j \{T_k(\rho_\delta)\mathcal{R}_{ij}(\rho_\delta u_\delta^i \eta) - \mathcal{R}_{ij}(T_k(\rho_\delta))\rho_\delta u_\delta^i \eta\}\, dx \\ &\qquad \to \int_\Omega u^j \{\overline{T_k(\rho)}\mathcal{R}_{ij}(\rho u^i \eta) - \mathcal{R}_{ij}(\overline{T_k(\rho)})\rho u^i \eta\}\, dx. \end{aligned} \tag{6.19}$$

We are now in position to pass to the limit in the identity (6.8). We get

$$\begin{aligned} &\int_\Omega \{\overline{\rho^\gamma T_k(\rho)} - (2\mu_1 + \mu_2)\overline{T_k(\rho)\operatorname{div} \boldsymbol{u}}\}\eta\, dx \\ &= -\int_\Omega \overline{\rho^\gamma}\mathcal{A}_i(\overline{T_k(\rho)})\partial_i \eta\, dx + (\mu_1+\mu_2)\int_\Omega \mathcal{A}_i(\overline{T_k(\rho)})\partial_i \eta \operatorname{div} \boldsymbol{u}\, dx \\ &\quad + \mu_1 \int_\Omega \partial_j u^i \mathcal{A}_i(\overline{T_k(\rho)})\partial_j \eta\, dx - \mu_1 \int_\Omega u^i \mathcal{R}_{ij}(\overline{T_k(\rho)})\partial_j \eta\, dx \\ &\quad + \mu_1 \int_\Omega u^i \overline{T_k(\rho)}\partial_i \eta\, dx - \int_\Omega \rho u^i u^j \mathcal{A}_i(\overline{T_k(\rho)})\partial_j \eta\, dx \\ &\quad - \int_\Omega (\rho f^i + g^i)\mathcal{A}_i(\overline{T_k(\rho)})\eta\, dx + \int_\Omega u^j \{\overline{T_k(\rho)}\mathcal{R}_{ij}(\rho u^i \eta) \\ &\quad - \mathcal{R}_{ij}(\overline{T_k(\rho)})\rho u^i \eta\}\, dx - \int_\Omega \rho u^i \mathcal{R}_{ij}(u^j \overline{T_k(\rho)})\eta\, dx. \end{aligned} \tag{6.20}$$

In accordance with Remark 2, we use in (4.12) the test function $\boldsymbol{\varphi} = \boldsymbol{\mathcal{A}}(\overline{T_k(\rho)})\eta$ with $\eta \in \mathcal{D}(\Omega)$. After a long but straightforward computation, we arrive at

$$\begin{aligned}
&\int_\Omega \{\overline{\rho^\gamma} - (2\mu_1+\mu_2)\operatorname{div}\boldsymbol{u}\}\overline{T_k(\rho)}\eta\,dx \\
&\quad= -\int_\Omega \overline{\rho^\gamma}\mathcal{A}_i(\overline{T_k(\rho)})\partial_i\eta\,dx + (\mu_1+\mu_2)\int_\Omega \mathcal{A}_i(\overline{T_k(\rho)})\partial_i\eta\operatorname{div}\boldsymbol{u}\,dx \\
&\quad+ \mu_1\int_\Omega \partial_j u^i\mathcal{A}_i(\overline{T_k(\rho)})\partial_j\eta\,dx - \mu_1\int_\Omega u^i\mathcal{R}_{ij}(\overline{T_k(\rho)})\partial_j\eta\,dx \\
&\quad+ \mu_1\int_\Omega u^i\overline{T_k(\rho)}\partial_i\eta\,dx - \int_\Omega \rho u^i u^j\mathcal{A}_i(\overline{T_k(\rho)})\partial_j\eta\,dx \\
&\quad- \int_\Omega (\rho f^i + g^i)\mathcal{A}_i(\overline{T_k(\rho)})\eta\,dx + \int_\Omega u^j\{\overline{T_k(\rho)}\mathcal{R}_{ij}(\rho u^i\eta) \\
&\quad- \mathcal{R}_{ij}(\overline{T_k(\rho)})\rho u^i\eta\}\,dx - \int_\Omega \rho u^i\mathcal{R}_{ij}(u^j\overline{T_k(\rho)})\eta\,dx.
\end{aligned} \tag{6.21}$$

Subtracting (6.20) and (6.21) yields the statement of Lemma 6.3. □

7. Boundedness of oscillations

The main result of this section reads:

Lemma 7.1. *It holds*

$$\sup_{k>0}\limsup_{\delta\to0^+}\|T_k(\rho_\delta) - T_k(\rho)\|_{0,\gamma+1} \le C. \tag{7.1}$$

Proof.

$$\begin{aligned}
\int_\Omega \overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\,\overline{T_k(\rho)}\,dx = \lim_{\delta\to0^+}\int_\Omega (\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) \\
- T_k(\rho))\,dx + \int_\Omega (\overline{\rho^\gamma} - \rho^\gamma)(T_k(\rho) - \overline{T_k(\rho)})\,dx.
\end{aligned}$$

Since $t\mapsto t^\gamma$, $t\ge0$, is convex and $t\mapsto T_k(t)$, $t\ge0$, is concave, the second term at the right hand side is nonnegative. Due to (5.3) and Lemma 6.3, we obtain

$$\begin{aligned}
&\limsup_{\delta\to0^+}\int_\Omega |T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1}\,dx \\
&\qquad\le (2\mu_1+\mu_2)\int_\Omega (\overline{T_k(\rho)\operatorname{div}\boldsymbol{u}} - \overline{T_k(\rho)}\operatorname{div}\boldsymbol{u})\,dx.
\end{aligned} \tag{7.2}$$

In virtue of (4.7), the right side is bounded by

$$\begin{aligned}
&\lim_{\delta\to0^+}\int_\Omega \operatorname{div}\boldsymbol{u}_\delta(T_k(\rho_\delta) - \overline{T_k(\rho)})\,dx \\
&\quad= \limsup_{\delta\to0^+}\int_\Omega \operatorname{div}\boldsymbol{u}_\delta\{(T_k(\rho_\delta) - T_k(\rho)) + (T_k(\rho) - \overline{T_k(\rho)})\}\,dx \\
&\quad\le C\limsup_{\delta\to0^+}\|\operatorname{div}\boldsymbol{u}_\delta\|_{0,2}\|T_k(\rho_\delta) - T_k(\rho)\|_{0,2} \\
&\quad\le C\limsup_{\delta\to0^+}\|T_k(\rho_\delta) - T_k(\rho)\|_{0,\gamma+1}.
\end{aligned}$$

This inequality together with (7.2) yields the result. □

8. Renormalized solutions of the weak limits

Theorem 8.1. *Let b satisfies* (1.8) *through* (1.10) *with* $p = s(\gamma)$. *Then*

$$\operatorname{div}(b(\rho)\boldsymbol{u}) + \{\rho b'(\rho) - b(\rho)\}\operatorname{div}\boldsymbol{u} = 0 \qquad in \quad \mathcal{D}'(\mathbb{R}^3). \tag{8.1}$$

Proof. First, we assume that $b \in C^1([0,+\infty))$. By Corollary 2.1 with $b(t) = t$

$$\operatorname{div}(T_k(\rho_\delta)\boldsymbol{u}_\delta) + \{\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta)\}\operatorname{div}\boldsymbol{u}_\delta = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \ \ \forall k > 0.$$

Since $\{\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta)\}\operatorname{div}\boldsymbol{u}_\delta$ is uniformly bounded in $L^2(\mathbb{R}^3)$ with respect to δ, we get as $\delta \to 0^+$

$$\operatorname{div}(\overline{T_k(\rho)}\boldsymbol{u}) = -\overline{\{\rho T_k'(\rho) - T_k(\rho)\}\operatorname{div}\boldsymbol{u}} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

Applying Corollary 2.1 to the above equation with b_R corresponding to arbitrary $b \in C^1([0,+\infty))$, one obtains

$$\begin{aligned}\operatorname{div}(b_R(\overline{T_k(\rho)})\boldsymbol{u}) &+ \{\overline{T_k(\rho)}b_R'(\overline{T_k(\rho)}) - b_R(\overline{T_k(\rho)})\}\operatorname{div}\boldsymbol{u}\\ &= -\overline{\{\rho T_k'(\rho) - T_k(\rho)\}\operatorname{div}\boldsymbol{u}}\, b_R'(\overline{T_k(\rho)}) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \ \ \forall R > 0, \ \ \forall k > 0.\end{aligned} \tag{8.2}$$

When $k \to +\infty$, (5.5) and the Lebesgue theorem yield the convergence in $\mathcal{D}'(\mathbb{R}^3)$ of the left side of (8.2) to $\operatorname{div}(b_R(\rho)\boldsymbol{u}) + \{\rho b_R'(\rho) - b_R(\rho)\}\operatorname{div}\boldsymbol{u}$. The $L^1(\Omega)$-norm of the right side can be estimated by

$$\max_{s\in[0,R]} |b'(s)| \int_{\Omega_R} |\overline{\{\rho T_k'(\rho) - T_k(\rho)\}\operatorname{div}\boldsymbol{u}}|\, dx$$

where $\Omega_R = \{x \in \Omega,\ \overline{T_k(\rho)}(x) \le R\}$. It holds

$$\|\overline{\{\rho T_k'(\rho) - T_k(\rho)\}\operatorname{div}\boldsymbol{u}}\|_{0,1,\Omega_R} \le \liminf_{\delta\to 0^+} \|\{\rho_\delta T_k'(\rho_\delta) - T_k(\rho_\delta)\}\operatorname{div}\boldsymbol{u}_\delta\|_{0,1,\Omega_R}.$$

By Hölder's inequality and (5.2), the second term of the right hand side of the last inequality is bounded by

$$\begin{aligned}C\|T_k(\rho_\delta)1_{\{\rho_\delta\ge k\}}\|_{0,2,\Omega_R}\|\operatorname{div}\boldsymbol{u}_\delta\|_{0,2,\Omega_R}&\\ \le C\|T_k(\rho_\delta)1_{\{\rho_\delta\ge k\}}\|_{0,1,\Omega_R}^{\frac{\gamma-1}{2\gamma}}&\|T_k(\rho_\delta)1_{\{\rho_\delta\ge k\}}\|_{0,\gamma+1,\Omega_R}^{\frac{\gamma+1}{2\gamma}}\end{aligned}$$

where we have used (4.7) and the interpolation of L^2 between L^1 and $L^{\gamma+1}$. Since $T_k(\rho_\delta) \le \rho_\delta$, it follows from (5.4) with $p = 1$, that

$$\limsup_{\delta\to 0^+} \|T_k(\rho_\delta)1_{\{\rho_\delta>k\}}\|_{0,1,\Omega_R} \to 0$$

as $k \to +\infty$. Further thanks to Lemma 7.1, $\forall k > 0$,

$$\begin{aligned}\limsup_{\delta\to 0^+} &\|T_k(\rho_\delta)1_{\{\rho_\delta > k\}}\|_{0,\gamma+1,\Omega_R}\\ &\leq \limsup_{\delta\to 0^+} \|T_k(\rho_\delta) - \overline{T_k(\rho)}\|_{0,\gamma+1,\Omega} + \|\overline{T_k(\rho)}\|_{0,\gamma+1,\Omega_R}\\ &\leq C + R|\Omega|^{\frac{1}{\gamma+1}}.\end{aligned}$$

Due to these facts $\overline{\{\rho T_k'(\rho) - T_k(\rho)\}\operatorname{div}\boldsymbol{u}} \to 0$ as $k \to +\infty$. We pass to the limit $k \to +\infty$ in (8.2) to obtain

$$\operatorname{div}(b_R(\rho)\boldsymbol{u}) + \{\rho b_R'(\rho) - b_R(\rho)\}\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{8.3}$$

Now, the aim is to send $R \to +\infty$. Due to (1.9), we have

$$\begin{aligned}\|b_R(\rho)1_{\{\rho\geq R\}}\|_{0,\frac{6}{5}} &\leq cR^{(\lambda_1+1)-\frac{5}{6}s(\gamma)}\|\rho\|_{0,s(\gamma)}^{\frac{5s(\gamma)}{6}}\\ \|\{\rho(b_R)'(\rho) - b_R(\rho)\}1_{\{\rho\geq R\}}\|_{0,2} &\leq cR^{\lambda_2-\frac{s(\gamma)}{2}}\|\rho\|_{0,s(\gamma)}^{\frac{s(\gamma)}{2}}.\end{aligned}$$

The term $\int_{\mathbb{R}^3} b_R(\rho)\boldsymbol{u}\cdot\nabla\varphi\,dx$, $\varphi \in \mathcal{D}(\mathbb{R}^3)$, can be written as the sum of $\int_{\mathbb{R}^3} b(\rho)1_{\{\rho\leq R\}}\boldsymbol{u}\cdot\nabla\varphi\,dx$ and $\int_{\mathbb{R}^3} b_R(\rho)1_{\{\rho>R\}}\boldsymbol{u}\cdot\nabla\varphi\,dx$. The first integral converges to $\int_{\mathbb{R}^3} b(\rho)\boldsymbol{u}\cdot\nabla\varphi\,dx$ while the second one tends to zero as $R \to +\infty$ because it is bounded by $cR^{(\lambda_1+1)-(5s(\gamma)/6)}\|\rho\|_{0,s(\gamma)}^{5s(\gamma)/6}\|\boldsymbol{u}\|_{0,6}\|\nabla\varphi\|_{0,\infty}$. The analysis of the convergence of the other term is similar and conclude the proof in the case $b \in C^1([0,+\infty))$.

In the case b satisfying (1.8), one copies word by word the reasoning of the proof of Lemma 2.2 starting from (2.12). □

9. Strong convergence of the density

For $k > 0$, let

$$L_k(t) = \begin{cases} t\ln t & \text{if} \quad t \in [0,k],\\ t\ln k + t - k & \text{if} \quad t > k.\end{cases} \tag{9.1}$$

We observe that L_k satisfies (1.8), (1.9) with $\lambda_1 = 0$ and $\lambda_2 = 1$ and that $tL_k'(t) - L_k(t) = T_k(t)$. Moreover L_k can be written as $L_k(t) = S_k(t) + (lnk+1)t$ with S_k satisfying (1.8) through (1.10). In virtue of (3.11)

$$\operatorname{div}(L_k(\rho_\delta)\boldsymbol{u}_\delta) + T_k(\rho_\delta)\operatorname{div}\boldsymbol{u}_\delta = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{9.2}$$

Integrating this identity over Ω and sending $\delta \to 0^+$, one obtains

$$\int_\Omega \overline{T_k(\rho)\operatorname{div}\boldsymbol{u}}\,dx = 0. \tag{9.3}$$

Thanks to Theorem 8.1, we also have

$$\operatorname{div}(L_k(\rho)\boldsymbol{u}) + T_k(\rho)\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \tag{9.4}$$

therefore

$$\int_\Omega T_k(\rho)\,\mathrm{div}\,\boldsymbol{u}\,dx = 0. \tag{9.5}$$

Inserting (9.3) and (9.5) in (7.2), we get

$$\begin{aligned}\limsup_{\delta\to0^+}\int_\Omega |T_k(\rho_\delta)-T_k(\rho)|^{\gamma+1}\,dx &\le \|T_k(\rho)-\overline{T_k(\rho)}\|_{0,2}\|\,\mathrm{div}\,\boldsymbol{u}\|_{0,2}\\ &\le C\|T_k(\rho)-\overline{T_k(\rho)}\|_{0,1}^{\frac{\gamma-1}{2\gamma}}\|T_k(\rho)-\overline{T_k(\rho)}\|_{0,\gamma+1}^{\frac{\gamma+1}{2\gamma}}\end{aligned}$$

which, in accordance with (5.5), (5.6) and (7.1), yields

$$\lim_{k\to+\infty}\limsup_{\delta\to0^+}\|T_k(\rho_\delta)-T_k(\rho)\|_{0,\gamma+1} = 0. \tag{9.6}$$

As

$$\|\rho_\delta-\rho\|_{0,1}\le\|\rho_\delta-T_k(\rho_\delta)\|_{0,1}+\|T_k(\rho_\delta)-T_k(\rho)\|_{0,1}+\|T_k(\rho)-\rho\|_{0,1},$$

(5.4), (5.6) and (9.6) imply strong convergence of a subsequence of $\{\rho_\delta\}_{\delta\in(0,1)}$ in $L^1(\Omega)$ and as a consequence in $L^p(\Omega)$, $\forall 1\le p<s(\gamma)$. Therefore $\overline{\rho^\gamma}=\rho^\gamma$. Theorem 1.1 is thus proved. □

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