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Topological characterization of extensor product on BSO

By

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1. Introduction

This paper is a serial work from [1] and [2]. In [1] we characterize the map $\hat{\otimes}_{\mathbb{C}} : \mathrm{BU} \wedge \mathrm{BU} \to \mathrm{BU}$ which is induced from the extensor product of complex bundles. The crucial point of this characterization is that $[\mathrm{BU} \wedge \mathrm{BU}, \mathrm{BU}]_*$ is a free $\mathbb{Z}$-module and $\widetilde{K}(\mathrm{BU} \wedge \mathrm{BU}) \otimes \mathbb{Q} = [\mathrm{BU} \wedge \mathrm{BU}, \mathrm{BU}]_* \otimes \mathbb{Q} \cong H^*(\mathrm{BU} \wedge \mathrm{BU}; \mathbb{Q})$. Noticing these facts, $\hat{\otimes}_{\mathbb{C}} : \mathrm{BU} \wedge \mathrm{BU} \to \mathrm{BU}$ can be characterized by the informations of integral cohomology, so that we can use Chern character.

The case of BSO is a little more complicated than that of BU because $H^*(\mathrm{BSO}; \mathbb{Z})$ has torsion. But this will be overcome by the striking fact that $[\mathrm{BSO} \wedge \mathrm{BSO}, \mathrm{BSO}]_*$ is a free $\mathbb{Z}$-module shown in Section 2. Then we can characterize the map $\hat{\otimes}_{\mathbb{R}} : \mathrm{BSO} \wedge \mathrm{BSO} \to \mathrm{BSO}$, which is induced from the extensor product of oriented real bundles, by the informations of $[\mathrm{BSO} \wedge \mathrm{BSO}, \mathrm{BSO}]_* \otimes \mathbb{Q} \cong H^*(\mathrm{BSO} \wedge \mathrm{BSO}; \mathbb{Q})$, which is obtained by the universality of the localization of H-spaces and that $\mathrm{BSO}_{(0)} \simeq \prod_n K(\mathbb{Q}, 4n)$.

Precisely the way of characterization is as follows. We consider a Hopf space X equivalent to BSO and a map $\tilde{\lambda} : X \wedge X \to X$ which holds some of the cohomological properties of $\hat{\otimes}_{\mathbb{R}} : \mathrm{BSO} \wedge \mathrm{BSO} \to \mathrm{BSO}$. Then we can construct another Hopf equivalence $g' : X \to \mathrm{BSO}$ which satisfies the following homotopy commutative diagram.

$$\begin{array}{ccc} X \wedge X & \xrightarrow{g' \wedge g'} & \mathrm{BSO} \wedge \mathrm{BSO} \\ {\scriptstyle \tilde{\lambda}} \downarrow & & \downarrow {\scriptstyle \hat{\otimes}_{\mathbb{R}}} \\ X & \xrightarrow[g']{} & \mathrm{BSO} \end{array}$$

2. Main theorem

We need some notations to state the main theorem and related lemmas clearly.

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We denote $H^*(\cdot;\mathbb{Z})/\operatorname{Tor}$ by $h^*(\cdot)$. Let ξ be the virtual vector bundle on BSO represented by $[\iota] \in [\mathrm{BSO}, \mathrm{BO}]_* = \widetilde{KO}(\mathrm{BSO})$, where $\iota : \mathrm{BSO} \to \mathrm{BO}$ is the natural inclusion. We know that $h^*(\mathrm{BSO}) = \mathbb{Z}[p_1, p_2, \ldots]$, where p_i is the i-th Pontryagin class of ξ. Let t_n be the n-th power sum in $\{p_i\}$.

Now we state the key lemmas.

Lemma 2.1. $[\mathrm{BSO} \wedge \mathrm{BSO}, \mathrm{BSO}]_*$ *is a free* $\mathbb{Z}$*-module.*

Proof. It is shown in [3] for a compact Lie group G that:

$$KO(\mathrm{BG}) \cong RO(\mathrm{G})^\wedge,$$

where $RO(\mathrm{G})$ is a real representation ring of G, $RO(\mathrm{G})^\wedge$ is $I(\mathrm{G})$-adic completion of $RO(\mathrm{G})$ and $I(\mathrm{G}) = \ker\{\deg : RO(\mathrm{G}) \to \mathbb{Z}\}$.

It is also shown in [4, p. 94] that $RO(\mathrm{SO}(2n+1)) \cong \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_n]$, where $\deg\lambda_i = 0$ and $\deg 1 = 1$. Therefore we have $\widetilde{KO}(\mathrm{BSO} \times \mathrm{BSO}) = [\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BO}]_*$ is a free $\mathbb{Z}$-module by the following.

$$\begin{aligned} KO(\mathrm{BSO} \times \mathrm{BSO}) &= \lim_{\leftarrow} KO(\mathrm{BSO}(2n+1) \times \mathrm{BSO}(2n+1)) \\ &= \lim_{\leftarrow} RO(\mathrm{SO}(2n+1) \times \mathrm{SO}(2n+1))^\wedge \end{aligned}$$

Since $\mathrm{BO} \simeq \mathrm{BSO} \times \mathbb{R}\mathrm{P}^\infty$ and $[\mathrm{BSO}, \mathbb{R}\mathrm{P}^\infty]_* \cong H^1(\mathrm{BSO}; \mathbb{Z}_2) = 0$, we have $[\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BO}]_* \cong [\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BSO} \times \mathbb{R}\mathrm{P}^\infty]_* \cong [\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BSO}]_*$. Hence $[\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BSO}]_*$ is a free $\mathbb{Z}$-module. Then $[\mathrm{BSO} \wedge \mathrm{BSO}, \mathrm{BSO}]_*$ is submodule of $[\mathrm{BSO} \times \mathrm{BSO}, \mathrm{BSO}]_*$, so that we have $[\mathrm{BSO} \wedge \mathrm{BSO}, \mathrm{BSO}]_*$ is a free $\mathbb{Z}$-module. □

Lemma 2.2.

$$\hat{\otimes}_{\mathbb{R}}^*(t_n) = 2\sum_{i=1}^{n-1} \binom{2n}{2i} t_i \otimes t_{n-i} \in h^*(\mathrm{BSO} \wedge \mathrm{BSO})$$

Proof. We have the following relation for a real vector bundle η.

$$\begin{aligned} \hat{\otimes}_{\mathbb{C}}^*(\eta \otimes_{\mathbb{R}} \mathbb{C}) &= (\eta \otimes_{\mathbb{R}} \mathbb{C}) \hat{\otimes}_{\mathbb{C}} (\eta \otimes_{\mathbb{R}} \mathbb{C}) \\ &= (\eta \,\hat{\otimes}_{\mathbb{R}}\, \eta) \otimes_{\mathbb{R}} \mathbb{C} \\ &= (\hat{\otimes}_{\mathbb{R}}^*\, \eta) \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

Let $s_n(\zeta)$ be the n-th power sum in Chern classes of a complex bundle ζ. By considering the above relation and Chern character, we obtain the below.

$$s_n((\hat{\otimes}_{\mathbb{R}}^*\, \eta) \otimes_{\mathbb{R}} \mathbb{C}) = \sum_{i=1}^{n-1} \binom{n}{i} s_i(\eta \otimes_{\mathbb{R}} \mathbb{C}) \otimes s_{n-i}(\eta \otimes_{\mathbb{R}} \mathbb{C})$$

Since $s_{2n-1}(\eta \otimes_{\mathbb{R}} \mathbb{C}) = 0$ and $s_{2n}(\eta \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^n 2t_n$ in $h^*(\mathrm{BSO})$, we obtain the equation in the statement. □

Now we come to state the main theorem.

Theorem 2.1. *Let $\mu : X \times X \to X$ be a Hopf space which is of finite type CW-complex and has a Hopf equivalence $g : X \to \mathrm{BSO}$. Suppose there exists a map $\tilde{\lambda} : X \wedge X \to X$ holding the following properties.*

(1) *$\lambda^* : Qh^n(X) \to h^n(X^{(4)} \wedge X)$ is split monic ($n > 4$), where $X^{(4)}$ is 4-skeleton of X and λ is the restriction of $\tilde{\lambda}$ to $X^{(4)} \wedge X$.*

(2) *The torsion part of* $\mathrm{coker}\{\lambda^* : Qh^{4n}(X) \to h^{4n}(X^{(4)} \wedge X)\}$ *is isomorphic to $\mathbb{Z}_{2n-1}$ ($n > 1$).*

(3) *$\tilde{\lambda} \circ T \simeq \tilde{\lambda}$, where $T : X \wedge X \to X \wedge X$ is the twisting map.*

(4) *$\tilde{\lambda} \circ (1 \wedge \tilde{\lambda}) \simeq \tilde{\lambda} \circ (\tilde{\lambda} \wedge 1)$*

(5) *The following diagram is homotopy commutative, where $\triangle$ is the diagonal map.*

$$\begin{tikzcd}
X \wedge (X \times X) \arrow[r, "\triangle\wedge 1"] \arrow[d, "1\wedge\mu"'] & (X \times X) \wedge (X \times X) \arrow[r, "1\times T\times 1"] & (X \wedge X) \times (X \wedge X) \arrow[d, "\tilde{\lambda}\times\tilde{\lambda}"] \\
X \wedge X \arrow[r, "\tilde{\lambda}"'] & X & X \times X \arrow[l, "\mu"]
\end{tikzcd}$$

Then there exists another Hopf equivalence $g' : X \to \mathrm{BSO}$ which satisfies the following homotopy commutative diagram.

$$\begin{tikzcd}
X \wedge X \arrow[r, "g'\wedge g'"] \arrow[d, "\tilde{\lambda}"'] & \mathrm{BSO} \wedge \mathrm{BSO} \arrow[d, "\hat{\otimes}_{\mathbb{R}}"] \\
X \arrow[r, "g'"'] & \mathrm{BSO}
\end{tikzcd}$$

Proof. By Lemma 2.1 we only need to construct a Hopf equivalence $g' : X \to \mathrm{BSO}$ which satisfies the following.

$$[\hat{\otimes}_{\mathbb{R}} \circ (g' \wedge g')] = [g' \circ \tilde{\lambda}] \in [X \wedge X, \mathrm{BSO}]_* \otimes \mathbb{Q}$$

This is equal to saying the following for all n.

$$(\hat{\otimes}_{\mathbb{R}} \circ (g' \wedge g'))^*(p_n) = (g' \circ \tilde{\lambda})^*(p_n) \in H^{4n}(X \wedge X; \mathbb{Q})$$

Therefore we will construct another Hopf equivalence $g' : X \to \mathrm{BSO}$ which satisfies the following for all n.

$$(\hat{\otimes}_{\mathbb{R}} \circ (g' \wedge g'))^*(p_n) = (g' \circ \tilde{\lambda})^*(p_n) \in h^{4n}(X \wedge X)$$

Let x_n and u_n be $g^*(p_n)$ and $g^*(t_n)$. We know that t_n is a generator of $Ph^{4n}(X)$.

First of all we calculate $\lambda^*(t_n) \in h^{4n}(X^{(4)} \wedge X)$. Since $h^4(X^{(4)})$ is a free $\mathbb{Z}$-module generated by u_1 and $h^n(X^{(4)}) = 0$ ($n \neq 0, 4$), there exists a unique element $v_{n-1} \in h^{4n-4}(X)$ such that $\lambda^*(u_n) = u_1 \otimes v_{n-1} \in h^{4n}(X^{(4)} \wedge X)$. By property 5 we can calculate $\lambda^*(u_n) \in h^{4n}(X^{(4)} \wedge X)$ as follows.

$$
\begin{array}{ccccc}
u_1 \otimes (v_{n-1} \otimes 1 + 1 \otimes v_{n-1}) & \longleftarrow & \begin{array}{c} u_1 \otimes 1 \otimes v_{n-1} \otimes 1 \\ + 1 \otimes u_1 \otimes 1 \otimes v_{n-1} \end{array} & \longleftarrow & \begin{array}{c} u_1 \otimes v_{n-1} \otimes 1 \otimes 1 \\ + 1 \otimes 1 \otimes u_1 \otimes v_{n-1} \end{array} \\
{\scriptstyle 1\otimes\mu^*}\Big\uparrow & & & & \Big\uparrow \\
u_1 \otimes v_{n-1} & \longleftarrow & u_n & \longrightarrow & u_n \otimes 1 + 1 \otimes u_n
\end{array}
$$

Then we see that v_{n-1} is primitive. Since u_{n-1} is a generator of $Ph^{4n-4}(X)$ and $\lambda^*(u_n) = \pm 2n\lambda^*(x_n)$, we obtain the following by properties 1 and 2.

$$\lambda^*(u_n) = \epsilon_n 2n(2n-1) u_1 \otimes u_{n-1} \in h^{4n}(X^{(4)} \wedge X),$$

where $\epsilon_n = \pm 1$.

Note that $\wp^1 s_{2n} \equiv 2n s_{2n+p-1} \mod p$. Then we obtain the following by naturality of $\wp^1$.

$$\wp^1 u_n \equiv 2n u_{n+(p-1)/2} \mod p$$

$$
\begin{aligned}
&\epsilon_n 2n(2n-1)(2n-2) u_1 \otimes u_{n-1+(p-1)/2} \\
&\quad \equiv \wp^1(\epsilon_n 2n(2n-1) u_1 \otimes u_{n-1}) \\
&\quad \equiv \wp^1 \lambda^*(u_n) \\
&\quad \equiv \lambda^*(\wp^1 u_n) \\
&\quad \equiv \lambda^*(2n u_{n+(p-1)/2}) \\
&\quad \equiv \epsilon_{n+(p-1)/2} 2n(2n+p-1)(2n+p-2) u_1 \otimes u_{n-1+(p-1)/2} \mod p
\end{aligned}
$$

Then we obtain the following relations.

$$
\begin{aligned}
n(n-1)(2n-1)\epsilon_n &\equiv n(n-1)(2n-1)\epsilon_{n+(p-1)/2} && \mod p \\
n(2n-1)(2n+1)\epsilon_n &\equiv n(2n-1)(2n+1)\epsilon_{n-(p-1)/2} && \mod p
\end{aligned}
$$

When n is even, there exists an odd prime p greater than 3 which divides either $n-1$ or $n+1$. Note that $3\epsilon_2 \equiv 3\epsilon_4 \pmod 5$, that is $\epsilon_2 = \epsilon_4$. Then we obtain $\epsilon_2 = \epsilon_4 = \epsilon_6 = \cdots$ by induction using the following.

$$3\epsilon_n \equiv 3\epsilon_{n-(p-1)/2} \mod p \qquad \text{if} \quad p|n-1 \quad \text{or} \quad p|n+1$$

When n is odd, we can choose prime p greater than $2n-1$ and equal to $4N-1$ for some N. Then we have the following.

$$\epsilon_n \equiv \epsilon_{n+(p-1)/2} \equiv \epsilon_{n+2N-1} \mod p$$

Therefore we obtain $\epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \cdots$.

We set a new Hopf equivalence $g' : X \to \mathrm{BSO}$ as follows, where $I : \mathrm{BSO} \to \mathrm{BSO}$ is the homotopy inverse map.

$$g' = \begin{cases} g, & \epsilon_n = 1 \\ I \circ g, & \epsilon_n = -1 \end{cases}$$

Let x'_n and u'_n be $g'^*(p_n)$ and $g'^*(t_n)$. It is easily verified the below.

$$\lambda^*(u'_n) = 2n(2n-1)u'_1 \otimes u'_{n-1} \in h^{4n}(X^{(4)} \wedge X)$$

Next we calculate the $h^{4j}(X) \otimes h^{4n-4j}(X)$-part of $\tilde{\lambda}^*(u'_n) \in h^{4n}(X \wedge X)$. Let $\alpha_1, \ldots, \alpha_l$ be basis of $h^{4j}(X)$ and the $h^{4j}(X) \otimes h^{4n-4j}(X)$-part of $\tilde{\lambda}^*(u'_n)$ be $\sum_{k=1}^{l} \alpha_k \otimes w_{k,n}$, where $w_{k,n} \in h^{4n-4j}(X)$. Similar to the case of $\lambda^*(u_n) \in h^{4n}(X^{(4)} \wedge X)$, we see that $w_{k,n}$ is primitive, that is $w_{k,n} = \delta_k u'_{n-j}$ for some $\delta_k \in \mathbb{Z}$. Therefore we obtain the $h^{4j}(X) \otimes h^{4n-4j}(X)$-part of $\tilde{\lambda}^*(u'_n) \in h^{4n}(X \wedge X)$ is $(\sum_{k=1}^{l} \delta_k \alpha_k) \otimes u'_{n-j}$.

By property 3 we see that $\sum_{k=1}^{l} \delta_k \alpha_k$ is also primitive. Therefore we obtain the following.

$$\tilde{\lambda}^*(u'_n) = \sum_{j=1}^{n-1} \delta_{n,j} u'_j \otimes u'_{n-j} \in h^{4n}(X \wedge X),$$

where $\delta_{n,j} \in \mathbb{Z}$ and $\delta_{n,1} = \delta_{1,n} = 2n(2n-1)$.

By property 4 we have the equations below.

$$\begin{aligned}
\text{coefficient of } u'_1 \otimes u'_1 \otimes u'_{n-2} &= \delta_{n,1}\delta_{n-1,1} = \delta_{n,2}\delta_{2,1} \\
\text{coefficient of } u'_1 \otimes u'_2 \otimes u'_{n-3} &= \delta_{n,1}\delta_{n-1,2} = \delta_{n,3}\delta_{3,1} \\
&\vdots \qquad\qquad \vdots \\
\text{coefficient of } u'_1 \otimes u'_{j-1} \otimes u'_{n-j} &= \delta_{n,1}\delta_{n-1,j-1} = \delta_{n,j}\delta_{j,1} \\
&\vdots \qquad\qquad \vdots \\
\text{coefficient of } u'_1 \otimes u'_{n-2} \otimes u'_1 &= \delta_{n,1}\delta_{n-1,n-2} = \delta_{n,n-1}\delta_{n-1,1}
\end{aligned}$$

Therefore we obtain the below.

$$\delta_{n,j} = \frac{2n(2n-1)}{2j(2j-1)}\delta_{n-1,j-1} = \cdots = 2 \cdot \frac{(2n)!}{(2j)!(2n-2j)!} = 2\binom{2n}{2j}$$

By Lemma 2.2, we have the following for all n.

$$(\hat{\otimes}_{\mathbb{R}} \circ (g' \wedge g'))^*(t_n) = (g' \circ \tilde{\lambda})^*(t_n) \in h^{4n}(X \wedge X)$$

By Newton's formula $t_n = \sum_{i=1}^{n-1}(-1)^{i-1}p_i t_{n-i} + (-1)^{n-1} n p_n$, the proof is completed. $\square$

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