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Non effectively hyperbolic operators, Hamilton map and bicharacteristics

Dedicated to Professor Mitsuru IKAWA on his sixties birthday

By

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1. Introduction

Let $P(x, D)$ be a differential operator of order m on Ω, an open set in $\mathbf{R}^{n+1}$ with a system of local coordinates $x = (x_0, x_1, \dots, x_n) = (x_0, x')$. Let $p(x, \xi)$ be the principal symbol of $P(x, D)$ and we assume that p admits at most double characteristics. Let H_p be the Hamilton vector field of p and let $\rho \in T^*\Omega \setminus \{0\}$ be a double characteristic of p. Then it is expected that the behavior of (null) bicharacteristics, that is integral curves of H_p on which p vanishes, near ρ plays a definitive role in the correctness of the (microlocal) Cauchy problem for P.

To study the behavior of bicharacteristics we linearize H_p at ρ which is a singular point of H_p: recall

$$dp(\rho)(X) = \sigma(X, H_p(\rho)), \qquad X \in T_\rho T^*\Omega,$$

where σ is the standard symplectic two form on $T^*\Omega$:

$$\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j = d\xi \wedge dx$$

and (x, ξ) is a system of symplectic coordinates on $T^*\Omega$. Then the linearization of H_p at ρ, called the Hamilton map (matrix) of p at ρ, denoted by $F_p(\rho)$ is given by

$$\frac{1}{2} \operatorname{Hess} p(\rho)(X, Y) = \sigma(X, F_p(\rho)Y), \qquad X, Y \in T_\rho T^*\Omega.$$

It is well known that $F_p(\rho)$ has only pure imaginary eigenvalues with a possible exception of a pair of non zero real eigenvalues $\pm\lambda$ (see [3], [6]). If $F_p(\rho)$ has a pair of non zero real eigenvalues we say that p is effectively hyperbolic at ρ and the microlocal Cauchy problem is well posed for any lower order term (see [12], [7], [4], [9]). We recall that p is effectively hyperbolic at ρ if and only if

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every bicharacteristic issuing from simple characteristics having a limit point ρ arrive at ρ transversally to the doubly characteristic set, Σ (see [13]). If $F_p(\rho)$ has only pure imaginary eigenvalues and moreover if

$$\mathrm{Im}\, F_p(\rho)^2 \cap \mathrm{Ker}\, F_p(\rho)^2 = \{0\}, \qquad \rho \in \Sigma, \tag{1.1}$$

then there is no bicharacteristic issuing from simple characteristics having a limit point on Σ. In this case, under some assumptions on the stability of the symplectic structure of $F_p(\rho)$ when ρ varies in Σ, necessary and sufficient conditions required on the lower order terms (Levi conditions) for the correctness of the Cauchy problem of P, are known (see [3], [5]).

In this paper we study the case

$$\mathrm{Im}\, F_p(\rho)^2 \cap \mathrm{Ker}\, F_p(\rho)^2 \neq \{0\}, \qquad \rho \in \Sigma. \tag{1.2}$$

In this case the behavior of bicharacteristics near ρ can not be determined completely by F_p. To determine the complete behavior of bicharacteristics we need the third order term of the Taylor expansion of p around ρ.

To be more precise we fix the notation. We are working in a conic neighborhood of a double characteristic $\bar{\rho} = (\bar{x}, \bar{\xi})$. Without restrictions we may assume that $P(x, D)$ is of second order. We assume that p is hyperbolic with respect to dx_0, i.e., $p(x, \xi_0, \xi') = 0$ has only real zeros ξ_0 for (x, ξ') near $\bar{\rho}' = (\bar{x}, \bar{\xi}')$.

We introduce the following hypotheses: the doubly characteristic set

$$\Sigma = \{(x, \xi) \mid p(x, \xi) = dp(x, \xi) = 0\}$$

is a smooth manifold near $\bar{\rho}$ such that

$$\dim T_\rho \Sigma = \mathrm{dimKer}\, F_p(\rho), \qquad \rho \in \Sigma \tag{1.3}$$

(the codimension of Σ is equal to the rank of the Hessian of p at every point on Σ) and

$$\mathrm{rank}\, \sigma|_\Sigma = \mathrm{constant}, \qquad \rho \in \Sigma \tag{1.4}$$

and finally

$$\sigma(F_p(\rho)) \subset i\mathbf{R}, \qquad \mathrm{Ker}\, F_p(\rho)^2 \cap \mathrm{Im}\, F_p(\rho)^2 \neq \{0\}, \qquad \forall \rho \in \Sigma, \tag{1.5}$$

where $\sigma(F_p(\rho))$ denotes the spectrum of $F_p(\rho)$. This implies that p is not effectively hyperbolic and the Hamilton map $F_p(\rho)$ has a Jordan block of size four at every $\rho \in \Sigma$.

Let S be a smooth real function vanishing on Σ such that $H_S(\rho) \in \mathrm{Im}\, F_p(\rho)^3 \cap \mathrm{Ker}\, F_p(\rho)$, $\rho \in \Sigma$ then we prove that there is no bicharacteristic issuing from simple characteristics admitting a limit point on Σ if and only if $H_S^3\, p(\rho) = 0$ for every $\rho \in \Sigma$. The same result has been proved in [10] when the codimension of Σ is 3. Actually in this case the assumptions (1.3) and (1.4) are not needed. In this paper we prove this assertion in full generality.

The proof of this equivalence is carried out by using another equivalence: $H_S^3 p$ vanishes on Σ if and only if p can be factorized in the sense of Ivrii [4]. This equivalence has been proved in [13] under unnecessary restrictions and was proved in full generality, removing these restrictions, by Bernardi-Bove-Parenti [2]. Then to prove the equivalence it suffices to show that there is a bicharacteristic issuing outside Σ which admits a limit point on Σ if $H_S^3 p(\rho) \neq 0$ at some $\rho \in \Sigma$ since it was proved in [5] that no such bicharacteristic exists if p admits an elementary decomposition. This generalization has been tried in [1] also, but it seems that the proof there is insufficient. Here to prove the existence of a bicharacteristic having a limit point on Σ we employ a different method from that in [10] and [1].

Every result in this paper is microlocal in its nature: the arguments take place in a conical neighborhood of a point of Σ, which can be possibly shrunken, during the course of the proof. For the sake of brevity there is no mention of the neighborhood if there is no confusion. Without restrictions we may assume that $p(x,\xi)$ has the form

$$p(x,\xi) = -\xi_0^2 + q(x,\xi'), \tag{1.6}$$

where $q(x,\xi') \geq 0$ near $\bar\rho' = (\bar x, \bar\xi')$.

We recall Proposition 2.2 of [1]:

Proposition 1.1 ([1]). *Assume that p satisfies* (1.3), (1.4) *and* (1.5). *Then there exist two smooth sections of $T_\Sigma T^*\Omega$, z_1, z_2 such that*

$$z_1(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p(\rho)^3, \qquad \forall \rho \in \Sigma, \tag{1.7}$$

$$z_2(\rho) \in \operatorname{Ker} F_p(\rho)^2 \cap \operatorname{Im} F_p(\rho)^2, \qquad \forall \rho \in \Sigma, \tag{1.8}$$

$$\forall w \in \langle z_1(\rho)\rangle^\sigma \Longrightarrow \sigma(w, F_p(\rho)w) \geq 0, \tag{1.9}$$

$$w \in \langle z_1(\rho)\rangle^\sigma,\ \sigma(w, F_p(\rho)w) = 0 \Longrightarrow w \in \operatorname{Ker} F_p(\rho) \oplus \langle z_2(\rho)\rangle. \tag{1.10}$$

Let $S(x,\xi)$ be a smooth real function defined on $T^*\Omega$, homogeneous of degree 0, such that

$$S(x,\xi) = 0, \qquad (x,\xi) \in \Sigma, \tag{1.11}$$

$$H_S(\rho) = \theta_S(\rho) z_2(\rho) + v(\rho), \qquad \theta_S(\rho) \neq 0, \qquad \rho \in \Sigma \tag{1.12}$$

with $v(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p(\rho)$.

We now state our result:

Theorem 1.1. *Assume that p satisfies* (1.3), (1.4) *and* (1.5). *Then the following assertions are equivalent*:

(i) $H_S^3 p(\rho) = 0, \quad \forall \rho \in \Sigma$,

(ii) *there is no null bicharacteristic of p issuing from a simple characteristic having a limit point on Σ.*

To relate the result to correctness results of the Cauchy problem, we first recall

Definition (see [4], [5]). We say that p admits an elementary decomposition if there exist λ, μ, Q real valued symbols in (x,ξ') smoothly depending on x_0, homogeneous of degree 1, 1, 2 respectively, $Q(x,\xi')\geq 0$ such that with $\Lambda(x,\xi)=\xi_0-\lambda(x,\xi')$ and $M(x,\xi)=\xi_0-\mu(x,\xi')$

$$p(x,\xi)=-\Lambda(x,\xi)M(x,\xi)+Q(x,\xi'), \tag{1.13}$$

$$|\{\Lambda,M\}(x,\xi)|\leq C[|\Lambda(x,\xi)-M(x,\xi)|+\sqrt{Q(x,\xi')}], \tag{1.14}$$

$$|\{\Lambda,Q\}(x,\xi')|\leq C'Q(x,\xi') \tag{1.15}$$

with some positive constants C, C' where $\{f,g\}$ denotes the Poisson bracket of f and g.

For a class of operators admitting an elementary decomposition, Ivrii [4] (see also [3]) derived an a priori estimate, assuming (Levi) conditions on lower order terms, yielding the correctness of the Cauchy problem. Thus the next result relates Theorem 1.1 to the correctness of the Cauchy problem:

Theorem 1.2 ([2], [13]). *Assume that p verifies* (1.3), (1.4) *and* (1.5). *Then the following assertions are equivalent*:

(i) $H_S^3\,p(\rho)=0,\quad \forall\rho\in\Sigma$,

(ii) *p admits an elementary decomposition.*

Theorems 1.1 and 1.2 show

Theorem 1.3. *Assume that p verifies* (1.3), (1.4) *and* (1.5). *Then p admits an elementary decomposition if and only if there is no bicharacteristic of p issuing from a simple characteristic having a limit point on Σ.*

As mentioned above, this result holds without the assumptions (1.3) and (1.4) if the codimension of Σ is 3 (Theorem 2.1 in [10]).

As far as the correctness of the Cauchy problem near a double characteristic is concerned, we may say that if there is no bicharacteristic having a limit point on Σ then the situation is fairly well understood while almost nothing is known in the case if there is such a bicharacteristic.

To prove Theorem 1.1, assuming that the condition (i) is violated, we look for a bicharacteristic $(x(s),\xi(s))$ such that

$$\lim_{s\to\infty}s^2(x(s),\xi(s))=v\neq 0,$$
$$v\in\operatorname{Ker}F_p^2\cap\operatorname{Im}F_p^2,\qquad 0\neq F_pv\in\operatorname{Ker}F_p\cap\operatorname{Im}F_p^3.$$

To put the above conditions in evidence, in Section 2, we choose symplectic coordinates so that the line spanned by $z(\rho)$:

$$z(\rho)\in\operatorname{Ker}F_p(\rho)^2\cap\operatorname{Im}F_p(\rho)^2,\qquad 0\neq F_p(\rho)z(\rho)\in\operatorname{Ker}F_p(\rho)\cap\operatorname{Im}F_p(\rho)^3$$

(actually $z(\rho)$ is unique up to a multiple factor so that it is proportional to v) is given by $m_j(x,\xi)=0$ on Σ and the expression of p, in these coordinates, contains the sum of squares of m_j. This suggests that our expecting solution satisfies approximately the Hamilton system with Hamiltonian $\tilde{p}$ which is obtained from p removing the terms m_j^2. In Section 3 we write down our Hamilton system supposing that m_j were unknowns. We look for a solution $(x(s), \xi(s))$ of the Hamilton system such that $\xi(s) = O(s^{-2})$, $x'(s) = O(s^{-3})$ and $m_j(x(s), \xi(s)) = O(s^{-4})$. To do so, in Section 4, we first transform the thus obtained Hamilton system to another system by the change of independent variable $t = s^{-1}$ and suitable change of unknowns. The resulting system is a coupled system consists of a system which has the zero as an irregular singularity and a system which has the zero as a regular singularity. The main feature of the system is that all eigenvalues of the leading term of the irregular singularity (the coefficient matrix of t^{-2}) are pure imaginary and different from zero. In Section 5 we show that if the condition (i) is not verified then there is a unique, up to $\operatorname{Ker} F_p / \operatorname{Ker} F_p \cap \operatorname{Im} F_p^3$, formal series solution in t and $\log 1/t$ of the Hamilton system. In Section 6 we prove the existence result of solutions to the coupled system, modelled by this Hamilton system, by successive approximations assuming the existence of a formal solution. Finally in Section 7 we prove that there exists a solution which is asymptotically equal to this formal series solution applying the results in Sections 5 and 6.

2. Symplectic coordinates

Following [13], we choose special symplectic coordinates so that the condition (i) in Theorem 1.1 comes clear. We assume that the condition (i) in Theorem 1.1 is violated at some $\bar{\rho} \in \Sigma$. Then there is a neighborhood W of $\bar{\rho}$ such that

$$H_S^3 p(\rho) \neq 0, \qquad \rho \in W \cap \Sigma. \tag{2.1}$$

Without restrictions we may assume that $\bar{\rho} = (0, e_n)$. Let us denote $x^{(p)} = (x_p, \ldots, x_n)$, $\xi^{(p)} = (\xi_p, \ldots, \xi_n)$. We recall Lemma 4.1 in [13].

Lemma 2.1 ([13]). *Assume* (1.5) *at* $\bar{\rho}$. *Then there is a symplectic local coordinates* $(x^{(1)}, \xi^{(1)})$ *around* $(0, e_n^{(1)})$ *such that*

$$\begin{aligned} p(x,\xi) = -\xi_0^2 + \sum_{i=1}^{p} (x_{i-1} - x_i)^2 q_i(x, \xi^{(1)}) \\ + \sum_{i=1}^{p} r_i(x, \xi^{(1)}) \xi_i^2 + r_{p+1}(x, \xi^{(1)}) g(x^{(p)}, \xi^{(p+1)}), \end{aligned}$$

where

$$\{\xi_p, \{\xi_p, g\}\}(0, e_n^{(p+1)}) = 0, \qquad \sum_{i=1}^{p} r_i(0, e_n^{(p+1)})^{-1} = 1 \tag{2.2}$$

and $r_{p+1}(0, e_n^{(1)}) > 0$, $g(x^{(p)}, \xi^{(p+1)}) \geq 0$, *vanishing at* $(0, e_n^{(p+1)})$.

From the hypothesis (1.3) and the Morse lemma there are $n_i(x^{(p)}, \xi^{(p+1)})$ such that

$$g(x^{(p)}, \xi^{(p+1)}) = \sum_{i=1}^{h} n_i(x^{(p)}, \xi^{(p+1)})^2,$$

where $dn_i(0, e_n^{(p+1)})$ are linearly independent. Note that (2.2) implies

$$\frac{\partial}{\partial x_p} n_i(0, e_n^{(p+1)}) = 0, \qquad i = 1, \ldots, h. \tag{2.3}$$

Proposition 2.1. *Assume* (1.5). *For any small conic neighborhood V of $(0, e_n)$ there exist $\hat{\rho} \in V$, $1 \le p \le n-1$ and a symplectic local chart $\{U, (x, \xi)\}$ around $\hat{\rho}$, such that*

$$\begin{aligned} p(x, \xi) = -\xi_0^2 &+ \sum_{i=1}^{p} q_i(x, \xi^{(1)})(x_{i-1} - x_i)^2 + \sum_{i=1}^{p} r_i(x, \xi^{(1)})\xi_i^2 \\ &+ r_{p+1}(x, \xi^{(1)}) \sum_{i=1}^{h} n_i(x^{(p)}, \xi^{(p+1)})^2, \end{aligned} \tag{2.4}$$

where

$$\frac{\partial}{\partial x_p} n_i(x^{(p)}, \xi^{(p+1)}) = 0 \qquad \text{on } \Sigma \cap U \tag{2.5}$$

and

$$\sum_{i=1}^{p} r_i(x, \xi^{(1)})^{-1} = 1 \qquad \text{on } \Sigma \cap U. \tag{2.6}$$

Proof. As observed after Lemma 2.1, (2.4) holds in a conic neighborhood V of $(0, e_n)$. Assume

$$\frac{\partial}{\partial x_p} n_i(\hat{x}^{(p)}, \hat{\xi}^{(p+1)}) = 0, \qquad 1 \le i \le h \tag{2.7}$$

at some $(\hat{x}, \hat{\xi}) \in V \cap \Sigma$. It is clear that $(\hat{x}, \hat{\xi}) = (\hat{x}_p, \ldots, \hat{x}_p, \hat{x}^{(p+1)}, 0, \ldots, 0, \hat{\xi}^{(p+1)})$ and hence the Taylor expansion of p around $(\hat{x}, \hat{\xi})$ starts with

$$\begin{aligned} P = -\xi_0^2 &+ \sum_{i=1}^{p} q_i(\hat{x}, \hat{\xi})(x_{i-1} - x_i)^2 + \sum_{i=1}^{p} r_i(\hat{x}, \hat{\xi})\xi_i^2 \\ &+ r_{p+1}(\hat{x}, \hat{\xi}) \sum_{i=1}^{h} dn_i(x^{(p)}, \xi^{(p+1)})^2, \end{aligned}$$

where dn_i is the linear part of n_i at $(\hat{x}, \hat{\xi})$. By (2.5) we have

$$\left\{\xi_p, \left\{\xi_p, \sum_{i=1}^{h} dn_i^2\right\}\right\} = 0$$

and then it follows that

$$F_P = F_{\hat{P}} \oplus F_E,$$

where E is a non negative quadratic form in $(x^{(p+1)}, \xi^{(p+1)})$ and

$$\hat{P} = -\xi_0^2 + \sum_{i=1}^{p} q_i(x_{i-1} - x_i)^2 + \sum_{i=1}^{p} r_i \xi_i^2, \qquad q_i = (\hat{x}, \hat{\xi}), \quad r_i = r_i(\hat{x}, \hat{\xi}).$$

Since $\det(\lambda - F_{\hat{P}}) = \lambda^2 \psi(\lambda)$ with

$$\psi(0) = -\left(\prod_{j=1}^{p} 4q_j\right)\left(\prod_{j=1}^{p} r_j\right)\left(\sum_{j=1}^{p} r_j^{-1} - 1\right),$$

where $\hat{P}$ is considered in $(x^{(p)}, \xi^{(p)})$ space (see Proposition 2.2 in [11]) we have $\sum_{j=1}^{p} r_j^{-1} \le 1$ otherwise $\psi(0) < 0$ and hence $F_{\hat{P}}$ had a non zero real eigenvalue contradicting (1.5). If $\sum_{j=1}^{p} r_j^{-1} < 1$ so that $\psi(0) > 0$, then

$$\operatorname{Ker} F_{\hat{P}}^2 \cap \operatorname{Im} F_{\hat{P}}^2 = \{0\}$$

because the eigenvalue 0 is at most double. On the other hand from Theorem 1.3.8 in [3] it follows that

$$\operatorname{Ker} F_E^2 \cap \operatorname{Im} F_E^2 = \{0\} \tag{2.8}$$

and hence we have a contradiction to (1.5). Thus we conclude that

$$\sum_{j=1}^{p} r_j(\hat{x}, \hat{\xi})^{-1} = 1$$

provided (2.7) holds. Thus if (2.5) holds in V then nothing to be proved. Assume that (2.5) is not fulfilled in V. Then there are an idex i and a point $(\hat{x}, \hat{\xi}) \in V \cap \Sigma$ such that

$$\frac{\partial}{\partial x_p} n_i(\hat{x}^{(p)}, \hat{\xi}^{(p+1)}) \neq 0,$$

where $(\hat{x}, \hat{\xi}) = (\hat{x}_p, \ldots, \hat{x}_p, \hat{x}^{(p+1)}, 0, \ldots, 0, \hat{\xi}^{(p+1)})$. By the translation of the coordinates $x \to x - \hat{x}$ and a linear change of coordinates $x^{(p+1)}$ we may assume that $(\hat{x}, \hat{\xi}) = (0, e_n)$ again and p takes the same form as (2.4) with

$$\left\{\xi_p, \left\{\xi_p, \sum_{i=1}^{h} n_i^2\right\}\right\}(0, e_n) \neq 0.$$

Now we can repeat the proof of Lemma 2.1 in [11] and we conclude that, in a new homogeneous symplectic coordinates around $(\hat{x}, \hat{\xi})$, p takes the form

(2.4) with a larger p. Repeating the same arguments as above we conclude the desired assertion unless we reach $p = n - 1$;

$$p = -\xi_0^2 + \sum_{i=1}^{n-1} q_i(x_{i-1} - x_i)^2 + \sum_{i=1}^{n-1} r_i\xi_i^2 + r_n n(x_{n-1}, x_n, \xi_n)^2.$$

Since $n(x_{n-1}, x_n, \xi_n)$ is homogeneous of degree 1 in ξ_n and $dn \neq 0$ at $(0, e_n^{(1)})$ one can write

$$n(x_{n-1}, x_n, \xi_n) = \alpha(x)(x_n - \phi(x_{n-1}))\xi_n$$

with $\phi'(0) = 0$. We show that $\phi'(x_{n-1})$ vanishes identically. If not, say $\phi'(\epsilon) \neq 0$, then the Taylor expansion of p around $(\epsilon, \ldots, \epsilon, \phi(\epsilon), 0, \ldots, 0, 1)$ starts with

$$\hat{P} = -\xi_0^2 + \sum_{i=1}^{n-1} q_i(x_{i-1} - x_i)^2 + \sum_{i=1}^{n-1} r_i\xi_i^2 + r_n\alpha(\hat{x})(x_n - \phi'(\epsilon)x_{n-1})^2.$$

It is easy to check that

$$\det(-F_{\hat{P}}) = -\left(\prod_{j=1}^{n} 4q_j\right)\left(\prod_{j=1}^{n-1} r_j\right),$$

where $\hat{P}$ is considered in $(x^{(n-1)}, \xi^{(n-1)})$ space and $q_n = r_n\alpha(\hat{x})\phi'(\epsilon)^2$ and hence $F_{\hat{P}}$ is non singular. This together with (2.8) contradicts our assumption and hence the assertion. □

Working in U we may assume that p verifies (2.5) and (2.6) on Σ. Making a linear change of coordinates x;

$$y_0 = x_0, \quad y_i = x_{i-1} - x_i, \quad i = 1, \ldots, p, \quad y_i = x_i, \quad i = p+1, \ldots, n,$$

one can write p in the form

$$p(x,\xi) = -(\xi_0 + \xi_1)^2 + \sum_{j=1}^{p} q_j(x,\xi')x_j^2 + \sum_{j=1}^{p-1} r_j(x,\xi')(\xi_j - \xi_{j+1})^2 + r_p(x,\xi')\xi_p^2$$
$$+ r_{p+1}(x,\xi')\sum_{j=1}^{h} n_j^2\left(x_0 - \sum_{s=1}^{p} x_s, x^{(p+1)}, \xi^{(p+1)}\right),$$

where (2.5) and (2.6) still hold. We now explicitly write down $\operatorname{Im} F_p(\rho)^3 \cap \operatorname{Ker} F_p(\rho)$ and $\operatorname{Im} F_p(\rho)^2 \cap \operatorname{Ker} F_p(\rho)^2$ for $\rho \in \Sigma$.

Lemma 2.2. *We have*

$$\operatorname{Im} F_p(\rho)^3 \cap \operatorname{Ker} F_p(\rho) = \langle H_{\xi_0} \rangle.$$

Proof. From (2.3) it is clear that $F_p(\rho) = F_P(\rho) \oplus F_E$ where

$$P = -(\xi_0 + \xi_1)^2 + \sum_{j=1}^{p} q_j x_j^2 + \sum_{j=1}^{p-1} r_j(\xi_j - \xi_{j+1})^2 + r_p \xi_p^2$$

and E is non negative. From Theorem 1.4.6 in [3] it follows that $\operatorname{Im} F_E^3 \cap \operatorname{Ker} F_E = \{0\}$ and hence it is enough to study F_P. To simplify notations we denote F_P by F. By Theorem 1.4.6 in [3] the space $\mathbf{R}^{2p+2}$ is a direct sum

$$\mathbf{R}^{2p+2} = \sum_i \oplus V_i \oplus W, \tag{2.9}$$

where V_i and W are subspaces of dimension 2 and 4 respectively which are invariant under F. Moreover one has

$$\operatorname{Im} F^3 \cap \operatorname{Ker} F = \{0\} \qquad \text{in } V_i.$$

By Theorem 1.4.6 in [3] again W is spanned by v, Fv, F^2v, F^3v, where $F^j v \neq 0$ for $j \leq 3$ and $F^4 v = 0$. Then it is clear that

$$\begin{aligned} &\operatorname{Ker} F^2 \cap \operatorname{Im} F^2 = \operatorname{span}\{F^2 v, F^3 v\} = \operatorname{Ker} F^2, \\ &\operatorname{Im} F^3 \cap \operatorname{Ker} F = \operatorname{span}\{F^3 v\}. \end{aligned} \tag{2.10}$$

Let us denote

$$\begin{aligned} &\pi\xi = (\xi_0 + \xi_1, \xi_1 - \xi_2, \ldots, \xi_{p-1} - \xi_p, \xi_p), \\ &R = \operatorname{diag}(-1, r_1, \ldots, r_p), \quad D = \operatorname{diag}(0, q_1, \ldots, q_p). \end{aligned}$$

Then one can write $P = \langle R\pi\xi, \pi\xi\rangle + \langle Dx, x\rangle$ and hence

$$F^2 = \begin{pmatrix} -{}^t\pi R\pi D & 0 \\ 0 & -D\,{}^t\pi R\pi \end{pmatrix}.$$

It is clear that ${}^t\pi R\pi DX = 0$ implies that $X_1 = \cdots = X_p = 0$. It is also clear from (2.10) that

$$F(\operatorname{Im} F^2 \cap \operatorname{Ker} F^2) = \operatorname{Im} F^3 \cap \operatorname{Ker} F.$$

Let $(X, \Xi) \in \operatorname{Ker} F^2 \cap \operatorname{Im} F^2$ and consider

$$F(X, \Xi) = ({}^t\pi R\pi\Xi, -DX).$$

From $(X, \Xi) \in \operatorname{Ker} F^2 \cap \operatorname{Im} F^2$ it follows that $DX = 0$ and hence

$${}^t\pi R\pi\Xi = (-(\Xi_0 + \Xi_1), 0, \ldots, 0)$$

for $D\,{}^t\pi R\pi\Xi = 0$. This proves the assertion. $\square$

We turn to $\operatorname{Im} F^2 \cap \operatorname{Ker} F^2$. We now write

$$\sum_{j=1}^{p-1} r_j(x,\xi')(\xi_j - \xi_{j+1})^2 + r_p(x,\xi')\xi_p^2 - \xi_1^2 = \langle A(x,\xi')\xi_{(p)}, \xi_{(p)}\rangle \tag{2.11}$$

with $\xi_{(p)} = (\xi_1, \dots, \xi_p)$ so that one has

$$P = -\xi_0^2 - 2\xi_0\xi_1 + \sum_{j=1}^{p} q_j(x,\xi')x_j^2 + \langle A(x,\xi')\xi_{(p)}, \xi_{(p)}\rangle. \tag{2.12}$$

Lemma 2.3. *Let $0 \neq v \in \langle H_{x_1}, \dots, H_{x_p}\rangle$ be such that $v \in \operatorname{Ker} A(\rho)$. Then*

$$\operatorname{Im} F_p(\rho)^2 \cap \operatorname{Ker} F_p(\rho)^2 = \langle H_{\xi_0}, v\rangle$$

and $F_p(\rho)v$ is proportional to H_{ξ_0}. Moreover $z_2 = v$ satisfies (1.10).

Proof. Recall that

$$\begin{aligned} F_P(\rho)w = &-\sigma(w, H_{\xi_0})H_{\xi_0} - \sigma(w, H_{\xi_1})H_{\xi_0} - \sigma(w, H_{\xi_0})H_{\xi_1} \\ &+ \sum_{j=1}^{p} q_j(\rho)\sigma(w, H_{x_j})H_{x_j} + F_A(\rho)w. \end{aligned} \tag{2.13}$$

Since $v \in \langle H_{x_1}, \dots, H_{x_p}\rangle \cap \operatorname{Ker} A(\rho)$ it follows that $F_P(\rho)v = -\sigma(v, H_{\xi_1})H_{\xi_0}$. Inserting this into w in (2.13) we obtain $F_P(\rho)^2 v = 0$. Thanks to (2.10), this proves the first assertion. If $w = (X, \Xi) \in \langle H_{\xi_0}\rangle^\sigma$ and $\sigma(w, F_p(\rho)w) = p_\rho(w) = 0$ then we have

$$\Xi_0 = 0, \quad \sum_{j=1}^{p} q_j(\rho)X_j^2 + \langle A(\rho)\Xi_{(p)}, \Xi_{(p)}\rangle = 0, \quad E(X^{(p+1)}, \Xi^{(p+1)}) = 0$$

and hence $X_1 = \cdots = X_p = 0$, $\Xi_{(p)} \in \operatorname{Ker} A(\rho)$ and $(X^{(p+1)}, \Xi^{(p+1)}) \in \operatorname{Ker} F_E$. This shows that

$$(X, \Xi) = (0, 0, \Xi_{(p)}, 0) + (X_0, 0, X^{(p+1)}, 0, \Xi^{(p+1)}) \in \operatorname{Ker} F_P \oplus \langle v\rangle.$$

This proves the second assertion. $\square$

We make more precise looks on $\operatorname{Ker} A(\rho)$ for later use. Consider $\langle A(x, \xi')\xi_{(p)}, \xi_{(p)}\rangle$. It is easy to see that

$$\langle A(x,\xi')\xi_{(p)}, \xi_{(p)}\rangle = \sum_{j=2}^{p} a_j \left(\xi_j - \frac{r_{j-1}}{a_j}\xi_{j-1}\right)^2 + \left(r_1 - \frac{r_1^2}{a_2} - 1\right)\xi_1^2,$$

where

$$a_i = r_i + r_{i-1} - \frac{r_i^2}{a_{i+1}}, \qquad 1 \le i \le p-1, \qquad a_p = r_{p-1} + r_p. \tag{2.14}$$

We examine that

$$a_i = \frac{r_{i-1}\cdots r_p}{a_{i+1}\cdots a_p}\left(\frac{1}{r_{i-1}}+\cdots+\frac{1}{r_p}\right). \tag{2.15}$$

Indeed assume (2.15) for $i+1$. Plugging (2.15) with $i+1$ into (2.14) to get

$$\begin{aligned} a_i &= r_i + r_{i-1} - \frac{r_i^2}{a_{i+1}} \\ &= \frac{1}{a_{i+1}}\left((r_i+r_{i-1})\frac{r_i\cdots r_p}{a_{i+2}\cdots a_p}\left(\frac{1}{r_i}+\cdots+\frac{1}{r_p}\right)-r_i^2\right). \end{aligned}$$

The induction hypothesis (2.15) with $i+2$;

$$a_{i+2}\cdots a_p = r_{i+1}\cdots r_p\left(\frac{1}{r_{i+1}}+\cdots+\frac{1}{r_p}\right)$$

shows that

$$\begin{aligned} a_i &= \frac{1}{a_{i+1}}\left((r_i+r_{i-1})\frac{r_i\cdots r_p}{a_{i+2}\cdots a_p}\frac{1}{r_i}+(r_i+r_{i-1})r_i-r_i^2\right) \\ &= \frac{1}{a_{i+1}}\left((r_i+r_{i-1})\frac{r_{i+1}\cdots r_p}{a_{i+2}\cdots a_p}+r_{i-1}r_i\right). \end{aligned}$$

Thus we have

$$\begin{aligned} a_i\cdots a_p &= (r_i+r_{i-1})r_{i+1}\cdots r_p + r_{i-1}r_i a_{i+2}\cdots a_p \\ &= r_{i+1}\cdots r_p\left((r_i+r_{i-1})+r_{i-1}r_i\left(\frac{1}{r_{i+1}}+\cdots+\frac{1}{r_p}\right)\right) \\ &= r_{i-1}\cdots r_p\left(\frac{1}{r_{i-1}}+\frac{1}{r_i}+\cdots+\frac{1}{r_p}\right), \end{aligned}$$

which proves the assertion.

From (2.5) it is easy to see that

$$a_j(x,\xi') = \frac{r_{j-1}(\frac{1}{r_{j-1}}+\cdots+\frac{1}{r_p})}{\frac{1}{r_j}\cdots+\frac{1}{r_p}}.$$

We define $c_j(x,\xi')$ by

$$c_j(x,\xi') = \frac{\sum_{s=j}^p r_s(x,\xi')^{-1}}{\sum_{s=j-1}^p r_s(x,\xi')^{-1}}, \qquad 2\le j\le p$$

so that $a_j(x,\xi') = r_{j-1}(x,\xi')/c_j(x,\xi')$. We now summarize:

Lemma 2.4. *We have*

$$\langle A(x,\xi')\xi_{(p)},\xi_{(p)}\rangle = \sum_{j=2}^p a_j m_j(x,\xi')^2 + R(x,\xi')\xi_1^2,$$

where $m_j(x,\xi')=\xi_j-c_j(x,\xi')\xi_{j-1}$ *and*

$$R(x,\xi')=r_1(x,\xi')-1-\frac{r_1(x,\xi')^2}{a_2(x,\xi')}=0 \qquad \textit{on} \quad \Sigma.$$

In particular

$$\operatorname{Ker} A(\rho)=\langle(1,c_2(\rho),(c_2c_3)(\rho),\ldots,(c_2\cdots c_p)(\rho))\rangle$$

is given by $m_j(\rho)=0$, $j=1,\ldots,p$, *for* $\rho\in\Sigma$.

Proof. We just check the assertion for R. Note that

$$\begin{aligned}r_1-1-\frac{r_1^2}{a_2}&=r_1-1-\frac{r_1(\frac{1}{r_2}+\cdots+\frac{1}{r_p})}{\frac{1}{r_1}+\cdots+\frac{1}{r_p}}\\&=\frac{1-(\frac{1}{r_1}+\cdots+\frac{1}{r_p})}{\frac{1}{r_1}+\cdots+\frac{1}{r_p}}=R(x,\xi'),\end{aligned}\tag{2.16}$$

which vanishes on Σ by (2.6). This proves the assertion. □

As observed above we can write

(2.17)

$$\begin{aligned}p(x,\xi)=-\xi_0^2-2\xi_0\xi_1&+\sum_{j=1}^{p}q_j(x,\xi')x_j^2+\sum_{j=2}^{p}a_j(x,\xi')m_j(x,\xi')^2\\&+R(x,\xi')\xi_1^2+r_{p+1}(x,\xi')\sum_{j=1}^{h}n_j^2\left(x_0-\sum_{s=1}^{p}x_s,x^{(p+1)},\xi^{(p+1)}\right),\end{aligned}$$

where $m_1(x,\xi')=\xi_1$ and $R=0$ on Σ hence

$$R=2\sum_{j=1}^{p}\beta_jm_j(x,\xi')+2\sum_{j=1}^{p}\gamma_jx_j+2\sum_{j=1}^{h}\delta_jn_j(x_0,x^{(p+1)},\xi^{(p+1)})$$

because Σ is given by

$$\begin{aligned}\Sigma=\{x_1=\cdots=x_p=0,\xi_0=\cdots=\xi_p=0,&\\ n_j(x_0,x^{(p+1)},\xi^{(p+1)})=0,1\le j\le h\}.&\end{aligned}$$

Since

$$\frac{\partial}{\partial x_p}n_j\left(x_0-\sum_{j=1}^{p}x_j,x^{(p+1)},\xi^{(p+1)}\right)=0 \qquad \text{on} \quad \Sigma,$$

then one has

$$\begin{aligned}&\frac{\partial}{\partial x_p}n_j\left(x_0-\sum_{s=1}^{p}x_s,x^{(p+1)},\xi^{(p+1)}\right)\\&\qquad=\sum_{i=1}^{p}a_{ji}x_i+\sum_{i=1}^{p}b_{ji}\xi_i+\sum_{i=1}^{h}c_{ji}n_i(x_0,x^{(p+1)},\xi^{(p+1)}).\end{aligned}$$

It is clear that $b_{ji} = 0$. Putting $x_i = 0$, $1 \le i \le p$ one has

$$\frac{\partial}{\partial x_p} n_j(x_0, x^{(p+1)}, \xi^{(p+1)}) = \sum_{i=1}^{p} c_{ji} n_i(x_0, x^{(p+1)}, \xi^{(p+1)})$$

and this proves that $n_j(x_0, x^{(p+1)}, \xi^{(p+1)})$, $1 \le j \le h$ are independent of x_0. Let us denote them by $n_j(x^{(p+1)}, \xi^{(p+1)})$ so that we have

$$\begin{aligned} p(x,\xi) = -\xi_0^2 - 2\xi_0\xi_1 &+ \sum_{j=1}^{p} q_j(x,\xi')x_j^2 + \sum_{j=2}^{p} a_j(x,\xi')m_j^2 \\ &+ \sum_{j=1}^{h} b_j(x,\xi')n_j(x^{(p+1)}, \xi^{(p+1)})^2 + R(x,\xi')\xi_1^2, \end{aligned} \tag{2.18}$$

where

$$R(x,\xi') = 2\sum_{j=1}^{p} \beta_j m_j + 2\sum_{j=1}^{p} \gamma_j x_j + 2\sum_{j=1}^{h} \delta_j n_j.$$

Here we recall

Proposition 2.2 ([2]). *Let S_1, S_2 be two smooth functions verifying* (1.11) *and* (1.12). *Then there exists $C \neq 0$ such that*

$$H_{S_1}^3 p|_\Sigma = C H_{S_2}^3 p|_\Sigma.$$

Let us define

$$S = -\frac{1}{c_1} \sum_{j=1}^{p} (c_1 \cdots c_j) x_j \tag{2.19}$$

so that $\langle H_S(\rho)\rangle = \operatorname{Ker} A(\rho)$. Due to Lemma 2.3, S verifies (1.11) and (1.12).

Lemma 2.5. *The condition $H_S^3 p(\rho) \neq 0$ implies that $\beta_1(\rho) \neq 0$.*

Proof. Recall that

$$H_S = \frac{1}{c_1} \sum_{j=1}^{p} (c_1 \cdots c_j) \frac{\partial}{\partial \xi_j} + \sum_{j=1}^{p} x_j L_j,$$

where L_j are some vector fields. Note that $H_S m_j$, $2 \le j \le p$, $H_S x_j$, $1 \le j \le p$ and $H_S n_j$, $1 \le j \le h$ vanish on Σ and hence they can be written as

$$\sum_{j=2}^{p} a_j m_j + \sum_{j=1}^{p} b_j x_j + \sum_{j=1}^{h} c_j n_j + d\xi_1.$$

Then the assertion is clear. ∎

3. Hamilton system

We study the Hamilton system with the Hamiltonian p of (2.18). Let $n_j(\bar{x}^{(p+1)}, \bar{\xi}^{(p+1)}) = 0$, $1 \le j \le h$ so that $(x_0, 0, \ldots, 0, \bar{x}^{(p+1)}, 0, \ldots, 0, \bar{\xi}^{(p+1)}) \in \Sigma$. In what follows, since the homogeneity in ξ is irrelevant in the study of bicharacteristics, replacing $(x^{(p+1)}, \xi^{(p+1)})$ by $(\bar{x}^{(p+1)} + x^{(p+1)}, \bar{\xi}^{(p+1)} + \xi^{(p+1)})$ we are led to study the Hamilton system with Hamiltonian p where $n_j(0,0) = 0$, $1 \le j \le h$. Making a linear symplectic change of coordinates we may assume that

$$\begin{aligned} n_j(x^{(p+1)}, \xi^{(p+1)}) &= x_{p+j} + O(n^2), \qquad 1 \le j \le k, \\ n_{k+j}(x^{(p+1)}, \xi^{(p+1)}) &= \xi_{p+j} + O(n^2), \qquad 1 \le j \le k+\ell, \end{aligned}$$

where $2k + \ell = h$ and $n^2 = |x^{(p+1)}|^2 + |\xi^{(p+1)}|^2$.

We start with

Lemma 3.1. *One can write*

$$p = -\xi_0^2 - 2\xi_0\xi_1 + \sum_{j=1}^{p} q_j \ell_j^2 + \sum_{j=2}^{p} r_j m_j^2 + \sum_{j=1}^{h} b_j n_j^2 - \beta^* \xi_1^3 + \Phi(x, \xi'),$$

where q_j, r_j, b_j, $\beta^* \in \mathbf{R}$ *and*

$$\begin{aligned} m_j &= \xi_j - c_j \xi_{j-1} - g_j(x, \xi'), \qquad \ell_j = x_j - d_j \xi_1^2, \\ n_j &= x_{p+j} - e_j \xi_1^2, \qquad 1 \le j \le k, \\ n_{k+j} &= \xi_{p+j} - e_{k+j}\xi_1^2, \qquad 1 \le j \le k + \ell \end{aligned}$$

with $c_j, d_j, e_j \in \mathbf{R}$. *Here* $g_j(x, \xi') = O(\rho^2)$, $g_j(x, 0) = 0$ *with* $\rho = |(x, \xi')|$. *Moreover*

$$\begin{aligned} \Phi(x, \xi') = \sum_{j=2}^{p} \alpha_{j0}(x, \xi') m_j^2 + \alpha_{j1}(x, \xi') m_j + \sum_{j=1}^{p} \beta_{j0}(x, \xi') \ell_j^2 + \beta_{j1}(x, \xi') \ell_j \\ + \sum_{j=1}^{h} \gamma_{j0}(x, \xi') n_j^2 + \gamma_{j1}(x, \xi') n_j + \delta(x, \xi'), \end{aligned}$$

where

$$\begin{aligned} &\alpha_{j0} = O(\rho), \qquad \beta_{j0} = O(\rho), \qquad \gamma_{j0} = O(\rho), \\ &\alpha_{j1} = O(\rho^3) O(|\xi|), \qquad \beta_{j1} = O(\rho) O(|\xi|^2), \qquad \gamma_{j1} = O(n^2) + O(\rho) O(|\xi|^2), \\ &\quad \delta = O(\rho^4) O(|\xi|^2) + O(\rho) O(|\xi|^3) \end{aligned}$$

with $m^2 = \sum_{j=2}^{p} m_j(x, \xi')^2$, $\ell^2 = \sum_{j=1}^{p} \ell_j(x, \xi')^2$.

Proof. Recall that we can write, changing the previous notations,

$$\begin{aligned} p = {} & -\xi_0^2 - 2\xi_0\xi_1 + \sum_{j=1}^{p} q_j(x,\xi)x_j^2 + \sum_{j=1}^{p} r_j(x,\xi')\tilde{m}_j(x,\xi')^2 \\ & + \sum_{j=1}^{h} b_j(x,\xi')\tilde{n}_j(x^{(p+1)},\xi^{(p+1)})^2 + \left[2\sum_{j=1}^{p}\gamma_j(x,\xi')\tilde{m}_j(x,\xi') \right. \\ & \left. + 2\sum_{j=1}^{p}\delta_j(x,\xi')x_j + 2\sum_{j=1}^{h}\mu_j(x,\xi')\tilde{n}_j\right]\xi_1^2, \end{aligned}$$

where $\tilde{m}_1 = \xi_1$, $\tilde{m}_j = \xi_j - c_j(x,\xi')\xi_{j-1}$. Let us write

$$\begin{aligned} r_j\tilde{m}_j^2 + 2\gamma_j\tilde{m}_j\xi_1^2 &= r_j\left(\tilde{m}_j + \frac{\gamma_j}{r_j}\xi_1^2\right)^2 - \frac{\gamma_j^2}{r_j}\xi_1^4, \\ q_jx_j^2 + 2\delta_jx_j\xi_1^2 &= q_j\left(x_j + \frac{\delta_j}{q_j}\xi_1^2\right)^2 - \frac{\delta_j^2}{q_j}\xi_1^4, \\ b_j\tilde{n}_j^2 + 2\mu_j\tilde{n}_j\xi_1^2 &= b_j\left(\tilde{n}_j + \frac{\mu_j}{b_j}\xi_1^2\right)^2 - \frac{\mu_j^2}{b_j}\xi_1^4. \end{aligned}$$

Let g_j be the sum of the quadratic and the cubic part of the Taylor expansion of $c_j(x,\xi')\xi_{j-1} - (\gamma_j(x,\xi')/r_j(x,\xi'))\xi_1^2$ around $0 = (0,0)$ so that

$$\tilde{m}_j + \frac{\gamma_j}{r_j}\xi_1^2 = m_j(x,\xi') + O(\rho^3)O(|\xi|), \qquad m_j = \xi_j - c_j\xi_{j-1} - g_j(x,\xi'),$$

where $c_j = c_j(0)$. Taking $d_j = -\delta_j(0)/q_j(0)$, $e_j = -\mu_j(0)/b_j(0)$ one has

$$\begin{aligned} x_j + \frac{\delta_j}{q_j}\xi_1^2 &= \ell_j(x,\xi) + O(\rho)O(|\xi|^2), \qquad \ell_j = x_j - d_j\xi_1^2, \\ \tilde{n}_j + \frac{\mu_j}{b_j}\xi_1^2 &= n_j(x^{(p+1)},\xi^{(p+1)}) + O(n^2) + O(\rho)O(|\xi|^2), \\ n_j &= x_{p+j} - e_j\xi_1^2, \qquad 1 \le j \le k, \\ n_{k+j} &= \xi_{p+j} - e_{k+j}\xi_1^2, \qquad 1 \le j \le k+\ell. \end{aligned}$$

Then with $\beta^* = -2\gamma_1(0) \neq 0$ we can write

$$\begin{aligned} p = {} & -\xi_0^2 - 2\xi_0\xi_1 + \sum_{j=1}^{p} q_j(x,\xi')[\ell_j + O(\rho)O(|\xi|^2)]^2 \\ & + \sum_{j=2}^{p} r_j(x,\xi')[m_j + O(\rho^3)O(|\xi|)]^2 \\ & + \sum_{j=1}^{h} b_j(x,\xi')[n_j + O(n^2) + O(\rho)O(|\xi|^2)]^2 - \beta^*\xi_1^3 + O(\rho)O(|\xi|^3). \end{aligned}$$

Note that

$$\begin{aligned} r_j(x,\xi')[m_j+O(\rho^3)O(|\xi|)]^2 &= [r_j(0)+O(\rho)][m_j+O(\rho^3)O(|\xi|)]^2\\ &= r_j(0)m_j^2+O(\rho)O(m^2)+O(\rho^3)O(m)O(|\xi|)+O(\rho^6)O(|\xi|^2)\end{aligned}$$

and

$$\begin{aligned} q_j(x,\xi')[\ell_j+O(\rho)O(|\xi|^2)]^2 &= [q_j(0)+O(\rho)][\ell_j+O(\rho)O(|\xi|^2)]^2\\ &= q_j(0)\ell_j^2+O(\rho)O(\ell^2)+O(\rho)O(\ell)O(|\xi|^2)+O(\rho^2)O(|\xi|^4),\end{aligned}$$

and

$$\begin{aligned} b_j(x,\xi')[n_j+O(n^2)+O(\rho)O(|\xi|^2)]^2 &= b_j(0)n_j^2+O(\rho)O(n^2)\\ &\quad +O(\rho)O(n)O(|\xi|^2)+O(\rho^2)O(|\xi|^4)+O(n^3).\end{aligned}$$

This proves the assertion for $\Phi(x,\xi')$. □

Our Hamilton system is:

(3.1)

$$\begin{cases} \dot{x}_0 = -2\xi_0-2\xi_1,\\ \dot{x}_1 = -2\xi_0-4\sum_{k=1}^{p} q_kd_k\xi_1\ell_k-2c_2r_2m_2-3\beta^*\xi_1^2\\ \qquad -2\sum_{k=2}^{p} r_km_k\dfrac{\partial g_k}{\partial \xi_1}-4\sum_{k=1}^{h} b_ke_k\xi_1n_k+\dfrac{\partial\Phi}{\partial\xi_1},\\ \dot{x}_j = 2r_jm_j-2r_{j+1}c_{j+1}m_{j+1}-2\sum_{k=2}^{p} r_km_k\dfrac{\partial g_k}{\partial \xi_j}+\dfrac{\partial\Phi}{\partial\xi_j}, \quad 2\le j\le p,\\ \dot{\xi}_0 = 2\sum_{k=2}^{p} r_km_k\dfrac{\partial g_k}{\partial x_0}-\dfrac{\partial\Phi}{\partial x_0},\\ \dot{\xi}_j = -2q_jx_j+2q_jd_j\xi_1^2+2\sum_{k=2}^{p} r_km_k\dfrac{\partial g_k}{\partial x_j}-\dfrac{\partial\Phi}{\partial x_j}, \qquad 1\le j\le p,\\ \dot{x}_{p+j} = 2b_{k+j}(\xi_{p+j}-e_{k+j}\xi_1^2)-2\sum_{i=2}^{p} r_im_i\dfrac{\partial g_i}{\partial\xi_{p+j}}+\dfrac{\partial\Phi}{\partial\xi_{p+j}},\\ \qquad\qquad 1\le j\le k+\ell,\\ \dot{\xi}_{p+j} = -2b_j(x_{p+j}-e_j\xi_1^2)+2\sum_{i=2}^{p} r_im_i\dfrac{\partial g_i}{\partial x_{p+j}}-\dfrac{\partial\Phi}{\partial x_{p+j}},\\ \qquad\qquad 1\le j\le k,\\ \dot{x}_{p+j} = -2\sum_{i=2}^{p} r_im_i\dfrac{\partial g_i}{\partial\xi_{p+j}}+\dfrac{\partial\Phi}{\partial\xi_{p+j}}, \qquad k+\ell+1\le j,\\ \dot{\xi}_{p+j} = 2\sum_{i=2}^{p} r_im_i\dfrac{\partial g_i}{\partial x_{p+j}}-\dfrac{\partial\Phi}{\partial x_{p+j}}, \qquad k+1\le j.\end{cases}$$

It is easy to see that

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_j} = \sum_{k=2}^{p} \beta_k^{(j)} m_k + O(\rho)O(\ell) + O(\rho)O(|\xi|^2) \\
+ O(\rho)O(n)O(|\xi|) + O(n^2)
\end{aligned}
\tag{3.2}
$$

for $1 \le j \le p$, where $\beta_k^{(j)} = O(\rho^2) + O(|\xi|)$ and

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \xi_j} = \sum_{k=2}^{p} \alpha_k^{(j)} m_k + O(\ell^2) + O(\ell)O(\rho)O(|\xi|) \\
+ O(n^2) + O(n)O(\rho)O(|\xi|) + O(\rho^3)O(|\xi|) + O(\rho)O(|\xi|^2)
\end{aligned}
\tag{3.3}
$$

for $1 \le j \le p$, where $\alpha_k^{(j)} = O(\rho)$. It is also easy to see that

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \xi_j} &= 2\gamma_{k-p+j,0}(\xi_j - e_{k-p+j}\xi_1^2) \\
&\quad + O(m^2) + O(m)O(\rho)O(|\xi|) + O(\rho^3)O(m) \\
&\quad + O(\ell^2) + O(\ell)O(\rho)O(|\xi|) + O(n^2) + O(n)O(\rho)O(|\xi|) \\
&\quad + O(\rho^4)O(|\xi|) + O(\rho)O(|\xi|^2)
\end{aligned}
\tag{3.4}
$$

for $p+1 \le j$, where $\gamma_{k-p+j} = 0$ for $p+k+\ell+1 \le j$ and

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_j} &= 2\gamma_{j-p,0}(x_j - e_{j-p}\xi_1^2) + \gamma_{j-p,1} \\
&\quad + O(m^2) + O(m)O(\rho)O(|\xi|) + O(\ell^2) \\
&\quad + O(\ell)O(|\xi|^2) + O(n^2) + O(n)O(|\xi|^2) + O(\rho^3)O(|\xi|^2) + O(|\xi|^3)
\end{aligned}
\tag{3.5}
$$

for $p+1 \le j$, where $\gamma_{j-p,0} = 0$, $\gamma_{j-p,1} = 0$ for $p+k+1 \le j$.

Suppose that m_j are also unknowns and $(x(s), \xi(s), m(s))$ verifies (3.1). From (3.1) one can write

$$
2r_j m_j - 2r_{j+1}c_{j+1}m_{j+1} = \dot{x}_j + r_j(x, \xi, m), \qquad 2 \le j \le p,
\tag{3.6}
$$

where we have set $m_{p+1} = 0$ and

$$
r_j = 2\sum_{k=2}^{p} r_k m_k \frac{\partial g_k}{\partial \xi_j} - \frac{\partial \Phi}{\partial \xi_j}.
$$

Similarly from (3.1) we can write

$$
x_j = -\frac{1}{2q_j}\dot{\xi}_j + s_j(x, \xi), \qquad 2 \le j \le p
\tag{3.7}
$$

with

$$
s_j = \frac{1}{q_j}\left[q_j d_j \xi_1^2 + \sum_{k=2}^{p} r_k m_k \frac{\partial g_k}{\partial x_j} - \frac{1}{2}\frac{\partial \Phi}{\partial x_j} \right].
\tag{3.8}
$$

From (3.6) and (3.7) it follows that

$$2r_j m_j - 2r_{j+1}c_{j+1}m_{j+1} = -\frac{1}{2q_j}\ddot{\xi}_j + r_j + \frac{d}{ds}s_j. \tag{3.9}$$

Let us set

$$\phi_j = \xi_j - c_j\xi_{j-1} - g_j(x,\xi), \qquad 2 \le j \le p,$$

then it is easy to see by induction that

$$\xi_j = \sum_{\ell=2}^{j}\frac{1}{c_\ell}[c_\ell\cdots c_j]\phi_\ell + \sum_{\ell=2}^{j}\frac{1}{c_\ell}[c_\ell\cdots c_j]g_\ell + (c_2\cdots c_j)\xi_1, \qquad 2 \le j \le p.$$

Then if (x,ξ,m) satisfies (3.1) one has

(3.10)

$$\begin{aligned}2r_j m_j - 2r_{j+1}c_{j+1}m_{j+1} = &-\frac{1}{2q_j}\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)\ddot{\phi}_\ell\\ &-\frac{1}{2q_j}(c_2\cdots c_j)\ddot{\xi}_1 + r_j + \frac{d}{ds}s_j\\ &-\frac{1}{2q_j}\left(\frac{d}{ds}\right)^2\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)g_\ell, \qquad 2 \le j \le p.\end{aligned}$$

Here we rewrite d^2g_ℓ/ds^2. In the expression

$$\frac{d}{ds}g_j(x,\xi') = \sum_{i=0}^{n}\frac{\partial g_j}{\partial x_i}\dot{x}_i + \sum_{i=1}^{n}\frac{\partial g_j}{\partial \xi_i}\dot{\xi}_i$$

we substitute the right-hand side of the equation (3.1) into $\dot{x}_i$ and $\dot{\xi}_i$ to get

$$\frac{d}{ds}g_j(x,\xi') = h_j(x,\xi'), \qquad 2 \le j \le p. \tag{3.11}$$

Let us put

$$k_j(x,\xi') = \sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)h_\ell. \tag{3.12}$$

Taking (3.10) into account, we introduce the following equations for m_j:

$$\begin{aligned}&2r_j m_j(s) - 2r_{j+1}c_{j+1}m_{j+1}(s)\\ &\quad = -\frac{1}{2q_j}\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)\ddot{m}_\ell(s)\\ &\qquad -\frac{1}{2q_j}(c_2\cdots c_j)\ddot{\xi}_1(s) + r_j + \frac{d}{ds}s_j(x(s),\xi(s),m(s))\\ &\qquad -\frac{1}{2q_j}\left(\frac{d}{ds}\right)k_j(x(s),\xi(s)), \qquad 2 \le j \le p.\end{aligned} \tag{3.13}$$

If (x, ξ, m) verifies (3.1) and (3.13) then we have

$$-\frac{1}{2q_j}\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)\left(\frac{d}{ds}\right)^2(m_\ell-\phi_\ell)=0, \qquad 2\le j\le p$$

and hence we conclude that

$$m_j(s)=\phi_j(s), \qquad 2\le j\le p$$

provided

$$(x,\xi)=O(s^{-1}), \quad m_j=O(s^{-1}). \tag{3.14}$$

Thus $(x(s), \xi(s))$ is a solution to the Hamilton system (3.1). Now our question is reduced to look for (x, ξ, m) verifying (3.1) and (3.13) with (3.14).

4. Reduction of Hamilton system

We further simplify the equations (3.1) and (3.13). We make the change of the independent variable s:

$$s=\frac{1}{t} \tag{4.1}$$

and put

$$\begin{array}{lll}
x_0(s)=tX_0(t), & \xi_0(s)=t^4\Xi_0(t), & m(s)=t^4M(t),\\
x_j(s)=t^3X_j(t), & \xi_j(s)=t^2\Xi_j(t), & 1\le j\le p,\\
x_j(s)=t^3X_j(t), & \xi_j(s)=t^3\Xi_j(t), & p+1\le j\le p+k,\\
x_j(s)=t^3X_j(t), & \xi_j(s)=t^4\Xi_j(t), & p+k+1\le j\le p+k+\ell,\\
x_j(s)=t^4X_j(t), & \xi_j(s)=t^4\Xi_j(t), & p+k+\ell+1\le j
\end{array} \tag{4.2}$$

and denote $V=(X,\Xi)$, $V_{(p)}=(X_0,\ldots,X_p,\Xi_0,\ldots,\Xi_p)$ and for $f(x,\xi,m)$ we put

$$\begin{aligned}f^\sharp(t,V,M)=f(&tX_0,t^3X',t^3X_{p+1},\ldots,t^3X_{p+k+\ell},t^4X_{p+k+\ell+1},\ldots,t^4X_n,\\
&t^4\Xi_0,t^2\Xi',t^3\Xi_{p+1},\ldots,t^3\Xi_{p+k},t^4\Xi_{p+k+1},\ldots,t^4\Xi_n,t^4M),\end{aligned}$$

where $X'=(X_1,\ldots,X_p)$, $\Xi'=(\Xi_1,\ldots,\Xi_p)$.

Lemma 4.1. *We have*

$$
\begin{aligned}
&\left(\frac{\partial\Phi}{\partial x_j}\right)^\sharp = O(t^4), \qquad 1 \le j \le p+k, \qquad \left(\frac{\partial\Phi}{\partial x_j}\right)^\sharp = O(t^6), \qquad p+k+1 \le j,\\
&\left(\frac{\partial\Phi}{\partial \xi_j}\right)^\sharp = O(t^5), \qquad 1 \le j \le p, \qquad \left(\frac{\partial\Phi}{\partial \xi_j}\right)^\sharp = O(t^4), \qquad p+1 \le j \le p+k,\\
&\left(\frac{\partial\Phi}{\partial \xi_j}\right)^\sharp = O(t^5), \qquad p+k+1 \le j \le p+k+\ell,\\
&\left(\frac{\partial\Phi}{\partial \xi_j}\right)^\sharp = O(t^6), \qquad p+k+\ell+1 \le j,\\
&\left(\frac{\partial\Phi}{\partial x_0}\right)^\sharp = O(t^6),
\end{aligned}
$$

where by $O(t^s)$ we denote a term which is of the form

$$t^s R(t,V,M)$$

with a smooth function $R(t,V,M)$.

Proof. Noting $\ell^\sharp = O(t^3)$, $\xi^\sharp = O(t^2)$, $\rho^\sharp = O(t)$, $m^\sharp = O(t^4)$, $n^\sharp = O(t^3)$ and

$$
\begin{aligned}
&\frac{\partial \ell}{\partial \xi_j} = O(|\xi|),\ \forall j, \qquad \frac{\partial \ell}{\partial x_0} = 0, \quad \xi_i^\sharp = O(t^4), \qquad p+k+1 \le i,\\
&\frac{\partial g_j}{\partial \xi_i} = O(|\xi|), \qquad p+1 \le i, \qquad \frac{\partial g_j}{\partial x_i} = O(|\xi|),\ \forall i,
\end{aligned}
$$

the assertion follows from (3.2), (3.3), (3.4) and (3.5). □

We study the Hamilton system (3.1). Let us set

$$D = t\frac{d}{dt}.$$

Then since

$$tD(t^\ell G) = t^{\ell+1}(DG + \ell G), \qquad \frac{d}{ds} = -tD,$$

thanks to Lemma 4.1 the equation (3.1) is transformed to

$$
(4.3) \quad \begin{cases}
DX_0 = -X_0 + 2\Xi_1 + t^2\phi_0(t,V), \\
DX_1 = -3X_1 + 2\Xi_0 + 2r_2c_2M_2 + 3\beta^*\Xi_1^2 + t\phi_1(t,V,M), \\
DX_j = -3X_j - 2r_jM_j + 2r_{j+1}c_{j+1}M_{j+1} + t\phi_j(t,V,M), \\
\qquad\qquad\qquad 2 \le j \le p, \\
D\Xi_0 = -4\Xi_0 + t\psi_0(t,V,M), \\
D\Xi_j = -2\Xi_j + 2q_jX_j + t\psi_j(t,V,M), \qquad 1 \le j \le p, \\
tDX_{p+j} = -3tX_{p+j} - 2b_{k+j}\Xi_{p+j} + t\phi_{p+j}(t,V,M), \qquad 1 \le j \le k, \\
tD\Xi_{p+j} = -3t\Xi_{p+j} + 2b_jX_{p+j} + t\psi_{p+j}(t,V,M), \qquad 1 \le j \le k, \\
DX_{p+j} = -3X_{p+j} - 2b_{k+j}(\Xi_{p+j} - e_{k+j}\Xi_1^2) + t\phi_{p+j}(t,V,M), \\
\qquad\qquad\qquad k+1 \le j \le k+\ell, \\
DX_{p+j} = -4X_{p+j} + t\phi_{p+j}(t,V,M), \qquad k+\ell+1 \le j, \\
D\Xi_{p+j} = -4\Xi_{p+j} + t\psi_{p+j}(t,V,M), \qquad k+1 \le j.
\end{cases}
$$

We turn to (3.13). In view of (3.3) one can write

$$
\left(\frac{\partial\Phi}{\partial\xi_j}\right)^\sharp = t^5\sum_{k=2}^p R_{jk}(t,V)M_k + t^5R(t,V)
$$

for $1 \le j \le p$ and then

$$
(4.4) \qquad t^{-4}r_j^\sharp = tR(t,V,M).
$$

Note that

$$
(4.5) \qquad [O(\rho)O(\ell)]^\sharp = t^4R_1(t,V_{(p)}) + t^5R_2(t,V)
$$

because $O(\ell) = O(|(x_1,\ldots,x_p)|) + O(|\xi|^2)$. Thus one sees from (3.2)

$$
\left(\frac{\partial\Phi}{\partial x_j}\right) = t^4R_1(t,V_{(p)}) + t^5R_2(t,V,M), \qquad 1 \le j \le p,
$$

where and below R_j may change from line to line. This shows that

$$
s_j^\sharp = t^4R_1(t,V_{(p)}) + t^5R_2(t,V,M)
$$

and hence one obtains

$$
(4.6) \qquad t^{-4}(tD)s_j^\sharp = R_1(t,V_{(p)},tDV_{(p)}) + tR_2(t,V,M,tDV,tDM),
$$

where

$$
(4.7) \qquad R_1(t,V_{(p)},0) = 0
$$

since

(4.8)
$$tD[t^k f(t,V,M)] = kt^{k+1}\frac{\partial f}{\partial t}(t,V,M) + t^k \sum \frac{\partial f}{\partial V_j}(t,V,M)tDV_j + t^k \sum \frac{\partial f}{\partial M_j}(t,V,M)tDM_j.$$

We finally show that one can write

$$t^{-4}(tD)h_j^\sharp = R_1(t,V_{(p)},tDV_{(p)}) + tR_2(t,V,tDV), \tag{4.9}$$

where

$$R_1(t,V_{(p)},0) = 0.$$

To examine this we first recall that

$$h_j(x,\xi') = \sum_{i=0}^{n} \frac{\partial g_j}{\partial x_i}\dot{x}_i + \sum_{i=1}^{n} \frac{\partial g_j}{\partial \xi_i}\dot{\xi}_i.$$

It is clear from the definition of g_j that

$$\frac{\partial g_j}{\partial x_i} = O(|\xi_{(p)}|). \tag{4.10}$$

We see from (3.1), (3.3) and (3.4) that

$$\dot{x}_i = O(|\xi_{(p)}|) + O(\rho)O(\ell) + O(\rho)O(n) + O(|\xi|^2) + O(\rho^3)O(|\xi|)$$

for $0 \le j \le p$ and

$$\dot{x}_i = O(\xi_i) + O(\rho)O(|\xi|) + O(\rho)O(\ell) + O(\rho)O(n) + O(|\xi|^2)$$

for $p+1 \le i$. Thus we have

$$\left(\frac{\partial g_j}{\partial x_i}\dot{x}_i\right)^\sharp = t^4 R_1(t,V_{(p)}) + t^5 R_2(t,V).$$

We turn to $(\partial g_j/\partial \xi_i)\dot{\xi}_i$. It is easy to see from (3.1), (3.2) and (3.5) that

$$\dot{\xi}_i = O(x_i) + O(\rho)O(\ell) + O(\rho)O(n) + O(|\xi|^2)$$

for $1 \le i$. Since

$$\frac{\partial g_j}{\partial \xi_i} = O(|x|), \qquad 1 \le i \le p, \qquad \frac{\partial g_j}{\partial \xi_i} = O(|\xi|), \qquad p+1 \le i,$$

we see that

$$\left(\frac{\partial g_j}{\partial \xi_i}\dot{\xi}_i\right)^\sharp = t^4 R_1(t,V_{(p)}) + t^5 R_2(t,V).$$

Combining these expressions one gets

$$h_j^\sharp = t^4 R_1(t, V_{(p)}) + t^5 R_2(t, V),$$

which proves (4.9).

It is also easy to see that

$$t^{-4}\left(\frac{d}{ds}\right)^2 (t^2\Xi_1) = 6\Xi_1 + 5D\Xi_1 + D^2\Xi_1 = \mathcal{L}\Xi_1 \tag{4.11}$$

and

$$t^{-4}\left(\frac{d}{ds}\right)^2 (t^4 M) = t^{-4}(tD)^2(t^4 M) = (tD+4t)^2 M. \tag{4.12}$$

Thus the equation (3.13) turns to

$$\begin{aligned} 2r_j M_j - 2r_{j+1}c_{j+1}M_{j+1} = &-\frac{1}{2q_j}\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)(tD+4t)^2 M \\ &-\frac{1}{2q_j}(c_2\cdots c_j)\mathcal{L}\Xi_1 + R_1(t, V_{(p)}, tDV_{(p)}) \\ &+ tR_2(t, V, M, tDV, tDM), \qquad 2\le j\le p, \end{aligned} \tag{4.13}$$

where

$$R_1(t, V_{(p)}, 0) = 0. \tag{4.14}$$

We further rewrite (4.13) removing the $D^2\Xi_1$ term in $\mathcal{L}\Xi_1$. Let us denote $W = (V, M)$ and recall that

$$DX_1 = -3X_1 + 2\Xi_0 + 2r_2c_2M_2 + 3\beta^*\Xi_1^2 + t\phi_1(t, W). \tag{4.15}$$

Noting (3.2) and (4.5) we get

$$D\Xi_1 = -2\Xi_1 + 2q_1X_1 + t\psi_1(t, V_{(p)}) + t^2\tilde{\psi}(t, V, M). \tag{4.16}$$

From (4.16) we have

$$\begin{aligned} DX_1 = &\frac{1}{2q_1}D^2\Xi_1 + \frac{1}{q_1}D\Xi_1 + \theta(t, V_{(p)}, tDV_{(p)}) \\ &+ t\tilde{\theta}(t, V, M, tDV, tDM), \end{aligned} \tag{4.17}$$

where $\theta(t, V_{(p)}, 0) = 0$. Equating (4.15) and (4.17) one obtains

$$\begin{aligned} 2r_2c_2M_2 = &\frac{1}{2q_1}D^2\Xi_1 + \frac{1}{q_1}D\Xi_1 + 3X_1 - 2\Xi_0 \\ &- 3\beta^*\Xi_1^2 + \theta_1(t, V_{(p)}, tDV_{(p)}) + t\tilde{\theta}_1(t, V, M, tDV, tDM). \end{aligned}$$

Using (4.16) we rewrite this as

$$
\begin{aligned}
2r_2c_2M_2 &= \frac{1}{2q_1}\mathcal{L}\Xi_1 - 2\Xi_0 - 3\beta^*\Xi_1^3 \\
&\qquad + \theta_2(t, V_{(p)}, tDV_{(p)}) + t\tilde{\theta}_2(t, V, M, tDV, tDM).
\end{aligned}
\tag{4.18}
$$

We insert (4.18) into (4.13) to get

$$
\begin{aligned}
2r_jM_j &- 2r_{j+1}c_{j+1}M_{j+1} + \frac{2q_1r_2}{q_j}(c_2^2c_3\cdots c_j)M_2 \\
&= -\frac{1}{2q_j}\sum_{\ell=2}^{j}\frac{1}{c_\ell}(c_\ell\cdots c_j)(tD+4t)^2M_\ell \\
&\quad + R_1(t, V_{(p)}, tDV_{(p)}) + tR_2(t, V, M, tDV, tDM).
\end{aligned}
\tag{4.19}
$$

For later use we give another less precise expression of (4.13). From (4.14) one can write

$$R_1(t, V_{(p)}, tDV_{(p)}) = t\tilde{R}_1(t, V, DV)$$

and hence we can rewrite (4.13) in the form

$$
\begin{aligned}
2r_jM_j - 2r_{j+1}c_{j+1}M_{j+1} &= -\frac{1}{2q_j}(c_2\cdots c_j)\mathcal{L}\Xi_1 \\
&\qquad + t\theta_j(t, V, M, DV, DM).
\end{aligned}
\tag{4.20}
$$

If we have a solution (X, Ξ, M) of (4.3) and (4.20) which is bounded as $t \downarrow 0$ then (x, ξ, m), defined by (4.2), satisfies (3.1) and (3.13) with (3.14) and hence (x, ξ) is a solution to the original Hamilton system.

5. Formal solutions

We first look for a formal solution to (4.3) and (4.20). Let us define the class of formal series in t and $\log 1/t$ in which we look for formal solutions:

Definition. For $k \in \mathbf{N}$ we set

$$\mathcal{E}_k = \left\{ t^k \sum_{0\le j\le i} t^i(\log 1/t)^j F_{ij} \mid F_{ij} \in \mathbf{C}^N \right\}.$$

The followings are checked immediately:

- $\mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_k \supset \cdots$,
- $t^p\mathcal{E}_k \subset \mathcal{E}_{p+k}$,
- $D\mathcal{E}_k \subset \mathcal{E}_k$,
- $\mathcal{E}_p\mathcal{E}_q \subset \mathcal{E}_{p+q}$.

We further rewrite the equation (4.20). From (4.20) it follows that

$$
\begin{aligned}
M_j &= -\frac{c_2\cdots c_j}{4r_j}\left[\frac{1}{q_j} + \frac{c_{j+1}^2}{q_{j+1}} + \cdots + \frac{c_{j+1}^2\cdots c_p^2}{q_p}\right]\mathcal{L}\Xi_1 \\
&\quad + t\tilde{f}_j(t, W, DW), \qquad 2 \le j \le p.
\end{aligned}
\tag{5.1}
$$

Let us set

$$(5.2)\qquad \begin{cases} \kappa_j = -\dfrac{c_2\cdots c_j}{4r_j}\left[\dfrac{1}{q_j}+\dfrac{c_{j+1}^2}{q_{j+1}}+\cdots+\dfrac{c_{j+1}^2\cdots c_p^2}{q_p}\right], \\ \kappa = \dfrac{1}{q_1}+\dfrac{c_2^2}{q_2}+\cdots+\dfrac{c_2^2\cdots c_p^2}{q_p} \end{cases}$$

so that

$$(5.3)\quad M_j = \kappa_j\mathcal{L}\Xi_1 + t\tilde{f}_j(t,W,DW), \qquad 2\le j\le p, \qquad 4r_2c_2\kappa_2 = \frac{1}{q_1}-\kappa.$$

Note that $\tilde{f}_j$ has the form

$$(5.4)\qquad \tilde{\theta}_j = \sum_{k=2}^{p} a_{0k}^{(j)}M_k + a_{1k}^{(j)}DM_k + a^{(j)},$$

where $a^{(j)}$, $a_{ik}^{(j)}$ are smooth in (t,V,DV).

We now assume that (5.3), (4.15) and (4.16) hold. Then we have from (4.18) and (5.3) that

$$\left(\frac{1}{q_1}-\kappa\right)\mathcal{L}\Xi_1 + 4r_2c_2t\tilde{f}_2 = \frac{1}{q_1}\mathcal{L}\Xi_1 - 4\Xi_0 - 6\beta^*\Xi_1^2 + 2tf_2$$

so that $\mathcal{L}\Xi_1 = [6\beta^*\kappa^{-1}\Xi_1^2 + 4\kappa^{-1}\Xi_0] + tf_3$ with $f_3 = \kappa^{-1}(4r_2c_2\tilde{f}_2 - 2f_2)$. Here we have set

$$tf_2(t,W,DW) = \theta_2(t,V_{(p)},tDV_{(p)}) + t\tilde{\theta}_2(t,V,M,tDV,tDM).$$

Thus one has

$$(5.5)\qquad M_j = \kappa_j[6\beta^*\kappa^{-1}\Xi_1^2 + 4\kappa^{-1}\Xi_0] + tf_j'(t,W,DW), \qquad 2\le j\le p$$

with $f_j' = \kappa_j f_3 + \tilde{f}_j$ where f_j' has the same form as (5.4).

Conversely assume that (4.15), (4.16) and (5.5) hold. From (5.5) and (4.18) one has

$$2r_2c_2\kappa_2[6\beta^*\kappa^{-1}\Xi_1^2 + 4\kappa^{-1}\Xi_0] + 2r_2c_2tf_2' = \frac{1}{2q_1}\mathcal{L}\Xi_1 - 2\Xi_0 - 3\beta^*\Xi_1^2 + tf_2$$

and hence

$$[6\beta^*\kappa^{-1}\Xi_1^2 + 4\kappa^{-1}\Xi_0] = \mathcal{L}\Xi_1 - tf_3.$$

Thus we get (5.3) from (5.5). We conclude that our problem is reduced to find a solution (X,Ξ,M) verifying (4.3) and (5.5).

Lemma 5.1. *Assume that $(X,\Xi,M)\in\mathcal{E}_0$ satisfies (4.3) and (5.5) formally and $\Xi_1(0)\neq 0$. Then $X(0)$, $\Xi(0)$ and $M(0)$ are uniquely determined.*

Proof. Let us set

$$X_\mu = \sum_{0\le j\le i} t^i(\log 1/t)^j \beta_{ij}^{(\mu)}, \qquad \Xi_\mu = \sum_{0\le j\le i} t^i(\log 1/t)^j \alpha_{ij}^{(\mu)},$$
$$M_\mu = \sum_{0\le j\le i} t^i(\log 1/t)^j m_{ij}^{(\mu)}.$$

Equating the constant terms of both sides of (5.5) and recalling that $\mathcal{L} = 6+5D+D^2$ one has

$$m_{00}^{(j)} = 6\kappa_j\kappa^{-1}\beta^*(\alpha_{00}^{(1)})^2 + 4\kappa^{-1}\kappa_j\alpha_{00}^{(0)}. \tag{5.6}$$

From $D\Xi_j = -2\Xi_j + 2q_jX_j + t\psi_j(t,W)$ we have

$$\alpha_{00}^{(j)} = q_j\beta_{00}^{(j)}, \qquad 1\le j\le p. \tag{5.7}$$

From $DX_1 = -X_0 + 2\Xi_1 + t^2\phi_0(t,W)$ and $D\Xi_0 = -4\Xi_0 + t\psi_0(t,W)$ it follows that

$$\alpha_{00}^{(0)} = 0, \qquad \beta_{00}^{(0)} = 2\alpha_{00}^{(1)}. \tag{5.8}$$

Now $DX_1 = -3X_1 + 2\Xi_0 + 2r_2c_2M_2 + 3\beta^*\Xi_1^2 + t\phi_1(t,W)$ with (5.8) gives

$$3\beta_{00}^{(1)} = 2r_2c_2m_{00}^{(2)} + 3\beta^*(\alpha_{00}^{(1)})^2. \tag{5.9}$$

Then from (5.7), (5.6) and (5.9) it follows that

$$\alpha_{00}^{(1)} = \frac{1}{\beta^*}\left[\frac{1}{q_1} + \frac{c_2^2}{q_2} + \cdots + \frac{c_2^2\cdots c_p^2}{q_p}\right] = \frac{\kappa}{\beta^*} \tag{5.10}$$

for $\alpha_{00}^{(1)}\neq 0$. Thus $\alpha_{00}^{(1)}$ is uniquely determined provided $\alpha_{00}^{(1)}\neq 0$. The equation (5.6) determines $m_{00}^{(j)}$, $2\le j\le p$ uniquely. From $DX_j = -3X_j - 2r_jM_j + 2r_{j+1}c_{j+1}M_{j+1} + t\phi_j(t,W)$ it follows that

$$\beta_{00}^{(j)} = \frac{1}{3}\left[2r_{j+1}c_{j+1}m_{00}^{(j+1)} - 2r_jm_{00}^{(j)}\right], \qquad 2\le j\le p. \tag{5.11}$$

Then (5.7) determines $\alpha_{00}^{(j)}$, $2\le j\le p$.

We turn to $\beta_{00}^{(p+j)}$, $\alpha_{00}^{(p+j)}$ for $j\ge 1$. From (4.3) it is clear that

$$\alpha_{00}^{(p+j)} = 0, \quad j\ge 1, \qquad \beta_{00}^{(p+j)} = 0, \quad j\neq k+1,\ldots,k+\ell.$$

It is also clear that

$$\beta_{00}^{(p+j)} = -\frac{2e_{k+j}}{3}(\alpha_{00}^{(1)})^2, \qquad k+1\le j\le k+\ell.$$

This proves the assertion. $\square$

We now show that there exists a formal solution $(X, \Xi, M) \in \mathcal{E}_0$ verifying $\Xi_1(0) \neq 0$ and (4.3), (5.5). If such a solution exists then $(X(0), \Xi(0), M(0))$ is uniquely determined by Lemma 5.1. Taking this fact into account let us put

$$(\bar{X}, \bar{\Xi}, \bar{M}) = (X(0), \Xi(0), M(0))$$

and

$$\mathcal{E}^\sharp = \left\{ \sum_{1 \le i, 0 \le j \le i} t^i (\log 1/t)^j F_{ij} \right\},$$

we substitute $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M)$ for (X, Ξ, M) in (5.5) to get the equation for (X, Ξ, M). Since $\beta^* \alpha_{00}^{(1)} = \kappa$ we get

$$M_j = 12\kappa_j \Xi_1 + 4\kappa_j \kappa^{-1} \Xi_0 + a_j \Xi_1^2 + tF_j + tf_j(t, W, DW), \tag{5.12}$$

where a_j and F_j are constants and $f_j(t, W, DW)$ has the same form as (5.4) and

$$f_j(0, 0, 0) = 0.$$

Hence if $(X, \Xi, M) \in \mathcal{E}^\sharp$ verifies (5.12) then $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M)$ satisfies (5.5).

We turn to the equation (4.3). Let us substitute $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M)$ for (X, Ξ, M) in (4.3). Then we have, thanks to $\beta^* \alpha_{00}^{(1)} = \kappa$,

$$\left\{\begin{aligned}
DX_0 &= -X_0 + 2\Xi_1 + t^2\beta_0 + t^2\phi_0(t, W),\\
DX_1 &= -3X_1 + 2\Xi_0 + 2r_2c_2M_2 + 6\kappa\Xi_1 + a_1\Xi_1^2\\
&\qquad + t\beta_1 + t\phi_1(t, W),\\
DX_j &= -3X_j - 2r_jM_j + 2r_{j+1}c_{j+1}M_{j+1} + t\beta_j + t\phi_j(t, W),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 2 \le j \le p,\\
D\Xi_0 &= -4\Xi_0 + t\gamma_0 + t\psi_0(t, W),\\
D\Xi_j &= -2\Xi_j + 2q_jX_j + t\gamma_j + t\psi_j(t, W), \qquad 1 \le j \le p,\\
tDX_{p+j} &= -3tX_{p+j} - 2b_{k+j}\Xi_{p+j} + t\beta_{p+j} + t\phi_{p+j}(t, W),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 1 \le j \le k,\\
tD\Xi_{p+j} &= -3t\Xi_{p+j} + 2b_jX_{p+j} + t\gamma_{p+j} + t\psi_{p+j}(t, W),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 1 \le j \le k,\\
DX_{p+j} &= -3X_{p+j} - 2b_{k+j}\Xi_{p+j} + 4b_{k+j}e_{k+j}\alpha_{00}^{(1)}\Xi_1\\
&\qquad + a_{p+j}\Xi_1^2 + t\beta_{p+j} + t\phi_{p+j}(t, W), \qquad k+1 \le j \le k+\ell,\\
DX_{p+j} &= -4X_{p+j} + t\beta_{p+j} + t\phi_{p+j}(t, W), \qquad k+\ell+1 \le j,\\
D\Xi_{p+j} &= -4\Xi_{p+j} + t\gamma_{p+j} + t\psi_{p+j}(t, W), \qquad k+1 \le j,
\end{aligned}\right. \tag{5.13}$$

where ϕ_j, ψ_j are polynomials in M such that $\phi_j(0, 0) = 0$, $\psi_j(0, 0) = 0$ with coefficients which are smooth in (t, V). If $(X, \Xi, M) \in \mathcal{E}^\sharp$ verifies (5.13) then $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M)$ satisfies (4.3).

Theorem 5.1. *There exists a formal solution $(X, \Xi, M) \in \mathcal{E}_0$ verifying $\Xi_1(0) \neq 0$ and (4.3), (5.5).*

We start with

Lemma 5.2. *For any $V = (X, \Xi) \in \mathcal{E}^\sharp$ there is a unique $M \in \mathcal{E}^\sharp$ such that $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M)$ satisfies (5.5) where M has the form*

$$M_j = 12\kappa_j \Xi_1 + 4\kappa_j \kappa^{-1} \Xi_0 + tF_j + C_j, \qquad 2 \leq j \leq p$$

with a constant F_j and

$$C_j = \sum_{2 \leq p, 0 \leq q \leq p-1} C_{pq}^{(j)} t^p (\log 1/t)^q, \tag{5.14}$$
$$C_{pq}^{(j)} = C_{pq}^{(j)}(V_{\mu\nu} \mid \nu \leq \mu \leq p-1).$$

Proof. Note that

$$\Xi_1^2 = \sum_{2 \leq i, 0 \leq j \leq i} \eta_{ij} t^i (\log 1/t)^j, \qquad \eta_{ij} = \eta_{ij}(\alpha_{pq}^{(1)} \mid q \leq p \leq i-1).$$

Then with

$$M_\mu = \sum_{1 \leq i, 0 \leq j \leq i} m_{ij}^{(\mu)} t^i (\log 1/t)^j$$

it is easy to see that (5.12) implies that

$$\begin{aligned} m_{pq}^{(j)} &= 12\kappa_j \alpha_{pq}^{(1)} + 4\kappa^{-1}\kappa_j \alpha_{pq}^{(0)} + \delta_{p1}\delta_{q0} F_j \\ &\quad + G_{pq}^{(j)}(\alpha_{\mu\nu}^{(1)}, \nu \leq \mu \leq p-1, V_{\mu\nu}, \nu \leq \mu \leq p-1, m_{\mu\nu}^{(i)}, \nu \leq \mu \leq p-1). \end{aligned}$$

By induction we get the desired assertion. $\square$

Substitute $(\bar{X} + X, \bar{\Xi} + \Xi, \bar{M} + M(X, \Xi))$ for (X, Ξ, M) in (5.5). Here $M(X, \Xi)$ is given by Lemma 5.2. Let us denote

$$\begin{aligned} V^I &= {}^t(X_0, \ldots, X_p, \Xi_0, \ldots, \Xi_p) = V_{(p)}, \\ V^{II} &= {}^t(X_{p+1}, \ldots, X_{p+k}, \Xi_{p+1}, \ldots, \Xi_{p+k}), \\ V^{III} &= {}^t(X_{p+k+1}, \ldots, X_n, \Xi_{p+k+1}, \ldots, \Xi_n). \end{aligned}$$

Then (5.13) becomes

$$\begin{aligned} DV^I &= A_I V^I + F_I t + G_I(t, V), \\ 0 &= A_{II} V^{II} + F_{II} t + G_{II}(t, V), \\ DV^{III} &= A_{III} V^{III} + K\Xi_1 + F_{III} t + G_{III}(t, V), \end{aligned} \tag{5.15}$$

where

$$G_*(t, V) = \sum_{2 \leq i, 0 \leq j \leq i} G_{*ij} t^i (\log 1/t)^j, \tag{5.16}$$
$$G_{*ij} = G_{*ij}(V_{pq} \mid q \leq p \leq i-1)$$

and F_*, K are constant vectors. Indeed tDW has the form (5.16) if $W \in \mathcal{E}^\sharp$. Make more precise looks on A_*. Let us study the linear part of the second equation (5.13):

$$2r_2c_2M_2 + 6\kappa\Xi_1 - 3X_1 + 2\Xi_0.$$

By Lemma 5.2 it turns out to be

$$\begin{aligned}
&-3X_1 + 2r_2c_2[12\kappa_2\Xi_1 + 4\kappa^{-1}\kappa_2\Xi_0] + 2\Xi_0 + 6\kappa\Xi_1 + G_2 + tF_2\\
&\quad = -3X_1 + 6\left(\frac{1}{q_1} - \kappa\right)\Xi_1 + 2\kappa^{-1}\left(\frac{1}{q_1} - \kappa\right)\Xi_0 + 2\Xi_0 + 6\kappa\Xi_1 + G_2 + tF_2\\
&\quad = -3X_1 + 6q_1^{-1}\Xi_1 + 2\kappa^{-1}q_1^{-1}\Xi_0 + G_2 + tF_2
\end{aligned}$$

where G_2 verifies (5.16). We note that

$$-3X_j - 2r_jM_j + 2r_{j+1}c_{j+1}M_{j+1} = -3X_j + 24\tau_j\Xi_1 + 8\kappa^{-1}\tau_j\Xi_0 + tF_j + G_j$$

with G_j verifying (5.16) where $\tau_j = r_{j+1}c_{j+1}\kappa_{j+1} - r_j\kappa_j$. Thus we get the expression of A_I:

$$A_IV^I = \begin{pmatrix} -X_0 + 2\Xi_1 \\ -3X_1 + (6q_1^{-1})\Xi_1 + 2\kappa^{-1}q_1^{-1}\Xi_0 \\ -3X_j + 24\tau_j\Xi_1 + 8\tau_j\kappa^{-1}\Xi_0 \\ -4\Xi_0 \\ -2\Xi_j + 2q_jX_j \end{pmatrix}. \tag{5.17}$$

On the other hand, it is easy to see that

$$A_{II} = \begin{pmatrix} & & & \vdots & -2b_{k+1} & & \\ & O & & \vdots & & \ddots & \\ & & & \vdots & & & -2b_{2k} \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 2b_1 & & & \vdots & & & \\ & \ddots & & \vdots & & O & \\ & & 2b_k & \vdots & & & \end{pmatrix}. \tag{5.18}$$

Turn to A_{III}. We see that

(5.19)

$$A_{III}V^{III} = \begin{pmatrix} -3X_{p+k+1} - 2b_{2k+1}\Xi_{p+k+1} \\ \vdots \\ -3X_{p+k+\ell} - 2b_{2k+\ell}\Xi_{p+k+\ell} \\ -4X_{p+k+\ell+1} \\ \vdots \\ -4X_n \\ -4\Xi_{p+k+1} \\ \vdots \\ -4\Xi_n \end{pmatrix}, \qquad K = \begin{pmatrix} 4b_{2k+1}e_{2k+1}\alpha_{00}^{(1)} \\ \vdots \\ 4b_{2k+\ell}e_{2k+\ell}\alpha_{00}^{(1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let us write

$$HDV = AV + tF + G(t,V), \tag{5.20}$$

where

$$H = \begin{pmatrix} E & O & O \\ O & O & O \\ O & O & E \end{pmatrix}, \qquad A = \begin{pmatrix} A_I & O & O \\ O & A_{II} & O \\ \star & O & A_{III} \end{pmatrix}.$$

Lemma 5.3. *We have*

$$\sigma(A_I) = \{-6, 0, 1\}, \quad \sigma(A_{II}) \subset i\mathbf{R} \setminus \{0\}, \quad \sigma(A_{III}) = \{-3, -4\}.$$

Proof. It is enough to show the first assertion. It is easy to see that

$$\begin{aligned} \det(\lambda - A_I) &= (\lambda+1)(\lambda+4) \begin{vmatrix} \lambda+3 & D_1 \\ D_2 & \lambda+2 \end{vmatrix} \\ &= (\lambda+1)(\lambda+4)|(\lambda+3)(\lambda+2) - D_2 D_1|, \end{aligned}$$

where $D_2 = -\operatorname{diag}(2q_1, 2q_2, \ldots, 2q_p)$ and

$$D_1 = -\begin{pmatrix} 6q_1^{-1} & 0 & \cdots & 0 \\ 24\tau_2 & 0 & \cdots & 0 \\ \vdots & & & \\ 24\tau_p & 0 & \cdots & 0 \end{pmatrix}.$$

From this we conclude that the eigenvalues are 0 and λ:

$$\lambda^2 + 5\lambda - 6 = 0.$$

This proves the assertion. □

Proof of Theorem 5.1. Note that (5.20) implies that

$$H(iV_{ij} - (j+1)V_{ij+1}) = AV_{ij} + \delta_{i1}\delta_{j0}F + G_{ij}, \tag{5.21}$$

where $G_{ij} = 0$ for $i = 0, 1$. Then we have

$$\begin{cases} (H-A)V_{11} = 0, \\ (H-A)V_{10} = V_{11} + F. \end{cases} \tag{5.22}$$

Choose $V_{11} \in \operatorname{Ker}(H-A)$ so that

$$F + V_{11} \in \operatorname{Im}(H-A).$$

Then we can take $V_{10} \neq 0$ so that

$$(H-A)V_{10} = F + V_{11}$$

since $\mathrm{Ker}(H-A) \neq \{0\}$ by Lemma 5.3. We turn to the case $i \geq 2$:

$$(iH - A)V_{ij} = (j+1)HV_{ij+1} + G_{ij}. \tag{5.23}$$

With $j = i$, (5.23) turns

$$(iH - A)V_{ii} = G_{ii}(V_{pq} \mid q \leq p \leq i-1).$$

Since $iH - A$ is non singular for $i \geq 2$ by Lemma 5.3 one has

$$V_{ii} = (iH - A)^{-1} G_{ii}(V_{pq} \mid q \leq p \leq i-1).$$

Recurrently one can solve V_{ij} by

$$V_{ij} = (iH - A)^{-1}\left[(j+1)HV_{ij+1} + G_{ij}(V_{pq} \mid q \leq p \leq i-1)\right]$$

for $j = i-1,\, i-2, \dots, 0$. This proves the assertion. □

6. A coupled system of ODEs

In this section we study the next system of ordinary differential equations

$$\begin{cases} \left(t^2 \dfrac{d}{dt} - i\Lambda\right) u = -tK_1 u + L_1(t)v + Q_1(t,u,v) \\ \qquad\qquad\qquad + tR_1(t,u,v) + tF_1, \\ \qquad t\dfrac{d}{dt} v = -K_2 v + Lu + L_2(t)v + Q_2(t,u,v) \\ \qquad\qquad\qquad + tR_2(t,u,v) + tF_2, \end{cases} \tag{6.1}$$

where $Q_j(t,u,v)$ and $R_j(t,u,v)$ are C^1 functions defined near $(0,0,0) \in \mathbf{R} \times \mathbf{C}^{N_1} \times \mathbf{C}^{N_2}$ such that

$$\begin{cases} |Q_j(t,u,v)| \leq B_{j0}(|u|^2 + |v|^2), \\ |R_j(t,u,v)| \leq \tilde{B}_{j0}(|u| + |v|) \end{cases} \tag{6.2}$$

for $(t,u,v) \in \{|t| \leq T_1\} \times \{|u| \leq C_1 T_1\} \times \{|v| \leq C_1 T_1\}$ and $L_2(t) \in C^1((0,T])$, $L_1(t) \in C^1((0,T])$ are $N_2 \times N_2$ and $N_1 \times N_2$ matrix valued function respectively which verifies

$$\|L_j(t)\|_{C((0,T])}, \qquad \|tL_j'(t)\|_{C((0,T])} \leq B$$

while L is a constant $N_2 \times N_1$ matrix. To simplify notations we write $\|f\|_T$ for $\|f\|_{C([0,T])}$. We assume that Λ is a constant nonsingular real diagonal matrix;

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{N_1}), \qquad \lambda_j \in \mathbf{R} \setminus \{0\} \tag{6.3}$$

and K_i are real diagonal matrices;

$$K_1 = \mathrm{diag}(m_{11}, \dots, m_{1N_1}), \qquad K_2 = \mathrm{diag}(m_{21}, \dots, m_{2N_2}).$$

We also assume that

$$|K_1|,\ |K_2| \leq 2m, \qquad m = \min\{m_{11}, \dots, m_{1N_1}, m_{21}, \dots, m_{2N_2}\}. \tag{6.4}$$

Our aim in this section is to prove:

Theorem 6.1. *If m is sufficiently large then* (6.1) *has a solution* (u, v) *such that* $u(0) = 0$, $v(0) = 0$.

Let $m > 0$. For $f \in C([0,T])$ with $f(t) = O(t)$ as $t \downarrow 0$ we define

$$\mathcal{H}[f] = \int_0^t e^{-\frac{i}{t}\Lambda + \frac{i}{s}\Lambda} \left(\frac{t}{s}\right)^{-K_1} \frac{1}{s^2} f(s) ds$$

and for $h \in C([0,T])$ we set

$$\mathcal{G}[h] = \int_0^t \left(\frac{t}{s}\right)^{-K_2} \frac{1}{s} h(s) ds$$

so that

$$\left(t^2 \frac{d}{dt} - i\Lambda\right) \mathcal{H}[f] = -tK_1 \mathcal{H}[f] + f \tag{6.5}$$

and

$$t\frac{d}{dt}\mathcal{G}[h] = -K_2 \mathcal{G}[h] + h. \tag{6.6}$$

We start with

Lemma 6.1. *Let* $f(t) \in C^1((0,T])$ *be such that* $f(t) = O(t)$ *and* $tf'(t) = O(1)$ *as* $t \downarrow 0$ *and let* $h \in C([0,T])$. *Assume* $m > 0$. *Then we have*

$$\begin{aligned}
&\mathcal{H}[f](t) = -(i\Lambda)^{-1} f(t) + K_1 (i\Lambda)^{-1} \mathcal{H}[tf](t) + (i\Lambda)^{-1} \mathcal{H}[t^2 f'](t),\\
&|\mathcal{H}[f](t)| \le \frac{1}{m} \|s^{-1} f\|_{C((0,t])},\\
&|\mathcal{G}[h](t)| \le \frac{1}{m} \|h\|_{C([0,t])}.
\end{aligned}$$

Proof. Let $m > 0$. Note that

$$\mathcal{H}[f] = e^{-\frac{i}{t}\Lambda} \int_{\frac{1}{t}}^{\infty} e^{i\rho\Lambda} \left(\frac{1}{t\rho}\right)^{K_1} f\left(\frac{1}{\rho}\right) d\rho.$$

Then the integration by parts gives

$$\begin{aligned}
\mathcal{H}[f] &= -(i\Lambda)^{-1} f(t) + K_1 (i\Lambda)^{-1} e^{-\frac{i}{t}\Lambda} \int_{\frac{1}{t}}^{\infty} e^{i\rho\Lambda} \left(\frac{1}{t\rho}\right)^{K_1 - I} \frac{1}{t\rho^2} f\left(\frac{1}{\rho}\right) d\rho\\
&\quad + (i\Lambda)^{-1} e^{-\frac{i}{t}\Lambda} \int_{\frac{1}{t}}^{\infty} e^{i\rho\Lambda} \left(\frac{1}{t\rho}\right)^{K_1} \frac{1}{\rho^2} f'\left(\frac{1}{\rho}\right) d\rho\\
&= -(i\Lambda)^{-1} f(t) + K_1 (i\Lambda)^{-1} \int_0^t e^{-\frac{i}{t}\Lambda + \frac{i}{s}\Lambda} \left(\frac{t}{s}\right)^{-K_1} \frac{1}{s^2} s f(s) ds\\
&\quad + (i\Lambda)^{-1} \int_0^t e^{-\frac{i}{t}\Lambda + \frac{i}{s}\Lambda} \left(\frac{t}{s}\right)^{-K_1} \frac{1}{s^2} s^2 f'(s) ds,
\end{aligned}$$

which proves the first assertion. Since

$$\left|e^{-\frac{i}{t}\Lambda+\frac{i}{s}\Lambda}\right| \le 1,$$

we have

$$\begin{aligned}|\mathcal{H}[f](t)| &\le \int_0^1 \left(\frac{1}{s}\right)^{-K_1} \frac{1}{s}|(ts)^{-1}f(ts)|ds \\ &\le \|t^{-1}f\|_{C((0,t])} \int_0^1 \left(\frac{1}{s}\right)^{-K_1} \frac{1}{s}ds = \frac{1}{m}\|t^{-1}f\|_{C((0,t])},\end{aligned}$$

which is the second assertion. The third assertion is clear because

$$|\mathcal{G}[h](t)| \le \int_0^1 \left(\frac{1}{s}\right)^{-K_2} \frac{1}{s}|h(ts)|ds \le \frac{1}{m}\|h\|_{C([0,t])}.$$

□

Using (6.5) and (6.6) we rewrite (6.1) as an integral equation:

$$\begin{cases} u = \mathcal{H}[L_1(t)v + Q_1(t,u,v) + tR_1(t,u,v) + tF_1], \\ v = \mathcal{G}[Lu + L_2(t)v + Q_2(t,u,v) + tR_2(t,u,v) + tF_2]. \end{cases} \tag{6.7}$$

Let $u_0(t) = 0$, $v_0(t) = 0$ and define $u_n(t)$, $v_n(t)$ successively by

$$\begin{aligned} u_{n+1}(t) &= \mathcal{H}[L_1(t)v_n + Q_1(t,u_n,v_n) + tR_1(t,u_n,v_n) + tF_1], \\ v_{n+1}(t) &= \mathcal{G}[Lu_n + L_2(t)v_n + Q_2(t,u_n,v_n) + tR_2(t,u_n,v_n) + tF_2]. \end{aligned}$$

Lemma 6.2. *There exist positive constants C, C^* ($C^* < C$) and $T > 0$ such that we have*

$$|u_n(t)| \le Ct, \quad |v_n(t)| \le C^* t, \qquad n = 0, 1, 2, \dots \tag{6.8}$$

for $0 \le t \le T$.

Proof. Assume (6.8) holds for n and $n-1$. Write

$$u_{n+1} = \mathcal{H}[L_1(t)v_n] + \mathcal{H}[Q_1(t,u_n,v_n)] + \mathcal{H}[tR_1(t,u_n,v_n)] + \mathcal{H}[tF_1].$$

From Lemma 6.1 we see

$$|\mathcal{H}[tF_1]| \le \frac{1}{m}|F_1|. \tag{6.9}$$

Noting that

$$|Q_1(t,u_n,v_n)| \le 2B_{10}C^2t^2, \qquad |tR_1(u_n,v_n)| \le 2\tilde{B}_{10}Ct^2,$$

which follows from the inductive hypothesis and (6.2), we have from Lemma 6.1 that

$$|\mathcal{H}[Q_1(t,u_n,v_n)]| \le \frac{2B_{10}C^2t}{m}, \qquad |\mathcal{H}[tR_1(t,u_n,v_n)]| \le \frac{2\tilde{B}_{10}Ct}{m}. \tag{6.10}$$

We study $\mathcal{H}[L_1(t)v_n]$. By Lemma 6.1 one can write

$$\begin{aligned}\mathcal{H}[L_1(t)v_n] = &-(i\Lambda)^{-1}L_1(t)v_n + K_1(i\Lambda)^{-1}\mathcal{H}[tL_1(t)v_n]\\ &+ (i\Lambda)^{-1}\mathcal{H}[t^2L_1'(t)v_n] + (i\Lambda)^{-1}\mathcal{H}[t^2L_1(t)v_n']\end{aligned} \tag{6.11}$$

provided $tv_n' = O(1)$ as $t \downarrow 0$ which will be examined below. Let us write

$$|\Lambda^{-1}| = \lambda, \qquad A_k = |L| + \|L_2\|_T + 2(B_{2k}C + \tilde{B}_{2k})T, \qquad k = 0, 1$$

so that one has

$$|(i\Lambda)^{-1}L_1v_n| \le \lambda\|L_1\|_T C^* t \tag{6.12}$$

while Lemma 6.1 gives

$$|K_1(i\Lambda)^{-1}\mathcal{H}[tL_1(t)v_n]| \le |K_1|\lambda\frac{1}{m}\|L_1(t)v_n\|_{C([0,t])} \le 2\lambda\|L_1\|_T C^* t \tag{6.13}$$

and

$$|(i\Lambda)^{-1}\mathcal{H}[t^2L_1'(t)v_n]| \le \lambda\frac{1}{m}\|tL_1'(t)v_n\|_{C([0,t])} \le \frac{\lambda}{m}\|tL_1'(t)\|_T C^* t. \tag{6.14}$$

Recall that

$$\begin{aligned}tv_n' = &-K_2v_n + Lu_{n-1} + L_2(t)v_{n-1} + Q_2(t, u_{n-1}, v_{n-1})\\ &+ tR_2(t, u_{n-1}, v_{n-1}) + tF_2.\end{aligned}$$

This with the inductive hypothesis gives that

$$\begin{aligned}|tv_n'| &\le |K_2||v_n| + |L|Ct + \|L_2\|_T C^* t + 2B_{20}C^2t^2 + 2\tilde{B}_{20}Ct^2 + t|F_2|\\ &\le 2m|v_n| + \|L_2\|_T C^* t + A_0Ct + t|F_2|,\end{aligned}$$

which shows that $tv_n' = O(t)$ as $t \downarrow 0$. Moreover thanks to Lemma 6.1, one gets

(6.15)
$$\begin{aligned}|(i\Lambda)^{-1}\mathcal{H}[t^2L_1(t)v_n']| &\le \lambda\frac{1}{m}\|L_1\|_T\{2mC^* + A_0C + \|L_2\|_T C^* + |F_2|\}t\\ &\le 2\lambda\|L_1\|_T C^* t + \frac{\lambda\|L_1\|_T}{m}\{\|L_2\|_T C^* + A_0C + |F_2|\}\,t.\end{aligned}$$

From (6.12), (6.13), (6.14) and (6.15) it follows that

$$\begin{aligned}|\mathcal{H}[L_1(t)v_n]| \le 5\lambda\|L_1\|_T C^* t + \frac{\lambda}{m}\{&\|tL_1'\|_T C^*\\ &+ \|L_1\|_T(\|L_2\|_T C^* + A_0C + |F_2|)\}t.\end{aligned} \tag{6.16}$$

Combining the estimates (6.9), (6.10) and (6.16) we conclude that

$$\begin{aligned}|u_{n+1}(t)| \le 5\lambda\|L_1\|_T C^* t + \frac{1}{m}\{&|F_1| + 2C(B_{10}C + \tilde{B}_{10})\\ &+ \lambda\|tL_1'\|_T C^* + \lambda\|L_1\|_T(\|L_2\|_T C^* + A_0C + |F_2|)\}t.\end{aligned}$$

Fix a $C^* > 0$ and choose $C > 0$ so that $C/2 > 5\lambda\|L_1\|_T C^*$. Then if m is chosen such that

$$\frac{1}{m}\{|F_1| + 2C(B_{10} + \tilde{B}_{10}) + \lambda\|tL_1'\|_T C^* + \lambda\|L_1\|_T(\|L_2\|_T C^* + A_0 C + |F_2|)\} \leq C/2, \tag{6.17}$$

then we have

$$|u_{n+1}(t)| \leq Ct.$$

We turn to v_{n+1}:

$$v_{n+1} = \mathcal{G}[Lu_n] + \mathcal{G}[L_2(t)v_n] + \mathcal{G}[Q_2(t, u_n, v_n)] + \mathcal{G}[tR_2(t, u_n, v_n)] + \mathcal{G}[tF_2].$$

By Lemma 6.1 and the induction hypothesis one has

$$|\mathcal{G}[Lu_n]| \leq \frac{|L|}{m}Ct, \quad |\mathcal{G}[L_2(t)v_n]| \leq \frac{\|L_2\|_T}{m}C^* t, \quad |\mathcal{G}[tF_2]| \leq \frac{1}{m}|F_2|t.$$

Since

$$|Q_2(t, u_n, v_n)| \leq 2B_{20}C^2t^2, \quad |tR_2(t, u_n, v_n)| \leq 2\tilde{B}_{20}Ct^2,$$

we have by Lemma 6.1 that

$$|v_{n+1}| \leq \frac{1}{m}\{\|L_2\|_T C^* + |F_2| + A_0 C\}t. \tag{6.18}$$

Hence to conclude the proof it suffices to take m so that both (6.17) and

$$\frac{1}{m}\{\|L_2\|_T C^* + |F_2| + A_0 C\} \leq C^* \tag{6.19}$$

hold. □

Let us assume that

$$\begin{cases} \left|\dfrac{\partial Q_j}{\partial u}\right|, \quad \left|\dfrac{\partial Q_j}{\partial v}\right| \leq B_{j1}(|u| + |v|), \\ \left|\dfrac{\partial R_j}{\partial u}\right|, \quad \left|\dfrac{\partial R_j}{\partial v}\right| \leq \tilde{B}_{j1} \end{cases} \tag{6.20}$$

for $(t, u, v) \in \{|t| \leq T_1\} \times \{|u| \leq C_1T_1\} \times \{|v| \leq C_1T_1\}$. We now show

Lemma 6.3. *For large m we have*

$$|v_n - v_{n-1}| \leq \frac{1}{m}A_1\{\|u_{n-1} - u_{n-2}\|_{C([0,t])} + \|v_{n-1} - v_{n-2}\|_{C([0,t])}\},$$
$$t|v_n' - v_{n-1}'| \leq 2A_1\{\|u_{n-1} - u_{n-2}\|_{C([0,t])} + \|v_{n-1} - v_{n-2}\|_{C([0,t])}\}.$$

Proof. We first note that from (6.20) and the induction hypothesis one has

$$(6.21)\quad \begin{aligned}&|Q_j(t,u_{n-1},v_{n-1})-Q_j(t,u_{n-2},v_{n-2})|\\ &\qquad\le 2B_{j1}C\{|u_{n-1}-u_{n-2}|t+|v_{n-1}-v_{n-2}|\}t.\end{aligned}$$

Then Lemma 6.1 shows that

$$(6.22)\quad \begin{aligned}&|\mathcal{G}[Q_j(t,u_{n-1},v_{n-1})-Q_j(t,u_{n-2},v_{n-2})]|\\ &\qquad\le \frac{2B_{j1}CT}{m}\{\|u_{n-1}-u_{n-2}\|_{C([0,t])}+\|v_{n-1}-v_{n-2}\|_{C([0,t])}\}.\end{aligned}$$

Similarly from

$$(6.23)\quad \begin{aligned}&|tR_j(t,u_{n-1},v_{n-1})-tR_j(t,u_{n-2},v_{n-2})|\\ &\qquad\le 2\tilde{B}_{j1}t\{|u_{n-1}-u_{n-2}|+|v_{n-1}-v_{n-2}|\}\end{aligned}$$

one gets

$$(6.24)\quad \begin{aligned}&|\mathcal{G}[tR_j(t,u_{n-1},v_{n-1})-tR_j(t,u_{n-2},v_{n-2})]|\\ &\qquad\le \frac{2\tilde{B}_{j1}T}{m}\{\|u_{n-1}-u_{n-2}\|_{C([0,t])}+\|v_{n-1}-v_{n-2}\|_{C([0,t])}\}.\end{aligned}$$

It is also clear that

$$(6.25)\quad |\mathcal{G}[L_2(t)(v_{n-1}-v_{n-2})]|\le \frac{\|L_2\|_T}{m}\|v_{n-1}-u_{v-2}\|_{C([0,t])},$$

$$(6.26)\quad |\mathcal{G}[L(u_{n-1}-u_{n-2})]|\le \frac{|L|}{m}\|u_{n-1}-n_{n-2}\|_{C([0,t])}.$$

Since

$$\begin{aligned}v_n-v_{n-1}&=\mathcal{G}[L(u_{n-1}-u_{n-2})]+\mathcal{G}[L_2(t)(v_{n-1}-v_{n-2})]\\ &\quad+\mathcal{G}[Q_2(t,u_{n-1},v_{n-1})-Q_2(t,u_{n-2},v_{n-2})]\\ &\quad+\mathcal{G}[tR_2(t,u_{n-1},v_{n-1})-tR_2(t,u_{n-2},v_{n-2})]\end{aligned}$$

from (6.22), (6.24), (6.25) and (6.26), the first assertion follows.

We turn to $t(v_n'-v_{n-1}')$. Recall that

$$\begin{aligned}t(v_n'-v_{n-1}')&=-K_2(v_n-v_{n-1})+L(u_{n-1}-u_{n-2})+L_2(t)(v_{n-1}-v_{n-2})\\ &\quad+Q_2(t,u_{n-1},v_{n-1})-Q_2(t,u_{n-2},v_{n-2})\\ &\quad+tR_2(t,u_{n-1},v_{n-1})-tR_2(t,u_{n-2},v_{n-2}).\end{aligned}$$

This shows that

$$\begin{aligned}|t(v_n'-v_{n-1}')|&\le|K_2||v_n-v_{n-1}|+|L||u_{n-1}-u_{n-2}|+\|L_2\|_T|v_{n-1}-v_{n-2}|\\ &\quad+(2B_{21}C+2\tilde{B}_{21})t\{|u_{n-1}-u_{n-2}|+|v_{n-1}-v_{n-2}|\}\\ &\le 2m|v_n-v_{n-1}|+A_1\{|u_{n-1}-u_{n-2}|+|v_{n-1}-v_{n-2}|\}.\end{aligned}$$

Here we apply the first assertion to estimate $|v_n - v_{n-1}|$ and get

$$|t(v_n' - v_{n-1}')| \le 2A_1\{\|u_{n-1} - u_{n-2}\|_{C([0,t])} + \|v_{n-1} - v_{n-2}\|_{C([0,t])}\},$$

which is the desired assertion. ☐

Proof of Theorem 6.1. We show that u_n, v_n converge to some u, v in $C([0,T])$. Since $|v_n(t)| \le C^*t$ by Lemma 6.2 this proves that (u,v) verifies (6.7). Let us write

$$\begin{aligned} u_{n+1} - u_n = &\mathcal{H}[L_1(t)(v_n - v_{n-1})] + \mathcal{H}[Q_1(t,u_n,v_n) - Q_1(t,u_{n-1},v_{n-1})] \\ &+ \mathcal{H}[tR_1(t,u_n,v_n) - tR_1(t,u_{n-1},v_{n-1})] \end{aligned}$$

and set

$$W_n(t) = \|u_n - u_{n-1}\|_{C([0,t])} + \|v_n - v_{n-1}\|_{C([0,t])}.$$

From (6.21), (6.23) and Lemma 6.1 it follows that

$$\begin{aligned} |\mathcal{H}[Q_1(t,u_n,v_n) - Q_1(t,u_{n-1},v_{n-1})]| + |\mathcal{H}[tR_1(t,u_n,v_n) - tR_1(t,u_{n-1},v_{n-1})]| \\ \le \frac{2}{m}(B_{11}C + \tilde{B}_{11})W_n(t). \end{aligned}$$

By Lemmas 6.1 through 6.3 one can write

$$\begin{aligned} \mathcal{H}[L_1(t)(v_n - v_{n-1})] &= -(i\Lambda)^{-1}L_1(t)(v_n - v_{n-1}) \\ &+ K_1(i\Lambda)^{-1}\mathcal{H}[tL_1(t)(v_n - v_{n-1})] + (i\Lambda)^{-1}\mathcal{H}[t^2L_1'(t)(v_n - v_{n-1})] \\ &+ (i\Lambda)^{-1}\mathcal{H}[t^2L_1(t)(v_n' - v_{n-1}')]. \end{aligned}$$

From Lemma 6.3 one obtains

$$|(i\Lambda)^{-1}L_1(t)(v_n - v_{n-1})| \le \frac{\lambda}{m}\|L_1\|_T A_1 W_{n-1}(t) \tag{6.27}$$

while

$$\begin{aligned} &|K_1(i\Lambda)^{-1}\mathcal{H}[tL_1(v_n - v_{n-1})]| + |(i\Lambda)^{-1}\mathcal{H}[t^2L_1'(t)(v_n - v_{n-1})]| \\ &\le \left(2m\lambda\frac{1}{m}\|L_1\|_T + \frac{\lambda}{m}\|tL_1'\|_T\right)\|v_n - v_{n-1}\|_{C([0,t])} \\ &\le \frac{\lambda(2\|L_1\|_T + \|tL_1'\|_T)A_1}{m}W_{n-1}(t), \end{aligned}$$

where the last inequality follows from Lemma 6.3. Finally we see that from Lemmas 6.1 and 6.3

$$\begin{aligned} |(i\Lambda)^{-1}\mathcal{H}[t^2L_1(t)(v_n' - v_{n-1}')]| &\le \lambda\frac{1}{m}\|tL_1(t)(v_n' - v_{n-1}')\|_{C([0,t])} \\ &\le \frac{2\lambda A_1}{m}\|L_1\|_T W_{n-1}(t). \end{aligned}$$

Combining these estimates one gets

$$|u_{n+1} - u_n| \le \frac{2}{m}(B_{11}C + \tilde{B}_{11})W_n(t) + \frac{\lambda A_1}{m}(5\|L_1\|_T + \|tL_1'\|_T)W_{n-1}(t).$$

We turn to $v_{n+1} - v_n$: Recall that

$$\begin{aligned} v_{n+1} - v_n &= \mathcal{G}[L(u_n - u_{n-1})] + \mathcal{G}[L_2(t)(v_n - v_{n-1})] \\ &\quad + \mathcal{G}[Q_2(t, u_n, v_n) - Q_2(t, u_{n-1}, v_{n-1})] \\ &\quad + \mathcal{G}[tR_2(t, u_n, v_n) - tR_2(t, u_{n-1}, v_{n-1})]. \end{aligned}$$

From (6.22) and (6.24) it is easy to see that

$$\begin{aligned} |v_{n+1} - v_n| &\le \frac{(|L| + \|L_2\|_T)}{m} W_n(t) + \frac{2}{m}(B_{21}C + \tilde{B}_{21})TW_n(t) \\ &\le \frac{1}{m} A_1 W_n(t). \end{aligned}$$

We now assume that m is large so that we have

$$W_{n+1}(t) \le \delta\{W_n(t) + W_{n-1}(t)\}, \qquad 0 \le t \le T$$

with $0 < \delta < 1/2$. It is easy to check that

$$W_n(t) \le \sum_{k=1}^{n-2} (2\delta)^k (W_2 + W_1). \tag{6.28}$$

This proves that $\{u_n\}$, $\{v_n\}$ converges in $C([0, T])$ to some $u(t)$, $v(t) \in C([0, T])$. $\square$

7. Proof of Theorem 1.1

In this section we prove Theorem 1.1. To prove the existence of a bicharacteristic which falls into the doubly characteristic set, we show that we can apply Theorem 6.1 to conclude this. Let us set

$$A = \frac{1}{2} \begin{pmatrix} \frac{1}{q_2} & & & & \\ \frac{c_3}{q_3} & \frac{1}{q_3} & & & \\ \frac{c_3 c_4}{q_4} & \frac{c_4}{q_4} & \frac{1}{q_4} & & \\ \vdots & & & & \\ \vdots & & & & \\ \frac{c_3 \cdots c_p}{q_p} & & & & \frac{1}{q_p} \end{pmatrix}$$

and

$$B = 2\begin{pmatrix} r_2 + \frac{q_1}{q_2} r_2 c_2^2 & -r_3 c_3 & & & & \\ \frac{q_1}{q_3} r_2 c_2^2 c_3 & r_3 & -r_4 c_4 & & & \\ \frac{q_1}{q_4} r_2 c_2^2 c_3 c_4 & 0 & r_4 & -r_5 c_5 & & \\ \vdots & & & & & \\ \vdots & & & & r_{p-1} & -r_p c_p \\ \frac{q_1}{q_2} r_2 c_2^2 c_3 \cdots c_p & & & & 0 & r_p \end{pmatrix}$$

so that one can express the equation (4.19) as

$$BM = -A(tD+4t)^2 M + R_1(t, V_{(p)}, tDV_{(p)}) + tR_2(t, V, M, tDV, tDM)$$

and hence

$$(7.1) \qquad \begin{aligned} (tD+4t)^2 M = -A^{-1}BM + R_1(t, V_{(p)}, tDV_{(p)}) \\ + tR_2(t, V, M, tDV, tDM), \end{aligned}$$

where and below R_j may change from line to line.

Lemma 7.1. *Every eigenvalue of $A^{-1}B$ is positive and $A^{-1}B$ is diagonalizable.*

Proof. Note that

$$A^{-1} = 2\begin{pmatrix} q_2 & 0 & & & & \\ -c_3 q_2 & q_3 & & & & \\ 0 & -c_4 q_3 & q_4 & & & \\ & & & \ddots & & \\ & & & & q_{p-1} & \\ & & & & -c_p q_{p-1} & q_p \end{pmatrix}$$

and then one can see easily that $A^{-1}B$ is

$$\begin{pmatrix} q_2 r_2 + q_1 r_2 c_2^2 & -q_2 r_3 c_3 & & & \\ -q_2 r_2 c_3 & q_3 r_3 + q_2 r_3 c_3^2 & -q_3 r_4 c_4 & & \\ & & & \ddots & \\ & & & & -q_{p-1} r_p c_p \\ & & & -q_{p-1} r_{p-1} c_p & q_p r_p + q_{p-1} r_p c_p^2 \end{pmatrix}$$

which is a tridiagonal matrix. We show that this is symmetrizable and hence diagonalizable. Indeed if we take

$$D = \begin{pmatrix} \sqrt{\dfrac{r_3}{r_2}} & & & \\ 0 & \sqrt{\dfrac{r_4}{r_3}} & & \\ & & \ddots & \\ & & & \sqrt{\dfrac{r_p}{r_{p-1}}} \end{pmatrix},$$

then it is easy to see that $D^{-1}(A^{-1}B)D$ is equal to

$$\begin{pmatrix} q_2r_2 + q_1r_2c_2^2 & -q_2\sqrt{r_2r_3}c_3 & & \\ -q_2\sqrt{r_2r_3}c_3 & q_3r_3 + q_2r_3c_3^2 & & \\ & & \ddots & \\ & & & -q_{p-1}\sqrt{r_{p-1}r_p}c_p \\ & & -q_{p-1}\sqrt{r_{p-1}r_p}c_p & q_pr_p + q_{p-1}r_pc_p^2 \end{pmatrix}$$

which is symmetric. We now show that this is positive definite. To see this write $D^{-1}(A^{-1}B)D$ as

$$\begin{pmatrix} q_2r_2 & -q_2\sqrt{r_2r_3}c_3 & & \\ -q_2\sqrt{r_2r_3}c_3 & q_3r_3 + q_2r_3c_3^2 & & \\ & & \ddots & \\ & & & -q_{p-1}\sqrt{r_{p-1}r_p}c_p \\ & & -q_{p-1}\sqrt{r_{p-1}r_p}c_p & q_pr_p + q_{p-1}r_pc_p^2 \end{pmatrix}$$
$$+ \begin{pmatrix} q_1r_2c_2^2 & 0 & \\ 0 & 0 & \\ & & O \end{pmatrix} = H_1 + H_2.$$

By induction on the size of matrix, we see that the k-th principal minor of H_1 is equal to

$$(q_2 \cdots q_{k+1})(r_2 \cdots r_{k+1})$$

and hence H_1 is positive definite. Since H_2 is non negative definite we conclude that $D^{-1}(A^{-1}B)D$ is positive definite. This proves the assertion. □

Let us set

$$N = (tD + 4t)M \tag{7.2}$$

and denote

$$u = {}^t(N, M), \quad v_a = {}^t(V^I, tDV^I, V^{III}, tDV^{III}), \quad v_b = {}^t(V^{II}, tDV^{II}) \tag{7.3}$$

and $v = (v_a, v_b)$. Then one can rewrite (7.1) and (7.2) as

$$(tD + 4t)u = \begin{pmatrix} O & -A^{-1}B \\ I & O \end{pmatrix} u + R_1(t, v_a) + tR_2(t, u, v). \tag{7.4}$$

Lemma 7.2. *The matrix*

$$\begin{pmatrix} O & -A^{-1}B \\ I & O \end{pmatrix}$$

is diagonalizable and the all eigenvalues are non zero pure imaginary.

Proof. Easy. □

By Lemma 7.2 there is a nonsingular matrix T such that

$$T^{-1} \begin{pmatrix} O & -A^{-1}B \\ I & O \end{pmatrix} T = i \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2(p-1)} \end{pmatrix} = i\Lambda_1,$$

where $\lambda_i \in \mathbf{R} \setminus \{0\}$. Denoting $T^{-1}u$ by u again the equation (7.4) becomes

$$(tD + 4t)u = i\Lambda_1 u + \Phi_1(t, v_a) + t\Phi_2(t, u, v). \tag{7.5}$$

We turn to the equation (4.3) which can be written as

$$tDV^{II} = -3tV^{II} + A_{II}V^{II} + t\Psi_{II}(t, V, M) \tag{7.6}$$

and

$$\begin{cases} DV^I = A_I V^I + \tilde{A}_I M + Q_I(V^I) + t\Psi_I(t, V, M), \\ DV^{III} = A_{III}V^{III} + Q_{III}(V^I) + t\Psi_{III}(t, V, M), \end{cases} \tag{7.7}$$

where A_J, $\tilde{A}_I$ are constant matrices and Q_J are quadratic forms. Since A_{II} is diagonalizable and every eigenvalue of A_{II} is non zero pure imaginary there is a nonsingular constant matrix S such that

$$S^{-1}A_{II}S = i \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{2k} \end{pmatrix} = i\Lambda_2.$$

Denoting $S^{-1}V^{II}$ by V^{II} again we get

$$tDV^{II} = -3tV^{II} + i\Lambda_2 V^{II} + t\tilde{\Psi}_{II}(t, V, M). \tag{7.8}$$

Applying tD to (7.8) we obtain

$$tD(tDV^{II}) = -3t(tDV^{II}) + i\Lambda_2(tDV^{II}) + t\Psi'_{II}(t, u, v). \tag{7.9}$$

Combining (7.8) and (7.9) we get with $v_b = (V^{II}, tDV^{II})$ again

$$tDv_b = -3tv_b + i\Lambda_2 v_b + t\Psi_b(t,u,v). \tag{7.10}$$

We now multiply (7.7) by t and then apply D to get

$$\begin{cases} D(tDV^I) = A_I(tDV^I) + \tilde{A}_I N + \tilde{Q}_I(t, V^I, tDV^I) + t\tilde{\Psi}_I(t,u,v), \\ D(tDV^{III}) = A_{III}(tDV^{III}) + \tilde{Q}_{III}(t, V^I, tDV^I) + t\tilde{\Psi}_{III}(t,u,v). \end{cases} \tag{7.11}$$

Combining (7.7) and (7.11) one gets

$$Dv_a = Av_a + \tilde{A}u + Q(t, v_a) + t\Psi_a(t,u,v). \tag{7.12}$$

We now denote (u, v_b) by u and v_a by v to get

$$\begin{cases} tDu = -tKu + i\Lambda u + \Phi_1(t,v) + t\Phi_2(t,u,v), \\ Dv = A_1 u + A_2 v + Q(t,v) + t\Psi(t,u,v), \end{cases} \tag{7.13}$$

where A_i are constant matrices and

$$\Lambda = \begin{pmatrix} \Lambda_1 & O \\ O & \Lambda_2 \end{pmatrix}, \qquad K = \begin{pmatrix} 4I & O \\ O & 3I \end{pmatrix}.$$

Proof of Theorem 1.1. By Theorem 5.1 there exists a non trivial formal solution to (7.13):

$$u = \sum_{0 \le j \le i} u_{ij} t^i (\log 1/t)^j, \qquad v = \sum_{0 \le j \le i} v_{ij} t^i (\log 1/t)^j.$$

This shows that for any $m \in \mathbf{N}$ there is a $N = N(m)$ such that

$$u_N = \sum_{0 \le j \le i \le N} u_{ij} t^i (\log 1/t)^j, \qquad v_N = \sum_{0 \le j \le i \le N} v_{ij} t^i (\log 1/t)^j$$

verifies (7.13) modulo $O(t^{m+1})$, that is

$$\begin{aligned} tDu_N - [-tKu_N + i\Lambda u_N + \Phi_1(t, v_N) + t\Phi_2(t, u_N, v_N)] &= O(t^{m+1}), \\ Dv_N - [A_1 v_N + A_2 u_N + Q(t, v_N) + t\Psi(t, u_N, v_N)] &= O(t^{m+1}). \end{aligned}$$

We look for a solution in the form

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} + t^m \begin{pmatrix} u \\ v \end{pmatrix}.$$

Note that one can write

$$\begin{aligned} \Phi(t, u_N + t^m u, v_N + t^m v) &= \Phi(t, u_N, v_N) + t^m \sum u_j \frac{\partial \Phi}{\partial u_j}(t, u_N, v_N) \\ &\quad + t^m \sum v_j \frac{\partial \Phi}{\partial v_j}(t, u_N, v_N) + t^{2m} R(t,u,v) \\ &= \Phi(t, u_N, v_N) + t^m L_1(t) u + t^m L_2(t) v + t^{2m} R(t,u,v). \end{aligned}$$

It is clear that $L_j(t) = C_j + O(t \log 1/t)$ so that $L_j(t)$ and $tL_j'(t)$ are bounded in $(0, T]$. Since

$$tD(t^m u) = t^m(tD + mt)u, \qquad D(t^m v) = t^m(D + m)v$$

substituting $(u_N + t^m u, v_N + t^m v)$ into (7.13) and dividing the resulting equation by t^m one has

$$\begin{cases} (tD - i\Lambda)u = -t(mI + K)u + L_1(t)v + tR_1(t, u, v) + tF_1, \\ Dv = -mv + Lu + L_2(t)v + tR_2(t, u, v) + tF_2, \end{cases} \tag{7.14}$$

where L is a constant matrix. Since it is clear that (6.4) is verified for large m, we can now apply Theorem 6.1 to conclude that there exist u, v verifying (7.14). □

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