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Spherically symmetric flow of the compressible Euler equations

For the case including the origin

By

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Abstract

We study the Euler equations of compressible isentropic gas dynamics with spherical symmetry. Due to the presence of the singularity at the origin, little is known in the case including the origin. In this paper, we prove the existence of local solutions for the case including the origin. A modified Godunov scheme is introduced to construct approximate solutions and obtain L^∞ estimates. The convergence and consistency of the approximate solutions are proved with the aid of a compensated compactness framework.

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1. Introduction

We are concerned with local weak solutions of the compressible Euler equations with spherically initial data. The compressible Euler equations are of the following conservation forms

$$\begin{cases} \rho_t + \nabla \cdot \vec{m} = 0, \\ \vec{m}_t + \nabla \cdot \left(\dfrac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p = 0, \quad \vec{x} \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

where ρ, $\vec{m}$ and p are the density, the momentum and the pressure of the gas, respectively. On the non-vacuum state $\rho > 0$, $\vec{u} = \vec{m}/\rho$ is the velocity. For polytropic gas, $p(\rho) = \rho^\gamma/\gamma$, where $\gamma \in (1, 5/3]$ is the adiabatic exponent for usual gases.

Consider the initial value problem (1.1) and

$$(\rho, \vec{m})|_{t=0} = (\rho_0(\vec{x}), \vec{m}_0(\vec{x})), \tag{1.2}$$

with following geometric structure

$$(\rho_0(\vec{x}), \vec{m}_0(\vec{x})) = \left(\rho_0(|\vec{x}|), m_0(|\vec{x}|) \frac{\vec{x}}{|\vec{x}|} \right), \tag{1.3}$$

where $m_0(x)$ is a scalar function of $x = |\vec{x}| \geq 0$. We look for the solutions of the form

$$(\rho(\vec{x}), \vec{m}(\vec{x})) = \left(\rho(|\vec{x}|, t), m(|\vec{x}|, t) \frac{\vec{x}}{|\vec{x}|} \right). \tag{1.4}$$

Then (1.1) has a singularity at the origin (see (1.5)). Therefore little is known in the case including the origin. For the case outside the origin, a global weak entropy solution with spherical symmetry was constructed in [20] and [21] for the isothermal case $\gamma = 1$ and local existence of such a weak solution for the general case $1 < \gamma \leq \frac{5}{3}$ was also discussed in [19]. In [6], a numerical shock capturing scheme was developed and applied for constructing global solutions with geometric structure and large initial data in L^∞ for the general case, including both spherical symmetric flows and transonic nozzle flows. However the proofs of this result are incorrect (see Section 8). There are many serious defects, for example, the proofs of L^∞ (Section 5) and local entropy estimates (Lemma 4.1 and 4.2). Therefore no global existence has obtained in this case. They were considered outside the origin ($|\vec{x}| \geq 1$). On the other hand, for the special case including the origin, the global existence theorem with large L^∞ data having only nonnegative initial velocity was obtained in [2].

In this paper, we consider the Cauchy problem (1.1)–(1.2) for the case including the origin and not only nonnegative but also negative velocity as initial data. In addition, the functional space L^∞ may not be fit for this case. Therefore, we prove the local existence of solutions in another function space by using weights. Our idea is to introduce the equation (1.6). This function space is derived from a calculation to obtain (1.6). We introduce a modified Godunov scheme to obtain L^∞ estimates and compensated compactness of corresponding approximate solutions. The method incorporates natural building blocks from Riemann solutions and steady-state solutions. Such estimates lead to the convergence of the approximate solutions and to an existence theory of weak entropy solutions for measurable initial data in L^∞.

In Section 2 through 4 we develop a first order Godunov scheme, with piecewise constant building blocks replaced by piecewise steady ones. The main point is to use the steady-state solutions, which incorporate spherical source terms, to modify the wave strengths in the Riemann solutions. This construction yields better approximate solutions, and permits uniform L^∞ bounds. There are two technical difficulties which we overcome to achieve this goal, both due to transonic phenomena. One is that no smooth steady-state solution exists in each cell in general. This problem is easily solved by introducing two kinds of an auxiliary steady-state solutions, as discussed in Section 3. The other is that the constructed steady-state solution in each cell must satisfy the following requirements.

(a) The oscillation of the steady-state solution around the Godunov value must be of the same order as the cell length to obtain the L^∞ estimate for the convergence arguments.

(b) The difference between the average of the steady-state solution over each cell and the Godunov value must be higher than first order in the cell length to ensure the consistency of the corresponding approximate solutions with Euler equations.

(c) Local entropy errors of the discontinuities in the approximate solutions must be higher than first order in the cell length to obtain the H^{-1} compactness estimates.

The requirements are satisfied by smooth steady-state solutions bounded away from the sonic state in the cell. The general case must include the transonic steady-state solutions. The sonic difficulty is overcome by introducing an auxiliary steady-state solutions. These requirements also enable us to make H^{-1} compactness estimates for corresponding entropy dissipation measures to deduce the strong convergence of the approximate solutions with the aid of the compactness framework (see [3] and [4]).

We rewrite (1.1) as

$$\begin{cases} \rho_t + m_x = -\dfrac{2}{x}m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -\dfrac{2}{x}\dfrac{m^2}{\rho}, \quad p(\rho) = \rho^\gamma/\gamma, \end{cases} \tag{1.5}$$

where $\rho(x,t)$ and $m(x,t)$, $x = |\vec{x}| \geq 0$ are the scalar functions. Set $\rho = \tilde{\rho}x^{\frac{2}{\gamma-1}}$ and $m = \tilde{m}x^{\frac{\gamma+1}{\gamma-1}}$. Then (1.5) becomes

$$\begin{cases} \tilde{\rho}_t + x\tilde{m}_x = -a_1\tilde{m}, \\ \tilde{m}_t + x\left(\dfrac{\tilde{m}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_x = -a_2\dfrac{\tilde{m}^2}{\tilde{\rho}} - a_3 p(\tilde{\rho}), \quad p(\tilde{\rho}) = \tilde{\rho}^\gamma/\gamma, \end{cases}$$

where $\theta = \frac{\gamma-1}{2}$, $a_1 = \theta^{-1} + 3$, $a_2 = \theta^{-1} + 4$ and $a_3 = \theta^{-1} + 2$. Moreover, set $\xi = \log x$. Then after changing ξ to x, $\tilde{\rho}$ to ρ and $\tilde{m}$ to m, we have

$$\begin{cases} \rho_t + m_x = -a_1 m, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -a_2\dfrac{m^2}{\rho} - a_3 p(\rho), \quad p(\rho) = \rho^\gamma/\gamma. \end{cases} \tag{1.6}$$

This equation can be written as

$$\begin{cases} v_t + f(v)_x = -g(v), \quad x \in \mathbf{R}, \\ v|_{t=0} = v_0(x), \end{cases} \tag{1.7}$$

where $v = (\rho, m)^\top$, $u = \frac{m}{\rho}$, $f(v) = (m, \frac{m^2}{\rho} + p(\rho))^\top$, $g(v) = (a_1 m, a_2\frac{m^2}{\rho} + a_3 p(\rho))^\top$ and $v_0 \in L^\infty(\mathbf{R})$. We notice not only that singularity vanishes, but also that coefficients of inhomogeneous terms become constants. This fact is convenient for the later analysis.

Remark 1.1. The velocity u and the sound speed ρ^θ are multiplied by a same weight, that is, $u = \tilde{u}x$ and $\rho^\theta = \tilde{\rho}^\theta x$. Therefore the Mach number $M = u/\rho^\theta$ is invariant by the above transformation. Therefore, we can keep the divergence form.

A pair of mapping $(\eta, q) : \mathbf{R}^2 \to \mathbf{R}^2$ is called an entropy-entropy flux pair [15] if it satisfies an identity

$$\nabla q = \nabla\eta\nabla f. \tag{1.8}$$

Furthermore, if, for any fixed $\frac{m}{\rho} \in (-\infty, \infty)$, η vanishes on the vacuum $\rho = 0$, then η is called a weak entropy. For example, the mechanical energy-energy flux pair

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \frac{1}{\gamma(\gamma-1)}\rho^\gamma, \quad q_* = m\left(\frac{1}{2}\frac{m^2}{\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right) \tag{1.9}$$

is a strictly convex weak entropy-entropy flux pair. One can prove that, for $0 \le \rho \le C$, $|\frac{m}{\rho}| \le C$,

$$|\nabla \eta| \le \text{const.} \tag{1.10}$$

and

$$|\nabla^2 \eta(r,r)| \le \text{const.} \nabla^2 \eta_*(r,r), \tag{1.11}$$

for any weak entropy η, where r is any vector and the constant is independent of r.

Definition 1.1. A pair of measurable functions $v(x,t) = (\rho(x,t), m(x,t))$ is called a local weak solution of the Cauchy problem (1.7) if, there exists a $T > 0$ with the following property. For any test function $\phi \in C_0^1(\Omega \times [0,T])$ with $\Omega \subset \mathbf{R}$,

$$\int_0^T \int_\Omega (v\phi_t + f(v)\phi_x - g(v)\phi) dx dt + \int_{\text{supp } \phi(0,\cdot)} \phi(0,x) dx = 0 \tag{1.12}$$

and, along any shock wave with left state v_-, right state v_+, and speed σ,

$$\sigma(\eta(v_+) - \eta(v_-)) - (q(v_+) - q(v_-)) \ge 0, \tag{1.13}$$

for any convex weak entropy-entropy flux pair (η, q).

For the initial problem for the compressible Euler equations (1.1) with

$$\begin{cases} \vec{m}\Big|_{|\vec{x}|=0} = 0, \\ (\rho, \vec{m})\Big|_{t=0} = (\rho_0(\vec{x}), \vec{m}_0(\vec{x})) = \left(\rho_0(|\vec{x}|), m_0(|\vec{x}|)\dfrac{\vec{x}}{|\vec{x}|}\right), \quad |\vec{x}| \ge 0, \end{cases} \tag{1.14}$$

we introduce the following conventional notion of weak entropy solution.

Definition 1.2. A measurable vector function $(\rho(\vec{x},t), \vec{m}(\vec{x},t))$ is a called a local weak entropy solution of the initial-boundary problem (1.1) and (1.14) provided that, there exists a $T > 0$ with the following property.

(1) The vector function $(\rho(\vec{x},t), \vec{m}(\vec{x},t))$ satisfies the Euler equation (1.1) in the sense of distributions with respect to the test function space $\{\phi \in C_0^\infty(\{|\vec{x}| > 0\} \times [0,T]) | \phi(\vec{x},t) = \phi(|\vec{x}|,t)\}$.

(2)

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} \vec{m}(\vec{x},t)\cdot\frac{\vec{x}}{|\vec{x}|}dx \overset{*}{\rightharpoonup} 0 \quad \text{as } \varepsilon \downarrow 0, \quad \text{in } L^{\infty}_{\mathrm{loc}}(S^1\times(0,T)). \tag{1.15}$$

(3) Along any shock wave propagating in the direction $\vec{\nu}\in\mathbf{R}^3$, $|\vec{\nu}|=1$, with left and right states $(\rho_{\pm},\vec{m}_{\pm})$ and speed $s=s(\rho_-,\rho_+,\vec{m}_-,\vec{m}_+;\vec{\nu})$,

$$\begin{aligned} s\left\{\rho_+\left(\frac{|\vec{m}_+|^2}{2\rho_+^2}+e_+\right)-\rho_-\left(\frac{|\vec{m}_-|^2}{2\rho_-^2}+e_-\right)\right\}& \\ -\vec{\nu}\cdot\left\{\vec{m}_+\left(\frac{|\vec{u}_+|^2}{2\rho_+^2}+e_++\frac{p_+}{\rho_+}\right)-\vec{m}_-\left(\frac{|\vec{u}_-|^2}{2\rho_-^2}+e_-+\frac{p_-}{\rho_-}\right)\right\}&\ge 0, \end{aligned} \tag{1.16}$$

where $e=\frac{\rho^{\gamma-1}}{\gamma(\gamma-1)}$ is the internal energy.

In these definitions, the entropy conditions (1.13) and (1.16) are equivalent to the corresponding Lax entropy conditions along the shock waves (cf. [15] and [23]).

First we shall prove the following theorem.

Theorem 1.1. *Assume that the initial data satisfy*

$$0\le\rho_0(x)\le C_0,\quad \left|\frac{m_0(x)}{\rho_0(x)}\right|\le C_0 \tag{1.17}$$

for some $C_0>0$. Then there exists a local weak entropy solution $(\rho(x,t), m(x,t))$ of the Cauchy problem (1.7) *in the sense of Definition* 1.1 *satisfying*

$$0\le\rho(x,t)\le C(T),\quad \left|\frac{m(x,t)}{\rho(x,t)}\right|\le C(T) \tag{1.18}$$

for some $C(T)\ge C_0$ in the region $\mathbf{R}\times[0,T]$ for some $T\in(0,\infty)$.

Our main result of this paper is deduced from the above theorem immediately.

Theorem 1.2. *Assume that the initial data are of the form*

$$(\rho,\vec{m})|_{t=0}=(\rho_0(\vec{x}),\vec{m}_0(\vec{x}))=\left(\rho_0(|\vec{x}|)x^{\frac{2}{\gamma-1}},m_0(|\vec{x}|)x^{\frac{\gamma+1}{\gamma-1}}\frac{\vec{x}}{|\vec{x}|}\right),\quad |\vec{x}|\ge 0 \tag{1.19}$$

with $(\rho_0(x),m_0(x))\in L^{\infty}(\{x\ge0\})$ satisfying (1.17)*. Then there exists a local weak entropy solution $(\rho(t,\vec{x}),\vec{m}(t,\vec{x}))$ of the initial boundary problem* (1.1) *and* (1.19) *in th sense of Definition* 1.2*, which takes the form*

$$(\rho(\vec{x},t),\vec{m}(\vec{x},t))=\left(\rho(|\vec{x}|,t)x^{\frac{2}{\gamma-1}},m(|\vec{x}|,t)x^{\frac{\gamma+1}{\gamma-1}}\frac{\vec{x}}{|\vec{x}|}\right)$$

with $(\rho(x,t),m(x,t))\in L^{\infty}(\{x\ge0\}\times[0,T])$ satisfying (1.18)*.*

Note that it is sufficient to show that $v(x,t) = (\rho(x,t), m(x,t))$ is local weak entropy solution of the Cauchy problem (1.7), in the sense of Definition 1.1. To achieve these results, we also apply a compensated compactness framework (7.1)–(7.2) (Section 7) in [3] and [4] (also see [11], [12] and [13]) uniform boundedness (7.1) of the approximate solutions $(\rho^h(x,t), m^h(x,t))$ and H^{-1} compactness (7.2) of the corresponding entropy dissipation measures imply the strong convergence of the approximate solutions $(\rho^h(x,t), m^h(x,t))$ to the local weak entropy solution $(\rho(x,t), m(x,t)) \in L^\infty$ of the Cauchy problem (1.7), almost every where with the same property (7.1). The importance of this framework is that it takes the vacuum into account in correct physical variables (ρ, m) near the vacuum, rather than (ρ, m) that is physical incorrect on the vacuum. This framework was proved in [13] for the case $\gamma = 1 + \frac{2}{2m+1}, m \geq 2$ integer, and in [3], [4] and [11] for the general case of gases $1 < \gamma \leq \frac{5}{3}$. Further discussion on this framework for the case of $\gamma > 1$ can be found in [16].

In Section 2 we construct two solutions which will serve as building blocks for our construction: Riemann solutions for the homogeneous system of gas dynamics and (exact and auxiliary) steady-state solutions for the inhomogeneous system (1.6). We discuss their basic properties in Sections 2 and 3.

Section 4 is devoted to the construction of the modified Godunov scheme and the corresponding approximate solution of the problem (1.7). Some basic properties of the approximate solutions are discussed. It is proved in Sections 5 and 6 that the approximate solutions satisfy the compensated compactness framework (7.1)–(7.2) (see [3] and [4]). The existence theory is established in Section 7.

2. Nonlinear waves and Riemann solutions

In this section, we first review some nonlinear waves in gas dynamics and construct Riemann solutions for the homogeneous system of gas dynamics, before further discussion. Then we discuss their basic properties for use in subsequent developments.

2.1. Shock waves and rarefaction waves for 1-D gas dynamics

Consider the Riemann problem for one dimensional system of isentropic gas dynamics

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = 0, \quad p(\rho) = \rho^\gamma/\gamma \end{cases} \tag{2.1}$$

with

$$(\rho, m) = \begin{cases} (\rho_-, m_-), & x < x_0, \\ (\rho_+, m_+), & x > x_0, \end{cases} \tag{2.2}$$

where $x_0 \in (-\infty, \infty)$, $\rho_\pm \geq 0$ and $m_\pm$ are constants satisfying $|m_\pm/\rho_\pm| < \infty$.

The eigenvalues of the system are

$$\lambda_1 = \frac{m}{\rho} - c \equiv c(M-1), \quad \lambda_2 = \frac{m}{\rho} + c \equiv c(M+1),$$

where the sound speed $c = \rho^\theta$, the Mach number $M = \frac{m}{\rho c}$, and $\theta = \frac{\gamma-1}{2}$. Corresponding Riemann invariants are

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta} \equiv \frac{c}{\theta}(\theta M + 1), \quad z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta} \equiv \frac{c}{\theta}(\theta M - 1). \tag{2.3}$$

Any discontinuity in the weak solutions to (2.1) must satisfy the Rankine-Hugoniot condition

$$\sigma(v - v_0) = f(v) - f(v_0),$$

where σ is the propagation speed of the discontinuity, $v_0 = (\rho_0, m_0)$ and $v = (\rho, m)$ are the corresponding left state and right state. This means that

$$\begin{cases} m - m_0 = \dfrac{m_0}{\rho_0}(\rho - \rho_0) \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \\ \sigma = \dfrac{m - m_0}{\rho - \rho_0} = \dfrac{m_0}{\rho_0} \pm \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}. \end{cases} \tag{2.4}$$

A discontinuity is called a shock if it satisfies the entropy condition

$$\sigma(\eta(v) - \eta(v_0)) - (q(v) - q(v_0)) \geq 0 \tag{2.5}$$

for any convex entropy pair (η, q).

There are two distinct types of rarefaction waves and shock waves denoted by 1-Rw or 2-Rw and 1-shock or 2-shock, respectively, in the isentropic gases. If a state (ρ_0, m_0) or (ρ_0, u_0) is given, the possible states (ρ, m) or (ρ, u) that can be connected to (ρ_0, m_0) or (ρ_0, u_0) on the right by a Rw or shock are

$$\begin{cases} R_1(0): m - m_0 = \dfrac{m_0}{\rho_0}(\rho - \rho_0) - \dfrac{\rho}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho < \rho_0 \\ \text{or} \\ u - u_0 = -\dfrac{1}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho < \rho_0, \end{cases}$$

$$\begin{cases} R_2(0): m - m_0 = \dfrac{m_0}{\rho_0}(\rho - \rho_0) + \dfrac{\rho}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho > \rho_0 \\ \text{or} \\ u - u_0 = \dfrac{1}{\theta}(\rho^\theta - \rho_0^\theta), \quad \rho > \rho_0, \end{cases}$$

$$\begin{cases} S_1(0): m - m_0 = \frac{m_0}{\rho_0}(\rho - \rho_0) - \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \\ \rho > \rho_0 > 0 \\ \text{or} \\ u - u_0 = -\sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \quad \rho > \rho_0 > 0, \end{cases}$$

$$\begin{cases} S_2(0): m - m_0 = \dfrac{m_0}{\rho_0}(\rho-\rho_0) + \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \\ \rho < \rho_0 \\ \text{or} \\ u - u_0 = \sqrt{\dfrac{1}{\rho\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}(\rho-\rho_0), \quad \rho<\rho_0, \end{cases}$$

respectively.

Along the curve $R_1(0)$,

$$\frac{dm}{d\rho}\bigg|_{R_1(0)} = \frac{m_0}{\rho_0} + \frac{\rho_0^\theta}{\theta} - \frac{\theta+1}{\theta}\rho^\theta, \quad \frac{d^2m}{d^2\rho}\bigg|_{R_1(0)} = -(\theta+1)\rho^{\theta-1} \le 0,$$

and along the curve $R_2(0)$,

$$\frac{dm}{d\rho}\bigg|_{R_2(0)} = \frac{m_0}{\rho_0} - \frac{\rho_0^\theta}{\theta} + \frac{\theta+1}{\theta}\rho^\theta, \quad \frac{d^2m}{d^2\rho}\bigg|_{R_2(0)} = (\theta+1)\rho^{\theta-1} \ge 0.$$

This shows that the curve $R_1(0)$ is concave and the curve $R_2(0)$ is convex in the (ρ, m)-plane. Along the curve $S_1(0)$,

$$\begin{cases} \dfrac{m-m_0}{\rho-\rho_0}\bigg|_{S_1(0)} = \dfrac{m_0}{\rho_0} - \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}, \\ \dfrac{d}{d\rho}\left(\dfrac{m-m_0}{\rho-\rho_0}\right)\bigg|_{S_1(0)} = -\dfrac{\frac{\rho}{\rho_0}p'(\rho) - \frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}{2\sqrt{\frac{\rho}{\rho_0}(p(\rho)-p(\rho_0))(\rho-\rho_0)}} \le 0, \quad \rho > \rho_0 > 0, \end{cases}$$

and along the curve $S_2(0)$,

$$\begin{cases} \dfrac{m-m_0}{\rho-\rho_0}\bigg|_{S_2(0)} = \dfrac{m_0}{\rho_0} + \sqrt{\dfrac{\rho}{\rho_0}\dfrac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}, \\ \dfrac{d}{d\rho}\left(\dfrac{m-m_0}{\rho-\rho_0}\right)\bigg|_{S_1(0)} = -\dfrac{\frac{\rho}{\rho_0}p'(\rho) - \frac{p(\rho)-p(\rho_0)}{\rho-\rho_0}}{2\sqrt{\frac{\rho}{\rho_0}(p(\rho)-p(\rho_0))(\rho-\rho_0)}} \ge 0, \quad \rho < \rho_0. \end{cases}$$

This shows that the curve $S_1(0)$ is concave and $S_2(0)$ is convex with respect to (ρ_0, m_0) in the (ρ, m)-plane.

On the (ρ, m)-plane and (ρ, u)-plane, the loci of the shock waves and rarefaction waves are depicted in Figures 2.1 and 2.2. Figure 2.1 shows that both the curves R_1 and R_2 lie on one side of the line connecting the points $(0,0)$ and (ρ_0, m_0), and S_1 and S_2 lie on another side of the line.

Along the curve $R_1(0)$, $\frac{dw}{dz}\big|_{R_1} = 0$ and along the curve $R_2(0)$, $\frac{dw}{dz}\big|_{R_2} = 0$.

Along the curve $S_1(0)$,

$$\frac{dw}{dz}\bigg|_{S_1(0)} = \frac{\gamma\rho^{\gamma+1} - (\gamma-1)\rho_0\rho^\gamma - \rho_0^{\gamma+1} - 2\sqrt{(\rho^\gamma - \rho_0^\gamma)(\rho-\rho_0)\gamma\rho_0\rho^\gamma}}{\gamma\rho^{\gamma+1} - (\gamma-1)\rho_0\rho^\gamma - \rho_0^{\gamma+1} + 2\sqrt{(\rho^\gamma - \rho_0^\gamma)(\rho-\rho_0)\gamma\rho_0\rho^\gamma}},$$

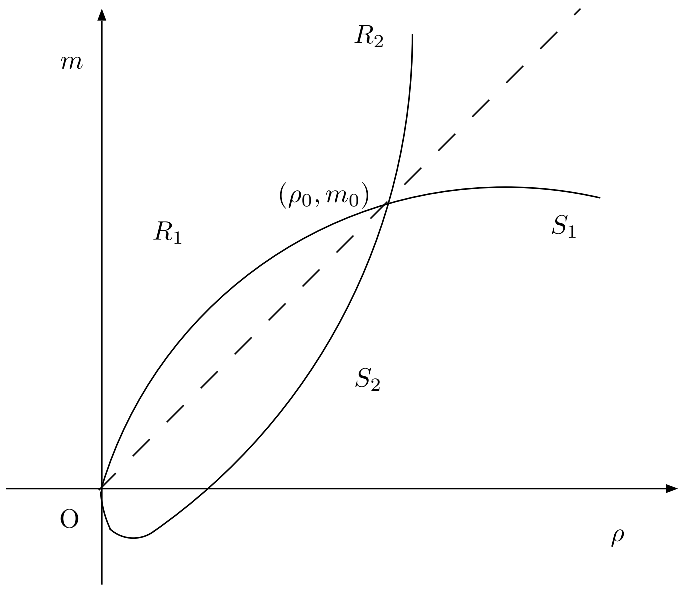

Figure 2.1. The rerafaction curves and the shock curves in (ρ, m)-plane.

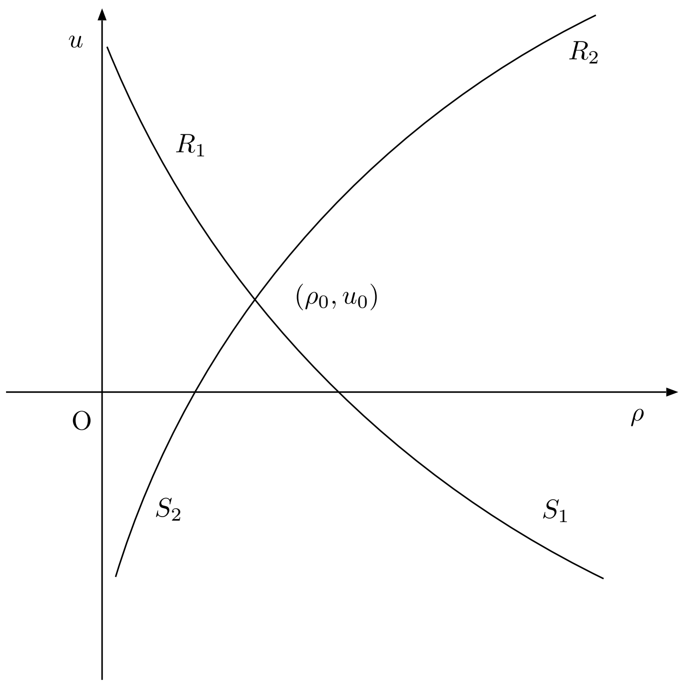

Figure 2.2. The rerafaction curves and the shock curves in (ρ, u)-plane.

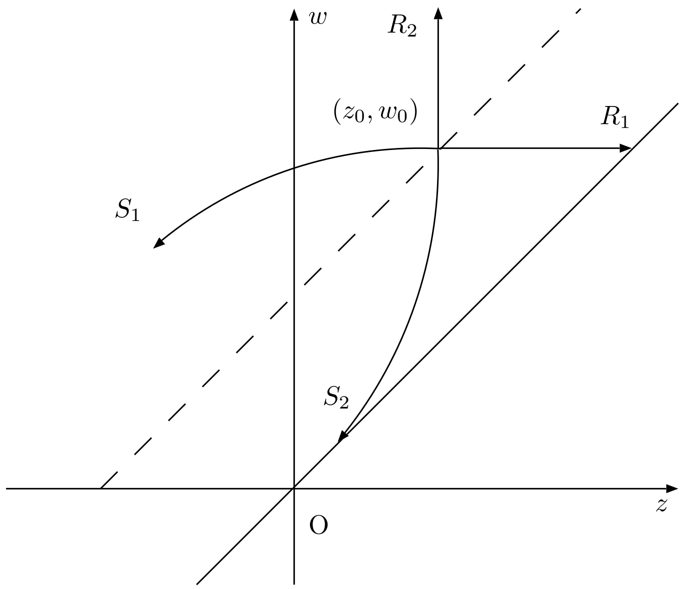

Figure 2.3. The rerafaction curves and the shock curves in (z, w)-plane.

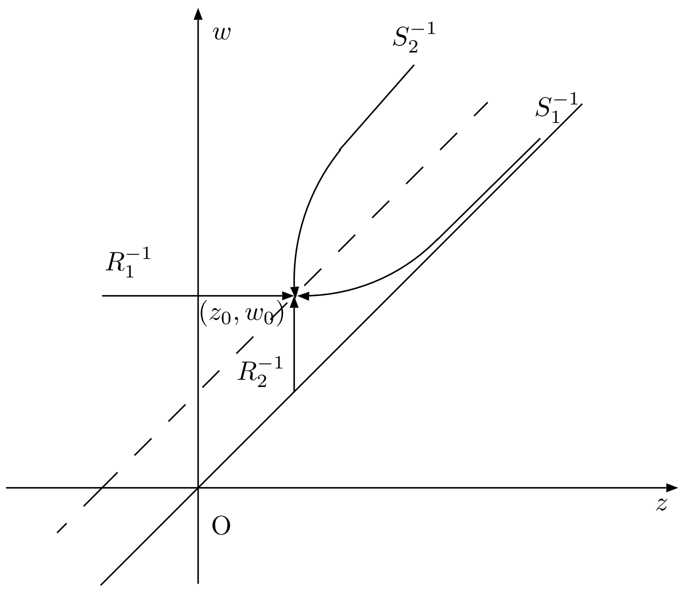

Figure 2.4. The inverse rerafaction curves and the inverse shock curves in (z, w)-plane.

and along the curve $S_2(0)$,

$$\left.\frac{dw}{dz}\right|_{S_2(0)} = \frac{\gamma\rho^{\gamma+1} - (\gamma-1)\rho_0\rho^{\gamma} - \rho_0^{\gamma+1} + 2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}}{\gamma\rho^{\gamma+1} - (\gamma-1)\rho_0\rho^{\gamma} - \rho_0^{\gamma+1} - 2\sqrt{(\rho^{\gamma}-\rho_0^{\gamma})(\rho-\rho_0)\gamma\rho_0\rho^{\gamma}}}.$$

Notice that

$$0 \le \left.\frac{dw}{dz}\right|_{S_1(0)} \le 1, \quad \lim_{\rho\to\infty} \left.\frac{dw}{dz}\right|_{S_1(0)} = 1$$

and

$$1 \le \left.\frac{dw}{dz}\right|_{S_2(0)}, \quad \lim_{\rho\to 0} \left.\frac{dw}{dz}\right|_{S_2(0)} = 1.$$

2.2. Riemann solution

Similarly, given a state (ρ_0, m_0) or (ρ_0, u_0) for $\rho_0 > 0$, the locus of possible states (ρ, m) or (ρ, u) for $\rho > 0$ that can be connected to the state in the left by a shock wave S^{-1} or rarefaction wave R^{-1} defines what is called an inverse shock wave curve or inverse rarefaction wave curve. It has behavior similar to that of S or R.

From the behavior of these curves in phase plane (ρ, m), we can construct the unique solution for the Riemann problem

$$v|_{t=0} = \begin{cases} v_-, & x < x_0, \\ v_+, & x > x_0 \end{cases} \tag{2.6}$$

and the Riemann initial boundary problem

$$v|_{t=0} = v_+, \quad m|_{x=0} = 0. \tag{2.7}$$

For the problem (2.8), we can get a diagram of the first family of elementary wave curves given left state v_- and a diagram of the second family of inverse elementary wave curves for given right state v_+ to determine a unique intersection point to obtain the unique solution. For the problem (2.9), we can draw a diagram of the second family of inverse elementary wave curves for given right state v_+ to determine a unique intersection point with the line $m = 0$ to obtain the unique solution.

Theorem 2.1. *There exists a unique piecewise entropy solution* $(\rho(x,t), m(x,t))$ *containing the vacuum state* $(\rho = 0)$ *on the upper plane* $t > 0$ *for each problem of* (2.8) *and* (2.9) *satisfying*

(1) *For the Riemann problem* (2.8),

$$\begin{cases} w(\rho(x,t), m(x,t)) \le \max(w(\rho_-, m_-), w(\rho_+, m_+)), \\ z(\rho(x,t), m(x,t)) \ge \min(z(\rho_-, m_-), z(\rho_+, m_+)), \\ w(\rho(x,t), m(x,t)) - z(\rho(x,t), m(x,t)) \ge 0. \end{cases}$$

(2) *For the Riemann initial boundary problem* (2.9),

$$\begin{cases} w(\rho(x,t), m(x,t)) \le \max(w(\rho_-, m_-), -z(\rho_+, m_+)), \\ z(\rho(x,t), m(x,t)) \ge \min(z(\rho_+, m_+), 0), \\ w(\rho(x,t), m(x,t)) - z(\rho(x,t), m(x,t)) \ge 0. \end{cases}$$

Such solutions have the following properties

Lemma 2.2. *The regions $\sum = \{(\rho, m) : w \leq w_0, z \geq z_0, w - z \geq 0\}$ are invariant with respect to both of the Riemann problem (2.8) and the average of the Riemann solutions in x. More preciously, if the Riemann date lie in $\sum$, the corresponding Riemann solutions $(\rho(x,t), m(x,t))$ lie in $\sum$, and their corresponding averages in x also in $\sum$*

$$\left(\frac{1}{b-a}\int_a^b \rho(x,t)dx, \frac{1}{b-a}\int_a^b m(x,t)dx\right) \in \sum.$$

Furthermore, for the Riemann initial-boundary problem (2.9), the region $\sum = \{(\rho, m) : w \leq w_0, z \geq z_0, w - z \geq 0\}$, $z_0 \leq 0 \leq \frac{w_0+z_0}{2}$, are invariant with respect to both of the Riemann problem (2.9) and the average of the corresponding Riemann solution in x.

The proof of Lemma 2.2 can be found in [4] and [19].

Lemma 2.3. *The rate of entropy production of a shock with left state v_- and right v_+ for an arbitrary weak entropy η is dominated by the associated rate of entropy production for η_* in the following sense*

$$\begin{aligned}&|\sigma(\eta(v_+) - \eta(v_-)) - (q(v_+) - q(v_-))| \\ &\qquad \leq C\{\sigma(\eta_*(v_+) - \eta_*(v_-)) - (q_*(v_+) - q_*(v_-))\},\end{aligned}$$

where the constant C depends only on η and $\max(|\rho_\pm| + |\frac{m_\pm}{\rho_\pm}|)$.

The proof of this fact can be found in [4].

3. Steady-state solutions

The purpose of this section is to provide important estimates on steady-state solutions of the inhomogeneous problem (1.6) determined by the following system of ordinary differential equations

$$\begin{cases} m_x = -a_1 m, \\ \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = -a_2\dfrac{m^2}{\rho} - a_3 p(\rho), \quad p(\rho) = \rho^\gamma/\gamma, \end{cases} \tag{3.1}$$

subject to the boundary condition

$$(\rho, m)|_{x=x_0} = (\rho_0, m_0). \tag{3.2}$$

The nonsonic and transonic cases are distinct, as the former produces smooth solutions and the latter may contain the auxiliary steady-state solutions. The L^∞ estimates are derived based on Riemann invariant inequalities and are required for the compensated compactness framework. The L^1 estimates are needed for consistency and verification of the entropy condition.

3.1. Smooth steady-state solutions for the nonsonic case

We first consider the nonsonic case $M_0^2 \not\approx 1$, where $M_0 = M(x = x_0) = \frac{m_0}{\rho_0 c_0}$.

From the first equation we obtain

$$m = m_0 e^{-a_1(x-x_0)}. \tag{3.3}$$

Using (3.3), the second equation can be rewritten as

$$\left(\frac{m^2}{\rho^{\gamma+1}} - 1\right)\frac{\rho_x}{\rho} = -a_3\left(\frac{m^2}{\rho^{\gamma+1}} - \frac{1}{\gamma}\right). \tag{3.4}$$

Observing $M = m/\rho^{\theta+1}$, we have

$$M_x = \frac{2M(\theta M^2+1)}{M^2-1}. \tag{3.5}$$

In terms of M and M_0, (3.5) becomes

$$\frac{Me^{2x}}{(\theta M^2+1)^{\frac{\theta+1}{2\theta}}} = \frac{M_0 e^{2x_0}}{(\theta M_0^2+1)^{\frac{\theta+1}{2\theta}}}. \tag{3.6}$$

The solution of the steady-state equations are thus reduced to the following procedure: solve Eq. (3.6) for M and use (3.3) to obtain ρ. Equation (3.6) can be written as

$$F(M) = \frac{e^{2x_0}}{e^{2x}}F(M_0), \tag{3.7}$$

where the function F is defined by

$$F(M) = \frac{M}{(\theta M^2+1)^{\frac{\theta+1}{2\theta}}}$$

satisfying

$$\begin{cases} F(0) = 0,\ F(M) \to 0 & \text{when} \quad M \to \pm\infty, \\ F'(M) \geq 0 & \text{when} \quad M \in [-1,1], \\ F'(M) \leq 0 & \text{when} \quad M \in (-\infty,-1] \cup [1,\infty). \end{cases}$$

Thus we see that there are two difficulties in solving function equation (3.7). If $F(1) < e^{2(x_0-x)}|F(M_0)|$, no smooth solution exists, since the right side of (3.7) exceeds the maximum value of $|F|$. If $F(1) > e^{2(x_0-x)}|F(M_0)|$, there are two solutions of (3.7), one with $|M| > 1$ and the others $|M| < 1$ since the line $F = const.$ intersects the graph of F at two points. As long as $|M| \neq 1$ is maintained, exactly one of these solutions is smooth for the problem (3.1)–(3.2).

Lemma 3.1. *Let $v(x)$ be a smooth steady-state solution satisfying $v|_{x=x_0} = v_0$, with $\rho_0 \geq 0$, in an interval $[a,b]$ containing x_0. Then*

$$\rho(x) \geq 0, \quad x \in [a,b].$$

The next two lemmas will be used in deriving L^∞ estimates. The main idea is that the quadrant in the Riemann invariant plane, which is invariant for the homogeneous hyperbolic equations, is approximately invariant for the steady inhomogeneous equations. Let $M = M(v(x))$ and $M_0 = M(v(x_0))$ be the Mach numbers. An important intermediate step is to establish Lipschitz continuity of a relative Mach number M/M_0.

Lemma 3.2. *Let $v(x)$ be the smooth steady-state solution satisfying $v|_{x=x_0} = v_0$. Then, given a $\varepsilon_0 \in (0,1)$, there exist $h_1 = h_1(\varepsilon_0) \in (0,1]$ and $C > 0$ such that, in any interval $[x_0-\frac{h}{2}, x_0+\frac{h}{2}], h \le h_1$, when $|M_0^2-1| \ge \varepsilon_0 M_0^2$,*

$$\left|\frac{M - M_0}{M_0}\right| \le C|x - x_0|.$$

Proof. First, we react the relative Mach number $N = M/M_0$. By (3.6)

$$\frac{e^{2x_0}}{e^{2x}} = \frac{M}{M_0}\left(\frac{\theta M_0^2 + 1}{\theta M^2 + 1}\right)^{\frac{\theta+1}{2\theta}}.$$

Define

$$G(N; M_0) = N\left(\frac{\theta M_0^2 + 1}{\theta M_0^2 N^2 + 1}\right)^{\frac{\theta+1}{2\theta}}.$$

Then $N = N(x)$ satisfies

$$\begin{cases} G(N; M_0) = \dfrac{e^{2x_0}}{e^{2x}} = 1 + \dfrac{e^{2x_0} - e^{2x}}{e^{2x}}, \\ N|_{x=x_0} = 1. \end{cases} \tag{3.8}$$

Our purpose is to control $N(x) - 1$ by $|x - x_0|$. For the case $|M_0^2 - 1| \ge \varepsilon_0 M_0^2$, $G(N; M_0)$ is uniformly monotone near $N = 1$ with respect to M_0, and so (3.8) provides an upper bound on $N - 1$ as we now establish. In this case,

$$M_0^2 \le \frac{1}{1+\varepsilon_0} \quad \text{or} \quad M_0^2 \ge \frac{1}{1-\varepsilon_0}.$$

Notice that, in the interval $N \in \left[\sqrt{1 - \frac{\varepsilon_0}{2}}, \sqrt{1 + \frac{\varepsilon_0}{2}}\right]$, there exists $c(\varepsilon_0) > 0$, independent of M_0, such that

$$|G'(N; M_0)| = \frac{|1 - M_0^2 N^2|}{1 + \theta M_0^2 N^2}\left(\frac{1 + \theta M_0^2}{1 + \theta M_0^2 N^2}\right)^{\frac{\theta+1}{2\theta}} \ge c(\varepsilon_0) > 0, \tag{3.9}$$

which means that $G(N; M_0)$ is uniformly monotone in $N \in \left[\sqrt{1 - \frac{\varepsilon_0}{2}}, \sqrt{1 + \frac{\varepsilon_0}{2}}\right]$. Therefore, we obtain that there exist $\tilde{h}(\varepsilon_0) > 0$ and $C_1(\varepsilon_0) > 0$, independent of M_0, such that, whenever $|x - x_0| \le \frac{\tilde{h}}{2}$,

$$|N(x) - 1| \le C_1(\varepsilon_0)|x - x_0|$$

using (3.9). We prove the lemma. $\square$

We now estimate the Riemann invariants in order to derive L^∞ bound on the approximate solutions.

Lemma 3.3. *Let $v(x)$ be a smooth steady-state solution satisfying $v|_{x=x_0} = v_0$ in $[x_0-\frac{h}{2}, x_0+\frac{h}{2}], h \le h_1$, with $|M_0^2-1| \ge \varepsilon_0 M_0^2$. Then, when $|x-x_0| \le \frac{h}{2}$, we have*

$$\begin{cases} w(v(x)) \le w(v_0)(1+C|x-x_0|), & \text{when} \quad M_0 > 0, \\ z(v(x)) \ge z(v_0)(1+C|x-x_0|), & \text{when} \quad M_0 < 0, \end{cases} \tag{3.10}$$

where (w,z) are the Riemann invariants, $h_1 > 0$ is the constant determined in Lemma 3.2, and C is a constant depending only on ε_0.

Proof. First we observe

$$\frac{\rho^\theta - \rho_0^\theta N^{-\frac{\theta}{\theta+1}}}{\rho_0^\theta} = \frac{1}{\rho_0^\theta}\left\{\left(\frac{m_0 e^{-a_1(x-x_0)}}{M_0 N}\right)^{\frac{\theta}{\theta+1}} - \left(\frac{m_0}{M_0 N}\right)^{\frac{\theta}{\theta+1}}\right\} = \mathbf{O}(|x-x_0|).$$

In this case $|M_0^2-1| \ge \varepsilon_0 M_0^2$, we use the estimate for $N-1$ in Lemma 3.2, when $|x-x_0| \le \frac{h}{2}, h \le h_1$, to obtain

$$\begin{aligned} \frac{w(v)}{w(v_0)} &= 1 + \frac{\rho^\theta(\theta M_0 N + 1) - \rho_0^\theta(\theta M_0 + 1)}{\rho_0^\theta(\theta M_0 + 1)} \\ &= 1 + \frac{\rho_0^\theta N^{-\frac{\theta}{\theta+1}}(\theta M_0 N + 1) - \rho_0^\theta(\theta M_0 + 1)}{\rho_0^\theta(\theta M_0 + 1)} + \mathbf{O}(|x-x_0|) \\ &= 1 + \frac{(\rho_0^\theta N^{-\frac{\theta}{\theta+1}} - \rho_0^\theta)(\theta M_0 N + 1) + \rho_0^\theta \theta M_0 (N-1)}{\rho_0^\theta(\theta M_0 + 1)} + \mathbf{O}(|x-x_0|) \\ &\le 1 + C|x-x_0|. \end{aligned}$$

Similarly, for the case $M_0 < 0$, we have

$$\begin{aligned} \frac{z(v)}{z(v_0)} &= 1 + \frac{\rho^\theta(\theta M_0 N - 1) - \rho_0^\theta(\theta M_0 - 1)}{\rho_0^\theta(\theta M_0 - 1)} \\ &\ge 1 + C|x-x_0| \end{aligned}$$

by using Lemma 3.2. The estimate (3.10) follows. $\square$

Now we derive L^1 estimates on the difference between the smooth steady-state solution over $[x_0-\frac{h}{2}, x_0+\frac{h}{2}]$ and the boundary data at x_0 for sufficiently small h. The following lemma can be checked easily by taking Taylor's expansion for x at x_0.

Lemma 3.4. *There exist $h_2 > 0$, $0 < h_2 \le h_1$, and a smooth steady-state solution in $[x_0-\frac{h}{2}, x_0+\frac{h}{2}]$, $h \le h_2$, with boundary condition $v|_{x=x_0} = v_0$ such that, when $|M_0^2-1| \ge \varepsilon_0 M_0^2$,*

$$|v(x)-v_0| = |v_0|\mathbf{O}(h), \qquad \frac{1}{h}\left|\int_{x_0-\frac{h}{2}}^{x_0+\frac{h}{2}} v(x) - v_0 dx\right| = |v_0|\mathbf{O}(h^2),$$

where bounds $\mathbf{O}(h)$ and $\mathbf{O}(h^2)$ depend only on ε_0 and are independent of M_0.

3.2. Auxiliary steady-state solutions near the sonic state

In the case $|M_0^2-1|<\varepsilon_0 M_0^2$, no smooth steady-state solution exists in general. Therefore we introduce auxiliary steady-state solutions. The gap the exact steady-state solutions and the auxiliary ones shall be fill up in Section 4. These constructions permit L^∞ estimates.

In the case $|M_0^2-1|<\varepsilon_0 M_0^2$ and $M_0<0$, we consider the following auxiliary steady-state equations

$$\begin{cases} m_x=-a_1 m, \\ \left(\dfrac{m^2}{\rho}+p(\rho)\right)_x=-a_2\dfrac{m^2}{\rho}-a_3\gamma p(\rho), \quad p(\rho)=\rho^\gamma/\gamma, \end{cases} \tag{3.11}$$

subject to the boundary condition

$$(\rho,m)|_{x=x_0}=(\rho_0,m_0). \tag{3.12}$$

From above equations, we have

$$m=m_0e^{-a_1(x-x_0)}, \quad \rho=\rho_0e^{-a_3(x-x_0)}.$$

Similarly, in the case $|M_0^2-1|<\varepsilon_0 M_0^2$ and $M_0>0$, we consider the following auxiliary steady-state equations

$$\begin{cases} m_x=bm, \\ \left(\dfrac{m^2}{\rho}+p(\rho)\right)_x=(2b+a_3/\gamma)\dfrac{m^2}{\rho}-a_3p(\rho), \\ \qquad\qquad p(\rho)=\rho^\gamma/\gamma,\ b=a_1-a_2-a_3/\gamma, \end{cases} \tag{3.13}$$

subject to the boundary condition

$$(\rho,m)|_{x=x_0}=(\rho_0,m_0). \tag{3.14}$$

From above equations, we have

$$m=m_0e^{b(x-x_0)}, \quad \rho=\rho_0e^{-a_3/\gamma(x-x_0)}.$$

In these cases, we can easily check the following lemmas.

Lemma 3.5. *There exist $h_3>0$, and an auxiliary steady-state solution $v(x)$ satisfying $v|_{x=x_0}=v_0$ in $[x_0-\frac{h}{2},x_0+\frac{h}{2}]$, $h\le h_3$, with $|M_0^2-1|<\varepsilon_0M_0^2$. Then, when $|x-x_0|\le\frac{h}{2}$, we have*

$$\begin{cases} w(v(x))\le w(v_0)(1+C|x-x_0|), & \textit{when} \quad M_0>0, \\ z(v(x))\ge z(v_0)(1+C|x-x_0|), & \textit{when} \quad M_0<0, \end{cases} \tag{3.15}$$

where (w,z) are the Riemann invariants, C is a constant.

Lemma 3.6. *There exist $h_4 > 0$, $0 < h_4 \le h_3$, and an auxiliary steady-state solution in $[x_0 - \frac{h}{2}, x_0 + \frac{h}{2}]$, $h \le h_4$, with boundary condition $v|_{x=x_0} = v_0$ such that, when $|M_0^2 - 1| < \varepsilon_0 M_0^2$,*

$$|v(x) - v_0| = |v_0|\mathbf{O}(h), \qquad \frac{1}{h}\left|\int_{x_0-\frac{h}{2}}^{x_0+\frac{h}{2}} v(x) - v_0 dx\right| = |v_0|\mathbf{O}(h^2),$$

where bounds $\mathbf{O}(h)$ and $\mathbf{O}(h^2)$ are independent of M_0.

4. Approximate solutions

In this section we construct approximate solutions $v^h = (\rho^h, m^h) = (\rho^h, \rho^h u^h)$ in the strip $0 \le t \le T$ for some fixed $T \in (0, \infty)$, where

$$h \le h_0 \equiv \min_{1 \le i \le 4} h_i \tag{4.1}$$

is the space mesh length, together with the time mesh length Δt, satisfying the following Courant-Friedrichs-Lewy condition

$$2\Lambda \equiv 2 \max_{i=1,2}\left(\sup_{0 \le t \le T} |\lambda_i(\rho^h, m^h)|\right) \le \frac{h}{\Delta t} \le 3\Lambda. \tag{4.2}$$

We will prove that the approximate solutions are bounded uniformly in the mesh length $h > 0$ and $\rho^h(x,t) \ge 0$ to guarantee the construction of (ρ^h, m^h). Since the speeds of propagation of the approximate solutions will be finite because of (4.2), we can assume that the initial data have compact support (i.e. are constant off some compact set) without loss of generality.

4.1. Construction of approximate solutions

We construct the approximate solutions (ρ^h, m^h) for the Cauchy problem. Let

$$t_n = n\Delta t, \quad x_j = jh, \quad (n, j) \in \mathbf{Z}_+ \times \mathbf{Z}.$$

Assume that $v^h(x,t)$ is defined for $t < n\Delta t$. Then we define $v_j^n \equiv (\rho_j^n, m_j^n)$ as

$$\begin{cases} \rho_j^n \equiv \dfrac{1}{h}\displaystyle\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \rho^h(x, n\Delta t - 0)dx, & (j-1/2)h \le x \le (j+1/2)h, \\ m_j^n \equiv \dfrac{1}{h}\displaystyle\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} m^h(x, n\Delta t - 0)dx, & (j-1/2)h \le x \le (j+1/2)h. \end{cases} \tag{4.3}$$

In the strip $n\Delta t \le t < (n+1)\Delta t$, $v^h(x,t)$ is defined as an approximate solution of the Cauchy problem

$$\begin{cases} v_t + f(v)_x = -g(v), \quad jh \le x \le (j+1)h, \\ v|_{t=n\Delta t} = \begin{cases} v_-(x), & x < (j+1/2)h, \\ v_+(x), & x > (j+1/2)h, \end{cases} \end{cases} \tag{4.4}$$

where $v_-(x)$ and $v_+(x)$ are smooth solutions of the steady-state equations (3.1) with boundary conditions

$$v_-(jh) = v_j^n, \quad v_+((j+1)h) = v_{j+1}^n,$$

constructed in Section 3.1 when $|M^2(v_k^n) - 1| \geq \varepsilon_0 M^2(v_k^n)$, $k = j, j+1$, and, otherwise, are auxiliary steady-solutions constructed in Section 3.2. Then the difference between the average of the exact or auxiliary steady-state solutions over $[(j-\frac{1}{2})h, (j+\frac{1}{2})h]$ and the Godunov value v_j^n is higher than first order in h by Lemmas 3.4 and 3.6. Such a construction ensures the consistency of the corresponding approximate solutions with the Euler equations.

We will solve the problem for small time approximately. This is done by perturbing about the solution of the Riemann problem at $x = (j+\frac{1}{2})h$

$$\begin{cases} v_t + f(v)_x = 0, \quad jh \leq x < (j+1)h, \\ v|_{t=n\Delta t} = \begin{cases} v_-((j+1/2)h - 0), & x < (j+1/2)h, \\ v_+((j+1/2)h + 0), & x > (j+1/2)h. \end{cases} \end{cases} \tag{4.5}$$

Our goal in this section is to construct the approximate solution satisfying the Rankine-Hugoniot conditions at the time center of the cell, $t = (n+1/2)\Delta t$.

We first introduce two Implicit function theorems.

Theorem 4.1. *Let $U \in \mathbf{R}^n, V \in \mathbf{R}$ be open sets. Let $(\vec{x}, y) \mapsto f(\vec{x}, y)$ be a C^1 map from $U \times V$ into $\mathbf{R}$. Let us consider the equation*

$$y = f(x_1, \ldots, x_n, y) \tag{4.6}$$

near $(\vec{x}, y) = (\vec{x}_0, y_0)$. Suppose $|y_0 - f(\vec{x}_0, y_0)| \leq \varepsilon$, and if $|y - y_0| \leq \delta$ and $|\vec{x} - \vec{x}_0| \leq \delta$ then $|\partial_y f(\vec{x}, y)| \leq L \leq \frac{1}{3}$ and $|\partial_{\vec{x}} f| \leq M$.

Assume $\varepsilon \leq \delta/3$ and $\delta_0 \leq \delta/3M$. Then there exists a unique solution $y = y(\vec{x})$ of (4.6) for $|y - y_0| \leq \delta$ and $|\vec{x} - \vec{x}_0| \leq \delta_0$ such that

$$|y(\vec{x}) - f(\vec{x}_0, y_0)| \leq \frac{1}{1-L}(M|\vec{x} - \vec{x}_0| + |y_0 - f(\vec{x}_0, y_0)|). \tag{4.7}$$

Moreover y is C^1.

Theorem 4.2. *Let $U \in \mathbf{R}^n$ be an open set. Let $\vec{x} \mapsto f(\vec{x})$ be a C^1 map from U into $\mathbf{R}^n$. Suppose*

$$|f(\vec{x}_0)| \leq \varepsilon,$$

$Df(\vec{x}_0)$ is invertible and $|\{Df(\vec{x}_0)\}^{-1}| \leq M$. Let if $|\vec{x} - \vec{x}_0| \leq \delta$ then $|Df(\vec{x}) - Df(\vec{x}_0)| \leq \frac{1}{2M}$.

Suppose

$$\varepsilon \leq \frac{\delta}{2M}.$$

Then there exists a unique solution $\vec{x}$ of

$$f(\vec{x}) = 0$$

in $|\vec{x} - \vec{x}_0| \leq \delta$ such that

$$|\vec{x} - \vec{x}_0| \leq 2M\varepsilon.$$

We now consider the case 1-Rw and 2-shock arise as Riemann solutions especially. Call constants $v_l(=v_-((j+1/2)h-0)), v_m, v_r(=v_+((j+1/2)h+0))$ the left, middle and right states, where v_l, v_m and v_m, v_r are connected by 1-Rw and 2-shock respectively. Let σ_s be a propagation speed of 2-shock.

We consider separately two cases. Choosing $\beta \in (0,1)$ small enough, we first consider the case $\rho_m \geq h^\beta$.

Case 1. $\rho_m \geq h^\beta$. We now construct approximate rarefaction wave. Let α be a constant such that $1/2 + \frac{(\gamma-1)}{2}\beta < \alpha < 1$. We first introduce the rays $x = (j+1/2)h + \lambda_1(w_l, z_i^*, z_{i+1}^*)(t - n\Delta t)$ separating finite constant states $w_i^*, z_i^*,\ i = 0, 1, \ldots, p$, with $w_i^* = w_l\ (i = 0, \ldots, p)$, $z_0^* = z_l$ and $z_p^* = z_m$, such that

$$\begin{aligned} z_{i+1}^* &= z_i^* + h^\alpha\ (i = 0, \ldots, p-2) && \text{when} \quad z_m - z_l > h^\alpha, \\ p &= 1 && \text{when} \quad z_m - z_l \leq h^\alpha, \end{aligned}$$

where $w_l = w(v_l), z_l = z(v_l), z_m = z(v_m)$ and $\lambda_1(w_l, z_i^*, z_{i+1}^*)$ is the propagation speed of 1-rarefaction shock (see [1]) with left state (w_l, z_i^*) and right state (w_l, z_{i+1}^*). We shall perturb this approximate rarefaction wave in order h. Set $\bar{v}_l(x) = v_-(x)$ and $\bar{v}_r(x) = v_+(x)$.

We determine propagation speeds $\sigma_i,\ i = 1, \ldots, p$ and steady-state solutions $\bar{v}_i(x),\ i = 0, 1, \ldots, p$ between $l_i : x = (j+1/2)h + \sigma_i(t - n\Delta t)$ and $l_{i+1} : x = (j+1/2)h + \sigma_{i+1}(t - n\Delta t)$ in the following manner.

Set $\bar{v}_0(x) = \bar{v}_l(x)$ and $x_0 = jh$. First, notice that, along 1-rarefaction shock,

$$w - w_0 = \mathbf{O}(h^{-(\gamma-1)\beta})(z - z_0)^3 \tag{4.8}$$

holds, where (w, z) is connected to (w_0, z_0) on the right by 1-rarefaction shock. Then applying Theorem 4.2 with $\vec{x} = \sigma_1$ and $\vec{x}_0 = \lambda_1(w_l, z_0^*, z_1^*)$, we have solution σ_1 such that

$$\sigma_1 = \bar{u}_0 - S(\rho_1, \bar{\rho}_0) \text{ at } x = x_1 \equiv (j+1/2)h + 1/2\sigma_1\Delta t$$

and

$$|\sigma_1 - \lambda_1(w_l, z_0^*, z_1^*)| = \mathbf{O}(h),$$

where
(4.9)

$$S(\rho, \rho_0) = \begin{cases} \sqrt{\dfrac{\rho(p(\rho) - p(\rho_0))}{\rho_0(\rho - \rho_0)}} = \rho_0^{\frac{\gamma-1}{2}} + \dfrac{\gamma+1}{4}\rho_0^{\frac{\gamma-3}{2}}(\rho - \rho_0) \\ \qquad\qquad + \dfrac{5\gamma^2 - 6\gamma - 11}{96}\rho_0^{\frac{\gamma-5}{2}}(\rho - \rho_0)^2 + \cdots, \quad \text{if } \rho \neq \rho_0, \\ \sqrt{p'(\rho_0)} \quad \text{if } \rho = \rho_0 \end{cases}$$

and v_1 be the possible state that can be connected $\bar{v}_0(x_1)$ by 1-rarefaction shock such that $z_1 = z_1^*$. Then let $\bar{v}_1$ be a steady-state solution satisfying the boundary condition

$$\bar{v}_1(x_1) = v_1.$$

Next we define $v_i, \bar{v}_i(x)$ and σ_i inductively. We assume that, for $1 \le j \le i$, v_j and $\bar{v}_j(x)$ satisfying

(4.10)
$$z_j = z_j*,\ |\bar{w}_j(x_j) - \bar{w}_{j-1}(x_{j-1})| \le C|\bar{w}_{j-1}(x_{j-1})||x_j - x_{j-1}| + \mathbf{O}(h^{3\alpha-(\gamma-1)\beta})$$

and

$$|\sigma_j - \lambda_1(w_l, z_{j-1}^*, z_j^*)| = \mathbf{O}(h) \tag{4.11}$$

are defined. Then, applying Theorem 4.2 with $\vec{x} = \sigma_{i+1}$ and $\vec{x}_0 = \lambda_1(w_l, z_i^*, z_{i+1}^*)$, we have solution σ_{i+1} such that

$$\sigma_{i+1} = \bar{u}_i - S(\rho_{i+1}, \bar{\rho}_i) \qquad \text{at} \quad x = x_{i+1} \equiv (j+1/2)h + \frac{1}{2}\sigma_{i+1}\Delta t$$

and

$$|\sigma_{i+1} - \lambda_1(w_l, z_i^*, z_{i+1}^*)| = \mathbf{O}(h),$$

where v_{i+1} be the possible state that can be connected $\bar{v}_i(x_{i+1})$ by 1-rarefaction shock such that $z_{i+1} = z_{i+1}^*$. Here notice that, from (4.8), (4.10) and the construction of v_{i+1},

$$\begin{aligned}
&\bar{w}_i((j+1/2)h + 1/2\lambda_1(w_l, z_i^*, z_{i+1}^*)\Delta t) = w_l + \mathbf{O}(h),\\
&\bar{z}_i((j+1/2)h + 1/2\lambda_1(w_l, z_i^*, z_{i+1}^*)\Delta t) = z_i^* + \mathbf{O}(h),\\
&w_{i+1} = w_l + \mathbf{O}(h), \quad z_{i+1} = z_{i+1}^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_1(w_l, z_i^*, z_{i+1}^*) - \bar{u}_i + S(\rho_{i+1}, \bar{\rho}_i) = \mathbf{O}(h)&\\
\text{at} \quad x = (j+1/2)h + \frac{1}{2}\lambda_1(w_l, z_i^*, z_{i+1}^*)\Delta t.&
\end{aligned}$$

Moreover, observing (4.9), since $\lambda_1(w_l, z_i^*, z_{i+1}^*) - \lambda_1(w_l, z_{i-1}^*, z_i^*) > Ch^\alpha$ for some positive number C, from (4.11), it follows $\sigma_{i+1} > \sigma_i$.

Then let $\bar{v}_{i+1}(x)$ be a steady-state solution satisfying the boundary condition

$$\bar{v}_{i+1}(x_{i+1}) = v_{i+1}.$$

For w_{i+1}, the estimate of Riemann invariant also holds. In fact, since, from (4.8),

$$\bar{w}_{i+1}(x_{i+1}) = \bar{w}_i(x_{i+1}) + \mathbf{O}(h^{3\alpha-(\gamma-1)\beta}),$$

by using the similar argument of Lemmas 3.3 and 3.5,

$$|\bar{w}_{i+1}(x_{i+1}) - \bar{w}_i(x_i)| \le C|\bar{w}_i(x_i)||x_{i+1} - x_i| + \mathbf{O}(h^{3\alpha-(\gamma-1)\beta}).$$

Noting $p = \mathbf{O}(h^{-\alpha})$, we have

$$|\bar{w}_{i+1}(x_{i+1}) \le |\bar{w}_0(x_1)|(1 + C|x_{i+1} - x_1| + \mathbf{O}(h^2)) + \mathbf{O}(h^{2\alpha-(\gamma-1)\beta}).$$

Now we fix $\bar{v}_r(x)$ and $\bar{v}_{p-1}(x)$. Choosing $\bar{\sigma}$ near to σ_p and $\bar{\sigma_s}$ near to σ_s, we fill up by a steady state solution $\bar{v}(x)$ such that $\bar{v}((j+1/2)h) = v \equiv (\rho, m)^\top$ the gap between $x = (j+1/2)h + \bar{\sigma}(t - n\Delta t)$ and $x = (j+1/2)h + \bar{\sigma}_s(t - n\Delta t)$. First let $\bar{\sigma} = \bar{\sigma}(\rho, m)$ and $\bar{\sigma}_s = \bar{\sigma}_s(\rho, m)$ be solutions of the equations

$$\begin{aligned} &\bar{\sigma} = \bar{u}_{p-1} - S(\bar{\rho}, \bar{\rho}_{p-1}) \quad \text{at} \quad x = \bar{x}_p \equiv (j+1/2)h + \bar{\sigma}\Delta t/2, \\ &\bar{\sigma}_s = \bar{u}_r + S(\bar{\rho}, \bar{\rho}_r) \quad \text{at} \quad x = \bar{x}_s \equiv (j+1/2)h + \bar{\sigma}_s\Delta t/2. \end{aligned}$$

Then, from Theorem 4.1,

$$\begin{aligned} &\bar{\sigma} = \bar{\sigma}^* + \mathbf{O}(h) + \mathbf{O}(|\rho - \rho_m| + |m - m_m|), \\ &\bar{\sigma}_s = \bar{\sigma}_s^* + \mathbf{O}(h) + \mathbf{O}(|\rho - \rho_m| + |m - m_m|), \end{aligned}$$

where

$$\bar{\sigma}^* = u_{p-1} - S(\rho_m, \rho_{p-1}), \quad \bar{\sigma}_s^* = u_r + S(\rho_m, \rho_r).$$

Furthermore

$$\begin{aligned} &\frac{\partial\bar{\sigma}}{\partial\rho} = -\frac{\partial}{\partial\bar{\rho}}S(\bar{\rho}, \bar{\rho}_{p-1})(1 + \mathbf{O}(h)), \ \frac{\partial\bar{\sigma}}{\partial u} = \mathbf{O}(h), \\ &\frac{\partial\bar{\sigma}_s}{\partial\rho} = \frac{\partial}{\partial\bar{\rho}}S(\bar{\rho}, \bar{\rho}_r)(1 + \mathbf{O}(h)), \ \frac{\partial\bar{\sigma}_s}{\partial u} = \mathbf{O}(h). \end{aligned}$$

The Rankine-Hugoniot conditions are reduced to $\Lambda_1 = \Lambda_2 = 0$, where

$$\begin{aligned} &\Lambda_1 \equiv \bar{m} - \bar{m}_{p-1} - \bar{\sigma}(\bar{\rho} - \bar{\rho}_{p-1}) \quad \text{at} \quad x = \bar{x}_p, \\ &\Lambda_2 \equiv \bar{m}_r - \bar{m} - \bar{\sigma}_s(\bar{\rho}_r - \bar{\rho}) \quad \text{at} \quad x = \bar{x}_s. \end{aligned}$$

Then we have

$$\begin{aligned} &\Lambda_1\big|_{\rho=\rho_m, m=m_m} = m_m - m_{p-1} - \bar{\sigma}^*(\rho_m - \rho_{p-1}) + \mathbf{O}(h) = \mathbf{O}(h), \\ &\Lambda_2\big|_{\rho=\rho_m, m=m_m} = m_r - m_m - \bar{\sigma}_s^*(\rho_r - \rho_m) + \mathbf{O}(h) = \mathbf{O}(h). \end{aligned}$$

Moreover

$$\begin{aligned} &\frac{\partial\Lambda_1}{\partial\rho} = (\bar{\rho} - \bar{\rho}_{p-1})\frac{\partial}{\partial\bar{\rho}}S(\bar{\rho}, \bar{\rho}_{p-1}) - \bar{\sigma}^* + \bar{u} + \mathbf{O}(h), \ \frac{\partial\Lambda_1}{\partial u} = \bar{\rho} + \mathbf{O}(h), \\ &\frac{\partial\Lambda_2}{\partial\rho} = (\bar{\rho}_r - \bar{\rho})\frac{\partial}{\partial\bar{\rho}}S(\bar{\rho}, \bar{\rho}_r) + \bar{\sigma}_s^* - \bar{u} + \mathbf{O}(h), \ \frac{\partial\Lambda_2}{\partial u} = -\bar{\rho} + \mathbf{O}(h). \end{aligned}$$

Hence

$$\det\begin{bmatrix} \dfrac{\partial\varLambda_1}{\partial\rho} & \dfrac{\partial\varLambda_1}{\partial u} \\ \dfrac{\partial\varLambda_2}{\partial\rho} & \dfrac{\partial\varLambda_2}{\partial u} \end{bmatrix} = \bar{\rho}\Delta + \mathbf{O}(h^{\alpha - \frac{3-\gamma}{2}\beta}),$$

where

$$\Delta = -(\rho_r - \rho_m)\frac{\partial}{\partial\rho_m}S(\rho_m, \rho_r) + u_r + S(\rho_m, \rho_r) - u_m + S(\rho_m, \rho_m).$$

Since $\rho_m \geq \rho_r$, we have

$$\begin{aligned}\Delta &\geq u_r - u_m + S(\rho_m, \rho_r) + S(\rho_m, \rho_m)\\ &= -\sqrt{\frac{(\rho_m - \rho_r)(p(\rho_m) - p(\rho_r))}{\rho_m \rho_r}} + \sqrt{\frac{\rho_m(p(\rho_m) - p(\rho_r))}{\rho_r(\rho_m - \rho_r)}} + \sqrt{p'(\rho_m)}\\ &= \frac{\rho_r}{\rho_m}\sqrt{\frac{\rho_m(p(\rho_m) - p(\rho_r))}{\rho_r(\rho_m - \rho_r)}} + \sqrt{p'(\rho_m)}\\ &\geq Ch^{\frac{\gamma-1}{2}\beta}.\end{aligned}$$

Applying Theorem 4.2, we have a solution (ρ, m) satisfying

$$|\rho - \rho_m| + |m - m_m| = \mathbf{O}(h^{1-\frac{\gamma+1}{2}\beta}).$$

We denote the approximate steady-state solutions by $v_{0,i}(x) = (\rho_{0,i}(x), \rho_{0,i}(x)u_{0,i}(x))$, $0 \leq i \leq q$. The approximate solution $(\rho_0^h(x,t), m_0^h(x,t))$ of the Cauchy problem (4.4) in the rectangle $[jh, (j+1)h] \times [n\Delta t, (n+1)\Delta t]$ consists of the (exact or auxiliary) steady-state $v_{0,i}(x)$, $i = 0, 1, \ldots, q$, separated by the discontinuities $x = (j + 1/2)h + \sigma_i(t - n\Delta t)$ in this case.

Case 2. $\rho_m < h^\beta$. We now consider the case $\rho_m < h^\beta$. In this case, noting $1 \leq \frac{dw}{dz}|_{S_2(0)}$, $\rho_r < h^\beta$. If $\rho_l > h^\beta$, let v_l^* be the state connected to v_l on the right by the (exact or auxiliary) steady-state solutions and 1-rarefaction shocks in the fashion of the Case 1, such that $\rho_l^* = h^\beta$. If $\rho_l \leq h^\beta$, set $v_l^* = v_l$. Next we solve the Riemann problem (v_l^*, v_r). Then the approximate solution $(\rho_0^h(x,t), m_0^h(x,t))$ of the Cauchy problem (4.4) in the rectangle $[jh, (j+1)h] \times [n\Delta t, (n+1)\Delta t]$ is defined by joining this Riemann solution to that approximate rarefaction wave. Notice that, in the region $\rho \leq h^\beta$, no rarefaction shock is used and the order of inhomogeneous terms is $\mathbf{O}(h^\beta)$. The other cases (i.e. 1-shock and 2-shock, 1-shock and 2-Rw, etc.) can be considered in the same fashion.

Finally, set $v_0^h(x,t) = (\rho_0^h(x,t), m_0^h(x,t))$, which we construct. Then we define the approximate solution $v^h(x,t)$ of (4.4) in the strip $n\Delta t \leq t < (n+1)\Delta t$ by the fractional step procedure:

$$v^h(x,t) = v_0^h(x,t) + h(v_0^h(x,t))(t - n\Delta t),$$

where

$$h(v_0^h(x,t)) = \begin{cases} (0,0)^\top, \quad \text{when} \quad |M_0^2 - 1| \geq \varepsilon_0 M_0^2, \\ (0, a_3(\gamma-1)p(\rho_0^h(x,t)))^\top, \\ \qquad \text{when} \quad |M_0^2 - 1| \leq \varepsilon_0 M_0^2 \quad \text{and} \quad M_0 < 0, \\ \left(-(b+a_1)m_0^h(x,t), -(2b + a_2 + a_3/\gamma)\dfrac{\{m_0^h(x,t)\}^2}{\rho_0^h(x,t)}\right)^\top, \\ \qquad \text{when} \quad |M_0^2 - 1| \leq \varepsilon_0 M_0^2 \quad \text{and} \quad M_0 > 0. \end{cases}$$

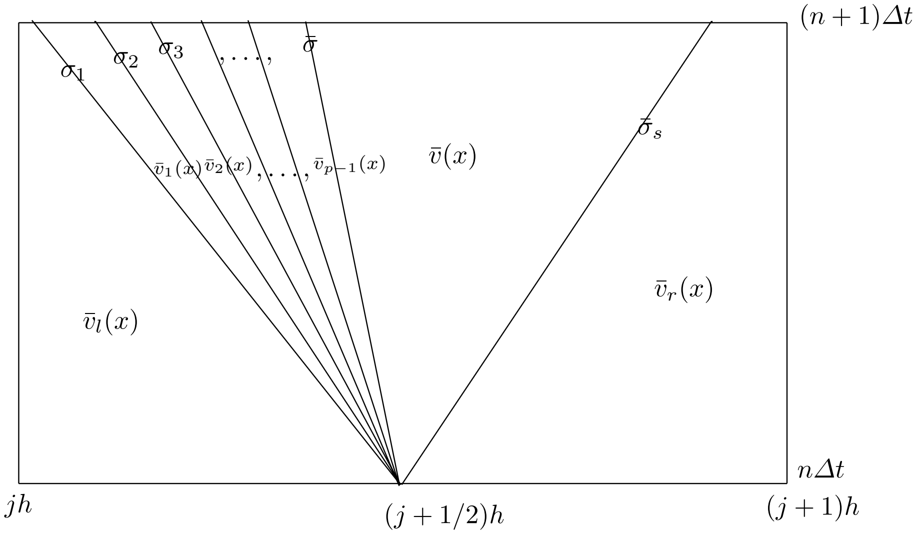

Figure 4.5. *Case* 1. The approximate solution v_0^h in the case 1-Rw and 2-shock arise.

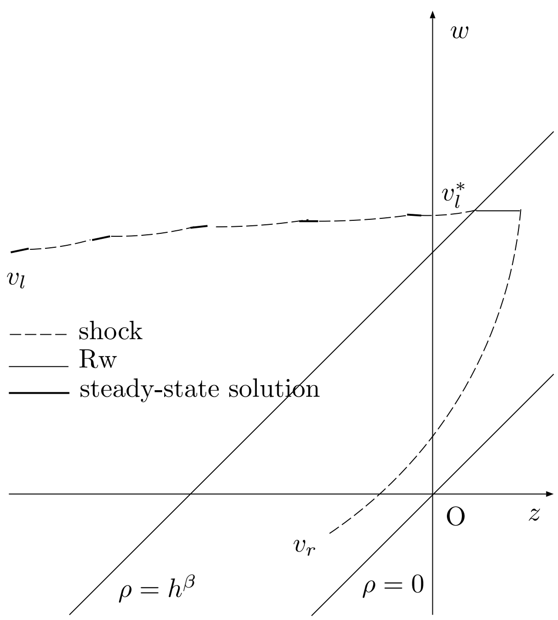

Figure 4.6. *Case* 2. The approximate solution v_0^h in the case 1-Rw and 2-shock arise.

4.2. Local entropy estimates

We now estimate local entropy errors of the discontinuities in the approximate solutions to allow the proof of vanishing of local entropy errors in the context of the H^{-1} compactness estimates in Section 6 and the consistency proof in Section 7 of the weak limit solution. First we prepare the following lemma.

Lemma 4.3. *Denote $v_0 = (\rho_0, m_0) \equiv v_i(x(t_0))$ as a left state of the discontinuity wave curve and denote $v = (\rho, m)$ as a point on the corresponding discontinuity wave curve with the left state v_0. Then, along the wave curve,*

$$\begin{aligned} &|\sigma(\rho)(\eta(v(\rho)) - \eta(v_0)) - (q(v(\rho)) - q(v_0))| \\ &\qquad \le C|\rho - \rho_0| \sup_{\rho \in [\rho_0, \rho]} |v(\rho) - v_0|^2 (\min(\rho, \rho_0))^{-2}, \end{aligned}$$

for any C^2 weak entropy-entropy flux pair (η, q), where C is a constant depending only on the uniform bound of v and v_0.

Proof. Along the wave curve, we have from (2.4),

$$\begin{aligned} m(\rho) &= \frac{m_0}{\rho_0}\rho \pm \sqrt{\frac{\rho}{\rho_0}\frac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \\ \sigma(\rho) &= \frac{m(\rho) - m_0}{\rho - \rho_0} = \frac{m_0}{\rho_0} \pm \sqrt{\frac{\rho}{\rho_0}\frac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}. \end{aligned}$$

Set

$$Q(\rho) = \sigma(\rho)(\eta(v(\rho)) - \eta(v_0)) - (q(v(\rho)) - q(v_0)).$$

Then

$$\dot{Q}(\rho) = \dot{\sigma}(\rho)(\eta(v(\rho)) - \eta(v_0)) + \sigma(\rho)\dot{\eta}(v(\rho)) - \dot{q}(v(\rho)).$$

Notice that

$$\begin{cases} \dot{\sigma}(\rho)(v(\rho) - v_0) + \sigma(\rho)\dot{v}(\rho) = \dot{f}(v(\rho)), & \text{(Rankine-Hugoniot condition)} \\ \dot{q}(v(\rho)) = \nabla q \cdot \dot{v}(\rho) = \nabla\eta \dot{f}(v(\rho)). \end{cases}$$

We have

$$\begin{aligned} \dot{Q}(\rho) &= \dot{\sigma}(\rho)\{\eta(v(\rho)) - \eta(v_0) - \nabla\eta(v(\rho))(v(\rho) - v_0)\} \\ &= -\dot{\sigma}(\rho)\int_0^1 \tau \frac{d^2}{d\tau^2}\eta(v_0 + \tau(v(\rho) - v_0))d\tau. \end{aligned}$$

Therefore, using the property of the Rankine-Hugoniot locus, we obtain

(4.12)
$$
\begin{aligned}
|Q(\rho)| &= \left|\int_{\rho_0}^{\rho} \dot{Q}(s)ds\right| \\
&= \left|\int_{\rho_0}^{\rho} (-\dot{\sigma}(s))\{\eta(v(s)) - \eta_0(v_0) - \nabla\eta(v(s))(v(s)-v_0)\}ds\right| \\
&= \left|\int_{\rho_0}^{\rho} \dot{\sigma}(s)ds \int_0^1 \tau(v(s)-v_0)^\top \nabla^2\eta(v_0+\tau(v(s)-v_0))(v(s)-v_0)d\tau\right| \\
&\le C\int_{\rho_0}^{\rho} |\dot{\sigma}(s)|ds \int_0^1 \tau(v(s)-v_0)^\top \nabla^2\eta_*(v_0+\tau(v(s)-v_0))(v(s)-v_0)d\tau \\
&\le C|\rho-\rho_0| \sup_{\rho\in[\rho_0,\rho]} |v(\rho)-v_0|^2(\min(\rho,\rho_0))^{-2}.
\end{aligned}
$$

□

The following lemmas can be checked easily by taking Taylor's expansion for t at $(n+1/2)\Delta t$ and using the above lemma.

Lemma 4.4. *On the discontinuous rays, $x = x_i$, $\sigma_i = \frac{dx_i(t)}{dt}$, of the approximate rarefaction waves constructed in Section* 4.1*,*

$$
\begin{aligned}
&\left|\int_{n\Delta t}^{(n+1)\Delta t} \sigma_i\{\eta(v_{0,i+1}(x(t))) - \eta(v_{0,i}(x(t)))\}\right. \\
&\qquad \left. - \{q(v_{0,i+1}(x(t))) - q(v_{0,i}(x(t)))\}dt\right| \le Ch^{1+3\alpha-\beta},
\end{aligned}
\tag{4.13}
$$

for any C^2 weak entropy-entropy flux pair (η, q), where C is a constant depending only on the uniform bound of $v_0^h(x,t)$.

Lemma 4.5. *There is a constant C depending only on the uniform bound of $v_0^h(x,t)$ such that, on the approximate shock waves,*

$$
\begin{aligned}
&\int_{n\Delta t}^{(n+1)\Delta t} \sigma_i(t)\{\eta(v_{0,i+1}(x(t))) - \eta(v_{0,i}(x(t))\} \\
&\qquad - \{q(v_{0,i+1}(x(t))) - q(v_{0,i}(x(t)))\}dt \ge -Ch^{3-\beta},
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
&\left|\int_{n\Delta t}^{(n+1)\Delta t} \sigma_i(t)\{\eta(v_{0,i+1}(x(t))) - \eta(v_{0,i}(x(t)))\}\right. \\
&\qquad \left. - \{q(v_{0,i+1}(x(t))) - q(v_{0,i}(x(t)))\}\right| \\
&\qquad \le C\int_{n\Delta t}^{(n+1)\Delta t} \sigma_i(t)\{\eta_*(v_{0,i+1}(x(t))) - \eta_*(v_{0,i}(x(t)))\} \\
&\qquad\quad - \{q_*(v_{0,i+1}(x(t))) - q_*(v_{0,i}(x(t)))\}dt + Ch^{3-\beta},
\end{aligned}
\tag{4.15}
$$

for any C^2 weak entropy-entropy pair (η, q) satisfying (1.8) *and the mechanical energy-energy flux (η_*, q_*) defined by* (1.9).

5. L^∞ estimates

We derive a L^∞ bound for the approximate solutions $v^h(x,t)$ of the Cauchy problem (1.7), with aid of the analysis of the approximate solutions $v^h(x,t) = (\rho^h(x,t), m^h(x,t))$. For some $T \in (0,\infty)$, define $\Pi_T = \mathbf{R} \times [0,T]$.

Theorem 5.1. *Assume that the initial velocity and nonnegative density data (ρ_0, u_0) are bounded in L^∞. Then there exists a constant $h_0 > 0$ such that, when $h \le h_0$ determined by* (4.1)*, the difference approximate solutions of the Cauchy problem* (1.7) *are uniformly bounded in the region Π_T. That is, there exists a constant $C(T) > 0$ such that*

$$|u^h(x,t)| \le C(T), \quad 0 \le \rho^h(x,t) \le C(T), \quad (x,t) \in \Pi_T. \tag{5.1}$$

Proof. Set $r_0 \equiv \max(\sup_x w(v_0(x)), -\inf_x z(v_0(x)), 1)$. Then we set $T = \frac{1}{3Cr_0e}$, where C is the constant determined in Lemmas 3.3 and 3.5. Let Λ in (4.2) be er_0.

First, using Lemma 2.2 and the construction of (ρ^h, m^h), one immediately concludes that

$$\rho^h(x,t) \ge 0, \quad \text{for} \quad -\infty < x < \infty, \quad 0 \le t \le T. \tag{5.2}$$

Now we make other estimates. Set

$$M_n = \max\left(\sup_x w(v^h(x, n\Delta t + 0)), -\inf_x z(v^h(x, n\Delta t + 0)), 1\right).$$

For $n\Delta t \le t < (n+1)\Delta t, n \ge 0$ integer, we assume that $M_n \le r_0 e^{3Cer_0T}$.

We use Lemma 2.2 and Lemma 3.3, Lemma 3.5 and the construction of (ρ_0^h, m_0^h) to get

$$\begin{cases} w(v_0^h(x,t)) \le \max\left(\sup_x w(v_0^h(x, n\Delta t + 0)), 1\right)(1 + 3Cer_0\Delta t), \\ z(v_0^h(x,t)) \ge \min\left(\inf_x z(v_0^h(x, n\Delta t + 0)), -1\right)(1 + 3Cer_0\Delta t) \end{cases}$$

for $h \le h_0$. In particular, this implies

$$\begin{cases} w(v_0^h(x, (n+1)\Delta t - 0)) \le \max\left(\sup_x w(v_0^h(x, n\Delta t + 0)), 1\right)(1 + 3Cer_0\Delta t), \\ z(v_0^h(x, (n+1)\Delta t - 0)) \ge \min\left(\inf_x z(v_0^h(x, n\Delta t + 0)), -1\right)(1 + 3Cer_0\Delta t). \end{cases}$$

Then one has

$$\begin{aligned} &\max\left(\sup_x w(v_0^h(x, (n+1)\Delta t - 0)), -\inf_x z(v_0^h(x, (n+1)\Delta t - 0))\right) \\ &\quad \le M_n(1 + 3Cer_0\Delta t). \end{aligned}$$

Now, in the case $|M_0^2-1|<\varepsilon_0 M_0^2$, $M_0<0$ and $\rho_0\geq h^\beta$, notice that

$$|z(v_0)|-|w(v_0)|>2\sqrt{\frac{1}{1+\varepsilon_0}}h^{\theta\beta}.$$

Then, choosing h_0 small enough, $|z(v^h(x,t))|>|w(v^h(x,t))|$. Since $z(v^h(x,t))>z(v_0^h(x,t))$, we have

$$|z(v_0^h(x,t))|>|z(v^h(x,t))|>|w(v^h(x,t))|>|w(v_0^h(x,t))|.$$

Similarly, in the case $|M_0^2-1|<\varepsilon_0 M_0^2$, $M_0>0$ and $\rho_0\geq h^\beta$, notice that

$$|w(v_0)|-|z(v_0)|>2\sqrt{\frac{1}{1+\varepsilon_0}}h^{\theta\beta}.$$

Then, choosing h_0 small enough, $|w(v^h(x,t))|>|z(v^h(x,t))|$. Since

$$\begin{aligned}
&w(v^h(x,t))\\
&\quad=\frac{m_0^h(x,t)}{\rho_0^h(x,t)}+\{\rho_0^h(x,t)\}^\theta/\theta-(b+a_1)\{\rho_0^h(x,t)\}^\theta u_0^h(x,t)(t-n\varDelta t)+\mathbf{O}(h^2)\\
&\quad<w(v_0^h(x,t)),
\end{aligned}$$

we have

$$|w(v_0^h(x,t))|>|w(v^h(x,t))|>|z(v^h(x,t))|>|z(v_0^h(x,t))|.$$

Therefore it follows from Lemma 2.2 that

$$M_{n+1}\leq M_n(1+3Cer_0\varDelta t),$$

that is,

$$\frac{M_{n+1}-M_n}{\varDelta t}\leq 3Cer_0M_n. \tag{5.3}$$

Consider the corresponding ordinary differential equation

$$\begin{cases}\dfrac{dr}{dt}=3Cer_0r,\\ r(0)=r_0.\end{cases} \tag{5.4}$$

It follows that

$$r_0\leq r(t)\leq\tilde{C}(T)\equiv r_0e^{3Cer_0T},\qquad\text{for}\quad 0\leq t\leq T. \tag{5.5}$$

Noting the integral curve $r=r(t)$ is convex curve, we obtain from (5.3)–(5.5) that

$$M_n\leq r(n\varDelta t)\leq\tilde{C}(T). \tag{5.6}$$

Furthermore, since $z \le \lambda_1$ and $w \ge \lambda_2$, we have

$$\max_{i=1,2}\left(\sup_{0\le t\le n\Delta t}|\lambda_i(\rho^h,m^h)|\right) \le er_0.$$

We derive from (5.2) and (5.6) that

$$\begin{cases} w(v^h(x,t)) \le \tilde{C}, \quad -z(v^h(x,t)) \le \tilde{C}, \\ w(v^h(x,t)) - z(v^h(x,t)) \ge 0, \end{cases}$$

that is, for $h \le h_0$, there is a constant $C(T) > 0$ such that

$$|u^h(x,t)| = \left|\frac{m^h(x,t)}{\rho^h(x,t)}\right| \le C, \quad 0 \le \rho^h(x,t) \le C.$$

□

6. H^{-1} compactness estimates

We prove the H^{-1} compactness for the approximate solutions (ρ^h, m^h) of the Cauchy problem (1.7). We first introduce a basic lemma of functional analysis (see [4]).

Lemma 6.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded and open set. Then*

$$\begin{aligned}(\text{compact set of } W^{-1,q}(\Omega)) \cap (\text{bounded set of } W^{-1,r}(\Omega))&\\ \subset (\text{compact set of } W^{-1,2}_{\text{loc}}(\Omega)),&\end{aligned}$$

where q and r are constants, $1 < q \le 2 < r < \infty$.

With Lemma 6.1, we have

Theorem 6.2. *Assume that (ρ^h, m^h) are the approximate solutions of the Cauchy problem* (1.7). *Then the measure sequence*

$$\eta(v^h)_t + q(v^h)_x$$

lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$ for all weak pairs (η, q), where $\Omega \subset \Pi_T$ is any bounded and open set.

Proof. For simplicity we will drop the index h of the approximate solutions $v^h(x,t)$.

Step 1. For any function $\phi \in C_0^1(\Pi_T)$, the entropy dissipation measures can be written in the form

$$\begin{aligned}\iint_{0\le t\le T=m\Delta t}&(\eta(v)\phi_t + q(v)\phi_x)dxdt\\ &= A(\phi) + L(\phi) + M(\phi) + N(\phi) + \sum(\phi) + E(\phi),\end{aligned} \tag{6.1}$$

where

$$A(\phi)=\iint_{\Pi_T}((\eta(v^h)-\eta(v_0^h))\phi_t+(q(v^h)-q(v_0^h))\phi_x)dxdt, \tag{6.2}$$

$$M(\phi)=\int_{-\infty}^{\infty}\phi(x,t)\eta(v_0^h(x,T))dx-\int_{-\infty}^{\infty}\phi(x,0)\eta(v_0^h(x,0))dx, \tag{6.3}$$

$$N(\phi)=\iint_{\Pi_T}\nabla\eta(v_0^h)(g(v_0^h)-h(v_0^h))\phi(x,t)dxdt, \tag{6.4}$$

$$L(\phi)=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta(v_{0-}^n)-\eta(v_{0+}^n))\phi(x,n\Delta t)dx\equiv L_1(\phi)+L_2(\phi), \tag{6.5}$$

$$L_1(\phi)=\sum_{j,n}\phi_j^n\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta(v_{0-}^n)-\eta(v_{0+}^n))dx, \tag{6.6}$$

$$L_2(\phi)=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta(v_{0-}^n)-\eta(v_{0+}^n))(\phi-\phi_j^n)dx, \tag{6.7}$$

$$\sum(\phi)=\int_0^T\sum(\sigma[\eta]-[q])\phi(x(t),t)dt, \tag{6.8}$$

$$|E(\phi)|\le Ch^{\beta}\|\phi\|_{H^1}, \tag{6.9}$$

where $v_{0\pm}^n=v_0^h(x,n\Delta t\pm 0)$, $\phi_j^n=\phi(jh,n\Delta t\pm 0)$, the summention in $\sum(\phi)$ is taken over all discontinuities in v_0^h at a fixed time t, σ is the propagating speed of the discontinuities, and $E(\phi)$ is the error term including the error in the steady-state solutions and the error near the vacuum in the construction of approximate solutions.

Let $S=(x(t),t)$ denote a discontinuity in $v_0^h(x,t)$, $[\eta]$ and $[q]$ denote the jump of $\eta(v_0^h(x,t))$ and $q(v_0^h(x,t))$ across S from left to right, respectively,

$$[\eta]=\eta(v_0^h(x(t)+0,t))-\eta(v_0^h(x(t)-0,t)),$$
$$[q]=q(v_0^h(x(t)+0,t))-q(v_0^h(x(t)-0,t)).$$

Step 2. Since the speeds of propagation of the approximate solutions $v^h(x,t)$ are finite, one can assume

$$(\rho_0^h,m_0^h)|_{x\ge K+\Lambda T}=(0,0)$$

for sufficiently large $K>0$, without loss of generality. This implies

$$\int_{-\infty}^{\infty}\eta_*(\rho^h(x,0),m^h(x,0))dx<\infty.$$

Noting that $(\rho^h,m^h)|_{x\ge K+\Lambda T}=(0,0)$, for sufficiently large $K>0$, we substitute $(\eta,q)=(\eta_*,q_*)$ and $\phi\equiv 1$ in the equality (6.1). Thus

$$\sum_{n=1}^{m}\int_{-\infty}^{\infty}[\eta_*^n]dx+\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\le C. \tag{6.10}$$

From Lemmas 3.4 and 3.6, we have

$$\begin{aligned}&\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)(v_+-v_j)^\top\nabla^2\eta_*(v_j+\tau(v_+-v_j))(v_+-v_j)d\tau dx\\&\qquad\le C\sum_{j,n}h\frac{(|\rho_j^n|^2+|m_j^n|^2)h^2}{\rho_j^n(1-\mathbf{O}(h))}.\end{aligned}$$

Using Lemmas 3.4 and 3.6 and the above estimate, we have

(6.11)

$$\begin{aligned}&\sum_{n=1}^{m}\int_{-\infty}^{\infty}[\eta_*^n]dx\\&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta_*(v_{0-}^n)-\eta_*(v_j^n))dx-\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta_*(v_{0+}^n)-\eta_*(v_j^n))dx\\&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)(v_{0-}^n-v_j^n)^\top\nabla^2\eta_*(v_j^n+\tau(v_{0-}^n-v_j^n))(v_{0-}^n-v_j^n)d\tau dx\\&\quad-\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)(v_{0+}^n-v_j^n)^\top\nabla^2\eta_*(v_j^n+\tau(v_{0+}^n-v_j^n))(v_{0+}^n-v_j^n)d\tau dx\\&\quad+\mathbf{O}(h)\\&=\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)(v_{0-}^n-v_j^n)^\top\nabla^2\eta_*(v_j^n+\tau(v_{0-}^n-v_j^n))(v_{0-}^n-v_j^n)d\tau dx\\&\quad+\mathbf{O}(h).\end{aligned}$$

Using Lemma 4.4, for approximate rarefaction waves, we have

$$\left|\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\right|\le CTh^{2\alpha-1-\beta}. \tag{6.12}$$

Similarly, using Lemma 4.5, for approximate shock waves,

$$\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\ge -CTh^{1-\beta}. \tag{6.13}$$

Therefore, choosing β small enough, we have

$$\int_0^T\sum(\sigma[\eta_*]-[q_*])dt\ge -CTh^{2\alpha-1-\beta} \tag{6.14}$$

for the convex entropy η_*. We have from (6.10)–(6.14) that

$$\int_0^T \sum(\sigma[\eta_*]-[q_*])dt \le C, \tag{6.15}$$

$$\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)(v_{0-}^n-v_j^n)^\top\nabla^2\eta_*(v_j^n+\tau(v_{0-}^n-v_j^n))(v_{0-}^n-v_j^n)d\tau dx \le C. \tag{6.16}$$

In particular, since $\nabla^2\eta_*(r,r)\ge c_0(r,r),\ c_0>0$ constant, one has

$$\sum_{\substack{j,n\\|jh|\le K}}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}|v_{0-}^n-v_j^n|^2dx \le C(K). \tag{6.17}$$

Step 3. For any bounded set $\Omega\subset\mathit{\Pi}_T$ and weak entropy pair (η,q), we derive from (6.1), (6.5)–(6.6), (6.15)–(6.16), and Lemma 2.3 that

$$\begin{aligned}
&|M(\phi)|\le C|\phi|_{C_0(\Omega)},\quad |N(\phi)|\le C\|\phi\|_{C_0(\Omega)},\\
&\left|\sum(\phi)\right|\le C\|\phi\|_{C_0(\Omega)}\int_0^T(h+\sum(\sigma[\eta_*]-[q_*])dt\le C\|\phi\|_{C_0(\Omega)},\\
&|L_1(\phi)|\le\left|\sum_{j,n}\phi_j^n\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta(v_{0-}^n)-\eta(v_j^n))dx\right|\\
&\qquad+\left|\sum_{j,n}\phi_j^n\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}(\eta(v_{0+}^n)-\eta(v_j^n))dx\right|\\
&\qquad\le\|\phi\|_{C_0(\Omega)}\left\{\begin{array}{l}\displaystyle\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)|(v_{0-}^n-v_j^n)^\top\nabla^2\eta\\
\quad\times(v_j^n+\tau(v_{0-}^n-v_j^n))(v_{0-}^n-v_j^n)|d\tau dx\\
\quad\displaystyle+\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)|(v_{0+}^n-v_j^n)^\top\nabla^2\eta\\
\quad\times(v_j^n+\tau(v_{0+}^n-v_j^n))(v_{0+}^n-v_j^n)|d\tau dx+\mathbf{O}(h)\end{array}\right\}\\
&\qquad\le C\|\phi\|_{C_0(\Omega)}\left\{\sum_{j,n}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\int_0^1(1-\tau)|(v_{0-}^n-v_j^n)^\top\right.\\
&\qquad\qquad\left.\times\nabla^2\eta_*(v_j^n+\tau(v_{0-}^n-v_j^n))(v_{0-}^n-v_j^n)|d\tau dx+\mathbf{O}(1)\right\}\\
&\qquad\le C\|\phi\|_{C_0(\Omega)},
\end{aligned}$$

where the constant C depends only on the support of ϕ. Hence

$$\left|\left(M+N+L_1+\sum\right)(\phi)\right|\le C\|\phi\|_{C_0},$$

that is

$$\left\| M + N + L_1 + \sum \right\|_{C_0^*} \leq C.$$

Therefore

$$M + N + L_1 + \sum \quad \text{compact in } W^{-1,q_1}(\Omega), \tag{6.18}$$

where $1 < q_1 < 2$.

Furthermore, for any $\phi \in C_0^\alpha(\Omega)$, $\frac{1}{2} < \alpha < 1$, we have

$$\begin{aligned}
|L_2(\phi)| &\leq \sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |\phi(n\Delta t, x) - \phi_j^n|(|\eta(v_{0-}^n) - \eta(v_j^n)| + |\eta(v_{0+}^n) - \eta(v_j^n)|)dx \\
&\leq h^\alpha \|\phi\|_{C_0^\alpha} \left\{ \sum_n \left(\sum_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |\eta(v_{0-}^n) - \eta(v_j^n)|^2 dx \right)^{\frac{1}{2}} + \mathbf{O}(h) \right\} \\
&\leq Ch^{\alpha-\frac{1}{2}} \|\nabla\eta\|_{L^\infty} \|\phi\|_{C_0^\alpha} \left\{ \left(\sum_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_{0-}^n - v_j^n|^2 dx \right)^{\frac{1}{2}} + \mathbf{O}(h) \right\} \\
&\leq 2Ch^{\alpha-\frac{1}{2}} \|\phi\|_{C_0^\alpha(\Omega)}.
\end{aligned}$$

Using the Sobolev theorem $W_0^{1,p}(\Omega) \subset C_0^\alpha(\Omega)$, $0 < \alpha < 1 - \frac{2}{p}$, one has

$$|L_2(\phi)| \leq Ch^{\alpha-\frac{1}{2}} \|\phi\|_{W_0^{1,p}(\Omega)}, \quad p > \frac{2}{1-\alpha},$$

that is

$$\|L_2\|_{W^{-1,q_2}(\Omega)} \leq Ch^{\alpha-\frac{1}{2}} \to 0, \quad h \to 0 \tag{6.19}$$

for $1 < q_2 < \frac{2}{1+\alpha}$. Moreover,

$$\|E\|_{H^{-1}} \leq Ch^\beta \to 0, \qquad \text{as} \quad h \to 0. \tag{6.20}$$

It follows from (6.18)–(6.20) that

$$M + N + L + \sum + E \text{ compact in } W_{\text{loc}}^{-1,q_0}, \tag{6.21}$$

where $1 < q_0 \equiv \min(q_1, q_2) < \frac{2}{1+\alpha}$. The fact $0 \leq \rho \leq C$ and $\left|\frac{m}{\rho}\right| \leq C$ implies

$$M + N + L + \sum + E \text{ bounded in } W_{\text{loc}}^{-1,r} (r > 1). \tag{6.22}$$

We derive from (6.20)–(6.22) and Lemma 6.1 that

$$M + N + L + \sum \text{ compact in } H_{\text{loc}}^{-1}. \tag{6.23}$$

Finally, for $A(\phi)$ we have

$$|A(\phi)| \leq \iint_{\Pi_T} (|\nabla\eta|_\infty + |\nabla q|_\infty)(|\phi_t| + |\phi_x|)|v^h - v_0^h| dxdt \leq Ch|\phi|_{H_0^1(\Omega)}.$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, then

$$|A|_{H_{\mathrm{loc}}^{-1}(\Omega)} \leq Ch \to 0, \qquad \text{as} \quad h \to 0,$$

so A is compact in $H_{\mathrm{loc}}^{-1}(\Omega)$.

Therefore $A + M + N + L + \sum + E$ is compact in $H_{\mathrm{loc}}^{-1}(\Omega)$, which means that

$$\eta(v^h)_t + q(v^h)_x \text{ compact in } H_{\mathrm{loc}}^{-1}(\Omega).$$

This completes the proof of Theorem 6.1.

□

7. Convergence and consistency

In Sections 5 and 6, it is proved that the approximate solutions (ρ^h, m^h) of the Cauchy problem (1.7) satisfy the following conditions.

(1) There is a constant $C(T) > 0$ such that

$$0 \leq \rho^h(x,t) \leq C, \quad \left|\frac{m^h(x,t)}{\rho^h(x,t)}\right| \leq C. \tag{7.1}$$

(2) The measure

$$\eta(v^h)_t + q(v^h)_x \text{ is compact in } H_{\mathrm{loc}}^{-1}(\Omega) \tag{7.2}$$

for all weak entropy pair (η, q), where $\Omega \subset \Pi_T$ is any bounded and open set.

The compensated compactness framework (see [3] and [4]) ensures the strong compactness of the approximate solution $v^h(x,t)$ in $L_{\mathrm{loc}}^1(\Pi_T)$ for $1 < \gamma \leq 5/3$.

We first introduce a lemma of Riemann solutions.

Lemma 7.1. *Let $v(x,t)$ be the approximate piecewise constant Riemann solution, which consist of boundary data of steady state solutions, and $K \subset \mathbf{R}$ be any bounded set. Then*

$$\sum_n \int_{(n-1)\Delta t}^{n\Delta t} \int_K |v(x,t) - v(x, n\Delta t - 0)|^2 dxdt = \mathbf{O}(h),$$

where $\mathbf{O}(h)$ depends on K. Notice that $v(x,t)$ is self-similar.

The proof of Lemma 7.1 can be found in [19].

Using Lemma 7.1, we have

Theorem 7.2. *Assume that (ρ^h, m^h) are the approximate solutions of the Cauchy problem* (1.3) *satisfying the conditions* (7.1)–(7.2). *Then there is a convergent subsequence in the approximate solutions $(\rho^h(x,t), m^h(x,t))$ such that*

$$(\rho^{h_n}(x,t), m^{h_n}(x,t)) \to (\rho(x,t), m(x,t)), \quad \text{a.e.} \tag{7.3}$$

The pair of functions $(\rho(x,t), m(x,t))$ is a local entropy solution of the Cauchy problem (1.7) *and satisfies*

$$0 \le \rho(x,t) \le C(T), \quad \left|\frac{m(x,t)}{\rho(x,t)}\right| \le C(T) \tag{7.4}$$

in the region Π_T for $T \in (0,\infty)$ determined Section 5.

Proof. It suffices to prove the limit functions (ρ, m) satisfy (1.12)–(1.13). Notice that for any convex weak entropy pair (η, q) and any nonnegative test function $\phi \in C_0^1(\Pi_T)$,

$$\begin{aligned}
&\iint_{0\le t\le T=m\Delta t} (\eta(v^h)\phi_t + q(v^h)\phi_x - \nabla\eta(v^h)g(v^h)\phi)dxdt \\
&\quad + \int_{-\infty}^{\infty} \eta(v_0^h(x))\phi(0,x)dx \\
&\quad = I(\phi) + J(\phi) + \int_0^T \sum(\sigma[\eta] - [q])\phi(x(t),t) + E(\phi),
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
I(\phi) = \iint_{\Pi_T} &\phi_t(\eta(v^h) - \eta(v_0^h)) + \phi_x(q(v^h) - q(v_0^h)) - \phi(\nabla\eta(v^h)g(v^h) \\
&- \nabla\eta(v_0^h)g(v_0^h))dxdt,
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
J(\phi) &= \sum_{j,n} \int_{(j-\frac12)h}^{(j+\frac12)h} (\eta(v_{0-}^n) - \eta(v_{0+}^n))\phi(x, n\Delta t)dx \\
&\quad + \iint_{\Pi_T} \nabla\eta(v_0^h)h(v_0^h)\phi(x,t)dxdt \\
&\equiv J_1(\phi) + J_2(\phi),
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
J_1(\phi) &= \sum_{j,n} \phi_j^n \int_{(j-\frac12)h}^{(j+\frac12)h} (\eta(v_{0-}^n) - \eta(v_{0+}^n))dx \\
&\quad + \iint_{\Pi_T} \nabla\eta(v_0^h)h(v_0^h)\phi(x,t)dxdt,
\end{aligned} \tag{7.8}$$

$$J_2(\phi) = \sum_{j,n} \int_{(j-\frac12)h}^{(j+\frac12)h} (\eta(v_{0-}^n) - \eta(v_{0+}^n))(\phi - \phi_j^n)dx, \tag{7.9}$$

where $v^n_{\pm} = v^h(x, n\Delta t \pm 0)$, $v^n_{0\pm} = v^h_0(x, n\Delta t \pm 0)$, $\phi^n_j = \phi(jh, n\Delta t)$, the summation is taken over all discontinuities in v^h_0 at a fixed t, σ is the propagating speed of the discontinuity, and

$$|E(\phi)| \le Ch^{\beta}\|\phi\|_{H^1}.$$

Since $v^h - v^h_0 = \mathbf{O}(h)$, $I \to 0$ as $h \to 0$ by Lebesgue's dominated convergence theorem.

Notice from Lemmas 4.4 and 4.5 that, for approximate rarefaction waves,

$$\left|\int_{n\Delta t}^{(n+1)\Delta t} \sigma[\eta] - [q]dt\right| \le Ch^{1+3\alpha-\beta},$$

and, for approximate shock waves,

$$\int_{n\Delta t}^{(n+1)\Delta t} \sigma[\eta] - [q]dt \ge -Ch^{3-\beta}$$

for the convex entropy η. Then one has

$$\begin{aligned}
&\int_0^T \sum(\sigma[\eta] - [q])\phi(x(t), t)dt \\
&\quad = \sum_n \min_{n\Delta t \le t \le (n+1)\Delta t}\{\phi(x(t), t)\} \int_{n\Delta t}^{(n+1)\Delta t} \sum(\sigma[\eta] - [q])dt \\
&\qquad + \sum_n \int_{n\Delta t}^{(n+1)\Delta t} \left(\phi(x(t), t) - \min_{n\Delta t \le t \le (n+1)\Delta t}\{\phi(x(t), t)\}\right) \sum(\sigma[\eta] - [q])dt \\
&\quad \ge -Ch^{2\alpha-1-\beta}\|\phi\|_C.
\end{aligned}$$

On the other hand, notice that for $(j - \frac{1}{2})h \le x \le (j + \frac{1}{2})h$

$$v^n_j = \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} v^h_{0-}dx + \frac{\Delta t}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} h(v^h_{0-})dx.$$

Then since η is convex, from the similar argument of (6.11),

$$\begin{aligned}
J_1(\phi) &= \sum_{j,n} \phi^n_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta(v^n_{0-}) - \eta(v^n_j))dx + \sum_{j,n} \phi^n_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\eta(v^n_j) - \eta(v^n_{0+}))dx \\
&\quad + \int\int_{\Pi_T} \nabla\eta(v^h_0)h(v^h_0)\phi(x, t)dxdt \\
&\ge \sum_{j,n} \phi^n_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \nabla\eta(v^n_j)(v^n_{0-} - v^n_j)dx + \int\int_{\Pi_T} \nabla\eta(v^h_0)h(v^h_0)\phi(x, t)dxdt \\
&\quad + \mathbf{O}(h) \\
&= -\Delta t \sum_{j,n} \phi^n_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \nabla\eta(v^n_j)h(v^n_{0-})dx
\end{aligned}$$

$$+\int\int_{\Pi_T} \nabla\eta(v_0^h)h(v_0^h)\phi(x,t)dxdt + \mathbf{O}(h)$$
$$= J_{11} + J_{12} + J_{13} + \mathbf{O}(h),$$

where

$$J_{11} = \sum_{j,n} \phi_j^n \int_{(n-1)\Delta t}^{n\Delta t} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\nabla\eta(v_0^h)h(v_0^h) - \nabla\eta(v_j^n)h(v_j^n))dxdt,$$

$$J_{12} = \sum_{j,n} \phi_j^n \int_{(n-1)\Delta t}^{n\Delta t} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \nabla\eta(v_j^n)(h(v_j^n) - h(v_{0-}^n))dxdt,$$

$$J_{13} = \sum_{j,n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (\nabla\eta(v_0^h)h(v_0^h) - \nabla\eta(v_j^n)h(v_{0-}^h))(\phi(x,t) - \phi_j^n)dxdt.$$

From Lemmas 3.4 and 3.6, the order of the difference between v_0^h and the corresponding piecewise constant approximate Riemann solution, which consist of boundary data of steady state solutions, is h. Therefore, noting $|\nabla^2\eta| \le C/\rho$, $|h| \le C\rho, |\nabla_v h| \le C, |\nabla\eta| \le C$, from (6.17) and Lemma 7.1,

$$|J_{11}| \le C\sum_{j,n} \phi_j^n \int_{(n-1)\Delta t}^{n\Delta t} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_0^h - v_j^n|dxddt$$
$$= \mathbf{O}(\sqrt{h}).$$

Since $|\nabla_v h| \le C, |\nabla\eta| \le C$, from (6.17), we can obtain

$$|J_{12}| \le C\sum_{j,n} \phi_j^n \int_{(n-1)\Delta t}^{n\Delta t} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_{0-}^h - v_j^n|dxdt = \mathbf{O}(\sqrt{h})$$

and

$$|J_{13}| = \mathbf{O}(h).$$

Therefore $J \to 0$ as $h \to 0$.

Furthermore, for any $\phi \in C_0^1(\Omega)$, we have,

$$\begin{aligned}
|J_2(\phi)| &\le \sum_{j,n} \phi_j^n \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |\phi(x,n\Delta t) - \phi_j^n|(|\eta(v_{0-}^n) - \eta(v_j^n)| \\
&\quad + |\eta(v_{0+}^n) - \eta(v_j^n)|)dx \\
&\le h\|\phi\|_{C_0^1} \left\{ \sum_n \left(\sum_j \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |\eta(v_{0-}^n) - \eta(v_j^n)|^2 dx \right)^{\frac{1}{2}} + \mathbf{O}(h) \right\} \\
&\le \sqrt{h}\|\nabla\eta\|_{L^\infty}\|\phi\|_{C_0^1} \left\{ \left(\sum_{j,n} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} |v_{0-}^n - v_j^n|^2 dx \right)^{\frac{1}{2}} + \mathbf{O}(h) \right\} \\
&\le 2C\sqrt{h}\|\phi\|_{C_0^1(\Omega)} \to 0, \quad \text{as } h \to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\iint_{0\le t\le T=m\Delta t}(\eta(v^h)\phi_t+q(v^h)\phi_x-\nabla\eta(v^h)g(v^h)\phi)dxdt\\
&\quad+\int_{-\infty}^{\infty}\eta(v_0^h(x))\phi(0,x)dx\\
&\quad\ge -C(h^{2\alpha-1-\beta}\|\phi\|_{C_0^1}+h^{\beta}\|\phi\|_{H^1})\to 0,\quad h\to 0.
\end{aligned}\tag{7.10}$$

Taking the limit $h\to 0$ on both of sides of (7.11) and using Lebesgue's dominated convergence theorem, we verify that the limit function $v=(\rho,m)$ satisfies

$$\eta(v)_t+q(v)_x+\nabla\eta(v)g(v)\le 0,\tag{7.11}$$

in the sense of distributions. Choosing $\eta(v)=\pm\rho,\pm m$, we immediately conclude that $v(x,t)$ is a weak solution. Using the standard procedure (cf. [23]), we conclude that the limit function $v(x,t)$ satisfies the entropy condition (1.12) along any shock wave. This completes the proof of Theorem 7.2. □

The rest of the proof of Theorem 1.2 is similar to the arguments in Section 9 of [6].

8. Defects in [6]

As mentioned above, the proofs of [6] are incorrect. Here we point out the defects using the notations and the equation number in [6].

8.1. Local entropy estimates

Noting that

the Rankine-Hugoniot condition

$$\Longleftrightarrow$$

(8.1)

$$\frac{dx(t)}{dt}=u_i(x(t))+(-1)^{\kappa}\sqrt{\frac{\rho_{i+1}(x(t))}{\rho_i(x(t))}\frac{p(\rho_{i+1}(x(t)))-p(\rho_i(x(t)))}{\rho_{i+1}(x(t))-\rho_i(x(t))}},\quad \kappa=1,2$$

and

$$\begin{aligned}
&\frac{\{m_{i+1}(x(t))-m_i(x(t))\}^2}{\{\rho_{i+1}(x(t))-\rho_i(x(t))\}}\\
&\qquad=\left\{\frac{m_{i+1}^2(x(t))}{\rho_{i+1}(x(t))}+p(\rho_{i+1}(x(t)))-\frac{m_i^2(x(t))}{\rho_i(x(t))}+p(\rho_i(x(t)))\right\},
\end{aligned}\tag{8.2}$$

the equation at the fourth line from the bottom p. 178 is an only necessary condition of the Rankine-Hugoniot condition. That is, their approximate solutions don't satisfy (8.2). However they use the Rankine-Hugoniot condition in the proof of Lemmas 4.1 and 4.2.

8.2. Standing shocks

The order of the error of the Rankine-Hugoniot condition for standing shock is $\sqrt{h}$. Therefore the H^{-1} compactness estimate in Section 6 is failed (they don't seem to consider standing shock at all in Section 6). In fact, since for standing shock

$$\int_{n\Delta t}^{(n+1)\Delta t} \sigma[\eta] - [q]dt \geq -C\sqrt{h},$$

(6.11) doesn't hold.

Moreover they don't consider the case standing shock and other waves interact.

8.3. L^∞ estimates

Constant C in Theorem 5.1 is exactly $C\Lambda$, where C and Λ are determined in Lemma 3.3, 3.6 and 3.7, (4.2) respectively. Then $\tilde{C}(T)$ in (5.5) becomes $r_0 e^{C\Lambda T}$ and we have

$$\begin{cases} w(v^h(x,t)) \leq \tilde{C}, \quad -z(v^h(x,t)) \leq \tilde{C}, \\ w(v^h(x,t)) - z(v^h(x,t)) \geq \dfrac{2h^{\beta\theta}}{\theta}. \end{cases} \tag{8.3}$$

Then, of course, the following Courant-Friedrichs-Lewy condition

$$\max_{i=1,2}\left(\sup_{0\leq t\leq n\Delta t} |\lambda_i(\rho^h, m^h)|\right) \leq 2\Lambda \tag{8.4}$$

must follow from (8.3). However, noting $z = u - \rho^\theta/\theta,\ w = u + \rho^\theta/\theta,\ \lambda_1 = u - \rho^\theta,\ \lambda_2 = u + \rho^\theta$, this doesn't hold in general.

8.4. The disposal of the vacuum

On pp. 177–178, they use cut-off technique in order to exclude the vacuum. However, if we use this method, in particular (4.3), the order of the difference v_- and v_j (v_j probably represents Godunov value v_j^n) at line 9-10 from the bottom on page 185 is h^β. Then this estimate of L_1 is failed. Moreover, although they introduce the constant region $\rho = h^\beta$ in solving the Riemann problem, on $\rho = h^\beta$ of this modified Riemann solution, since $u(x,t) = u\left(\frac{x-(j+\frac{1}{2})h}{t-n\Delta t}\right)$, for example, the first equation of (2.1) becomes

$$\rho_t + m_x = h^\beta \frac{1}{t-n\Delta t} u'\left(\frac{x-(j+\frac{1}{2})h}{t-n\Delta t}\right). \tag{8.5}$$

Therefore, noting $\beta < 1$, (8.5) $\to \infty$ as $t \to n\Delta t$.

8.5. A counterexample of their shock capturing scheme

Their scheme claims that, if steady-state solutions satisfy Lemma 3.3, 3.6 and 3.7, the corresponding equations have a global solution. This is the main point of this paper. However, consider

$$\begin{cases} \rho_t + m_x = am, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = 2a\dfrac{m^2}{\rho}, \quad p(\rho) = \rho^\gamma/\gamma, \end{cases} \tag{8.6}$$

where a is constant such that $a \neq 0$. Then steady-state solutions in (8.6) satisfy Lemmas. In fact, consider ordinary differential equations

$$\begin{cases} m_x = am, \\ \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = 2a\dfrac{m^2}{\rho}, \end{cases} \tag{8.7}$$

subject to the boundary condition

$$(\rho, m)|_{x=x_0} = (\rho_0, m_0). \tag{8.8}$$

From (8.7)–(8.8), we have $\rho(x) = \rho_0, m(x) = m_0 e^{a(x-x_0)}$ and $M(x) = M_0 e^{a(x-x_0)}$. Then there exist $h_1 \in (0,1]$ and $C > 0$ such that, in any interval $[x_0 - h/2, x_0 + h/2], h \leq h_1$, when $M_0 > 0$,

$$\begin{aligned} \frac{w(v)}{w(v_0)} &= 1 + \frac{\theta(M - M_0)}{\theta M_0 + 1} \\ &\leq 1 + C|x - x_0|. \end{aligned}$$

Similarly, when $M_0 < 0$,

$$\begin{aligned} \frac{z(v)}{z(v_0)} &= 1 + \frac{\theta(M - M_0)}{\theta M_0 - 1} \\ &\geq 1 + C|x - x_0|. \end{aligned}$$

However, assume that solutions depend on only time. From the following equations

$$\begin{cases} \rho_t = am, \\ m_t = 2a\dfrac{m^2}{\rho}, \end{cases} \tag{8.9}$$

we have $u_t = au^2$. This clearly has a blow up solution. Notice that this is caused by Subsection 8.3.

Remark 8.1. Since this result is used in [7], [8], [9] and [25], their proofs are also incorrect.

9. Open problems

Here we list some open problems related to this paper.

• We first introduce a example.

$$\rho(x,t) = \frac{C_2}{(t+C_1)^3}, \quad u(x,t) = \frac{x}{t+C_1}, \tag{9.1}$$

where C_1 and C_2 are constants. (8.1) is a solution of (1.5). If $C_1 > 0$ (the initial velocity is positive), this solution is global. On the other hand, if $C_1 < 0$ (the initial velocity is negative), this solution blows up. Therefore a blow up solution certainly exists. Then can another blow up solution be constructed, perfectly in more general?

• For the case initial Riemann invariant z is nonnegative, the global existence of solutions has obtained in [2]. Can the global existence of solutions (not necessarily including the origin) be proved except this result (of course, and (8.1))? In addition, since the proof of [6] is incorrect, the global existence theorem for the duct flow and self-gravitating gases isn't also obtained.

• The initial density of [2] and Theorem 1.2 is 0 at the origin. Can the existence (not necessarily global) with initial density, which isn't 0 at the origin, be proved (of course, except (8.1))?

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