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A certain class of distribution-valued additive functionals II —for the case of stable process

By

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1. Introduction

This paper is a sequel to [7].

Let B_s be a d-dimensional Brownian motion. In the previous paper [7], we gave a significance to the intuitive expression

$$A_T(a : t, \omega) = \int_0^t T(B_s - a)ds$$

for the certain distribution T and studied joint continuity on a and t and the energy of $A_T(a : t, \omega)$.

In this paper, we consider the property of $A_T(a : t, \omega)$ for one-dimensional stable process with index α or d-dimensional symmetric stable process with index α. Since we can prove these results in the similar way to the case of Brownian motion, we will omit the detail of the proof. For further details, refer to [7].

Furthermore we study some representation theorems. We get a unified method for the proof of representation theorems of occupation time formula including the special case of $T = v.p.(1/x)$ by M. Yor ([16]) and T. Yamada ([19]).

Our method is very simple. It is principally based on the Fourier transform theory in distribution sense. The concrete estimate of the characteristic function of stable process with index α plays an essential role in the proof of our main result.

The present paper is organized as follows. In Section 2, we define distribution valued additive functionals and prepare some notations.

In Section 3, we discuss the existence, (a, t)-joint continuity and the energy of $A_T(a : t, \omega)$ in the sense of M. Fukushima ([4]) for stable process with index α.

In Section 4, we discuss the representation theorems.

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2. Definitions and preliminary results

Throughout this paper, we shall use the same notations as those in the previous paper [7]. But we notice some notations.

We denote that q is Hölder conjugate of p.

We denote the Fourier transform of $\phi(a)$ by $\hat{\phi}(\lambda)$ and the Fourier inverse transform of $\psi(\lambda)$ by $\mathcal{F}^{-1}(\psi)(a)$:

$$\mathcal{F}^{-1}(\psi)(a) = \frac{1}{(2\pi)^d} \int \psi(\lambda) e^{-i\lambda \cdot a} d\lambda,$$

where $x \cdot y$ ($x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$) denotes the inner product.

Let $T \in \mathcal{S}'$. We denote the Fourier transform of T by $\hat{T}$.

Definition 2.1. We say that T is an element of H_p^β ($1 \le p \le \infty$, $-\infty < \beta < \infty$) if and only if T is an element of $\mathcal{S}'$ and the Fourier transform of T has a version as a function $\hat{T}(\lambda)$ on $\mathbb{R}^d$ such that

$$\hat{T}(\lambda)(1 + |\lambda|^2)^{\frac{\beta}{2}} \in L^p.$$

Then we set

$$\|T\|_{H_p^\beta} = \|\hat{T}(\lambda)(1 + |\lambda|^2)^{\frac{\beta}{2}}\|_{L^p}.$$

We note $\mathcal{F}^{-1}(T)(\lambda) = (2\pi)^{-d}\hat{T}(-\lambda)$ for $T \in H_p^\beta$.

Let (X_s) be the standard Brownian motion on $\mathbb{R}^d$ or one-dimensional real valued stable process with index α ($0 < \alpha < 2$) or d-dimensional real valued symmetric stable process with index α ($0 < \alpha < 2$).

We define τ_x and θ_t as following:

$$\tau_x : X_t(\tau_x \omega) = X_t(\omega) + x$$

and

$$\theta_t : X_s(\theta_t \omega) = X_{t+s}(\omega).$$

We remember preliminary results in [7].

Lemma 2.2. *Let $T \in \mathcal{D}'$, $\phi \in \mathcal{D}$ and set $T * \phi(x) = \langle T_y, \phi(x - y)\rangle_y$. Then*

$$\langle A_T(t, \omega), \phi\rangle = \int_0^t T * \phi(X_s(\omega)) ds$$

is well-defined and we have

$$A_T(t, \omega) \in \mathcal{D}'.$$

Lemma 2.3.

$$\langle A_T(t,\tau_x\omega),\phi\rangle = \langle A_T(t,\omega),\phi(\cdot + x)\rangle, \tag{2.1}$$

$$\langle A_T(s+t,\omega),\phi\rangle = \langle A_T(s,\omega),\phi\rangle + \langle A_T(t,\theta_s\omega),\phi\rangle. \tag{2.2}$$

Lemma 2.4. *Let T be an element of H_p^β. Then $A_T(t,\omega)$ is also an element of H_p^β.*

We remember the important lemma in [7]. In fact, using this lemma, we will prove the boundedness of certain integrals.

Lemma 2.5. *We set*

$$J = \int_{\mathbb{R}^d} \frac{d\mu}{(1+|\mu|^2)^p(1+|\mu+\lambda|^2)^q}.$$

Let $2p+2q>d$ and $p \geq q > 0$.

(1) *If $2p<d$ and $2q<d$, then*

$$J \asymp \frac{1}{(1+|\lambda|^2)^{p+q-\frac{d}{2}}}. \tag{2.3}$$

(2) *If $2p=d$, then*

$$J \asymp \frac{1+\log^+|\lambda|}{(1+|\lambda|^2)^q}, \tag{2.4}$$

where $\log^+|x| = \max(\log|x|,0)$.

(3) *If $2p>d$, then*

$$J \asymp \frac{1}{(1+|\lambda|^2)^q}. \tag{2.5}$$

Here we denote that "$f \asymp g$" means $k \leq f/g \leq K$ for some positive constants k and K, where $f,g \not\equiv 0$.

Now let ρ_ϵ be the molifier. We denote

$$A_T^\epsilon(t,\omega) = \langle A_T(t,\omega),\rho_\epsilon\rangle$$

and

$$A_T^\epsilon(a:t,\omega) = A_T^\epsilon(t,\tau_{-a}\omega).$$

We note that

$$\langle A_T^\epsilon(t,\omega),\phi\rangle = \langle A_T(t,\omega),\rho_\epsilon * \phi\rangle.$$

Here we emphasize $A_T^\epsilon(a:t,\omega)$ is a usual function of a. We can take ρ_ϵ such that $\rho_\epsilon \to \delta_0$ as $\epsilon \to 0$ and $\hat\rho_\epsilon$ uniformly converges to one in wider sense tending ϵ to zero and $\|\hat\rho_\epsilon\|_\infty \leq 1$.

Thus we will study the existence and the continuity of the limit $A_T(a:t,\omega)$ of $A_T^\epsilon(a:t,\omega)$ as ϵ to zero.

3. The case of 1-dimensional stable process with index α

3.1. Convergence and continuity theorems

Let P_x be the probability measure of the one-dimensional stable process $\{X_s\}$ with index $\alpha(0<\alpha<2)$ starting from x and we denote the transition probability density by $p(t,y)$. We notice that the characteristic function of X_s is

$$E_x[e^{i\lambda X_s}] = \exp\{-s\psi(\lambda) + i\lambda x\},$$

where $\psi(\lambda)$ is given in the following. For some constants $c>0$, $-1\le\gamma\le 1$ and $\gamma_0\in\mathbb{R}$, if $\alpha\neq 1$ then

$$\psi(\lambda) = c|\lambda|^\alpha\left(1 - i\gamma(\operatorname{sgn}\lambda)\tan\frac{\pi}{2}\alpha\right) + i\gamma_0\lambda$$

and if $\alpha = 1$ then

$$\psi(\lambda) = c|\lambda|\left(1 + i\gamma\frac{2}{\pi}(\operatorname{sgn}\lambda)\log|\lambda|\right) + i\gamma_0\lambda.$$

We prepare the following lemma to discuss the existence and the continuity of $A_T(a:t,\omega)$, which is the limit of $A_T^\epsilon(a:t,\omega)$ as ϵ goes to zero.

Lemma 3.1. *Let $F = |\int_0^t e^{-\psi(\lambda)s}ds|$. Then we get*

$$F \le \frac{C}{(1+|\lambda|^2)^{\frac{\eta}{2}}}, \tag{3.1}$$

where we take $\eta=\alpha$ but if $\alpha<1$ and $\gamma_0\neq 0$ then we take $\eta = 1$.

Proof. We can obtain this lemma by the following evaluations of $\psi(\lambda)$.
(1) $\alpha>1$ or $\alpha<1$ and $\gamma_0=0$, then

$$|\psi(\lambda)| \asymp |\lambda|^\alpha.$$

(2) $\alpha<1$ and $\gamma_0\neq 0$, then

$$|\psi(\lambda)| \asymp |\lambda|.$$

(3) $\alpha = 1$

$$\begin{aligned} |\psi(\lambda)| &\ge C|\lambda||\log|\lambda|| \\ &\ge C|\lambda| \qquad \text{if} \quad \lambda \text{ is large.} \end{aligned}$$

□

We get the following in the similar way to the case of Brownian motion ([7]).

First, we prove the convergence theorem.

Theorem 3.2. *We suppose that $1 < p \leq \infty$ and q satisfy $1/p+1/q = 1$. Suppose that $\beta > (1-q)/2q$ in the case where $\gamma_0 \neq 0$ and $\alpha < 1$ and that $\beta > (1-\alpha q)/q$ in the case where $\alpha q < 1$ and that $\beta > (1-\alpha q)/2q$ in the case where $\alpha q \geq 1$.*

For $T \in H_p^\beta$,

$$\lim_{\epsilon \to 0} A_T^\epsilon(a : t, \omega) = A_T(a : t, \omega) \qquad in \quad L^2(dP_x).$$

Outline of Proof. We set

$$\Gamma_N = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq N, |\lambda_2| \leq N\} \qquad \text{for any} \quad N > 0.$$

Without loss of generality, we can assume that the stable process starts from zero.

$$\begin{aligned}
|I| &= |E_0[(A_T^\epsilon(a : t, \omega))^2]| \\
&= \frac{2}{(2\pi)^2} \left| \int d\lambda_1 \int d\lambda_2 \overline{\hat{T}(\lambda_1)\hat{\rho}_\epsilon(\lambda_1)\hat{T}(\lambda_2)\hat{\rho}_\epsilon(\lambda_2)} e^{-i(\lambda_1+\lambda_2)a} \right. \\
&\quad \times \left. \int_0^t ds \int_s^t du e^{-\psi(\lambda_1+\lambda_2)s - \psi(\lambda_2)(u-s)} \right| \\
&\leq \frac{2}{(2\pi)^2} \left(\sup_{|\lambda| \leq N} |\hat{\rho}_\epsilon(\lambda)| \right)^2 t^2 \int\int_{\Gamma_N} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)| \\
&\quad + \frac{2}{(2\pi)^2} (\|\hat{\rho}_\epsilon\|_\infty)^2 \int\int_{\Gamma_N^c} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)| \\
&\quad \times \left| \int_0^t ds \int_0^{t-s} du e^{-\psi(\lambda_1+\lambda_2)s - \psi(\lambda_2)u} \right| \\
&= \frac{2}{(2\pi)^2} \left(\sup_{|\lambda| \leq N} |\hat{\rho}_\epsilon(\lambda)| \right)^2 t^2 I_1 + \frac{2}{(2\pi)^2} (\|\hat{\rho}_\epsilon\|_\infty)^2 I_2(\Gamma_N^c), \quad \text{say.}
\end{aligned}$$

For the proof of this theorem, we show that $I_2(\Gamma_0^c) = I_2$ is finite. By Hölder's inequality we get

$$I_2 \le (\|T\|_{H_p^\beta})^2 \left(\int d\lambda_1 \int d\lambda_2 (1+|\lambda_1|^2)^{-\frac{q\beta}{2}} (1+|\lambda_2|^2)^{-\frac{q\beta}{2}} \right. \tag{3.2}$$

$$\left. \times \left| \int_0^t ds \int_0^{t-s} du e^{-\psi(\lambda_1+\lambda_2)s - \psi(\lambda_2)u} \right|^q \right)^{\frac{1}{q}}$$

$$\le (\|T\|_{H_p^\beta})^2 \left(\int d\mu_1 \int d\mu_2 (1+|\mu_1-\mu_2|^2)^{-\frac{q\beta}{2}} (1+|\mu_2|^2)^{-\frac{q\beta}{2}} \right. \tag{3.3}$$

$$\left. \times \left| \int_0^t ds \int_0^{t-s} du e^{-\psi(\mu_1)s - \psi(\mu_2)u} \right|^q \right)^{\frac{1}{q}}$$

By (3.1) we get

$$I_2 \le C^2 (\|T\|_{H_p^\beta})^2 \left(\int d\mu_1 \int d\mu_2 (1+|\mu_1-\mu_2|^2)^{-\frac{q\beta}{2}} (1+|\mu_2|^2)^{-\frac{q\beta}{2}-\frac{q\eta}{2}} \right.$$

$$\left. \times (1+|\mu_1|^2)^{-\frac{q\eta}{2}} \right)^{\frac{1}{q}},$$

Then we apply (2.5) for the finiteness of this integral. We obtain sufficient condition:

$$q\beta + q\beta + \eta q > 1,$$
$$\eta q + q\beta > 1$$

and

$$\frac{q\beta}{2} + \frac{q\eta}{2} > \frac{q\beta}{2}.$$

Thus if β satisfies

$$\beta > \frac{1-\eta q}{2q} \qquad \text{for the case where} \quad \eta q \ge 1$$

and

$$\beta > \frac{1-\eta q}{q} \qquad \text{for the case where} \quad \eta q < 1,$$

then we can easily see that $\{A_T^\epsilon(a : t, \omega)\}$ is a Cauchy sequence in $L^2(dP_x)$ when ϵ goes to zero. □

If $p = 2$ then we can improve Theorem 3.2 as follows:

Theorem 3.3. *Suppose that $\beta > -1/2$ in the case where $\gamma_0 \ne 0$ and $\alpha < 1$ and that $\beta \ge -\alpha/2$ in the case where $\alpha > 1$ and that $\beta > 1/2 - \alpha$ in the case where $\alpha \le 1$.*

For $T \in H_2^\beta$,

$$\lim_{\epsilon \to 0} A_T^\epsilon(a : t, \omega) = A_T(a : t, \omega) \qquad in \quad L^2(dP_x).$$

Outline of Proof. We set

$$\Lambda_1 = \left\{(\lambda_1, \lambda_2) : \frac{|\lambda_1|}{2} \le |\lambda_1 + \lambda_2|\right\}$$

and

$$\Lambda_2 = \left\{(\lambda_1, \lambda_2) : \frac{|\lambda_1|}{2} \ge |\lambda_1 + \lambda_2|\right\}.$$

For the proof, it is sufficient to show that the following integral I_2 is finite.

$$I_2 = \int d\lambda_1 \int d\lambda_2 |\hat{T}(\lambda_1)||\hat{T}(\lambda_2)| \left| \int_0^t ds \int_0^{t-s} du e^{-\psi(\lambda_1+\lambda_2)s - \psi(\lambda_2)u} \right|.$$

Then by (3.1) we get

$$\begin{aligned} I_2 &\le C^2 \int\int_{\Lambda_1} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)||\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}(1+|\lambda_2|^2)^{-\frac{\eta}{2}} \\ &\quad + C^2 \int\int_{\Lambda_2} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)||\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}(1+|\lambda_2|^2)^{-\frac{\eta}{2}} \\ &= J_{\Lambda_1} + J_{\Lambda_2}, \quad \text{say.} \end{aligned}$$

First, we estimate J_{Λ_1}. By the definition of Λ_1, we immediately have

$$J_{\Lambda_1} \le C_1(\|T\|_{H_2^\beta})^2 \int (1+|\lambda|^2)^{-\eta-\beta} d\lambda.$$

Second, we estimate J_{Λ_2}.

$$\begin{aligned} J_{\Lambda_2} &= C^2 \int\int_{\Lambda_2} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)|(1+|\lambda_1|^2)^{\frac{\beta}{2}} |\hat{T}(\lambda_2)|(1+|\lambda_2|^2)^{\frac{\beta}{2}} \\ &\qquad \times (1+|\lambda_1|^2)^{-\frac{\beta}{2}}(1+|\lambda_2|^2)^{-\frac{\beta}{2}}(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}(1+|\lambda_2|^2)^{-\frac{\eta}{2}} \\ &\le C_2(\|T\|_{H_2^\beta})^2 \int d\mu (1+|\mu|^2)^{-\eta-\beta}. \end{aligned}$$

Thus, for the finiteness of I_2, we have

$$\beta > \frac{1-2\eta}{2} \quad \text{and} \quad \beta \ge -\frac{\eta}{2}.$$

□

If $p = 1$ we have the following theorem.

Theorem 3.4. *For $T \in H_1^\beta$,*

$$\lim_{\epsilon \to 0} A_T^\epsilon(a : t, \omega) = A_T(a : t, \omega) \qquad \text{in} \quad L^2(dP_x),$$

where we take $\beta \ge -\alpha/2$ but if $\alpha < 1$ and $\gamma_0 \ne 0$ we take $\beta \ge -1/2$.

Outline of Proof. To prove this result, it is sufficient to show (3.2) is finite. By Hölder's inequality and (3.1) we have

$$I_2 \leq C^2(\|T\|_{H_1^\beta})^2\|(1+|\lambda_1|^2)^{-\frac{\beta}{2}}(1+|\lambda_2|^2)^{-\frac{\beta}{2}-\frac{\eta}{2}}(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}\|_\infty,$$

If $\beta \geq 0$, then clearly $I_2 < \infty$. We consider the case $\beta < 0$. We set

$$L = (1+|\lambda_1|^2)^{-\frac{\beta}{2}}(1+|\lambda_2|^2)^{-\frac{\eta}{2}-\frac{\beta}{2}}(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}},$$

$$\Lambda_1 = \left\{(\lambda_1,\lambda_2) : |\lambda_1| \leq \frac{|\lambda_2|}{2}\right\},$$
$$\Lambda_2 = \left\{(\lambda_1,\lambda_2) : |\lambda_1+\lambda_2| \leq \frac{|\lambda_2|}{2}\right\},$$
$$\Lambda_3 = \{(\lambda_1,\lambda_2) : |\lambda_1| \leq 2|\lambda_2|\} - \Lambda_1 - \Lambda_2,$$

and

$$\Lambda_4 = (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)^c.$$

We will consider each case. First, we consider the case such that (λ_1,λ_2) belongs to Λ_4. We have

$$L \asymp (1+|\lambda_1|^2)^{-\frac{\beta}{2}-\frac{\eta}{2}}(1+|\lambda_2|^2)^{-\frac{\eta}{2}-\frac{\beta}{2}}. \tag{3.4}$$

Second, we consider the case such that (λ_1,λ_2) belongs to Λ_3. We have

$$L \asymp (1+|\lambda_1|^2)^{-\beta-\eta}. \tag{3.5}$$

Third, we consider the case such that (λ_1,λ_2) belongs to Λ_2. We have

$$L \asymp (1+|\lambda_1|^2)^{-\beta-\frac{\eta}{2}}(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}. \tag{3.6}$$

Last, we consider the case such that (λ_1,λ_2) belongs to Λ_1. We have

$$L \leq C_\infty(1+|\lambda_2|^2)^{-\beta-\frac{\eta}{2}}(1+|\lambda_1+\lambda_2|^2)^{-\frac{\eta}{2}}, \tag{3.7}$$

for some positive constant C_∞.

Therefore using from (3.4) to (3.7), for the finiteness of I_2 we take $\beta \geq -\eta/2$. □

Next we discuss the (a,t)-joint continuity of $A_T(a : t,\omega)$.

Theorem 3.5. *Let $T \in H_p^\beta$ $(1 < p \leq \infty)$, where we take β as Theorem 3.2 and q satisfy $1/p + 1/q = 1$.*

Suppose that

(1) (*in the case where $\alpha > 1$*)

$$\delta = \min\left(1, \frac{2q\beta - 1 + \alpha q}{2q}\right) \qquad \textit{if} \quad 1 \leq \alpha q.$$

(2) (*in the case where* $\alpha \le 1$)

$$\delta = \min\left(\alpha, \frac{q\beta - 1 + \alpha q}{q}\right) \qquad \textit{if} \quad 1 > \alpha q,$$
$$\delta = \min\left(1, \frac{2q\beta - 1 + \alpha q}{2q}\right) \qquad \textit{if} \quad 1 \le \alpha q.$$

(3) (*in the case where* $\alpha < 1$ *and* $\gamma_0 \neq 0$)

$$\delta = \min\left(1, \frac{2q\beta - 1 + q}{2q}\right).$$

Then $A_T(a : t, \omega)$ *has* (a, t)*-jointly continuous modification, which is locally Hölder-continuous with exponent* γ*, where* $0 < \gamma < \delta$*.*

Outline of Proof. Without loss of generality, we suppose that $t > s$ and the stable process starts from zero and $b = 0$.

We set

$$\begin{aligned} &E_0[(A_T^\epsilon(a : t, \omega) - A_T^\epsilon(0 : s, \omega))^{2n}] \\ &\quad \le 2^{2n}|E_0[(A_T^\epsilon(a : t, \omega) - (A_T^\epsilon(0 : t, \omega))^{2n}]| \\ &\qquad + 2^{2n}|E_0[(A_T^\epsilon(0 : t, \omega) - (A_T^\epsilon(0 : s, \omega))^{2n}]| \\ &\quad = 2^{2n}|I_a| + 2^{2n}|I_t|. \end{aligned}$$

First we estimate I_a. By the similar calculation of the case of Brownian motion ([7]) we obtain

$$\begin{aligned} |I_a| \le & \frac{(2n)!}{(2\pi)^{2n}} (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\epsilon\|_\infty)^{2n} \\ & \times \left(\int d\lambda_1 \cdots \int d\lambda_{2n} (1 + |\lambda_1|^2)^{-\frac{q\beta}{2}} \cdots (1 + |\lambda_{2n}|^2)^{-\frac{q\beta}{2}} \right. \\ & \times \left| \int_0^t du_1 \int_{u_1}^t du_2 \cdots \int_{u_{2n-1}}^t du_{2n} \right. \\ & \times e^{-\psi(\lambda_{2n})(u_{2n} - u_{2n-1}) - \psi(\lambda_{2n} + \lambda_{2n-1})(u_{2n-1} - u_{2n-2}) - \cdots - \psi(\lambda_{2n} + \cdots + \lambda_1) u_1} \Bigg|^q \\ & \times |e^{-i\lambda_{2n} a} - 1|^q |e^{-i(\lambda_{2n} + \lambda_{2n-1}) a} - 1|^q \cdots |e^{-i(\lambda_{2n} + \cdots + \lambda_1) a} - 1|^q \Bigg)^{\frac{1}{q}}. \end{aligned}$$

By the change of variables we have

$$
\begin{aligned}
|I_a| \le& \frac{(2n)!}{(2\pi)^{2n}} (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\epsilon\|_\infty)^{2n} \\
&\times \Bigg(\int d\mu_1 \cdots \int d\mu_{2n} (1+|\mu_1-\mu_2|^2)^{-\frac{q\beta}{2}} \cdots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{q\beta}{2}} \\
&\times (1+|\mu_{2n}|^2)^{-\frac{q\beta}{2}} \\
&\times \Bigg| \int_0^t du_1 \int_{u_1}^t du_2 \cdots \int_{u_{2n-1}}^t du_{2n} \\
&\times e^{-\psi(\mu_{2n})(u_{2n}-u_{2n-1})-\cdots-\psi(\mu_2)(u_2-u_1)-\psi(\mu_1)u_1} \Bigg|^q \\
&\times |e^{-i\mu_{2n}a}-1|^q |e^{-i(\mu_{2n-1}-\mu_{2n})a}-1|^q \cdots |e^{-i(\mu_1-\mu_2)a}-1|^q \Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Now we notice that for any $1 \ge l_a > 0$

$$
|e^{-i\mu\cdot a}-1| \le K_a|a|^{l_a}(1+|\mu|^2)^{l_a/2} \qquad \text{for some positive constant } K_a > 0.
$$

Then we apply this inequality and (3.1) to I_a:

$$
\begin{aligned}
|I_a| \le& C_a (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\epsilon\|_\infty)^{2n} |a|^{2nl_a} \\
&\times \Bigg(\int d\mu_1 \cdots \int d\mu_{2n} \\
&\times (1+|\mu_1-\mu_2|^2)^{-\frac{q\beta}{2}+\frac{ql_a}{2}} \cdots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{q\beta}{2}+\frac{ql_a}{2}} \\
&\times (1+|\mu_1|^2)^{-\frac{q}{2}\eta} \cdots (1+|\mu_{2n-1}|^2)^{-\frac{q}{2}\eta} (1+|\mu_{2n}|^2)^{-\frac{q}{2}(\eta-l_a+\beta)} \Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Now we apply (2.5) to the integral with respect to $d\mu_1 \ldots d\mu_{2n}$ of the above inequality. Then for the finiteness of I_a, we have

$$
\begin{aligned}
q(\eta - l_a + \beta) + q(\beta - l_a) &> 1, \\
q(\eta - l_a + \beta) &> 1.
\end{aligned}
$$

Thus we get

$$
\beta > \max\left(\frac{1-\eta q + ql_a}{q}, \frac{1-\eta q + 2ql_a}{2q} \right), \tag{3.8}
$$

$$
|I_a| \le C'_a |a|^{2nl_a} (\|T\|_{H_p^\beta})^{2n} \|\hat{\rho}_\epsilon\|_\infty^{2n}, \tag{3.9}
$$

where C'_a is a positive constant and only depends on n.

Next we estimate I_t in a similar way of I_a. But we notice that for any $l_t > 0$ and fixed $t > 0$, there exists a positive constant K_t such that

$$
\left| \int_0^s e^{-\psi(\mu)u} du \right| \le K_t \left(\frac{s^{l_t}}{(1+|\mu|^2)^{\frac{\eta}{2}}} \right)^{\frac{1}{l_t+1}} \qquad \text{for} \quad s \in [0,t].
$$

Then we have

$$
\begin{aligned}
|I_t| \le\ & C_t|t-s|^{2n\frac{l_t}{l_t+1}}(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\epsilon\|_\infty)^{2n} \\
& \times \Bigg(\int d\mu_1 \cdots \int d\mu_{2n}(1+|\mu_1-\mu_2|^2)^{-\frac{q\beta}{2}} \cdots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{q\beta}{2}} \\
& \times (1+|\mu_1|^2)^{-\frac{\eta q}{2(l_t+1)}} \cdots (1+|\mu_{2n-1}|^2)^{-\frac{\eta q}{2(l_t+1)}}(1+|\mu_{2n}|^2)^{-\frac{q\beta}{2}-\frac{\eta q}{2(l_t+1)}}\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

We apply (2.5) to the integral with respect to $d\mu_1 \dots d\mu_{2n}$ of the above inequality. Then we have

$$
\beta > \max\left(\frac{1-\frac{\eta q}{l_t+1}}{q}, \frac{1-\frac{\eta q}{l_t+1}}{2q}\right) \tag{3.10}
$$

and

$$
|I_t| \le C_t'|t-s|^{2n\frac{l_t}{l_t+1}}(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\epsilon\|_\infty)^{2n}, \tag{3.11}
$$

where C_t' is a positive constant and only depends on n and t.

Therefore by (3.8) and (3.10) we make l_a and l_t satisfy the following equalities:

$$
\begin{aligned}
1-\frac{\eta q}{l_t+1} &= 1-\eta q+2ql_a, \\
1-\frac{\eta q}{l_t+1} &= 1-\eta q+ql_a.
\end{aligned}
$$

That is, $l_t = 2l_a/(\eta-2l_a)$ and $l_t = l_a/(\eta-l_a)$. Since l_a is positive, β satisfies the condition in Theorem 3.2 and then we get

$$
\begin{aligned}
&|E_0[(A_T^\epsilon(a:t,\omega)-A_T^\epsilon(0:s,\omega)^{2n}]| \\
&\qquad \le C_{st}(|a|^{2n\delta}+|t-s|^{2n\delta})(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\epsilon\|_\infty)^{2n},
\end{aligned} \tag{3.12}
$$

where we denote l_a by δ and $C_{st} = \max(C_a', C_t')$.

Therefore we get the condition in the theorem.

Then tending ϵ to zero, we get (a,t)-jointly continuity of $A_T(a:t,\omega)$ by Kolmogorov-Čentsov theorem. □

But we cannot still get the result corresponding to Theorems 3.3 and 3.4.

By Theorem 3.5, we can take the (a,t)-jointly continuous modification of $A_T(a:t,\omega)$.

Now we discuss the existence and (a,t)-jointly continuity of $A_T(a:t,\omega)$ in the case of $p=\infty$ and $p=2$.

Example 3.6. Let $T=\delta_0$. Then T belongs to $H_\infty^0 \cap H_2^{-1/2-\epsilon}$, where $\epsilon>0$. $A_T(a:t,\omega)$ is the local time. $A_T(a:t,\omega)$ has (a,t)-jointly continuous modification which is locally Hölder continuous with exponent $0 < \gamma < (\alpha - 1)/2$, where $\alpha > 1$ applying the fact of $T = \delta_0 \in H^0_\infty$ and exponent $0 < \gamma < (2\alpha - 3)/4$, where $\alpha > 3/2$ applying the fact of $T = \delta_0 \in H_2^{-1/2-\epsilon}$, where $\epsilon > 0$.

Therefore we conclude that the local time for stable process with index $\alpha > 1$ exists, which agrees to the result in E. S. Boylan ([2]) and it has (a, t)-jointly continuous modification which is locally Hölder continuous with exponent $(\alpha - 1)/2 - \epsilon$.

Example 3.7. Let $T = v.p.(1/x)$. Then T also belongs to $H^0_\infty \cap H_2^{-1/2-\epsilon}$, where $\epsilon > 0$. Thus $A_T(a : t, \omega)$ has (a, t)-jointly continuous modification which has the same exponent in the case of $T = \delta_0$.

3.2. The energy of $A_T(a : t, \omega)$

In this section we will discuss the energy of $A_T(a : t, \omega)$. First we define the energy of additive functionals in M. Fukushima, Y. Oshima and M. Takeda ([4]).

Definition 3.8. For any additive functional $A_T(a : t, \omega)$, we set

$$e(A_T) = \lim_{t \downarrow 0} \frac{1}{2t} E_m[(A_T(a : t, \omega))^2]$$

whenever the limit exits. We call $e(A_T)$ the energy of $A_T(a : t, \omega)$.

For the stable processes, we take $m = dx$.

First, we show that the convergence of $A^\epsilon_T(a : t, \omega)$ in $L^2(dP_x \times dx)$.

Theorem 3.9. *We suppose that $2 < p \le \infty$ and q satisfy $1/p + 1/q = 1$. For $T \in H^\beta_p$,*

$$\lim_{\epsilon \to 0} A^\epsilon_T(a : t, \omega) = A_T(a : t, \omega) \qquad in \quad L^2(dP_x \times dx),$$

where we take $\beta > (1 - \alpha q)/2q$ but if $\gamma_0 \ne 0$ then for $\alpha < 1$ we take $\beta > (1 - q)/2q$.

Proof. We proceed on the similar way of Brownian motion ([7]).

$$\begin{aligned} I &= E_{dx}[(A^\epsilon_T(a : t, \omega))^2] \\ &= \int dx E_0 \left[2 \int_0^t ds \int_s^t du T * \rho_\epsilon(X_s - a - x) T * \rho_\epsilon(X_u - a - x) \right] \\ &\le 2(2\pi)^{-1} t \left(t \sup_{|\lambda| \le N} |\hat{\rho}_\epsilon(\lambda)|^2 \int_{|\lambda| \le N} d\lambda |\hat{T}(\lambda)|^2 \right. \\ &\quad \left. + C \|\hat{\rho}_\epsilon\|^2_\infty \int_{|\lambda| > N} d\lambda |\hat{T}(\lambda)|^2 \frac{1}{(1 + |\lambda|^2)^{\frac{\eta}{2}}} \right) \qquad (3.13) \\ &= 2(2\pi)^{-1} t \left(t \sup_{|\lambda| \le N} |\hat{\rho}_\epsilon(\lambda)|^2 I_1 + C \|\hat{\rho}_\epsilon\|^2_\infty I_2(|\lambda| > N) \right), \quad \text{say.} \end{aligned}$$

By Hölder's inequality we get

$$I_1 \le (\|T\|_{H_p^\beta})^2 \left(\int_{|\lambda| \le N} d\lambda (1+|\lambda|^2)^{-q\beta} \right)^{\frac{1}{q}}.$$

Therefore I_1 is finite for any β.

Now we estimate $I_2(|\lambda| \ge N)$. First we consider $I_2(|\lambda| \ge 0) = I_2$. By Hölder's inequality we get

$$I_2 \le (\|T\|_{H_p^\beta})^2 \left(\int d\lambda (1+|\lambda|^2)^{-q\beta - \frac{\eta q}{2}} \right)^{\frac{1}{q}}.$$

For the finiteness of this integral we get $\eta q + 2q\beta > 1$.

Thus if β satisfies $\beta > (1-\eta q)/2q$, then we can easily see that tending ϵ to zero, $\{A_T^\epsilon(a:t,\omega)\}$ is Cauchy's sequence in $L^2(dP_x \times dx)$ and $A_T^\epsilon(a:t,\omega)$ converges $A_T(a:t,\omega)$ in $L^2(dP_x \times dx)$. □

If $p=2$, then we can easily obtain

$$I_2 \le \|\hat{\rho}_\epsilon\|_\infty^2 (\|T\|_{H_2^\beta})^2 \|(1+|\lambda|^2)^{-\frac{\eta}{2}-\beta}\|_\infty.$$

Thus we have

Corollary 3.10. *For $T \in H_2^\beta$,*

$$\lim_{\epsilon \to 0} A_T^\epsilon(a:t,\omega) = A_T(a:t,\omega) \qquad \text{in} \quad L^2(dP_x \times dx),$$

where we take $\beta \ge -\alpha/2$ but if $\gamma_0 \ne 0$ then for $\alpha < 1$ we take $\beta \ge -1/2$.

These results guarantee the existence $A_T(a:t,\omega)$ for $T \in H_p^\beta$ wider than Theorems 3.2 and 3.3. Then we denote by $A_T^{dx}(a:t,\omega)$ in this sense.

Now we show that $A_T^{dx}(a:t,\omega)$ has 0-energy.

Theorem 3.11. *For $T \in H_p^\beta$, $e(A_T^{dx}) = 0$, where we take β satisfying the condition in Theorem* 3.9 *or Corollary* 3.10.

Proof. By (3.13) we know

$$\begin{aligned} &|E_{dx}[A_T^\epsilon(a:t,\omega)^2]| \\ &\le 2(2\pi)^{-1} t \left(t \sup_{|\lambda| \le N} |\hat{\rho}_\epsilon(\lambda)|^2 \int_{|\lambda| \le N} d\lambda |\hat{T}(\lambda)|^2 \right. \\ &\qquad \left. + C\|\hat{\rho}_\epsilon\|_\infty^2 \int_{|\lambda|>N} d\lambda \frac{|\hat{T}(\lambda)|^2}{(1+|\lambda|^2)^{\frac{\eta}{2}}} \right). \end{aligned}$$

Since $\|\hat{\rho}_\epsilon\|_\infty^2 \le 1$, letting ϵ to zero, we get

$$\begin{aligned} &|E_{dx}[A_T^\epsilon(a:t,\omega)^2]| \\ &\le 2(2\pi)^{-1} t \left(t \int_{|\lambda| \le N} d\lambda |\hat{T}(\lambda)|^2 + C \int_{|\lambda|>N} d\lambda \frac{|\hat{T}(\lambda)|^2}{(1+|\lambda|^2)^{\frac{\eta}{2}}} \right). \end{aligned}$$

Thus we get

$$e(A_T^{dx}) = 0.$$

□

3.3. The case of d-dimensional symmetric stable process with index α

We can apply the above method the d-dimensional symmetric stable process.

Let $\{X_s\}$ be the d-dimensional symmetric stable process with index α. That is,

$$E_x[e^{i\lambda\cdot X_s}] = \exp\{-c|\lambda|^\alpha s + i\lambda\cdot x\},$$

where c is positive constant and $x\cdot y(x\in\mathbb{R},\, y\in\mathbb{R})$ denotes the inner product.

Noting

$$\int_0^t e^{-c|\lambda|^\alpha s} ds \le \frac{C}{(1+|\lambda|^2)^{\frac{\alpha}{2}}}, \tag{3.14}$$

we get the following Theorems.

Theorem 3.12. *We suppose that* $1\le p\le\infty$.
Suppose that
(1) $p>1$

$$\beta > \frac{d-\alpha q}{q} \qquad \textit{if} \quad \alpha q < d,$$
$$\beta > \frac{d-\alpha q}{2q} \qquad \textit{if} \quad \alpha q \ge d.$$

(2) $p=1$

$$\beta > -\alpha/2.$$

For $T\in H_p^\beta$,

$$\lim_{\epsilon\to 0} A_T^\epsilon(a:t,\omega) = A_T(a:t,\omega) \qquad \textit{in} \quad L^2(dP_x).$$

If $p=2$ then we can improve above Theorem as follows:

Theorem 3.13. *Suppose that* $\beta > -\alpha/2$ *in the case where* $d<\alpha$ *and that* $\beta > (d-2\alpha)/2$ *in the case where* $d\ge\alpha$.
For $T\in H_2^\beta$,

$$\lim_{\epsilon\to 0} A_T^\epsilon(a:t,\omega) = A_T(a:t,\omega) \qquad \textit{in} \quad L^2(dP_x).$$

The results of the (a,t)-joint continuity of $A_T(a:t,\omega)$ are the following:

Theorem 3.14. *Let $T \in H_p^\beta$ $(1 < p \leq \infty)$, where we take β as Theorem 3.12 and q satisfy $1/p + 1/q = 1$.*

Suppose that $\delta = \min(\alpha/2, (q\beta - d + \alpha q)/2q)$ in the case where $d > \alpha q$ and that $\delta = \min(\alpha/2, (2q\beta - d + \alpha q)/2q)$ in the case where $d \leq \alpha q$.

Then $A_T(a : t, \omega)$ has (a, t)-jointly continuous modification, which is locally Hölder-continuous with exponent γ, where $0 < \gamma < \delta$.

Noting (3.14), we get the following Theorem in the similar way to Theorems 3.9, 3.11 and Corollary 3.10.

Theorem 3.15. *We suppose that $2 < p \leq \infty$ and q satisfy $1/p + 1/q = 1$.*

Suppose that $\beta > (d - \alpha q)/2q$ in the case where $p > 2$ and that $\beta > -\alpha/2$ in the case where $p = 2$.

For $T \in H_p^\beta$,

$$\lim_{\epsilon \to 0} A_T^\epsilon(a : t, \omega) = A_T^{dx}(a : t, \omega) \qquad \text{in} \quad L^2(dP_x \times dx)$$

and $e(A_T^{dx}) = 0$.

4. Representation theorems

We describe the occupation time formula and a representation theorem by T. Yamada ([17]) with respect to one-dimensional Brownian motion and one-dimensional stable process with index α, but with a little modification in our views.

Throughout this section, we suppose the following assumption.

Assumption 4.1.
(1) *$d = 1$.*
(2) *(X_t) stands for Brownian motion or stable process with index $\alpha(\alpha > 1)$.*

Let $L_a(t, \omega) = A_{\delta_0}(a; t, \omega)$ and $C_a(t, \omega) = A_{v.p.\frac{1}{x}}(a; t, \omega)$.

Theorem 4.2. *Let $f \in \mathcal{S}$. Then for* a.e.-ω,

$$\int_0^t f(X_s) ds = \int f(a) L_a(t, \omega) da, \tag{4.1}$$

$$\int_0^t f(X_s) ds = \frac{1}{\pi} \int Hf(a) C_a(t, \omega) da. \tag{4.2}$$

Here H is the Hilbert transform, that is,

$$H\phi(t) = \left(\frac{1}{\pi} v.p. \frac{1}{x} * \phi \right)(t).$$

Proof. First we show the equation (4.1). We denote $A^{\epsilon}_{\delta_0}(a : t, \omega)$ by $L^{\epsilon}_a(t, \omega)$. We notice

$$\begin{aligned}\widehat{L^{\epsilon}_{\cdot}}(t,\omega)(\lambda) &= \widehat{A^{\epsilon}_{\delta_0}}(\cdot : t,\omega) \\ &= \int_0^t e^{i\lambda X_s}\hat{\rho}_{\epsilon}(-\lambda)ds.\end{aligned}$$

By the Parseval's equality, letting $\epsilon \to 0$,

$$\begin{aligned}\int f(a)L^{\epsilon}_a(t,\omega)da &= \frac{1}{2\pi}\int d\lambda \hat{f}(\lambda)\int_0^t dse^{-i\lambda X_s}\hat{\rho}_{\epsilon}(\lambda) \\ &\to \frac{1}{2\pi}\int d\lambda\int_0^t ds\hat{f}(\lambda)e^{-i\lambda X_s} \\ &= \int_0^t f(X_s)ds.\end{aligned}$$

On the other hand, tending ϵ to zero, we get

$$\int f(a)L^{\epsilon}_a(t,\omega)da \to \int f(a)L_a(t,\omega)da$$

by Theorem 3.2 and [7, Theorem 3.1].

Next, we show (4.2). We denote $A^{\epsilon}_{v.p.\frac{1}{x}}(a : t, \omega)$ by $C^{\epsilon}_a(t, \omega)$. We notice

$$\widehat{Hf}(\lambda) = i\,\mathrm{sgn}(\lambda)\hat{f}(\lambda)$$

and

$$\begin{aligned}\widehat{C^{\epsilon}_{\cdot}}(t,\omega) &= \widehat{A^{\epsilon}_{v.p.\frac{1}{x}}}(\cdot : t,\omega) \\ &= \int_0^t dse^{i\lambda X_s}(i\pi\,\mathrm{sgn}(\lambda))\hat{\rho}_{\epsilon}(-\lambda).\end{aligned}$$

Since $Hf \in \mathcal{S}$, letting ϵ to zero, we get

$$\begin{aligned}&\int Hf(a)C^{\epsilon}_a(t,\omega)da \\ &= \frac{1}{2\pi}\int d\lambda\widehat{Hf}(\lambda)\overline{\widehat{C^{\epsilon}_{\cdot}}(t,\omega)} \\ &= \frac{1}{2\pi}\int d\lambda i\,\mathrm{sgn}(\lambda)\hat{f}(\lambda)\int_0^t dse^{-i\lambda X_s}(-\pi i\,\mathrm{sgn}(\lambda))\hat{\rho}_{\epsilon}(\lambda) \\ &\to \frac{1}{2\pi}\pi\int d\lambda\int_0^t ds\hat{f}(\lambda)e^{-i\lambda X_s} \\ &= \pi\int_0^t f(X_s)ds.\end{aligned}$$

On the other hand, tending ϵ to zero uniformly in wider sense, we get

$$\int Hf(a)C_a^{\epsilon}(t,\omega)da \to \int Hf(a)C_a(t,\omega)da$$

by Theorem 3.2 and [7, Theorem 3.1]. □

Corollary 4.3.

$$C_a(t,\omega) = H(L.(t,\omega))(a).$$

Here H is the Hilbert transform.

Proof. By Theorem 4.2 we have

$$\begin{aligned}\pi \int Hf(a)L_a(t,\omega) &= \pi \int_0^t Hf(X_s)ds \\ &= \pi \int H(Hf)(a)C_a(t,\omega)da \\ &= -\pi \int f(a)C_a(t,\omega)da.\end{aligned}$$

On the other hand for any $f, g \in L^2(da)$

$$\int Hf(a)g(a)da = -\int f(a)Hg(a)da.$$

Since $L.(t,\omega) \in L^2(da)$ by proof of [7, Theorem 4.2], [7, Corollary 4.3], Theorem 3.9 and Corollary 3.10, we have

$$\pi \int Hf(a)L_a(t,\omega)da = -\pi \int f(a)H(L.(t,\omega))(a)da.$$

Thus we have for any $f \in \mathcal{S}(\mathbb{R})$

$$\int f(a)C_a(t,\omega)da = \int f(a)H(L.(t,\omega))da.$$

Therefore we get

$$C_a(t,\omega) = H(L.(t,\omega))(a).$$

□

Remark 4.4. This Corollary is proved by M. Yor ([16]).

Remark 4.5. In the case of the symmetric stable process with index $\alpha(1 < \alpha < 2)$ or Brownian motion, Theorem 4.2 holds for $f \in L^2$.

Proof. By scaling property, we have

$$\begin{aligned}
&E_0\left[\left\{\int_0^t F(X_s)ds\right\}^2\right]\\
&= t^2 E_0\left[\left\{\int_0^1 F(X_{ut})du\right\}^2\right]\\
&\le t^2 E_0\left[\int_0^1 (F(X_{ut}))^2 du\right]\\
&= t^2 \int dy \int_0^1 du F^2(y) p(ut, y)\\
&= t^2 \int dy \int_0^1 du (tu)^{-\frac{1}{\alpha}} F^2(y) p\left(1, y(tu)^{-1/\alpha}\right).
\end{aligned}$$

Noting $|p(1, y(tu)^{-1/\alpha})| \le C$ for some positive constant C, we get

$$\begin{aligned}
&E_0\left[\left\{\int_0^t F(X_s)ds\right\}^2\right]\\
&\le t^2 C \int_0^1 du (tu)^{-1/\alpha} \int dy F^2(y)\\
&= t^2 C \|F\|_{L^2}^2 \int_0^1 du (tu)^{-1/\alpha}.
\end{aligned}$$

For $f \in L^2$, we can take $f_n \to f$ in L^2 with $\{f_n\} \subset \mathcal{S}$.

We set $F = f_n - f$ and by taking a subsequence, we get $\int_0^t f_{n'}(X_s)ds \to \int_0^t f(X_s)ds$ in $L^2(dP)$ and almost surely.

On the other hand, since we can easily see that $C_a(t, \omega)$ and $L_a(t, \omega)$ belong to $L^2(da)$, the right hand sides of (4.1) and (4.2) converge also. $\square$

Lemma 4.6. *For $T \in H_p^\beta$,*

$$\int A_T(a : t, \omega)\phi(a)da = \int_0^t T \check{*} \phi(X_s)ds.$$

Here

$$\phi \check{*} \psi(x) = \int \phi(u - x)\psi(u)du$$

and

$$T \check{*} \phi(x) = \langle T_y, \phi(y - x)\rangle_y,$$

where ϕ and ψ belong to $\mathcal{S}(\mathbb{R})$ and T, β are same in [7, *Theorem* 3.1] *and Theorem* 3.2.

Proof.

$$\begin{aligned}\int A_T^\epsilon(a:t,\omega)\phi(a)da &= \frac{1}{2\pi}\int\int_0^t \hat{T}(\lambda)\hat{\rho}_\epsilon(\lambda)e^{i\lambda X_s}\overline{\hat{\phi}(\lambda)}dsd\lambda \\ &= \frac{1}{2\pi}\int\int_0^t \hat{T}(\lambda)\hat{\phi}(-\lambda)e^{i\lambda X_s}\hat{\rho}_\epsilon(\lambda)dsd\lambda \\ &\to \frac{1}{2\pi}\int_0^t\int \hat{T}(\lambda)\hat{\phi}(-\lambda)e^{i\lambda X_s}d\lambda ds, \qquad \epsilon\to 0 \\ &= \int_0^t T\check{*}\phi(X_s)ds.\end{aligned}$$

On the other hand, tending ϵ to zero, we get

$$\int A_T^\epsilon(a:t,\omega)\phi(a)da \to \int A_T(a:t,\omega)\phi(a)da$$

by Theorem 3.2 and [7, Theorem 3.1]. □

Remark 4.7. This lemma holds for the d-dimensional Brownian motion and the d-dimensional symmetric stable process with index $\alpha(1<\alpha<2)$.

For T satisfying a certain assumption, this lemma is proved by T. Yamada ([19]) in the case of d-dimensional Brownian motion.

Theorem 4.8. *For any* $T\in H_p^\beta$,

$$A_T(a:t,\omega) = (T * L.(t,\omega))(a),$$

where β *are same in Theorem* 3.2 *and* [7, *Theorem* 3.1].

Proof.

$$\begin{aligned}\int A_T(a-x:t,\omega)\rho_\epsilon(x)dx &= \int_0^t T\check{*}\rho_\epsilon(X_s-a)ds \\ &= \int L_x(t,\omega)(T\check{*}\rho_\epsilon)(x-a)dx \\ &= \frac{1}{2\pi}\int \hat{L}.(t,\omega)(\lambda)\overline{\hat{T}(-\lambda)\hat{\rho}_\epsilon(\lambda)e^{i\lambda a}}d\lambda \\ &= \frac{1}{2\pi}\int \hat{L}.(t,\omega)(\lambda)\hat{T}(\lambda)\hat{\rho}_\epsilon(-\lambda)e^{-i\lambda a}d\lambda.\end{aligned}$$

Since $L.(t,\omega)$ has compact support by [7, Lemma 2.1], we have $|\hat{L}.(t,\omega)(\lambda)|\le tK$, where K is constant, Then we get

$$\hat{L}.(t,\omega)e^{-i\lambda a}\hat{T}(\lambda)(1+|\lambda|^2)^{\beta/2}\in L^p$$

and

$$(1+|\lambda|^2)^{-\beta/2}\hat{\rho}_\epsilon(\lambda)\in L^q.$$

Thus tending ϵ to zero, we get

$$A_T(a:t,\omega) = (T * L.(t,\omega))(a).$$

$\square$

Remark 4.9. For T satisfying a certain assumption, this theorem is proved by T. Yamada ([19]) in the case of one-dimensional Brownian motion and in the distribution sense.

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References

[1] R. Bass, Joint continuity and representations of additive functionals of d-dimensional Brownian motion, Stochastic Process Appl., **17** (1984), 211–227.

[2] E. S. Boylan, Local times for a class of Markoff Processes, Illinois J. Math., **8** (1964), 19–39.

[3] M. Fukushima, A Decomposition of additive functionals of finite energy, Nagoya Math. J., **74** (1979), 137–168.

[4] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov process, Walter de Gruyter, 1994.

[5] K. Ito and H. P. Mckean, Diffusion processes and their sample paths, Springer, 1965.

[6] I. Karatzas and S. Shreve, Brownian motion and stochastic calculus (second edition), Springer-Verlag, 1994.

[7] T. Nakajima, A certain class of distribution–valued additive functionals I—for the case of Brownian motion, J. Math. Kyoto Univ., **40**-2 (2000), 293–314.

[8] D. Revuz, Measures associées aux fonctionnelles additivees de Markov I, Trans. Amer. Math. Soc., **148** (1970), 501–531.

[9] D. Revuz and M. Yor, Continuous martingales and Brownian motion (second edition), Springer-Verlag, 1994.

[10] K. Sato, Lévy processes and infinitely divisible distributions, Cambridge Univ. Press, 1999.

[11] H. P. Mckean and H. Tanaka, Additive Functionals of the Brownian Motion Path, Mem. Coll. Sci. Univ. Kyoto, **33** (1961), 479–506.

[12] S. Mizohata, Theory of Partial Differential Equations, Cambridge Univ. Press, 1973.

[13] J. Takeuchi, T. Yamada and S. Watanabe, Stable process—Riesz's potential and the property of path, Seminar on Probability 13, 1962 (in Japanese).

[14] H. F. Trotter, A property of Brownian motion paths, Illinois J. Math., **2** (1958), 425–433.

[15] D. Revuz and M. Yor, Continuous martingales and Brownian motion (second edition), Springer-Verlag, 1994.

[16] M. Yor, Sur la transformée Hilbert des temps locaux Browniens et une extension de la formule d'Ito, Séminarie de Probabilités 16, 1982, pp. 238–247.

[17] T. Yamada, On some representations concerning the stochastic integrals, Prob. Math. Statist., **4**-2 (1984), 155–166.

[18] T. Yamada, On the fractional derivative of Brownian local time, J. Math. Kyoto Univ., **25** (1985), 49–58.

[19] T. Yamada, Representations of continuous additive functionals of zero energy via convolution type transforms of Brownian local times and the Radon transform, Stochastics and Stoch. Rep., **46**-1 (1994), 1–15.