J. Math. Kyoto Univ. (JMKYAZ)
42-3 (2002), 429–441

J-adic filtration of orders with application to orders of finite representation type

By

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For a ring Λ with the Jacobson radical J_Λ, we denote by $\mathrm{Gr}\,\Lambda$ the *associated completely graded ring* with respect to the J_Λ-adic filtration, namely $\mathrm{Gr}\,\Lambda := \prod_{i\geq 0} J_\Lambda^i/J_\Lambda^{i+1}$. In Section 1, for an order Λ over a complete discrete valuation ring R, we will study the associated ring $\mathrm{Gr}\,\Lambda$, which is not even noetherian in general (Remark 1.3 (2)). Our main theorem (Theorem 1.2) asserts that $\mathrm{Gr}\,\Lambda$ *is again an order over some complete discrete valuation ring if and only if* Λ *has the filtering overorder* Γ (*Section* 1.1)*, which is a hereditary overorder of* Λ *such that* $J_\Lambda^n = \Lambda \cap J_\Gamma^n$ *for any* $n \geq 0$*.*

Now, we explain background and application in Section 2. For an additive category $\mathcal{C}$ with the Jacobson radical $\mathcal{J}_\mathcal{C}$, we denote by $\mathrm{Gr}\,\mathcal{C}$ the *associated completely graded category* $\prod_{i\geq 0} \mathcal{J}_\mathcal{C}^i/\mathcal{J}_\mathcal{C}^{i+1}$. In study of representation theory of an order Δ over a complete regular local ring R of dimension $d \leq 2$, the associated category $\mathrm{Gr}(\mathrm{lat}\,\Delta)$ of $\mathrm{lat}\,\Delta$ plays an important role. Under the assumption that Δ is an isolated singularity, we can define a combinatorial invariant $\mathbb{A}(\mathrm{lat}\,\Delta)$ called the *Auslander-Reiten quiver* and its "algebraic realization" $\widehat{\mathbb{A}}(\mathrm{lat}\,\Delta)$ called the *Auslander-Reiten species*. It is important that we can recover $\mathrm{Gr}(\mathrm{lat}\,\Delta)$ from $\widehat{\mathbb{A}}(\mathrm{lat}\,\Delta)$, namely $\mathrm{Gr}(\mathrm{lat}\,\Delta)$ is equivalent to the *mesh category* $\widehat{\mathbb{M}}(\widehat{\mathbb{A}}(\mathrm{lat}\,\Delta))$ of $\widehat{\mathbb{A}}(\mathrm{lat}\,\Delta)$ ([I2], [IT], [BG]).

When Δ is of finite representation type, it is convenient to study the endomorphism ring $\Lambda := \mathrm{End}_\Delta(M)$ of an additive generator M of $\mathrm{lat}\,\Delta$, which is called the *Auslander order* of Δ. The category $\mathrm{pr}\,\Lambda$ of finitely generated projective Λ-modules is equivalent to $\mathrm{lat}\,\Delta$, and $\mathrm{pr}(\mathrm{Gr}\,\Lambda)$ is equivalent to $\mathrm{Gr}(\mathrm{lat}\,\Delta)$. For $d \leq 2$, it is surprising that we can characterize Auslander orders by some homological conditions ([ARS], [ARo], [RV] and Definition 2.1). It is also remarkable that, if R is an algebraically closed field ($d = 0$) with $\mathrm{chr}\,R \neq 2$, then $\mathrm{Gr}\,\Lambda$ is always isomorphic to Λ, so $\mathrm{lat}\,\Delta$ is completely recoverd by the combinatorial data $\mathbb{A}(\mathrm{lat}\,\Delta)$ ([BGRS]).

In Section 2, we will study the associated ring $\mathrm{Gr}\,\Lambda$ of an Auslander order Λ over a complete discrete valuation ring R ($d = 1$). In many important cases like $R = \mathbb{Z}_p$, the ring $\mathrm{Gr}\,\Lambda$ is not isomorphic to Λ. But, we know by

Received April 17, 2001
Revised July 12, 2001
*The author is supported by the JSPS Research Fellowships for Young Scientists.

a result of [I3] that $\mathrm{Gr}\,\Lambda$ *is again an Auslander order over the formal power series ring* $(R/J_R)[[t]]$ (Proposition 2.2.1), and consequently, Λ *has the filtering overorder* Γ (Section 2.2). This leads us to concept of the *filtering functor* $\mathbb{F}_\Delta : \mathrm{lat}\,\Delta \to \mathrm{lat}\,\Gamma$ of an order Δ of finite representation type (Definition 2.3), where Γ is the filtering overorder of the Auslander order Λ of Δ. We will study some properties of $\mathbb{F}_\Delta$ in Section 2. We also study the relationship with additive functions (Section 2.5) and the Grothendieck group of an order (Section 2.6).

0.1 Notations. In the rest of this paper, *assume that R is a complete discrete valuation ring unless explicitly stated otherwise.*

(1) For a commutative ring C, we denote by $\widetilde{C}$ the total quotient ring of C. For a ring Λ, we denote by $\mathrm{Cen}(\Lambda)$ the center of Λ, and put $\widetilde{\Lambda} := \widetilde{\mathrm{Cen}(\Lambda)} \otimes_{\mathrm{Cen}(\Lambda)} \Lambda$. We denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated (left) Λ-modules, by $\mathrm{pr}\,\Lambda$ the category of finitely generated projective Λ-modules, and by $\mathrm{len}_\Lambda(X)$ the length of a Λ-module X. We have a functor $\widetilde{(\ )} := \widetilde{\Lambda} \otimes_\Lambda\ : \mathrm{mod}\Lambda \to \mathrm{mod}\widetilde{\Lambda}$.

(2) Let Λ be an R-*order*, namely it is an R-algebra that is finitely generated free as an R-module. A (left) Λ-module L is called a Λ-*lattice* if it is finitely generated free as an R-module. We denote by $\mathrm{lat}\,\Lambda$ the category of Λ-lattices. Notice that $\widetilde{\Lambda} = \widetilde{R} \otimes_R \Lambda$ and $\widetilde{L} = \widetilde{R} \otimes_R L$ hold for any $L \in \mathrm{mod}\Lambda$ by the following easy fact 0.1.1, which will be used in the proof of 1.2.

0.1.1. Let R be a commutative noetherian domain, Λ an R-algebra, and $C := \mathrm{Cen}(\Lambda)$. Assume that Λ is a finitely generated torsionfree R-module. Then $\widetilde{R} \otimes_R \Lambda = \widetilde{C} \otimes_C \Lambda$ holds.

Proof. Since $\widetilde{R} \otimes_R \Lambda = (\widetilde{R} \otimes_R C) \otimes_C \Lambda$, we may assume $\Lambda = C$. Since C is a torsionfree R-module, we have an injective map $\widetilde{R} \otimes_R C \to \widetilde{C}$. We only have to show that $x^{-1} \in \widetilde{R} \otimes_R C$ holds for any non-zerodivisor x in C. Since C is a finitely generated R-module, there exist $n > 0$ and $r_i \in R$ such that $x^n + r_1 x^{n-1} + \cdots + r_{n-1} x + r_n = 0$ and $r_n \neq 0$. Then $x^{-1} = -r_n^{-1} y \in \widetilde{R} \otimes_R C$ holds for $y := x^{n-1} + r_1 x^{n-2} + \cdots + r_{n-1}$. □

1. J-adic filtration of orders

1.1. Let R be a complete discrete valuation ring with a residue field k and Λ an R-order.

(1) We call an R-order Γ a *filtering overorder* of Λ if Γ is a hereditary overorder of Λ such that $J_\Lambda^n = \Lambda \cap J_\Gamma^n$ holds for any $n \geq 0$. For example, any Bäckström order ([RR]) Λ has a filtering overorder $\Gamma = O_l(J_\Lambda)$.

(2) Assume that Λ has a filtering overorder Γ. Then Γ is the unique filtering overorder of Λ. In this case, there exists a subring S of $\mathrm{Cen}(\mathrm{Gr}\,\Lambda)$ such that S is isomorphic to the formal power series ring $k[[t]]$ and $\mathrm{Gr}\,\Lambda$ is an S-order in a semisimple $\widetilde{S}$-algebra $\widetilde{\mathrm{Gr}\,\Lambda}$.

(3) $J_{\mathrm{Gr}\,\Lambda}^n = \prod_{i \geq n} J_\Lambda^i / J_\Lambda^{i+1}$ holds for any $n \geq 0$.

1.1.1. Let R be a complete discrete valuation ring with a residue field k and a prime element π_R.

(1) Let Ω be a local maximal R-order with a residue k-algebra $D := \Omega/J_\Omega$ and a prime element π_Ω. Define $\sigma \in \mathrm{Aut}_k(D)$ by $\overline{a}^\sigma := \overline{\pi_\Omega a \pi_\Omega^{-1}}$ for $a \in \Omega$. Then $\mathrm{Gr}\,\Omega$ is isomorphic to the skew formal power series ring $D[[x;\sigma]]$ ($xd = d^\sigma x$ for $d \in D$). Take $l > 0$ such that $\pi_R \in J_\Omega^l - J_\Omega^{l+1}$ and put $t := \overline{\pi_R} \in J_\Omega^l/J_\Omega^{l+1} \subset \mathrm{Gr}\,\Omega$. Then $\mathrm{Gr}\,\Omega$ is a local maximal $k[[t]]$-order.

(2) Let Λ be a ring indecomposable hereditary R-order. Then Λ is Morita equivalent to $\mathrm{T}_n(\Omega)$ for some local maximal R-order Ω ([CR]), where $\mathrm{T}_n(\Omega)$ denotes the subring $\begin{pmatrix} \Omega & \Omega & \cdots & \Omega & \Omega \\ J_\Omega & \Omega & \cdots & \Omega & \Omega \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J_\Omega & J_\Omega & \cdots & \Omega & \Omega \\ J_\Omega & J_\Omega & \cdots & J_\Omega & \Omega \end{pmatrix}$ of $\mathrm{M}_n(\Omega)$. We can easily check that $\mathrm{Gr}\,\Lambda$ is Morita equivalent to $\mathrm{T}_n(\mathrm{Gr}\,\Omega)$. Thus $\mathrm{Gr}\,\Lambda$ is a hereditary $k[[t]]$-order by (1).

Proof of 1.1. (2) Put $O_l(L) := \{x \in \widetilde{\Gamma} \mid xL \subseteq L\}$ for $L \in \mathrm{lat}\,\Gamma$. We can take sufficiently large n such that $J_\Gamma^n \subseteq \Lambda$. Then $\Gamma = O_l(J_\Gamma^n) = O_l(J_\Lambda^n)$ holds since Γ is hereditary ([CR]). Thus the former assertion follows. Since we have a natural inclusion $J_\Lambda^i/J_\Lambda^{i+1} \to J_\Gamma^i/J_\Gamma^{i+1}$ for any $i \geq 0$, $\mathrm{Gr}\,\Lambda$ is a subring of $\mathrm{Gr}\,\Gamma$ containing $\prod_{i\geq n} J_\Gamma^i/J_\Gamma^{i+1}$. Thus the latter assertion follows from 1.1.1 (2).

(3) Put $I_n := \prod_{i\geq n} J_\Lambda^i/J_\Lambda^{i+1}$. Then $I_1 = J_{\mathrm{Gr}\,\Lambda}$ holds since I_1 is quasi-regular ([AF] Section 15) and $(\mathrm{Gr}\,\Lambda)/I_1 = \Lambda/J_\Lambda$ is semisimple. Since $I_1^n \subseteq I_n$ holds, we only have to show $I_n I_1 \supseteq I_{n+1}$. Take a finite subset $\{g_j\}_j$ of J_Λ such that $J_\Lambda = \sum_j \Lambda g_j$. Then $J_\Lambda^i = \sum_j J_\Lambda^{i-1} g_j$ holds hor any $i > 0$. For any $(x_i)_{i\geq n+1} \in I_{n+1}$, take $y_{i-1,j} \in J_\Lambda^{i-1}$ such that $x_i = \sum_j y_{i-1,j} g_j$. Put $y_j := (y_{i,j})_{i\geq n} \in I_n$ and regard g_j as an element $(0, g_j, 0, 0, \dots)$ of I_1. Then $(x_i)_{i\geq n+1} = \sum_j y_j g_j \in I_n I_1$ holds. □

Theorem 1.2. *Let R be a complete discrete valuation ring and Λ an R-order in a semisimple $\widetilde{R}$-algebra $\widetilde{\Lambda}$. Then the following conditions are equivalent.*

(1) *There exists a subring S of* $\mathrm{Cen}(\mathrm{Gr}\,\Lambda)$ *such that S is a complete discrete valuation ring and* $\mathrm{Gr}\,\Lambda$ *is an S-order in a semisimple $\widetilde{S}$-algebra* $\widetilde{\mathrm{Gr}\,\Lambda}$.

(2) Λ *has the filtering overorder* Γ (*cf.* 1.1).

1.2.1. Let $C = \prod_{i\geq 0} C_{(i)}$ be a commutative completely graded ring without nilpotent elements, e an idempotent of $\widetilde{C}$ and $n \geq 0$. If $eC_{(n)} \subseteq C$ holds, then $eC_{(n)} \subseteq C_{(n)}$ holds.

Proof. For any $x \in C$, we put $x = \sum_{i\geq m(x)} x_i$ ($x_i \in C_{(i)}$ and $x_{m(x)} \neq 0$), and put $m(0) := \infty$. Then $m(xy) \geq m(x) + m(y)$ and $m(x^l) = lm(x)$ hold for any $x, y \in C$ and $l > 0$ since C has no nilpotent element.

(i) Assume that $x \in C$ and an idempotent $f \in \widetilde{C}$ satisfy $fx \in C$. We will show that $m(fx) \geq m(x)$ holds, and the equality holds if $fx \neq 0$ and x is homogeneous.

The former assertion is immediate from $2m(fx) = m((fx)^2) = m((fx)x) \geq m(fx) + m(x)$. We will show the latter assertion. Since x is homogeneous,

$(fx^2)_{i+m(x)} = (fx)_i x$ and $(fx^3)_{i+2m(x)} = (fx)_i x^2$ hold for any $i \geq 0$. Since $(fx)_i x \neq 0$ is equivalent to $(fx)_i x^2 \neq 0$, we obtain $2m(fx) - m(x) = m(fx^2) - m(x) = m(fx^3) - 2m(x) = 3m(fx) - 2m(x)$. Since $m(fx) < \infty$ holds, we obtain $m(fx) = m(x)$.

(ii) For any $x \in C_{(n)}$, we will show $ex \in C_{(n)}$. We may assume $ex \neq 0$. Then $m(ex) = n$ holds by (i). Put $y := ex - (ex)_n$ and $\dot{e} := 1 - e$. If $\dot{e}(ex)_n \neq 0$, then $n = m((ex)_n) = m(\dot{e}(ex)_n) = m(\dot{e}y) \geq m(y) > n$ holds by $\dot{e}y = -\dot{e}(ex)_n \in C$ and (i), a contradiction. Hence we obtain $\dot{e}(ex)_n = 0$ and $e(ex)_n = (ex)_n$. If $e(x - (ex)_n) = ex - (ex)_n \neq 0$, then $n = m(x - (ex)_n) = m(ex - (ex)_n) > n$ holds by (i), a contradiction. Thus $ex = (ex)_n$ holds. □

Proof of Theorem 1.2. (2) implies (1) by 1.1 (2). We will show that (1) implies (2). For simplicity, put $\Lambda_i := J_\Lambda^i$ and $\Lambda_{(i)} := J_\Lambda^i / J_\Lambda^{i+1}$ for any $i \geq 0$.

(I) We will show that there exists $l > 0$ and $a \in \Lambda_{(l)} \cap \operatorname{Cen}(\operatorname{Gr}\Lambda)$ such that a is an invertible element in $\widetilde{\operatorname{Gr}\Lambda}$.

Put $C := \operatorname{Cen}(\operatorname{Gr}\Lambda)$ and $C_{(i)} := C \cap \Lambda_{(i)}$. Then $C = \prod_{i \geq 0} C_{(i)}$ holds. Since $\widetilde{\operatorname{Gr}\Lambda}$ is semisimple, C does not have nilpotent elements. Let $\mathbf{E}$ be a complete set of primitive idempotents of $\widetilde{C}$. We only have to show that there exists a homogeneous element $a \in C$ such that $ea \neq 0$ for any $e \in \mathbf{E}$.

Since $\prod_{i \geq n} \Lambda_{(i)} = J_{\operatorname{Gr}\Lambda}^n$ holds by 1.1 (3), we can take sufficiently large $n \geq 0$ such that $e \prod_{i \geq n} \Lambda_{(i)} \subseteq \operatorname{Gr}\Lambda$ holds for any $e \in \mathbf{E}$. For any $i \geq n$ and $e \in \mathbf{E}$, since $eC_{(i)} \subseteq \widetilde{C}$ holds, we obtain $eC_{(i)} \subseteq C_{(i)}$ by 1.2.1. For any $e \in \mathbf{E}$, we can take a non-zero element $a_e \in eC_{(l_e)} \subseteq C_{(l_e)}$ for some $l_e \geq n$. Then $a := \sum_{e \in \mathbf{E}} a_e^{l/l_e} \in C_{(l)}$ $(l := \prod_{e \in \mathbf{E}} l_e)$ satisfies the desired condition.

(II) Fix a lift $a \in \Lambda_l$ of $a \in \Lambda_{(l)}$ in (I). We will show that a is an invertible element of $\widetilde{\Lambda}$.

Since $\dim_{\widetilde{R}} \widetilde{\Lambda} < \infty$, we only have to show that a is a non-zerodivisor in $\widetilde{\Lambda}$, or equivalently, a is a non-zerodivisor in Λ. Assume that $x \in \Lambda_i - \Lambda_{i+1}$ satisfies $ax = 0$. Then $\overline{x} \in \Lambda_{(i)}$ satisfies $a\overline{x} = 0$, a contradiction to (I).

(III) We will show that there exists $N \geq 0$ such that $a\Lambda_i = \Lambda_i a = \Lambda_{i+l}$ for any $i > N$.

By (I), $(a\cdot)$ and $(\cdot a) : \Lambda_{(i)} \to \Lambda_{(i+l)}$ are injective for any $i \geq 0$. Since $\dim_{R/J_R} \Lambda_{(i)} \leq \operatorname{rank}_R \Lambda_i = \dim_{\widetilde{R}} \widetilde{\Lambda}$ holds for any $i \geq 0$, there exists $N \geq 0$ such that $(a\cdot)$ and $(\cdot a) : \Lambda_{(i)} \to \Lambda_{(i+l)}$ are bijective for any $i > N$. Hence $a\Lambda_i + \Lambda_{i+l+1} = \Lambda_i a + \Lambda_{i+l+1} = \Lambda_{i+l}$ holds for any $i > N$. By Nakayama's Lemma, we obtain the assertion.

(IV) Put $\Gamma_i := \{x \in \widetilde{\Lambda} \mid a^n x \in \Lambda_{i+nl}$ for sufficiently large $n\}$ for $i \in \mathbb{Z}$. We will show that the following (i)–(v) hold.

(i) If $i, n \in \mathbb{Z}$ satisfies $i + nl > N$, then $\Gamma_i = a^{-n}\Lambda_{i+nl} = \Lambda_{i+nl}a^{-n}$ holds.

(ii) $\Gamma_i \Gamma_j = \Gamma_{i+j}$ holds for any $i, j \in \mathbb{Z}$.

(iii) $\Gamma_{i+1} \cap \Lambda_i = \Lambda_{i+1}$ and $\Gamma_i \cap \Lambda = \Lambda_i$ hold for any $i \geq 0$.

(iv) $\Gamma_i = \Lambda_i$ holds for any $i > N$.

(v) $(a\cdot)$ and $(\cdot a) : \Gamma_i \to \Gamma_{i+l}$ are bijective for any $i \in \mathbb{Z}$.

(i) is immediate since $a^{-n}\Lambda_{i+nl} = a^{-n+1}\Lambda_{i+(n+1)l} = \cdots$ holds by (III).

(ii) Taking n such that $i + nl > N$ and $j + nl > N$, we obtain $\Gamma_i\Gamma_j = a^{-n}\Lambda_{i+nl}a^{-n}\Lambda_{j+nl} = a^{-2n}\Lambda_{i+nl}\Lambda_{j+nl} = a^{-2n}\Lambda_{i+j+2nl} = \Gamma_{i+j}$ by (III) and (i). (iii) Taking n such that $i + nl > N$, we obtain $\Gamma_{i+1} \cap \Lambda_i = a^{-n}\Lambda_{i+nl+1} \cap \Lambda_i = \Lambda_{i+1}$ since $(a^n\cdot) : \Lambda_{(i)} \to \Lambda_{(i+nl)}$ is injective by (I). Inductively, we obtain $\Gamma_i \cap \Lambda = \Gamma_i \cap (\Gamma_{i-1} \cap \Lambda) = \Gamma_i \cap \Lambda_{i-1} = \Lambda_i$. (iv) Put $n = 0$ in (i). (v) They are injective by (II) and surjective by (i).

(V) Put $\Gamma_{(i)} := \Gamma_i/\Gamma_{i+1}$ and $G := \prod_{i\in\mathbb{Z}} \Gamma_{(i)}$. Then G is a completely graded ring by (ii), and $\operatorname{Gr}\Lambda$ is a subring of G since we have a natural injection $\Lambda_{(i)} \to \Gamma_{(i)}$ $(i \geq 0)$ by (iii).

(vi) $\Gamma_{(i)}\Gamma_{(j)} = \Gamma_{(i+j)}$ $(i, j \in \mathbb{Z})$ holds by (ii), $\Gamma_{(i)} = \Lambda_{(i)}$ $(i > N)$ holds by (iv), and $(a\cdot)$ and $(\cdot a) : \Gamma_{(i)} \to \Gamma_{(i+l)}$ $(i \in \mathbb{Z})$ are bijective by (v).

(vii) We will show that $\widetilde{\operatorname{Gr}\Lambda}$ is isomorphic to G.

Regard $k := R/J_R$ as a subring of $\Lambda_{(0)} \subset \operatorname{Gr}\Lambda$ and put $T := k[[a]] \subset \operatorname{Gr}\Lambda$. Then T is a complete discrete valuation ring, which is contained in $\operatorname{Cen}(\operatorname{Gr}\Lambda)$. Moreover, $\operatorname{Gr}\Lambda$ is a finitely generated T-module since (III) shows that $\operatorname{Gr}\Lambda$ is generated by a finite dimensional k-space $\prod_{0\leq i\leq N+l} \Lambda_{(i)}$. Since $\operatorname{Gr}\Lambda$ is T-torsionfree by (I), $\widetilde{\operatorname{Gr}\Lambda} = \widetilde{T} \otimes_T \operatorname{Gr}\Lambda$ holds by 0.1.1. We will show $\widetilde{T} \otimes_T \operatorname{Gr}\Lambda = G$. Since a is invertible in G by (vi), any non-zero element of T is invertible in G. Hence we have an injection $\widetilde{T} \otimes_T \operatorname{Gr}\Lambda \to G$. It is surjective since $\Gamma_{(i)} = a^{-n}\Gamma_{(i+nl)} = a^{-n}\Lambda_{(i+nl)}$ holds for sufficiently large n by (vi).

(VI) We will show that $\Gamma_{(0)}$ is a semisimple k-algebra.

Assume $J := J_{\Gamma_{(0)}} \neq 0$. Since $G = \widetilde{\operatorname{Gr}\Lambda}$ is semisimple, we obtain $G = GJG$. Comparing degree 0 part, we obtain $\Gamma_{(0)} = \sum_{i\in\mathbb{Z}} \Gamma_{(i)}J\Gamma_{(-i)}$. By (vi), we obtain $\Gamma_{(0)} = \sum_{0\leq i<l} \Gamma_{(i)}J\Gamma_{(-i)}$. Then Nakayama's Lemma shows $\Gamma_{(0)} = \sum_{1\leq i<l} \Gamma_{(i)}J\Gamma_{(-i)}$. Hence $\Gamma_{(0)} = \Gamma_{(-1)}\Gamma_{(0)}\Gamma_{(1)} = \sum_{1\leq i<l} \Gamma_{(-1)}\Gamma_{(i)}J\Gamma_{(-i)}\Gamma_{(1)} = \sum_{0\leq i<l-1} \Gamma_{(i)}J\Gamma_{(-i)}$ holds by (vi). Repeating similar argument, we obtain $J = \Gamma_{(0)}$, a contradiction.

(VII) We will show the theorem. By (i), $\Gamma := \Gamma_0$ is an R-order. By (VI), $J_\Gamma \subseteq \Gamma_1$ holds. Since Γ_1 is a topologically nilpotent ideal of Γ by (ii)(iv), we obtain $J_\Gamma = \Gamma_1$. Since $\Gamma_1^l = \Gamma_l = \Gamma a$ holds by (ii)(v), we obtain $O_l(J_\Gamma) \subseteq O_l(\Gamma_1^l) = \Gamma$. Hence Γ is hereditary by [CR]. Morever, $J_\Lambda^i = \Lambda_i = \Gamma_i \cap \Lambda = J_\Gamma^i \cap \Lambda$ holds by (iii). □

Remark 1.3. Let Λ be an R-order.

(1) Although $R' := \prod_{i\geq 0}(R \cap J_\Gamma^i / R \cap J_\Gamma^{i+1})$ is a subring of $\operatorname{Cen}(\operatorname{Gr}\Lambda)$, an R'-module $\operatorname{Gr}\Lambda$ is not necessarily finitely generated even if Λ has the filtering overorder. For example, put $R := k[[t]] \subset \Lambda := k[[t]] \times k[[t]]$, $f(t) \mapsto (f(t), f(t^2))$. Then $R' = k[[t]] \subset \operatorname{Gr}\Lambda = k[[t]] \times k[[t]]$, $f(t) \mapsto (f(t), f(0))$.

(2) In general, $\operatorname{Gr}\Lambda$ is neither noetherian nor a finitely generated $\operatorname{Cen}(\operatorname{Gr}\Lambda)$-module. For example, put $R := k[[t^2]] \subset \Omega := k[[t]]$, $\Delta := k + t^2\Omega \subset \Omega$ and $\Lambda := \begin{pmatrix} \Omega & \Omega \\ J_\Omega^3 & \Delta \end{pmatrix}$. Then $J_\Lambda^n = \begin{pmatrix} J_\Omega^n & J_\Omega^{n-1} \\ J_\Omega^{n+2} & J_\Omega^{n+1} \end{pmatrix}$ holds for any $n > 0$. Put $K := \widetilde{\Omega}$ and $I := \begin{pmatrix} K\epsilon & K\epsilon \\ 0 & 0 \end{pmatrix} \subset A := \begin{pmatrix} K[\epsilon] & K[\epsilon] \\ K\epsilon & K[\epsilon] \end{pmatrix} \subset \mathrm{M}_2(K[\epsilon])$ $(\epsilon^2 = 0)$. It is easily checked that $\operatorname{Gr}\Lambda$ is isomorphic to a subring $\begin{pmatrix} \Omega & \Omega \\ J_\Omega\epsilon & k+\Omega\epsilon \end{pmatrix}$ of A/I, and $\operatorname{Cen}(\operatorname{Gr}\Lambda) = k + \begin{pmatrix} 0 & 0 \\ 0 & \Omega\epsilon \end{pmatrix}$.

2. Filtering functors of orders of finite representation type

For an R-order Δ, we denote by $\operatorname{ind}\Delta$ the set of isomorphism classes of indecomposable Δ-lattices. We call Δ *of finite representation type* if $\operatorname{ind}\Delta$ is a finite set. Let $\mathrm{F}(\Delta)$ (resp. $\mathrm{F_n}(\Delta)$, $\mathrm{F_p}(\Delta)$) be the free aberian group generated by the base set $\operatorname{ind}\Delta$ (resp. $\operatorname{ind}\Delta - \operatorname{pr}\Delta$, $\operatorname{ind}\Delta \cap \operatorname{pr}\Delta$).

Assume that $\widetilde{R}$-algebra $\widetilde{\Delta}$ is semisimple. Then $\operatorname{lat}\Delta$ has almost split sequences, and we denote by $0 \to \tau^+L \to \theta^+L \to L \to 0$ (resp. $0 \to L \to \theta^-L \to \tau^-L \to 0$) the complex of the *sink map* (resp. *source map*) of L ([ARS]). Define maps $\phi^+, \phi^- \in \operatorname{End}_{\mathbb{Z}}(\mathrm{F}(\Delta))$ by $\phi^+X := X - \theta^+X + \tau^+X$ and $\phi^-X := X - \theta^-X + \tau^-X$.

Definition 2.1. Let R be a complete discrete valuation ring. An R-order Λ is called an *Auslander order* if $\operatorname{gl.dim}\Lambda \leq 2$ and a minimal relative-injective resolution $0 \to \Lambda \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \to 0$ of Λ satisfies $I^0 \in \operatorname{pr}\Lambda$.

A main result of [ARo] shows that there exists a bijection between Morita equivalence classes of R-orders of finite representation type and Morita equivalence classes of Auslander R-orders in semisimple algebras. It is given by $\Delta \mapsto \operatorname{End}_\Delta(M)$, where M is an additive generator of an R-order Δ of finite representation type. In this case, $\Lambda := \operatorname{End}_\Delta(M)$ is called the *Auslander order of* Δ, and we have a natural equivalence $\mathbb{G}_\Delta := \operatorname{Hom}_\Delta(M, \ ) : \operatorname{lat}\Delta \to \operatorname{pr}\Lambda$.

2.2. The following theorem follows immediately from 1.2 and 2.2.1.

Theorem. *Let R be a complete discrete valuation ring and Λ an Auslander R-order in a semisimple $\widetilde{R}$-algebra $\widetilde{\Lambda}$. Then Λ has the filtering overorder* (1.1).

Proposition 2.2.1. ([I3, 3.3]) *Let R be a complete discrete valuation ring with a residue field k, $k[[t]]$ the formal power series ring and Λ an Auslander R-order in a semisimple $\widetilde{R}$-algebra $\widetilde{\Lambda}$. Then $\operatorname{Gr}\Lambda$ is an Auslander $k[[t]]$-order in a semisimple $k((t))$-algebra $\widetilde{\operatorname{Gr}\Lambda}$.*

Definition 2.3. (1) Let Δ be an R-order of finite representation type with its Auslander order Λ and Γ the filtering overorder of Λ (1.1). Then the *filtering functor* $\mathbb{F}_\Delta : \operatorname{lat}\Delta \to \operatorname{lat}\Gamma$ of Δ is defined as a composition of the natural equivalence $\operatorname{lat}\Delta \xrightarrow{\mathbb{G}_\Delta} \operatorname{pr}\Lambda$ and the functor $\operatorname{pr}\Lambda \to \operatorname{lat}\Gamma$, $P \mapsto \Gamma P$. We denote by $\mathbb{F}_\Delta \in \operatorname{Hom}_{\mathbb{Z}}(\mathrm{F}(\Delta), \mathrm{F}(\Gamma))$ the homomorphism induced by $\mathbb{F}_\Delta$.

(2) Let $\mathbb{Z}\langle x, y\rangle$ be a non-commutative polynomial ring. Put $x_0 := 1$, $x_1 := x$ and $x_n := xx_{n-1} - yx_{n-2}$ for $n \geq 2$, or equivalently, $\left(\begin{smallmatrix} 0 & -y \\ 1 & x \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} -yx_{n-2} & -yx_{n-1} \\ x_{n-1} & x_n \end{smallmatrix}\right)$. Define a ring morphism $\gamma : \mathbb{Z}\langle x, y\rangle \to \operatorname{End}_{\mathbb{Z}}(\mathrm{F}(\Delta))$ by $\gamma(x) := \theta^+$ and $\gamma(y) := \tau^+$. Put $\theta_n^+ := \gamma(x_n)$.

2.3.1. (1) We have the following exact sequence for any $L \in \operatorname{lat}\Delta$, $n > 0$ and $i \geq 0$, which gives a minimal projective resolution of $\mathcal{J}^n_{\operatorname{lat}\Delta}(\ , L)$ for $i = 0$.

$$0 \to \mathcal{J}^{i-1}_{\operatorname{lat}\Delta}(\ , \tau^+\theta^+_{n-1}L) \to \mathcal{J}^{i}_{\operatorname{lat}\Delta}(\ , \theta^+_{n}L) \to \mathcal{J}^{n+i}_{\operatorname{lat}\Delta}(\ , L) \to 0$$

(2) Let A be an abelian group, $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{F}(\Delta), A)$ and $a \in \mathrm{End}_{\mathbb{Z}}(A)$. If $f - af\theta^+ + a^2 f\tau^+ = 0$, then $f - a^n f\theta_n^+ + a^{n+1} f\tau^+\theta_{n-1}^+ = 0$ holds for any $n > 0$.

Proof. (1) By [I2, 4.2 and 7.1.]

(2) Immediate from $f - a^n f\theta_n^+ + a^{n+1} f\tau^+\theta_{n-1}^+ = f - a^n(af\theta^+ - a^2 f\tau^+)\theta_n^+ + a^{n+1} f\tau^+\theta_{n-1}^+ = f - a^{n+1} f(\theta^+\theta_n^+ - \tau^+\theta_{n-1}^+) + a^{n+2} f\tau^+\theta_n^+ = f - a^{n+1} f\theta_{n+1}^+ + a^{n+2} f\tau^+\theta_n^+$. □

Theorem 2.4. *Let Δ be an R-order of finite representation type with its Auslander order Λ and filtering functor $\mathbb{F}_\Delta : \mathrm{lat}\,\Delta \to \mathrm{lat}\,\Gamma$. We denote by $l_\Delta \geq 0$ the minimal integer such that $J_\Gamma^{l_\Delta} \subseteq \Lambda$. Take any L, $L' \in \mathrm{lat}\,\Delta$.*

(1) *By $\mathbb{F}_\Delta$, $\mathrm{Hom}_\Delta(L, L')$ is a full sub R-lattice of $\mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(L), \mathbb{F}_\Delta(L'))$. Thus $\mathbb{F}_\Delta$ induces an equivalence $\mathrm{mod}\widetilde{\Delta} \to \mathrm{mod}\widetilde{\Gamma}$.*

(2) *$\mathcal{J}^i_{\mathrm{lat}\,\Delta}(L, L') = \mathrm{Hom}_\Delta(L, L') \cap \mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(L), J_\Gamma^i \mathbb{F}_\Delta(L'))$ holds for any $i \geq 0$, and $\mathcal{J}^i_{\mathrm{lat}\,\Delta}(L, L') = \mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(L), J_\Gamma^i \mathbb{F}_\Delta(L'))$ holds for any $i \geq l_\Delta$,*

(3) *Let $\{L_j\}_{1 \leq j \leq c}$ be a finite subset of $\mathrm{ind}\,\Delta$ such that $\{\widetilde{L_j}\}_{1 \leq j \leq c}$ gives the set of isomorphism classes of simple $\widetilde{\Delta}$-modules. For any j, denote by $p_j > 0$ the minimal integer such that $J_\Gamma^{p_j} \mathbb{F}_\Delta(L_j)$ is isomorphic to $\mathbb{F}_\Delta(L_j)$. Then Γ is Morita equivalent to $\prod_{1 \leq j \leq c} \mathrm{T}_{p_j}(\Omega_j)$ (1.1.1 (2)) for some local maximal order Ω_j, and $\mathrm{ind}\,\Gamma = \{J_\Gamma^i \mathbb{F}_\Delta(L_j) \mid 1 \leq j \leq c,\ 0 \leq i < p_j\}$ holds.*

(4)(*Periodicity*) *Let p_Δ be the least common multiple of p_j $(1 \leq j \leq c)$. Then $\theta^+_{i+p_\Delta} = \theta^+_i$ holds for any $i \geq l_\Delta$.*

(5) *$0 \to \mathcal{J}^i_{\mathrm{lat}\,\Delta}(\ , \tau^+ L) \to \mathcal{J}^{i+1}_{\mathrm{lat}\,\Delta}(\ , \theta^+ L) \to \mathcal{J}^{i+2}_{\mathrm{lat}\,\Delta}(\ , L) \to 0$ is a split exact sequence for any $i \geq l_\Delta$.*

(6) *$\mathbb{F}_\Delta(L) \oplus J_\Gamma^{-n-1} \mathbb{F}_\Delta(\tau^+\theta^+_{n-1} L)$ is isomorphic to $J_\Gamma^{-n} \mathbb{F}_\Delta(\theta_n^+ L)$ for any $n > 0$.*

(7) *Take $X_i \in \mathrm{F}(\Delta)$. Then $\sum_{0 \leq i < p_\Delta} J_\Gamma^{-i} \mathbb{F}_\Delta(X_i) = 0$ holds if and only if $\sum_{0 \leq i < p_\Delta} \theta^+_{n-i} X_i = 0$ holds for any $n \geq l_\Delta + p_\Delta - 1$.*

Proof. (1) Since $\mathrm{Hom}_\Delta(L, L') = \mathrm{Hom}_\Lambda(\mathbb{G}_\Delta(L), \mathbb{G}_\Delta(L'))$ is a full sub R-lattice of $\mathrm{Hom}_\Lambda(\Gamma\mathbb{G}_\Delta(L), \Gamma\mathbb{G}_\Delta(L')) = \mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(L), \mathbb{F}_\Delta(L'))$, the first assertion follows. Since $\mathbb{G}_\Delta$ induces an equivalence $\mathrm{mod}\widetilde{\Delta} \to \mathrm{pr}\,\widetilde{\Lambda} = \mathrm{mod}\widetilde{\Lambda}$, the second assertion follows.

(2) Since Γ is the filtering overorder of Λ, we obtain $\mathcal{J}^i_{\mathrm{pr}\,\Lambda} = \mathrm{pr}\,\Lambda \cap \mathcal{J}^i_{\mathrm{pr}\,\Gamma} = \mathrm{pr}\,\Lambda \cap \mathcal{J}^i_{\mathrm{lat}\,\Gamma}$ for any $i \geq 0$. Since the equivalence $\mathbb{G}_\Delta : \mathrm{lat}\,\Delta \to \mathrm{pr}\,\Lambda$ induces an isomorphism $\mathcal{J}^i_{\mathrm{lat}\,\Delta} \to \mathcal{J}^i_{\mathrm{pr}\,\Lambda}$, we obtain $\mathcal{J}^i_{\mathrm{lat}\,\Delta} = \mathrm{lat}\,\Delta \cap \mathcal{J}^i_{\mathrm{lat}\,\Gamma}$.

(3) $\{\widetilde{\mathbb{F}_\Delta(L_j)}\}_{1 \leq j \leq c}$ gives the set of isomorphism classes of simple $\widetilde{\Gamma}$-modules by (1). Since Γ is hereditary, we obtain the assertion.

(4) Since $\mathcal{J}^{i+p_\Delta}_{\mathrm{lat}\,\Delta}(\ , L) = \mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(\ ), J_\Delta^{i+p_\Delta} \mathbb{F}_\Delta(L)) \simeq \mathrm{Hom}_\Gamma(\mathbb{F}_\Delta(\ ), J_\Delta^i \mathbb{F}_\Delta(L)) = \mathcal{J}^i_{\mathrm{lat}\,\Delta}(\ , L)$ holds by (2), we obtain the assertion by 2.3.1 (1).

(5) It is exact for any $i \geq 0$ by 2.3.1 (1). On $\mathrm{lat}\,\Delta$, it is isomorphic to an sequence $\mathbf{X} : 0 \to \mathrm{Hom}_\Gamma(\ , J_\Gamma^i \mathbb{F}_\Delta(\tau^+ L)) \to \mathrm{Hom}_\Gamma(\ , J_\Gamma^{i+1} \mathbb{F}_\Delta(\theta^+ L)) \to \mathrm{Hom}_\Gamma(\ , J_\Gamma^{i+2} \mathbb{F}_\Delta(L)) \to 0$ by (2). By (3), $\mathbf{X}$ is exact on $\mathrm{lat}\,\Gamma$. Since the functor $\mathrm{Hom}_\Gamma(\ , J_\Gamma^{i+2} \mathbb{F}_\Delta(L))$ is projective, $\mathbf{X}$ splits. Thus the assertion follows.

(6) $\mathbb{F}_\Delta(L) \oplus J_\Gamma^{-2} \mathbb{F}_\Delta(\tau^+ L)$ is isomorphic to $J_\Gamma^{-1} \mathbb{F}_\Delta(\theta^+ L)$ by the proof of

(5). Applying 2.3.1 (2) to $\mathbb{F}_\Delta \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Delta), \mathrm{F}(\Gamma))$ and $J^{-1} \in \mathrm{End}_\mathbb{Z}(\mathrm{F}(\Gamma))$ $(L \mapsto J_\Gamma^{-1} L)$, we obtain the assertion.

(7) "if" part follows from (6) since $\sum_i J_\Gamma^{-i}\mathbb{F}_\Delta(X_i) = \sum_i (J_\Gamma^{-n}\mathbb{F}_\Delta(\theta^+_{n-i}X_i) - J_\Gamma^{-n-1}\mathbb{F}_\Delta(\tau^+\theta^+_{n-i-1}X_i)) = 0$. Conversely, put $X_i = L_i - L'_i$ $(L_i, L'_i \in \mathrm{lat}\,\Delta)$ and assume $\bigoplus_i J_\Gamma^{-i}\mathbb{F}_\Delta(L_i) \simeq \bigoplus_i J_\Gamma^{-i}\mathbb{F}_\Delta(L'_i)$. Then, for any $n \geq l_\Delta + p_\Delta - 1$, we obtain $\bigoplus_i \mathcal{J}^{n-i}_{\mathrm{lat}\,\Delta}(\ , L_i) = \bigoplus_i \mathcal{J}^n_{\mathrm{lat}\,\Gamma}(\mathbb{F}_\Delta(\ ), J_\Gamma^{-i}\mathbb{F}_\Delta(L_i)) \simeq \bigoplus_i \mathcal{J}^n_{\mathrm{lat}\,\Gamma}(\mathbb{F}_\Delta(\ ), J_\Gamma^{-i}\mathbb{F}_\Delta(L'_i)) = \bigoplus_i \mathcal{J}^{n-i}_{\mathrm{lat}\,\Delta}(\ , L'_i)$ by (2). Thus $\bigoplus_i \theta^+_{n-i}L_i = \bigoplus_i \theta^+_{n-i}L'_i$ holds by 2.3.1 (1). □

2.5. Additive functions

We call $f \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Delta), \mathbb{Z})$ a *right additive function* (resp. *left additive function*) if $f\phi^+ = 0$ (resp. $f\phi^- = 0$) holds.

Corollary 2.5.1. *Let Δ be an R-order of finite representation type with its filtering functor $\mathbb{F}_\Delta : \mathrm{lat}\,\Delta \to \mathrm{lat}\,\Gamma$. Define $J^{-1} \in \mathrm{End}_\mathbb{Z}(\mathrm{F}(\Gamma))$ by $L \mapsto J_\Gamma^{-1}L$.*

(1) *Let A be an abelian group, $f \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Delta), A)$ and $a \in \mathrm{End}_\mathbb{Z}(A)$. If $f - af\theta^+ + a^2 f\tau^+ = 0$ holds, then there exists a unique element $g \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Gamma), A)$ such that $f = g\mathbb{F}_\Delta$ and $gJ^{-1} = ag$.*

(2)([I1, 4.1.1]) *Let $\{e_j\}_{1\leq j\leq c}$ be the complete set of central irreducible idempotents of $\widetilde{\Delta}$. Then $f \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Delta), \mathbb{Z})$ is a right additive function if and only if $f(L) = \sum_{1\leq j\leq c} l_j \,\mathrm{len}_{\widetilde{\Delta}}(\widetilde{L}e_j)$ for some $l_j \in \mathbb{Z}$ if and only if f is a left additive function.*

Proof. (1) Take $X_i \in \mathrm{F}(\Delta)$. If $\sum_{0\leq i<p_\Delta} J_\Gamma^{-i}\mathbb{F}_\Delta(X_i) = 0$ holds, then $\sum_{0\leq i<p_\Delta} a^i f(X_i) = \sum_{0\leq i<p_\Delta}(a^n f\theta^+_{n-i}X_i - a^{n+1}f\tau^+\theta^+_{n-i-1}X_i) = 0$ holds by 2.3.1 (2) and 2.4 (7). Hence, by 2.4 (3), $g \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Gamma), A)$ is well defined by $g(\sum_{0\leq i<p_\Delta} J_\Gamma^{-i}\mathbb{F}_\Delta(X_i)) := \sum_{0\leq i<p_\Delta} a^i f(X_i)$. Then g is a unique element which satisfies the desired properties.

(2) We only have to show the "only if" part of the first equivalence. By (1), there exists $g \in \mathrm{Hom}_\mathbb{Z}(\mathrm{F}(\Gamma), \mathbb{Z})$ such that $f = g\mathbb{F}_\Delta$ and $gJ^{-1} = g$. Take $L_j \in \mathrm{ind}(\Gamma e_j)$ and put $l_j := f(L_j)$. Then $gJ^{-1} = g$ implies that $g(L) = \sum_{1\leq j\leq c} l_j \,\mathrm{len}_{\widetilde{\Gamma}}(\widetilde{L}e_j)$ holds for any $L \in \mathrm{lat}\,\Gamma$. By 2.4 (1), f has the desired form. □

Remark 2.5.2. Above (2) shows that the triple $(\mathbb{F}_\Delta, \mathrm{F}(\Gamma), J^{-1})$ gives an initial object of the category $\mathcal{C}(\mathrm{F}(\Delta); 1, -\theta^+, \tau^+)$, which is defined by (1) below. In particular, we can construct the triple $(\mathbb{F}_\Delta, \mathrm{F}(\Gamma), J^{-1})$ by the manner in (2) below.

(1) Let F be an abelian group and $\eta_i \in \mathrm{End}_\mathbb{Z}(F)$ $(0 \leq i \leq n)$. Define a category $\mathcal{C} = \mathcal{C}(F; \eta_0, \dots, \eta_n)$ as follows. An object is (f, A, a), where A is an abelian group, $f \in \mathrm{Hom}_\mathbb{Z}(F, A)$, $a \in \mathrm{End}_\mathbb{Z}(A)$ such that $\sum_{0\leq i\leq n} a^i f\eta_i = 0$. Put $\mathrm{Hom}((f, A, a), (f', A', a')) := \{g \in \mathrm{Hom}_\mathbb{Z}(A, A') \mid f' = gf,\ ga = a'g\}$.

(2) $\mathcal{C}$ has an initial object $(f_F, \widehat{F}, a_F)$ defined as follows.

Define $a_F \in \mathrm{End}_\mathbb{Z}(\bigoplus_{i\geq 0} F)$ and a subgroup G of $\bigoplus_{i\geq 0} F$ by $a_F(x_0, x_1, \dots) := (0, x_0, x_1, \dots)$ and $G := \sum_{x\in F, i\geq 0} a_F^i(\eta_0(x), \eta_1(x), \dots, \eta_n(x), 0, 0, \dots)$. Put $\widehat{F} := (\bigoplus_{i\geq 0} F)/G$ and $f_F(x) := (x, 0, 0, \dots)$. Then, for any $(f, A, a) \in \mathcal{C}$, it is easy to show that $\mathrm{Hom}((f_F, \widehat{F}, a_F), (f, A, a))$ is a singleton set $\{g\}$, where g is defined by $g(x_0, x_1, \dots) := \sum_{i\geq 0} a^i f(x_i)$.

2.6. Appendix: Grothendieck groups

We denote by $\mathrm{K}_0(\mathcal{C})$ the Grothendieck group of an abelian category $\mathcal{C}$, and by $\mathrm{fl}\,\mathrm{mod}\Delta$ the category of finite length Δ-modules. It was well known (eg. [AR]) that there exists an exact sequence $(*) : \mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta) \xrightarrow{\mathrm{K}_0(\mathbb{I})} \mathrm{K}_0(\mathrm{mod}\Delta) \to \mathrm{K}_0(\mathrm{mod}\widetilde{\Delta}) \to 0$ of Grothendieck groups. When Δ is of finite representation type, a main result of [W] gave an explicit description of the kernel of $\mathrm{K}_0(\mathbb{I})$. The following 2.6.1 shows that his result holds for any order Δ.

On the other hand, when Δ is of finite representation type, 2.6.2 gives a connection between each terms in $(*)$ and $\mathrm{F}(\Delta)$, ϕ^+ etc. In particular, it gives another proof of 2.5.1 (2).

2.6.1. A bounded complex $\mathbf{X} : \cdots \to X_{i-1} \overset{d_{i-1}}{\to} X_i \overset{d_i}{\to} X_{i+1} \to \cdots$ on $\mathrm{lat}\,\Delta$ is called *rationally exact* if the induced complex $\widetilde{\mathbf{X}} : \cdots \to \widetilde{X}_{i-1} \overset{\widetilde{d}_{i-1}}{\to} \widetilde{X}_i \overset{\widetilde{d}_i}{\to} \widetilde{X}_{i+1} \to \cdots$ is exact. Then put $H(\mathbf{X}) := \sum_{i\in\mathbb{Z}}(-1)^i H_i(\mathbf{X}) \in \mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta)$. Let Z be the subgroup of $\mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta)$ generated by $[H(\mathbf{X})]$ for any rationally exact bounded complex $\mathbf{X}$ on $\mathrm{lat}\,\Delta$ such that $\bigoplus_{i\in\mathbb{Z}} X_{2i}$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} X_{2i-1}$.

Proposition. *Let Δ be an R-order in a semisimple $\widetilde{R}$-algebra $\widetilde{\Lambda}$. Then the natural inclusion $\mathbb{I} : \mathrm{fl}\,\mathrm{mod}\Delta \to \mathrm{mod}\Delta$ and $\mathbb{J} := \widetilde{(\ )} : \mathrm{mod}\Delta \to \mathrm{mod}\widetilde{\Delta}$ induce the following exact sequence.*

$$0 \to Z \to \mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta) \xrightarrow{\mathrm{K}_0(\mathbb{I})} \mathrm{K}_0(\mathrm{mod}\Delta) \xrightarrow{\mathrm{K}_0(\mathbb{J})} \mathrm{K}_0(\mathrm{mod}\widetilde{\Delta}) \to 0$$

Thus $\mathrm{K}_0(\mathrm{mod}\Delta)$ *is isomorphic to* $\mathrm{K}_0(\mathrm{mod}\widetilde{\Delta}) \oplus \mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta)/Z$*. Moreover, $Z = \langle [M] \rangle_{M\in \mathrm{fl}\,\mathrm{mod}\Omega}$ holds for any maximal overorder Ω of Δ.*

Proof. (i) Assume that $[M]-[M'] \in \mathrm{Ker}\,\mathrm{K}_0(\mathbb{J})$ holds for $M, M' \in \mathrm{mod}\,\Delta$. Then $\widetilde{M}$ is isomorphic to $\widetilde{M'}$ since $\widetilde{\Delta}$ is semisimple. Hence there exists an exact sequence $0 \to M' \to M \to M'' \to 0$ such that $M'' \in \mathrm{fl}\,\mathrm{mod}\Delta$. Thus $[M] - [M'] = [M''] \in \mathrm{Im}\,\mathrm{K}_0(\mathbb{I})$. Moreover, $\langle [M] \rangle_{M\in \mathrm{fl}\,\mathrm{mod}\Omega} \subseteq \mathrm{Ker}\,\mathrm{K}_0(\mathbb{I})$ holds since the Ω-projective resolution of M has the form $0 \to P \to P \to M \to 0$. Now, we will show $\mathrm{Ker}\,\mathrm{K}_0(\mathbb{I}) \subseteq Z$.

Assume $[M]-[M'] \in \mathrm{Ker}\,\mathrm{K}_0(\mathbb{I})$ holds for $M, M' \in \mathrm{fl}\,\mathrm{mod}\,\Delta$. By definition, we can easily obtain an exact sequence $\mathbf{X} : 0 \to X_1 \to X_2 \to X_3 \to X_4 \to 0$ in $\mathrm{mod}\,\Delta$ such that $X_1 \oplus X_3 \oplus M$ is isomorphic to $X_2 \oplus X_4 \oplus M'$. Let $\mathbb{T} : \mathrm{mod}\,\Delta \to \mathrm{fl}\,\mathrm{mod}\Delta$ be the functor such that $\mathbb{T}(X)$ is the torsion submodule of $X \in \mathrm{mod}\,\Delta$, and $\mathbb{L} : \mathrm{mod}\Delta \to \mathrm{lat}\,\Delta$ the functor defined by $\mathbb{L}(X) := X/\mathbb{T}(X)$. Since $\mathbb{T}(X_1) \oplus \mathbb{T}(X_3) \oplus M$ is isomorphic to $\mathbb{T}(X_2) \oplus \mathbb{T}(X_4) \oplus M'$, we obtain $H(\mathbb{T}(\mathbf{X})) = [M] - [M']$. Since we have an exact sequence $0 \to \mathbb{T}(\mathbf{X}) \to \mathbf{X} \to \mathbb{L}(\mathbf{X}) \to 0$ of complexes, we obtain $[M] - [M'] = H(\mathbb{T}(\mathbf{X})) - H(\mathbf{X}) = -H(\mathbb{L}(\mathbf{X}))$. Thus the assertion follows since $\mathbb{L}(\mathbf{X})$ is a rationally exact complex satisfying $\mathbb{L}(X_1) \oplus \mathbb{L}(X_3) \simeq \mathbb{L}(X_2) \oplus \mathbb{L}(X_4)$.

(ii) We will show $Z \subseteq \langle [M] \rangle_{M \in \mathrm{fl}\,\mathrm{mod}\,\Omega}$ for any maximal overorder Ω of Δ.

Let $\mathbf{X}$ be a rationally exact bounded complex on $\mathrm{lat}\,\Delta$ such that $\bigoplus_{i\in\mathbb{Z}} X_{2i}$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} X_{2i-1}$. Let $\Omega\mathbf{X}$ be the complex $\cdots \to \Omega X_i \xrightarrow{d_i} \Omega X_{i+1} \to \cdots$, and $\mathbf{Y}$ the complex $\cdots \to \Omega X_i/X_i \xrightarrow{d_i} \Omega X_{i+1}/X_{i+1} \to \cdots$. Since $\bigoplus_{i\in\mathbb{Z}} \Omega X_{2i}/X_{2i}$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} \Omega X_{2i-1}/X_{2i-1}$, we obtain $H(\mathbf{Y}) = 0$. Since we have an exact sequence $0 \to \mathbf{X} \to \Omega\mathbf{X} \to \mathbf{Y} \to 0$ of complexes, we obtain $H(\mathbf{X}) = H(\Omega\mathbf{X}) - H(\mathbf{Y}) = H(\Omega\mathbf{X}) \in \langle [M] \rangle_{M \in \mathrm{fl}\,\mathrm{mod}\,\Omega}$. □

Proposition 2.6.2. *Let Δ be an R-order of finite representation type. Then we have the following commutative diagram of exact sequences whose vertical maps are isomorphisms.*

$$\begin{array}{ccccccc}
\mathrm{F_p}(\Delta) & \xrightarrow{\phi^+} & \mathrm{F}(\Delta)/\phi^+(\mathrm{F_n}(\Delta)) & \longrightarrow & \mathrm{F}(\Delta)/\phi^+(\mathrm{F}(\Delta)) & \longrightarrow & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\
\mathrm{K}_0(\mathrm{fl}\,\mathrm{mod}\Delta) & \xrightarrow{\mathrm{K}_0(\mathbb{I})} & \mathrm{K}_0(\mathrm{mod}\Delta) & \xrightarrow{\mathrm{K}_0(\mathbb{J})} & \mathrm{K}_0(\mathrm{mod}\widetilde{\Delta}) & \longrightarrow & 0
\end{array}$$

Proof. Define f_i $(i = 0, 1, 2)$ by $f_2(L) := [L/J_\Delta L]$, $f_1(L) := [L]$ and $f_0(L) := [\widetilde{L}]$. Clearly f_2 is an isomorphism, and f_1 is an isomorphism by [AR, 1.1 Chapter 2]. Thus f_0 is also an isomorphism. □

2.7. We state a result concerning structure of orders of finite representation type (cf. [I3, 3.3]). We call a filtration $(\Omega = I_0 \supseteq I_{-1} \supseteq I_{-2} \supseteq \cdots)$ of a hereditary order Ω *almost J-adic* if there exists another hereditary order Γ, an idempotent e of Γ and an R-algebra isomorphism $f : e\Gamma e \to \Omega$ such that $I_i = f(eJ_\Gamma^{-i}e)$ holds for any $i \leq 0$.

Corollary. *Let R be a complete discrete valuation ring with a residue field k, $k[[t]]$ the formal power series ring and Δ an R-order of finite representation type. Then there exists a hereditary overorder Ω of Δ and an almost J-adic filtration $\{I_i\}_{i\leq 0}$ of Ω such that $\Delta' := \prod_{i\leq 0}(\Delta \cap I_i/\Delta \cap I_{i-1})$ is a $k[[t]]$-order whose Auslander-Reiten quiver coincides with that of Δ.*

Proof. Let Λ be the Auslander order of Δ and Γ the filtering overorder of Λ. Then there exists an idempotent e of Λ such that $e\Lambda e = \Delta$. Putting $\Omega := e\Gamma e$ and $I_i := eJ_\Gamma^{-i}e$, we obtain the assertion by 2.2.1. □

Examples 2.8. (1) Let $\Delta := \left(\begin{smallmatrix} \Omega & \Omega \\ J^n & \Omega \end{smallmatrix}\right)$ $(n = 2m - 1 > 0)$, where Ω is a local maximal order with the radical J. Then $\mathrm{ind}\,\Delta = \{\left(\begin{smallmatrix} \Omega \\ J^j \end{smallmatrix}\right)\}_{0\leq j\leq n}$ holds. The

Auslander order Λ of Δ and the filtering overorder Γ of Λ are the following.

$$\overset{\Lambda}{\begin{pmatrix} \Omega & \Omega & \Omega & \cdots & \Omega & \Omega & \Omega \\ J & \Omega & \Omega & \cdots & \Omega & \Omega & \Omega \\ J^2 & J & \Omega & \cdots & \Omega & \Omega & \Omega \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ J^{n-2} & J^{n-3} & J^{n-4} & \cdots & \Omega & \Omega & \Omega \\ J^{n-1} & J^{n-2} & J^{n-3} & \cdots & J & \Omega & \Omega \\ J^n & J^{n-1} & J^{n-2} & \cdots & J^2 & J & \Omega \end{pmatrix}}$$

$$\subset \overset{\Gamma}{\begin{pmatrix} \Omega & \Omega & J^{-1} & \cdots & J^{2-m} & J^{1-m} & J^{1-m} \\ J & \Omega & \Omega & \cdots & J^{2-m} & J^{2-m} & J^{1-m} \\ J & J & \Omega & \cdots & J^{3-m} & J^{2-m} & J^{2-m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ J^{m-1} & J^{m-2} & J^{m-2} & \cdots & \Omega & \Omega & J^{-1} \\ J^{m-1} & J^{m-1} & J^{m-2} & \cdots & J & \Omega & \Omega \\ J^m & J^{m-1} & J^{m-1} & \cdots & J & J & \Omega \end{pmatrix}}$$

Thus Γ is Morita equivalent to $\left(\begin{smallmatrix} \Omega & \Omega \\ J & \Omega \end{smallmatrix}\right)$. Put $L_{even} := \bigoplus_{0\le j<m} \binom{\Omega}{J^{2j}}$ and $L_{odd} := \bigoplus_{0<j\le m} \binom{\Omega}{J^{2j-1}}$. Then, for sufficiently large i, it can be checked that $\theta_i^+ \binom{\Omega}{J^j} = L_{even}$ (resp. L_{odd}) holds if $i+j$ is even (resp. odd).

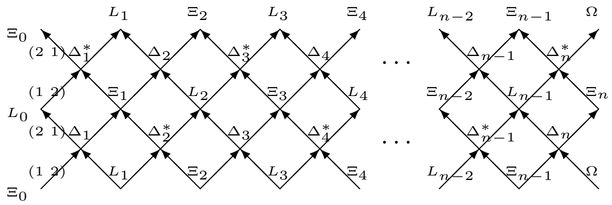
(2) Let $\{\Xi_j\}_{0\le j\le n}$ be a local Bass chain of type(IVa) ([HN1]) and $\Delta := \Xi_n$ ($n = 2m-1 > 0$). Then $\operatorname{ind}\Delta = \{\Xi_j\}_{0\le j\le n}$ holds. The Auslander order Λ of Δ is the following order ($J_{\Xi_n} = \Xi_{n-1}x = x\Xi_{n-1}$), and the filtering order Γ of Λ is the same order as in (1) above ($\Omega := \Xi_0$).

$$\Lambda = \begin{pmatrix} \Xi_n & \Xi_{n-1} & \Xi_{n-2} & \cdots & \Xi_2 & \Xi_1 & \Xi_0 \\ \Xi_{n-1}x & \Xi_{n-1} & \Xi_{n-2} & \cdots & \Xi_2 & \Xi_1 & \Xi_0 \\ \Xi_{n-2}x^2 & \Xi_{n-2}x & \Xi_{n-2} & \cdots & \Xi_2 & \Xi_1 & \Xi_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Xi_2x^{n-2} & \Xi_2x^{n-3} & \Xi_2x^{n-4} & \cdots & \Xi_2 & \Xi_1 & \Xi_0 \\ \Xi_1x^{n-1} & \Xi_1x^{n-2} & \Xi_1x^{n-3} & \cdots & \Xi_1x & \Xi_1 & \Xi_0 \\ \Xi_0x^n & \Xi_0x^{n-1} & \Xi_0x^{n-2} & \cdots & \Xi_0x^2 & \Xi_0x & \Xi_0 \end{pmatrix}$$

Put $L_{even} := \Xi_0 \oplus (\bigoplus_{0<j<m} \Xi_{2j}^2)$ and $L_{odd} := \bigoplus_{0<j\le m} \Xi_{2j-1}^2$. Then, for sufficiently large i, it can be checked that $\theta_i^+\Xi_j = L_{even}$ (resp. L_{odd}) holds if $i+j$ is even (resp. odd).

(3) In this example, we will compute the filtering functor $\mathbb{F}_\Delta : \operatorname{lat}\Delta \to \operatorname{lat}\Gamma$ without considering the Auslander order. Let $\{\Xi_j\}_{0\le j\le n}$ be a local Bass chain of type(IVa) again, Ω a local maximal order and $f : \Omega/J_\Omega \to \Xi_n/J_{\Xi_n}$ an R-algebra isomorphism. Put $\Delta = \Delta_n := \{(x,y) \in \Omega \times \Xi_n \mid f(\overline{x}) = \overline{y}\}$ ($n = 2m-1 > 0$). Then Δ is an order of finite representation type with the

following Auslander-Reiten quiver ([HN2]).



It is easily checked that $(\theta_i^+\Omega)_{i\geq 0}$ has the following period (1) of length $4n$, and $(\theta_i^+\Xi_n)_{i\geq 0}$ has the following period (2) of length 4 for sufficiently large i. Hence Γ is Morita equivalent to $\mathrm{T}_{4n}(\Omega)\times\mathrm{T}_4(\Xi_0)$ (1.1.1 (2)) by 2.4 (3).

$$(1)\quad \begin{aligned}&(\theta_i^+\Omega)_{0\leq i<4n}\\ &=(\Omega,\Delta_n^*,L_{n-1},\Delta_{n-1}^*,\dots,L_1,\Delta_1^*,L_0,\Delta_1,L_1,\dots,L_{n-2},\Delta_{n-1},L_{n-1},\Delta_n)\end{aligned}$$

(2)

$i \pmod 4$	$\theta_i^+\Xi_n$
0	$(\oplus_{0\leq j<m}\Xi_{n-2j}^2)\oplus(\oplus_{0<j<m}L_{n-2j+1}^2)\oplus L_0$
1	$(\oplus_{0\leq j<m}\Delta_{n-2j}^2)\oplus(\oplus_{0<j<m}\Delta_{n-2j+1}^{*2})$
2	$(\oplus_{0<j<m}L_{n-2j}^2)\oplus(\oplus_{0<j<m}\Xi_{n-2j+1}^2)\oplus\Xi_0$
3	$(\oplus_{0\leq j<m}\Delta_{n-2j}^{*2})\oplus(\oplus_{0<j<m}\Delta_{n-2j+1}^2)$

Putting $P:=\mathbb{F}_\Delta(\Omega)$ and $Q:=\mathbb{F}_\Delta(\Xi_n)$, we can obtain the following list of $\mathbb{F}_\Delta$ by using $\mathbb{F}_\Delta(L)\oplus J_\Gamma^{-2}\mathbb{F}_\Delta(\tau^+L)\simeq J_\Gamma^{-1}\mathbb{F}_\Delta(\theta^+L)$ (2.4 (6)) repeatedly.

$\mathbb{F}_\Delta(\Xi_{n-i})$	$\mathbb{F}_\Delta(\Delta_{n-i})$	$\mathbb{F}_\Delta(\Delta_{n-i}^*)$	$\mathbb{F}_\Delta(L_{n-i})$
$J_\Gamma^{2i}Q$	$J_\Gamma^{-2i-1}P\oplus J_\Gamma^{2i+1}Q$	$J_\Gamma^{2i+1}P\oplus J_\Gamma^{-2i-1}Q$	$J_\Gamma^{2i}P\oplus J_\Gamma^{-2i}P\oplus J_\Gamma^{2i+2}Q$

Added in proof: Professor W. Rump kindly informed the author that 2.6.1 was given in his paper [R, Proposition 10.2].

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