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Rational equivalence and Phantom map out of a loop space, II

Dedicated to Professor Goro Nishida on his 60th birthday

By

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1. Introduction

In this short paper we will prove the following theorems.

Theorem 1.A. *Let X be a simply connected finite complex. Then there exist no essential phantom maps from ΩX to a target of finite type.*

According to Theorem 8.7 of [3], this theorem is equivalent to the following theorem.

Theorem 1.B. *Let X be a simply connected finite complex. Then there exists a rational equivalence*

$$\Omega X \to \prod_{\alpha} S^{2n_\alpha+1} \times \prod_{\beta} \Omega S^{2n_\beta+1}.$$

Corollary 1.C. *Let X be a connected finite complex with finite fundamental group. Then there exist no essential phantom maps from ΩX to a target of finite type.*

These results give a negative answer to Question 4 of [3] at least for those spaces with finite fundamental groups.

2. Proof

In this paper, all spaces are assumed to have basepoints and all maps and homotopies are assumed to preserve them. In topological diagrams "commutative" means "commutative up to homotopy".

Let X and Y be spaces with minimal Sullivan models $(\Lambda V, d)$ and $(\Lambda W, d)$. For a map $f : X \to Y$, we denote by $\Lambda(f) : (\Lambda W, d) \to (\Lambda V, d)$ a Sullivan representative for f, see §12 of [1]. The degree of a homogeneous element x of ΛV is denoted by $|x|$.

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Lemma 2.1. *Let X be a simply connected finite complex and $(\Lambda V, d)$ be its minimal Sullivan model. If $x \in V^{2n}$ is a non-zero element with $dx = 0$, then there is a simply connected finite complex Y and a rational equivalence*

$$\Omega X \to \Omega Y \times S^{2n-1}.$$

The minimal model of Y, $(\Lambda W, d)$, is given by $W = V/(x)$ and there is the natural projection $(\Lambda V, d) \to (\Lambda W, d)$ of differential graded algebras.

Proof. By Corollary to Theorem 15.11 of [1], there is an element $\alpha \in \pi_{2n}(X)$ with $\langle x, H(\alpha)\rangle \neq 0$, where $\langle -, -\rangle$ denotes the Kronecker product and $H : \pi_*(X) \otimes \mathbb{Q} \to H_*(X; \mathbb{Q})$ is the Hurewicz homomorphism. Then by the same argument as in p. 781 of [2] there is a map $g : X \to BU(n)$ such that $g_*(\alpha) \in \pi_{2n}(BU(n)) \cong \mathbb{Z}$ is non-zero. Pulling back the fiber bundle $S^{2n-1} \to BU(n-1) \to BU(n)$ along the map $g : X \to BU(n)$, we obtain a fibration

$$\Omega X \xrightarrow{f} S^{2n-1} \xrightarrow{j} Y' \to X.$$

For $n = 1$ we must choose the map $g : X \to BU(1)$ carefully so that the above Y' is simply connected as in p. 783 of [2].

Since the order of $[j] \in \pi_{2n-1}(Y')$ is finite, say k, j can be extended to a map $j' : M^{2n-1}(k) = S^{2n-1} \cup_k e^{2n} \to Y'$. Put $Y = Y' \cup_{j'} CM^{2n-1}(k)$. As in p.783-4 of [2] we construct a map $\bar{h} : \Omega X \to \Omega Y$. Then it is easy to see that the composite of the maps

$$\Omega X \xrightarrow{\Delta} \Omega X \times \Omega X \xrightarrow{\bar{h} \times f} \Omega Y \times S^{2n-1}$$

is a rational equivalence, where Δ is the diagonal map.

It is easy to see that Y has the desired minimal model. □

Let n_i, $i = 1, \ldots, m$, be positive odd integers. We consider a finite complex $E = E(n_1, \ldots, n_m)$ with the following property, which will be referred as property (S):

There are maps $f_i : S^{n_i} \to E$ for $i = 1, \ldots, m$ such that the composite of maps

$$\prod_{i=1}^{m} \Omega S^{n_i} \xrightarrow{\prod \Omega f_i} \prod_{m} \Omega E \xrightarrow{\mu} \Omega E$$

is a homotopy equivalence, where μ is the iterated multiplication of loops.

An example of such a space is a product of odd dimensional spheres.

The minimal model of $E = E(n_1, \ldots, n_m)$, a finite complex with the property (S), is given by $(\Lambda(x_1, \ldots, x_m), d)$, where the degree of x_i is n_i. Let $x \in \Lambda(x_1, \ldots, x_m)$ be an even dimensional decomposable element such that $dx = 0$. We will kill the cohomology class represented by x and obtain another finite complex with the property (S):

Lemma 2.2. *Let E and x be as above and $|x| = 2n$. Then there is a fiber bundle*

$$S^{2n-1} \to E' \xrightarrow{\pi} E$$

such that

(i) *the minimal model of E' is given by*

$$(\Lambda(x_1, x_2, \ldots, x_m, x_{m+1}), \bar{d}),$$

where $\bar{d}x_i = dx_i$ for $i = 1, \ldots, m$ and $\bar{d}x_{m+1} = x$,

(ii) *$\Omega\pi : \Omega E' \to \Omega E$ has a cross section.*

Therefore the composite of maps

$$\Omega S^{2n-1} \times \Omega E \to \Omega E' \times \Omega E' \xrightarrow{\mu} \Omega E'$$

is a homotopy equivalence, where the first map is the product of the loop maps of the natural inclusion and the cross section. In particular, E' has also the property (S).

Proof. We may assume that x represents an integral cohomology class and that the map $\vee f_i : \vee S^{n_i} \to E$ is a cofibration. Let $p : E \to E/\vee S^{n_i}$ be the natural collapsing map. Then there is $y \in H^{2n}(E/\vee S^{n_i}; \mathbb{Z})$ such that $p^*(y) = x$. Since $E/\vee S^{n_i}$ is compact, by the same argument as in p. 781 of [2] there is a map

$$g : E/\vee S^{n_i} \to BU(n)$$

such that $g^*(c_n) = cy$ with $c \neq 0$, where $c_n \in H^{2n}(BU(n); \mathbb{Z})$ is the n-th Chern class.

Pulling back the fiber bundle $S^{2n-1} \to BU(n-1) \to BU(n)$ along the map $g \circ p : E \to BU(n)$, we obtain a desired fiber bundle

$$S^{2n-1} \to E' \xrightarrow{\pi} E.$$

The minimal model of E' is clearly the one described in (i).

Since $g \circ p|_{\vee S^{n_i}}$ is the constant map to the base point, there are lifts $s_i : S^{n_i} \to E'$ of the maps $f_i : S^{n_i} \to E$ for $i = 1, \ldots, m$. Consider the composite of maps

$$s : \Omega E \xrightarrow{g} \prod_{i=1}^{m} \Omega S^{n_i} \xrightarrow{\prod \Omega s_i} \prod_{m} \Omega E' \xrightarrow{\mu} \Omega E',$$

where $g : \Omega E \to \prod_{i=1}^{m} \Omega S^{n_i}$ is a homotopy inverse of the composite of maps

$$\prod_{i=1}^{m} \Omega S^{n_i} \xrightarrow{\prod \Omega f_i} \prod_{m} \Omega E \xrightarrow{\mu} \Omega E.$$

Then $\Omega\pi \circ s$ is homotopic to the identity map. By making use of the covering homotopy property we will obtain a cross section of $\Omega\pi$. $\square$

Lemma 2.3. *Let X be a simply connected finite complex. Then there is a minimal model of X, $(\Lambda(x_1, x_2, \ldots), d)$, with the following properties*:

(i) $|x_i| \leq |x_{i+1}|$ *for all* i,

(ii) *if* $|x_{\ell-1}| < |x_\ell| = \cdots = |x_s| = n < |x_{s+1}|$, *then a set of* $x_\ell, \ldots, x_k$, $k < s+1$, *and some decomposable elements is a basis of the kernel of the differential map* $d : \Lambda^n V \to \Lambda^{n+1} V$.

Moreover, if $|x_i|$ *is odd for* $1 \leq i \leq m$, *then there is a simply connected finite complex* E *with property* (S) *whose minimal model,* $(\Lambda(y_1, y_2, \ldots, y_m), d)$, *is isomorphic to* $(\Lambda(x_1, \ldots, x_m), d)$ *and a map* $f : X \to E$ *such that* $\Lambda(f)(y_i) = x_i$ *for* $i = 1, \ldots, m$. *In particular,*

$$\Omega f : \Omega X \to \Omega E \simeq \prod_{i=1}^{m} \Omega S^{|x_i|}$$

induces isomorphisms of rational homotopy groups up to dimension $|x_m| - 1$.

Proof. The first assertion is easy to prove by changing the given generators of a minimal model of X.

We show the second assertion by induction on m. For $m = 0$ there is nothing to be proved.

Let $m \geq 1$ and $|x_{\ell-1}| < |x_\ell| = \cdots = |x_m| = \cdots = |x_s| = 2n-1 < |x_{s+1}|$. By the property (ii), a set of $x_\ell, \ldots, x_k$, and some decomposable elements is a basis of the kernel of the differential map $d : \Lambda^{2n-1} V \to \Lambda^{2n} V$. By the induction hypothesis, we assume that there is a simply connected finite complex E with property (S) whose minimal model, $(\Lambda(y_1, \ldots, y_{m-1}), d)$, is isomorphic to $(\Lambda(x_1, \ldots, x_{m-1}), d)$ and a map $f : X \to E$ such that $\Lambda(f)(y_i) = x_i$ for $i = 1, \ldots, m-1$. Now the proof is divided into two cases.

Case I. $m \leq k$. Since $dx_m = 0$, it is easy to see that there is a map $f' : X \to S^{2n-1}$ such that $\Lambda(f')(y_m) = x_m$, where y_m is a generator of the minimal model of S^{2n-1}. Then $f \times f' : X \to E \times S^{2n-1}$ is a desired map.

Case II. $k < m$. Since dx_m is in $\Lambda(x_1, \ldots, x_{m-1})$ and $\Lambda(y_1, \ldots, y_{m-1})$ is mapped isomorphically onto $\Lambda(x_1, \ldots, x_{m-1})$, $\Lambda(f)^{-1}(dx_m)$ is an element of $\Lambda(y_1, \ldots, y_{m-1})$. Regarding dx_m and $\Lambda(f)^{-1}(dx_m)$ as k-invariants of X and E, we consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X^{(2n-2)} \arrow[r, "dx_m"] \arrow[d, "f^{(2n-2)}"] & K(\mathbb{Q}, 2n) \arrow[d, equal] & \\
E \arrow[r] & E^{(2n-2)} \arrow[r, "\Lambda(f)^{-1}(dx_m)"] & K(\mathbb{Q}, 2n) \arrow[r] & \prod_{k=1}^{n} K(\mathbb{Q}, 2k) \simeq BU(n)_{(0)},
\end{tikzcd}$$

where $X^{(n)}$ denotes the Postnikov approximation of X through dimension n and the map $K(\mathbb{Q}, 2n) \to \prod_{k=1}^{n} K(\mathbb{Q}, 2k) \simeq BU(n)_{(0)}$ is the canonical inclusion map. Then similarly to the proof of Lemma 2.1, it is easy to prove that there is a map

$$g : E \to BU(n)$$

such that $\Lambda(f)\Lambda(g)(c_n) = cdx_m$ with $c \neq 0$ and that $g \circ f \simeq *$. Pulling back the fiber bundle $S^{2n-1} \to BU(n-1) \to BU(n)$ along the map g we have a fiber bundle

$$S^{2n-1} \to E' \xrightarrow{\pi} E$$

and a lift $\tilde{f} : X \to E'$ of f. By Lemma 2.2 and its proof, E' is a simply connected finite complex with property (S) whose minimal model, $(\Lambda(y_1, \ldots, y_m), d)$, is isomorphic to $(\Lambda(x_1, \ldots, x_m), d)$. Since

$$d\Lambda(\tilde{f})(y_m) = \Lambda(\tilde{f})(dy_m) = \Lambda(f)(\Lambda(f)^{-1}(dx_m)) = dx_m,$$

we have $d(\Lambda(\tilde{f})(y_m) - x_m) = 0$, that is, $\Lambda(\tilde{f})(y_m) - x_m \in \mathrm{Ker}[d : \Lambda^{2n-1}V \to \Lambda^{2n}V]$. By changing y_m to $y_m + a_\ell y_\ell + \cdots + a_{k-1}y_{k-1}$ with suitable $a_\ell, \ldots, a_{k-1} \in \mathbb{Q}$ we may assume that $z = \Lambda(\tilde{f})(y_m) - x_m$ is decomposable and that $dz = 0$. By dimensional reason $z \in \Lambda(x_1, \ldots, x_{\ell-1})$. Since $\Lambda(\tilde{f}) = \Lambda(f) : \Lambda(y_1, \ldots, y_{\ell-1}) \to \Lambda(x_1, \ldots, x_{\ell-1})$ maps $\Lambda(y_1, \ldots, y_{\ell-1})$ isomorphically onto $\Lambda(x_1, \ldots, x_{\ell-1})$, there is an element $z' \in \Lambda(y_1, \ldots, y_{\ell-1})$ such that $\Lambda(\tilde{f})(z') = z$. Then we have $\Lambda(\tilde{f})(y_m - z') = x_m$. Thus the minimal model $(\Lambda(y_1, \ldots, y_{m-1}, y_m - z'), d)$ is a desired model. □

Proof of Theorem 1.*B.* Let $(\Lambda V, d) = (\Lambda(x_1, x_2, \ldots), d)$ be the minimal model of X, where we assume that $|x_i| \leq |x_{i+1}|$ for all i. We also assume that $|x_i|$ is odd for all $1 \leq i \leq m$ and that $|x_{m+1}| = 2n$, where $m \geq 0$. Then there is a simply connected finite complex E_0 with property (S) whose minimal model, $(\Lambda(y_1, \ldots, y_m), d)$, is isomorphic to $(\Lambda(x_1, \ldots, x_m), d)$ and a map $f : X \to E_0$ such that $\Lambda(f)(y_i) = x_i$ for $i = 1, \ldots, m$ by Lemma 2.3. Since even dimensional rational homotopy groups of a bouquet of odd dimensional spheres are zero, there is a minimal model $(\Lambda(y_1, \ldots, y_m, z_1, \ldots, z_s), d)$ which represents $S^{|x_1|} \vee \cdots \vee S^{|x_m|}$ up to dimension $2n$, where the degree of z_i is odd for all $1 \leq i \leq s$. By Lemma 2.2 there is a tower of fibrations

$$E_s \to \cdots \to E_k \to \cdots \to E_0,$$

such that the minimal model of E_k is isomorphic to $(\Lambda(y_1, \ldots, y_m, z_1, \ldots, z_k), d)$ and that $\Omega E_{k+1} \to \Omega E_k$ has a cross section for each k. By pulling back this tower of fibrations along the map $f : X \to E_0$ we have a tower of fibrations

$$X_s \to \cdots \to X_k \to \cdots \to X_0 = X.$$

Then the minimal model of X_s is isomorphic to that of a wedge of appropriate odd dimensional spheres up to dimension $2n$ except indecomposable elements of dimension $2n$. Therefore in the minimal model of X_s there is a decomposable element w such that $d(x_{m+1} + w) = 0$. By Lemma 2.1 there is a simply connected finite complex Y and a rational equivalence

$$\Omega X_s \to \Omega Y \times S^{2n-1}.$$

Let $(\Lambda W, d)$ be the minimal model of Y, then $\dim W^{2n} = \dim V^{2n} - 1$. By construction there is a map

$$\Omega X \to \Omega X_s$$

which induces monomorphisms of rational homotopy groups. Thus there is a map

$$\Omega X \to \Omega Y \times S^{2n-1}$$

which induces monomorphisms of rational homotopy groups. For any positive integer N, iterating this procedure we have a map

$$\Omega X \to \Omega X' \times \prod_{n_\alpha \leq N} S^{2n_\alpha+1}$$

such that it induces monomorphisms of rational homotopy groups and that the even dimensional rational homotopy groups of X' are zero up to dimension $2N+2$. For X' there is a map

$$\Omega X' \to \prod_{n_\beta \leq N} \Omega S^{2n_\beta+1}$$

which induces isomorphisms of rational homotopy groups up to dimension $2N+1$ by Lemma 2.3. Thus we have a map

$$\Omega X \to \prod_{n_\alpha \leq N} S^{2n_\alpha+1} \times \prod_{n_\beta \leq N} \Omega S^{2n_\beta+1}$$

which induces monomorphisms of rational homotopy groups up to dimension $2N+1$. Removing redundant factors in the right-hand side, we have a map

$$\Omega X \to \prod_{n_a} S^{2n_a+1} \times \prod_{n_b} \Omega S^{2n_b+1}$$

which induces isomorphisms of rational homotopy groups up to dimension $2N+1$. Since N is arbitrary, the proof of Theorem 1.B is completed. $\square$

Proof of Corollary 1.*C*. Let $\tilde{X}$ be the universal covering space of X. Since the fundamental group of X is finite, $\tilde{X}$ has also the structure of a finite complex. By Theorem 1.A there exist no essential phantom maps from $\Omega X \simeq \pi_0(X) \times \Omega \tilde{X}$ to a target of finite type. $\square$

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