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The image of cycle map of the classifying space of the exceptional group F_4

By

Nobuaki YAGITA

Abstract

The image of the cycle map $CH^*(BF_4)_{(3)} \to H^*(BF_4)_{(3)}$ is studied by using BP^*-theory and the motivic cohomology.

1. Introduction

Let X be a smooth algebraic variety over the complex number field $\mathbb{C}$. Let $CH^*(X)$ be its Chow ring and $BP^*(X)$ the Brown-Peterson cohomology localized at a prime p. Totaro [To1] defined the modified cycle map

$$\bar{cl} : CH^n(X)_{(p)} \to (BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)})^{2n}$$

such that its composition with the Thom map $(BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)})^{2n} \to H^{2n}(X)_{(p)}$ is the usual cycle map $cl : CH^n(X)_{(p)} \to H^{2n}(X)_{(p)}$. Totaro conjectures that the above map $\bar{cl}$ is isomorphic for any $X = BG$ where G is a linear algebraic group (e.g. finite group) over $\mathbb{C}$. While BG itself is not a smooth variety, it is a colimit of smooth varieties and we can define $CH^*(BG)$ naturally ([To1, 2]).

Totaro computed the Chow rings of classifying spaces of abelian groups and symmetric groups in [To1, 2], and he and Pandharipande [To2, P] determined the Chow rings of $BO(n)$, $BSO(2n+1)$ and $BSO(4)$. For these cases the cycle maps $\bar{cl}$ are of course isomorphisms. Vezzosi [Ve] has shown that $\bar{cl}$ is epimorphic for $X = BPGL_3(\mathbb{C})$, $p = 3$. Surjectivity of $\bar{cl}$ are also shown in [Ya] for the cases $X = BG_2, BSpin(7), BD_8$; the dihedral group of order 8 for $p = 2$, and Bp_+^{1+2} ; the extraspecial p-group of order p^3 and exponent p for odd primes.

In this paper, we consider the case that G is an algebraic group which corresponds the exceptional Lie group F_4 and $p = 3$. The mod 3 ordinary cohomology $H^*(BF_4; \mathbb{Z}/3)$ and its cohomology operations are completely determined by Toda [Toda]. The BP-theory $BP^*(BF_4)$ is computed by Kono-Yagita in [K-Y]. Using these results we get new information of the cycle map $\bar{cl}$, while our results are incomplete. The cohomology $H^*(BF_4; \mathbb{Z}/3)$ is generated by x_i of degree $i = 4, 8, 9, 20, 21, 25, 26, 36$ and 48.

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Proposition 1.1. *Let $X = BF_4$ and $p = 3$. If $x_8^2 \in \mathrm{Im}(cl)$, then $\bar{cl}$ is epic.*

2. mod 3 cohomology and BP-theory

We recall $H^*(BF_4;\mathbb{Z}/3)$ and $BP^*(BF_4)$ which are used in the preceding section. The mod 3 cohomology is completely determined by Toda.

Theorem 2.1 ([Toda]). *The cohomology $H^*(BF_4;\mathbb{Z}/3)$ is (additively) isomorphic to*

$$\mathbb{Z}/3[x_{36},x_{48}]\otimes(\mathbb{Z}/3[x_4,x_8]\otimes\{1,x_{20},x_{20}^2\}+\mathbb{Z}/3[x_{26}]\otimes\Lambda(x_9)\otimes\{1,x_{20},x_{21},x_{25}\}),$$

where the above two terms have the intersection $\{1,x_{20}\}$.

Remark. Toda also determines the multiplicative structure. See [Toda] for detail multiplicative relations ,e.g., $x_{21}x_8 + x_{20}x_9 = 0$.

This theorem is proved by use of the fiber bundle

$$\Pi \longrightarrow BSpin(9) \longrightarrow BF_4, \tag{2.1}$$

where $\Pi = F_4/Spin(9)$ is the Cayley plane. Let T be the maximal torus of $Spin(9) \subset F_4$, and $W(G)$ the Weyl group of G. Let $H^*(BT;\mathbb{Z}/3) \cong \mathbb{Z}/3[u_1,u_2,u_3,u_4]$. It is well-known that

$$H^*(BSpin(9);\mathbb{Z}/3) \cong H^*(BT;\mathbb{Z}/3)^{W(Spin(9))} \cong \mathbb{Z}/3[p_1,p_2,p_3,p_4],$$

where p_i is the Pontrjagin class of degree $4i$, which is the i-th elementary symmetric function of variables u_i^2. The Weyl group $W(F_4)$ is generated by $W(Spin(9))$ and by $R(u_i) = u_i - (u_1+u_2+u_3+u_4)/2$. The invariant ring for F_4 is also computed by Toda

$$\begin{aligned} &H^*(BT;\mathbb{Z}/3)^{W(F_4)} \cong \mathbb{Z}/3[p_1,\bar{p}_2,\bar{p}_5,\bar{p}_9,\bar{p}_{12}]/(r_{15}) \subset \mathbb{Z}/3[p_1,p_2,p_3,p_4],\\ &\quad \text{where}\quad \bar{p}_2 = p_2 - p_1^2,\ \ \bar{p}_5 = p_4p_1 + p_3\bar{p}_2,\ \ \bar{p}_9 \equiv p_3^3\ mod(I),\\ &\quad \bar{p}_{12} \equiv p_4^3\ mod(I),\ r_{15} \equiv \bar{p}_5^3\ mod(I) \quad \text{with } I = Ideal(p_1,\bar{p}_2). \end{aligned} \tag{2.2}$$

Let us write the inclusion $i : T \subset F_4$. The above elements correspond even degree generators (except for x_{26})

$$i^*(x_4) = p_1,\ i^*(x_8) = \bar{p}_2,\ i^*(x_{20}) = \bar{p}_5,\ i^*(x_{36}) = \bar{p}_9,\ i^*(x_{48}) = \bar{p}_{12}. \tag{2.3}$$

By using this fact, reduced power operations are also given by

$$\begin{aligned} &P^1(x_4) = -x_8 + x_4^2,\ P^1(x_8) = x_8x_4,\ P^1(x_{20}) = 0,\\ &P^3(x_4) = 0,\ P^3(x_8) = x_{20} - x_8^2x_4,\ P^3(x_{20}) = x_{20}(-x_8 + x_4^2)x_4,\\ &P^3x_{36} \equiv x_{48}\ mod(x_4,x_8), \quad \text{and so on.} \end{aligned} \tag{2.4}$$

The other generators are defined by using the spectral sequence induced from the fibering (2.1)

$$E_2^{*,*} = H^*(BF_4; \mathbb{Z}/3) \otimes H^*(\Pi; \mathbb{Z}/3) \Longrightarrow H^*(BSpin(9); \mathbb{Z}/3).$$

Let $H^*(\Pi; \mathbb{Z}/3) \cong \mathbb{Z}/3[w]/(w^3)$ with $|w| = 8$. Then odd dimensional generators and x_{26} are given by

$$d_9(w) = x_9,\ d_{17}(x_4 w^2) = x_{21},\ d_{17}(x_8 w^2) = x_{25},\quad d_{17}(x_9 w^2) = x_{26}.$$

We also know the cohomology operations

$$(2.5) \qquad \beta(x_i) = x_{i+1}\ (i = 8, 20, 25), \qquad P^1(x_{21}) = x_{25},\quad P^3(x_9) = x_{21}.$$

Now we consider the $BP^*(-)$-theory. Let us write its coefficient ring by $BP^* = \mathbb{Z}_{(p)}[v_1, \ldots, v_i, \ldots]$ with $|v_i| = -2(p^i - 1)$.

Theorem 2.2 ([K-Y]). *The BP-cohomology $BP^*(BF_4)$ has a filtration whose graded ring is*

$$grBP^*(BF_4) \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus BP^*/(3, v_1, v_2)[x_{26}]\{x_{26}\}),$$

where $D = \mathbb{Z}_{(3)}[x_{36}, x_{48}]$ and $E = \mathbb{Z}_{(3)}[x_4, x_8]\{ab | a, b \in \{x_4, x_8, x_{20}\}\}$.

This theorem is proved by use of the Atiyah-Hirzebruch spectral sequence

$$E_2 = H^*(BF_4) \otimes BP^* \Longrightarrow BP^*(BF_4).$$

There is no higher 3-torsion in $H^*(BF_4)$. Hence the integral cohomology can be written as

$$H^*(BF_4)_{(3)} \cong D \otimes (Z_{(3)}\{1, x_4\} \oplus E \oplus \mathbb{Z}/3[x_{26}]\{x_{26}, x_{21}, x_9, x_9 x_{21}\}).$$

The first nonzero differential is $d_{2p-1} = v_1 \otimes Q_1$. From (2.4) and (2.5), we have $Q_1(x_4) = x_9$, $Q_1(x_{21}) = x_{26}$. Here we note that x_4 is torsion free but x_9 is 3-torsion. Hence we get

$$E_{2p}^{*,*} \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus BP^*/(3, v_1)[x_{26}]\{x_{26}, x_9\}).$$

Since all odd dimensional elements are just v_1-torsion, it is proved that the next nonzero differential is $d_{2p^2-1} = v_2 \otimes Q_2$. We also know $Q_2 x_9 = x_{26}$ from (2.5). Therefore we get

$$E_{2p^2}^{*,*} \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus BP^*/(3, v_1, v_2)[x_{26}]\{x_{26}\}).$$

Since this algebra is generated by even dimensional elements, it is also the E_∞ -term of the spectral sequence. Thus we get the theorem.

3. Cohomology operations and representations

We will study elements in $BP^*(BF_4) \otimes_{BP^*} \mathbb{Z}_{(3)}$ which are represented by Chern classes or their reduced power operations. At first we know the following fact.

Lemma 3.1. *Let $RP \subset A_3^*$ be the subalgebra generated by reduced power operations. Then the ring $E/3 \subset H^*(BF_4;\mathbb{Z}/3)$ is generated as an RP-algebra by x_4^2, x_8^2 and x_4^3.*

Proof. Recall $E = \mathbb{Z}_{(3)}[x_4, x_8]\{ab|a, b \in \{x_4, x_8, x_{20}\}\}$. Hence it is generated as a ring by ab or abc with $a, b \in \{x_4, x_8, x_{20}\}$ and $c \in \{x_4, x_8\}$. Using (2.4), we easily get the following diagram of reduced powers actions.

$$\begin{tikzcd}
x_4^2 \arrow[r, "P^1"] & -x_4^3 + x_8x_4 \arrow[r, "P^1"] \arrow[d, "P^3"', "P^3"] & x_8^2 + x_8x_4^2 \arrow[r, "P^1"] \arrow[d, "P^3"', "P^3"] & x_8^2x_4 \arrow[d, "P^3"'] & \\
 & x_8^3,\ x_{20}x_4 & x_8x_{20},\ x_{20}x_4^2 \arrow[d, "P^3"'] & x_{20}x_4x_8 \arrow[r, "P^1"] \arrow[d, "P^3"'] & x_{20}x_8^2 \\
 & & x_{20}^2 & x_{20}^2x_4 \arrow[r, "P^1"] & x_{20}^2x_8
\end{tikzcd}$$

Here $a \xrightarrow{P^i} b$ means

$$P^i(a) = \pm b \quad mod(UL),$$

where UL is the subalgebra of E generated by elements which appeared upper or left side than the position of b. For example,

$$x_4^3 \xrightarrow{P^3} x_8^3 \quad \text{follows from} \quad P^3(x_4^3) = -x_8^3 + x_4^6.$$

In the diagram, all elements of form ab or abc appeared. Hence we get the lemma. □

Next we consider the Chern classes of complex representations. The representation ring of $Spin(8)$ and F_4 are give by ([Yo, page 281])

$$R(Spin(8)) \cong R(T)^{W(Spin(8))} = \mathbb{Z}[a, b, c, d] \subset \mathbb{Z}[z_1, z_1^{-1}, \dots, z_4, z_4^{-1}] = R(T),$$
$$R(F_4) \cong R(T)^{W(F_4)} = \mathbb{Z}[a + b + c, ab + bc + ca, abc, d] \subset R(Spin(8)),$$
$$\text{where} \quad a = \sum_{1 \le i \le 4} z_i^2 + z_i^{-2}, \quad b = \sum_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3} z_4^{\epsilon_4},$$
$$c = \sum_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1} z_1^{\epsilon_1} z_2^{\epsilon_2} z_3^{\epsilon_3} z_4^{\epsilon_4},$$
$$d = 4 + \sum_{1 \le i < j \le 4} z_i^{2\epsilon_i} z_j^{2\epsilon_j} \qquad \text{with } \epsilon_k = 1 \ or \ -1.$$

Let us write $H^*(BT) \cong \mathbb{Z}[u_1, u_2, u_3, u_4]$. Then the total Chern class of representations is given by

$$1 + c_1 + c_2 + \dots = c(\sum z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4}) = \Pi(1 + a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4).$$

Lemma 3.2. *We can take the generator $x_{36} \in H^*(BF_4;\mathbb{Z}/3)$ by a Chern class.*

Proof. Recall $x_{36} \equiv p_3^3\ mod(Ideal(p_1, \bar{p}_2))$. We consider the restriction to the 3-dimensional torus $(S^1)^3 = T_3$ with $H^*(BT_3; \mathbb{Z}/3) = \mathbb{Z}/3[u_1, u_2, u_3]$. Let us write $c(d)|T_3 = c(p_1, p_2, p_3)$. Since p_i is the elementary symmetric function of the variables u_j^2, if we take $u_1 = u_2 = u_3 = u$ and $u_4 = 0$, then

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = u^6 \quad \text{in } \mathbb{Z}/3[u].$$

Hence we have $c(p_1, p_2, p_3)_{\{u_i = u, 1 \le i \le 3\}} = c(0, 0, u^6)$ in $\mathbb{Z}/3[u]$.

The restriction of the representation d to T_3 is

$$d|T_3 = 4 + \sum_{1 \le i < j \le 3} z_i^{2\epsilon_i} z_j^{2\epsilon_j} + 2 \sum_{1 \le i \le 3} z_i^{2\epsilon_i}.$$

Hence letting $z_1 = z_2 = z_3 = z$, we have

$$d|T_3 = 4 + 3(z^4 + z^{-4} + 2) + 2 \times 3(z^2 + z^{-2}).$$

Thus letting $u_1 = u_2 = u_3 = u$, we get

$$c(d)|T_3 = ((1+4u)(1-4u))^3((1+2u)(1-2u))^6 = 1 - u^{18} \text{ in } \mathbb{Z}/3[u].$$

Therefore we get $c(d)_{19}|T_3 \equiv -p_3^3\ \ mod(p_1, \bar{p}_2)$. This means that x_{36} can be represented by a Chern class. □

Lemma 3.3. *In $H^*(BF_4; \mathbb{Z}/3)$, x_4^3, $x_4 x_8$ are represented by Chern classes but x_4^2, x_8^2 are not.*

Proof. Recall $i^*(x_4) = p_1$ and $i^*(x_8) = p_2 - p_1^2$. We consider the restriction to $T_2 = (S^1)^2$ with $H^*(BT_2; \mathbb{Z}/3) \cong \mathbb{Z}/3[u_1, u_2]$. Moreover we consider the representation with $z_i^3 = 1$, that is $R(T)/(z_i^3 = 1)$ because we only consider $mod(3)$ cohomology. Let us write $w_i = z_i + z_i^{-1}$. Then

$$\begin{aligned}
a|T_2 &= z_1^2 + z_1^{-2} + z_2^2 + z_2^{-2} + 4 = w_1 + w_2 + 4,\\
b|T_2 &= c|T_2 = 2 \sum_{\epsilon_1\epsilon_2 = \pm 1} z_1^{\epsilon_1} z_2^{\epsilon_2} = 2w_1w_2,\\
d|T_2 &= 4 + \sum_{\epsilon_1\epsilon_2 = \pm 1} z_i^{2\epsilon_i} z_j^{2\epsilon_j} + 4(z_1^2 + z_1^{-2} + z_2^2 + z_2^{-2}) + 4\\
&= 8 + w_1w_2 + 4(w_1 + w_2).
\end{aligned}$$

To simplify the notations, let us write $A = c(w_1 + w_2)$ and $B = c(w_1w_2)$. Then it is immediate that

$$\begin{aligned}
A &= (1 - u_1^2)(1 - u_2^2) = 1 - p_1 + p_2,\\
B &= (1 - (u_1 + u_2)^2)(1 - (u_1 - u_2)^2) = 1 + p_1 + p_1^2 - p_2.
\end{aligned}$$

Now we compute the total Chern class of representations of F_4.

$$
\begin{aligned}
c(d)|T_2 = A^4B, \qquad c(a+b+c)|T_2 &= c(w_1+w_2+4+4w_1w_2) = AB^4,\\
c(ab+ac+cb)|T_2 &= c(4(w_1+w_2+4)w_1w_2+4w_1^2w_2^2)\\
&= c(4(7w_1w_2+4(w_1+w_2)+4)) = A^{16}B^{28},\\
c(abc)|T_2 &= c(4(w_1+w_2+4)w_1^2w_2^2)\\
&= c(4(10w_1w_2+16(w_1+w_2)+24)) = A^{64}B^{40},
\end{aligned}
$$

where we use $w_i^2 = w_i+2$ and $w_i^3 = 3w_i+2$.

All above Chern classes have the form for some C

$$
\begin{aligned}
ABC^3 &= (1-p_1+p_2)(1+p_1+p_1^2-p_2)C^3\\
&= (1+(-p_1^3-p_1p_2)+(p_1^2p_2-p_2^2))C^3.
\end{aligned}
$$

Thus the Chern class c_8 is represented by only one generator

$$p_1^2p_2 - p_2^2 = i^*(-x_8^2 - x_4^2x_8).$$

For the Chern class c_6, we get two cases

$$p_1^3 - p_1p_2 = i^*(-x_4x_8), \qquad -p_1p_2 = i^*(-x_4^3 - x_4x_8).$$

The first (resp. second) case comes from the case $C^3 = (1-p_1^3+\cdots)$ (resp. $C^3 = (1+p_1^3+\cdots)$), e.g. $C = A^3$ (resp. $C = B^3$). □

Lemma 3.4. *The element $3x_4 \in H^*(BF_4)_{(3)}$ is represented by a Chern class.*

Proof. Consider the restriction to $S^1 = T_1$ with $H^*(BT_1)_{(3)}$. The restriction of the representation d is $d|T_1 = 16+6(z_1^2+z_1^{-2})$. Hence the total Chern class is

$$c(d)|T_1 = (1-4u^2)^6 = 1-24u^2+\cdots.$$

This means that the element $3x_4$ is represented by a Chern class in $H^*(BF_4)_{(3)}$. □

4. Cycle maps

For an algebraic variety X over $\mathbb{C}$, Suslin-Voevodsky constructed the motivic cohomology $H^{*,*}(X)$([Vo2]). This cohomology has the properties that if X is smooth, then $H^{m,n}(X) \cong 0$ for $m > 2n$, and $H^{2n,n}(X) \cong CH^n(X)$ the classical Chow ring of algebraic cycles modulo rational equivalence. There is the natural map (realization map)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X) \to H^m(X)$$

such that $t_{\mathbb{C}}^{2m,m}$ is the usual cycle map $cl : CH^m(X) \to H^{2m}(X)$. V. Voevodsky defines ([Vo1, 2]) the cohomology (reduced and Bockstein) operations

$$\begin{aligned}
\beta &: H^{m,n}(X;\mathbb{Z}/p) \to H^{m+1,n}(X;\mathbb{Z}/p),\\
P^i &: H^{m,n}(X;\mathbb{Z}/p) \to H^{m+2(p-1)i,n+(p-1)i}(X;\mathbb{Z}/p),\\
Q_i &: H^{m,n}(X;\mathbb{Z}/p) \to H^{m+2p^i-1,n+p^i-1}(X;\mathbb{Z}/p)
\end{aligned}$$

such that they commute with realization map $t_{\mathbb{C}}^{*,*}$. In particular mod p Chow ring $CH^*(X)/p$ is closed under the reduced power operations. We also know that ([H-K])

$$H^{*,*}(BGL_n) \cong \mathbb{Z}/p[c_1,\dots,c_n] \otimes H^{*,*}(pt) \qquad \text{with } \deg(c_i) = (2i,i).$$

Hence the Chow ring of BG also has Chern classes. Here $H^{*,*}(pt)$ is something complicated but the mod p cohomology is just $H^{*,*}(pt;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$ with $\tau \in H^{0,1}(pt;\mathbb{Z}/p)$ ([Vo2], [Ya]).

By extending Quillen's [Q] arguments, Levine and Morel defined the algebraic cobordism theory $\Omega^*(-)$ as the universal theory in theories having transfers and Chern classes [L-M1, 2] (We say that $h^*(X)$ is a theory having transfers and Chern classes if this theory satisfies the axioms A1 to A4 in [L-M 1]). Given a theory $h^*(-)$ having transfers and Chern classes, the universality induces the existence of the natural map

$$\rho_h : \Omega^*(-) \to h^*(-).$$

The theories $H^{2*,*}(X) = \oplus_n H^{2n,n}(X) \cong CH^*(X)$,
$MGL^{2*,*}(X) = \oplus_n MGL^{2n,n}(X)$ (motivic cobordism theory defined by V. Vedodsky) and $MU^*(X)$ are typical examples of theories having transfers and Chern classes. In particular, Levine and Morel proves that

$$\rho_{MU} : \Omega^{2n}(pt) \cong MU^{2n}(pt), \qquad \rho_{CH} \otimes_{\Omega^*} \mathbb{Z} : (\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z})^{2n} \cong CH^n(X).$$

Hence the Totaro's cycle map $\bar{cl}$ is represented as

$$\begin{aligned}
\rho_{MU} \otimes_{\Omega^*} \mathbb{Z} \circ (\rho_{CH} \otimes_{\Omega^*} \mathbb{Z})^{-1} : CH^n(X) &\to (\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z})^{2n}\\
&\to (MU^*(X) \otimes_{MU^*} \mathbb{Z})^{2n}.
\end{aligned}$$

Moreover they conjecture that ρ_{MGL} are always isomorphisms.

Let $K^0(X)$ be the Grothendieck group of algebraic vector bundles over X. Let $\tilde{K}(1)^*(X)$ be the integral K-theory, that is, $\tilde{K}(1)^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$. Then they showed that

$$\tilde{K}(1)^* \otimes K^0(X) \cong \Omega^*(X) \otimes_{\Omega^*} \tilde{K}(1)^*.$$

Remark. Hopkins and Morel announced the existence of Atiyah-Hirzebruch spectral sequences for generalized motivic theories. Then we have $MGL^{2*,*}(X) \otimes_{MU^*} \mathbb{Z} \cong CH^*(X)$ and $\tilde{K}(1)^* \otimes K^0(X) \cong MGL^{2*,*}(X) \otimes_{MU^*} \tilde{K}(1)^*$. Hence we can also prove all our results bellow using $MGL^{2*,*}(-)$ instead of $\Omega^*(X)$.

Lemma 4.1. *The element x_4^2 is in the image of the cycle map cl.*

Proof. Merkurjev showed ([To2]) that the Grothendieck group $K^0(BG)$ of algebraic vector bundles is isomorphic to the usual K-theory $K(BG)$. We also recall the Conner-Floyd relation

$$\Omega^*(BG)_{(p)} \otimes_{\Omega^*_{(p)}} \tilde{K}(1)^* \cong \tilde{K}(1)^* \otimes K^0(BG)$$
$$\cong \tilde{K}(1)^*(BG) \cong BP^*(BG) \otimes_{BP^*} \tilde{K}(1)^*.$$

Let $x = [x_4^2] \in BP^*(BF_4)$. Since $0 \neq x \in BP^*(BF_4) \otimes_{BP^*} \tilde{K}(1)^*$, there is an element x' in $\Omega^*(BF_4)_{(3)}$ such that

$$\rho_{BP}(x') = v_1^s x \qquad \text{for } \ s \geq 0.$$

Take the smallest s. Then x' is an $MU^*_{(3)}$-module generator and $x' \neq 0 \in \Omega(BF_4)^* \otimes_{\Omega^*} \mathbb{Z}_{(3)} \cong CH^*(BF_4)_{(3)}$.

Suppose $s \geq 1$. Then $|x'| = |v_1^s x_4^2| \leq 4$. But it is known from Totaro (Corollary 3.5 in [To2]) that

$$CH^i(BG)_{(p)} \cong (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)})^{2i} \qquad \text{if } \ i \leq 2.$$

This is a contradiction and $s = 0$. Hence we see that $\rho_{BP}(x') = x$ in $BP^*(BF_4)$ and so $cl(x') = x_4^2$ in $H^*(BF_4)_{(3)}$. □

Remark. By arguments similar to the above proof, we can see that there is $x' \in \Omega^*(BF_4)$ such that

$$\rho_{BP}(x') = v_1^s[x_8^2] \qquad \text{for some } 0 \leq s \leq 2.$$

However it does not seem easy to prove $s = 0$.

Remark. We still know that $3x_4, x_4x_8$ and $x_4^2x_8 + x_8^2$ are in $\mathrm{Im}(cl)$. Hence $3x_8^2 \in \mathrm{Im}(cl)$.

Lichtenbaum defined the cohomology $H_L^{*,*}(X;\mathbb{Z})$ by using the étale topology, while $H^{*,*}(X;\mathbb{Z})$ is defined by using Nisnevich topology. There is a natural map $H^{m,n}(X) \to H_L^{m,n}(X)$. We say that the condition $B(n,p)$ holds if

$$B(n,p) \quad : \quad H^{m,n}(X;\mathbb{Z}_{(p)}) \cong H_L^{m,n}(X;\mathbb{Z}_{(p)}) \qquad \text{for all } m \leq n+1$$

and all smooth X. Merkurjev-Suslin [M-S] and Voevodsky proved that $B(n,p)$ holds for $n \leq 2$ or $p = 2$ respectively. (Indeed, the Milnor conjecture is equivalent to hold $B(n, p = 2)$). M. Rost [R] proves that $B(3,3), B(4,3)$ are correct.

Moreover Suslin-Voevodsky proves $H_L^{m,n}(X;\mathbb{Z}/p) \cong H_{et}^m(X;\mu_p^{\otimes n})$. On the other hand, it is well known $H_{et}^m(X;\mu_p^{\otimes n}) \cong H_{et}^m(X;\mathbb{Z}/p) \cong H^m(X;\mathbb{Z}/p)$.

Suppose that $B(n,p)$ condition holds. Then we have isomorphisms

$$H^{n,n}(X;\mathbb{Z}/p) \cong H_L^{n,n}(X;\mathbb{Z}/p) \cong H_{et}^n(X;\mu_p^{\otimes n}) \cong H^n(X;\mathbb{Z}/p).$$

The composition of these isomorphisms also represents the realization map $t_{\mathbb{C}}^{n,n}$.

Remark. Quite recently, V. Voevodsky announced the proof that $B(n,p)$ hold for all $n \geq 0$ and all primes p ([Vo3]).

Here we recall some useful fact about multiplying τ. Write $H^i(X, H^j_{\mathbb{Z}/2})$ the Zarisky cohomology of X with the coefficient in presheaf $H^j_{et}(V; \mathbb{Z}/2)$ for open subset V of X. From the result of Voevodsky we have a long exact sequence (Lemma 2.4 in [Or-Vi-Vo])

$$H^{m,n-1}(X;\mathbb{Z}/2) \xrightarrow{\tau} H^{m,n}(X;\mathbb{Z}/2) \to H^{m-n}(X; H^n_{\mathbb{Z}/2}) \to H^{m+1,n-1}(X;\mathbb{Z}/2).$$

In particular we get

Lemma 4.2 (Lemma 2.4 in [Or-Vi-Vo]). *Let X be smooth. Then $\tau : H^{n,n-1}(X;\mathbb{Z}/2) \to H^{n,n}(X;\mathbb{Z}/2)$ is injective.*

Remark. If $B(n,p)$ condition is satisfied, then the facts similar to Lemma 4.2 hold for odd primes p. This is also explained by the Bloch-Ogus spectral sequence

$$E_2^{i,j} \cong H^i(X; H^j_{\mathbb{Z}/2}) \Longrightarrow H^{i+j}_{et}(X;\mathbb{Z}/2),$$

where $E_2^{i,j} = 0$ unless $0 \leq i \leq j$ (Theorem 1.3 in [To3]). See also [Vo3].

Lemma 4.3. *Suppose that $B(3,p)$ holds. Let x be an element in $H^4(BG;\mathbb{Z})$ such that $px \in \mathrm{Im}(cl)$. Then we can take $x' \in H^{4,3}(BG;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x$.*

Proof. Let $\{px\} = a \in H^{4,2}(BG) = CH^2(BG)$. We consider in the coefficient $\mathbb{Z}/p^2$. Let τ_{p^2} be a $\mathbb{Z}/p^2$-module generator of $H^{0,1}(pt;\mathbb{Z}/p^2)$. Then $\tau_{p^2}^2 a = px \in H^{4,4}(BG;\mathbb{Z}/p^2)$ defining $x \in H^{4,4}(BG;\mathbb{Z}/p)$ since so in the topological case. But the map $\tau : H^{4,3}(BG;\mathbb{Z}/p) \to H^{4,4}(BG;\mathbb{Z}/p)$ is injective from Remark of Lemma 4.2. This means $\tau a = 0 \in H^{4,3}(BG;\mathbb{Z}/p)$. Hence there is an element $x' \in H^{4,3}(BG;\mathbb{Z}/p^2)$ so that $\tau_{p^2} a = px'$. We get $t_{\mathbb{C}}(x') = x$ since $\tau_{p^2}(px') = px$. □

From Lemma 3.4, we know that there is an element $x' \in H^{4,3}(BF_4;\mathbb{Z}/3)$ such that $t_{\mathbb{C}}(x') = x_4$. So there is an element

$$x'_{26} = Q_1 Q_2 x' \in H^{26,13}(BF_4;\mathbb{Z}/3) \qquad \text{with } t_{\mathbb{C}}(x'_{26}) = x_{26}.$$

Proof of Proposition 1.1. From the above arguments, $x_{26} \in \mathrm{Im}(cl)$. From Lemma 3.2, x_{36} is represented by a Chern class. Since $CH^*(BG)$ has Chern classes, we get $x_{36} \in \mathrm{Im}(cl)$. Since $P^3 x_{36} = x_{48}$ and $CH^*(X)$ is closed under the reduced power operations, we get $x_{48} \in \mathrm{Im}(cl)$ and so $D = \mathbb{Z}[x_{36}, x_{48}] \subset \mathrm{Im}(cl)$. From Lemmas 3.3 and 3.4, we know $x_4^3, 3x_4 \in \mathrm{Im}(cl)$, which are Chern classes. From Lemma 4.1, we also know $x_4^2 \in \mathrm{Im}(cl)$.

Therefore from Lemma 3.1,

$$E = \mathbb{Z}_{(3)}[x_4, x_8]\{ab \mid a, b \in \{x_4, x_8, x_{20}\}\} \subset \mathrm{Im}(cl) \qquad \text{if } x_8^2 \in \mathrm{Im}(cl).$$

Thus from Theorem 2.2, we know that $\bar{cl}$ is epic if x_8^2 is in $\mathrm{Im}(cl)$. □

Department of Mathematics
Faculty of Education
Ibaraki University
Mito, Ibaraki, Japan
e-mail: yagita@mx.ibaraki.ac.jp

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