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On characterizations of general helices for ruled surfaces in the pseudo-Galilean space G_3^1-(Part-I)

By

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Abstract

T. Ikawa obtained in [5] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \qquad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve α on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacısalihoğlu generalized in [4] T. Ikawa's this result, i.e., k and τ are variable, but $\frac{k}{\tau}$ is constant.

In [1] H. Balgetir, M. Bektaş and M. Ergüt obtained a geometric characterization of null Frenet curve with constant ratio of curvature and torsion (called null general helix).

In this paper, making use of method in [1, 4, 5] , we obtained characterizations of a curve with respect to the Frenet frame of ruled surfaces in the 3-dimensional pseudo-Galilean space G_3^1.

1. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0, 0, +, -)$). The absolute figure of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the a plane in three-dimensional real projective space $P_3(\mathbf{R})$ (the absolute plane), f is a line in w (the absolute line) and I is the fixed hyperbolic involution of points of f ([2]).

A vector $X(x, y, z)$ is said to non isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: space-like ($y^2 - z^2 > 0$), time-like ($y^2 - z^2 < 0$) and two types of lightlike ($y = \pm z$) vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

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A trihedron $(T_o;\, e_1, e_2, e_3)$ with a proper origin

$$T_o(x_o, y_o, z_o) \sim (1 : x_o : y_o : z_o)$$

is orthonormal in pseudo-Galilean sense iff the vectors e_1, e_2, e_3 are of following form: $e_1 = (1, y_1, z_1)$, $e_2 = (0, y_2, z_2)$, $e_3 = (0, \varepsilon z_2, \varepsilon y_2)$ with $y_2^2 - z_2^2 = \delta$, where ε, δ is $+1$ or -1.

Such trihedron $(T_o; e_1, e_2, e_3)$ is called positively oriented if for its vectors $\det(e_1, e_2, e_3) = 1$ holds, i.e., if $y_2^2 - z_2^2 = \varepsilon$.

2. Ruled surfaces in the Galilean space

A general equation of a ruled surface G_3^1 is

$$x(u, v) = r(u) + va(u),\ v \in \mathbf{IR};\ r, a \in \mathbf{C}^3, \tag{2.1}$$

where the curve r does not line in a pseudo-Euclidean plane and is called a directix. The curve r is given by

$$r(u) = (u, y(u), z(u)). \tag{2.2}$$

This means that the curve r is parametrized by the pseudo-Galilean arc length. Further, the generator vector field is of the form

$$a(u) = (1, a_2(u), a_3(u)). \tag{2.3}$$

Notice that under the given assumptions all tangent planes of ruled surfaces are isotropic.

According to the absolute figure, we distinguish two types of ruled surfaces in G_3^1. More about ruled surface in G_3^1 can be found in [3].

Type I: The equation of a ruled surface of type I in G_3^1 is

$$\begin{cases} x(u, v) = (u, y(u), z(u)) + v(1, a_2(u), a_3(u)), \\ y, z, a_2, a_3 \in C^3, \quad u \in \mathbf{I} \subseteq \mathbf{IR}, \quad v \in \mathbf{IR}. \end{cases} \tag{2.4}$$

The ruled surfaces of type I are non-conoidal and conoidal surfaces whose directional straight line at infinity is not the absolute line. The striction curve of these surfaces does not lie in a pseudo-Euclidean plane.

The associated trihedron of a ruled surface of type I in G_3^1 is defined by

$$T(u) = a(u), \quad N(u) = \frac{1}{k(u)} a'(u), \quad B(u) = \frac{1}{k(u)} (0, a_3'(u), a_2'(u)).$$

The curvature is given by $k(u) = \sqrt{|a_2'^2 - a_3'^2|}$.

Type II: The equation of ruled surface of type II in G_3^1 is

$$\begin{cases} x(u, v) = (0, y(u), z(u)) + v(1, a_2(u), a_3(u)), \\ y, z, a_2, a_3 \in C^3, \quad u \in \mathbf{I} \subseteq \mathbf{IR}, \quad v \in \mathbf{IR}, \\ \left|y'^2 - z'^2\right| = 1, \quad y' a_2' - z' a_3' = 0. \end{cases} \tag{2.5}$$

A ruled surface of type II is a surface whose striction curve lies in a pseudo-Euclidean plane.

The associated trihedron of ruled surface of type II in G_3^1 is defined by

$$\begin{aligned} T(u) &= a(u) = (1, a_2(u), a_3(u)), \\ N(u) &= (0, z'(u), y'(u)), \\ B(u) &= (0, y'(u), z'(u)), \end{aligned}$$

where

$$k(u) = \frac{a_2(u)}{z'(u)}, \qquad \tau(u) = \frac{y''(u)}{z'(u)}.$$

The Frenet's formulas are in type I or type II as follows.

$$\begin{aligned} \nabla_{T(u)} T(u) &= k(u) N(u), \\ \nabla_{T(u)} N(u) &= \tau(u) B(u), \\ \nabla_{T(u)} B(u) &= \tau(u) N(u). \end{aligned} \tag{2.6}$$

The function

$$\tau(u) = \frac{\det(a(u), a'(u), a''(u))}{k^2(u)}$$

is called the torsion of ruled surfaces.

3. The characterizations of curves on ruled surfaces

Definition 3.1. Let α be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along α. If both k and τ are positive constants along α, then α is called a circular helix with respect to the Frenet frame.

Definition 3.2. Let α be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along α. A curve α such that

$$\frac{k(u)}{\tau(u)} = \text{const}$$

is called a general helix with respect to Frenet frame.

Theorem 3.1. *Let α be a curve of a ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$ if and only if*

$$\nabla_{T(u)} \nabla_{T(u)} \nabla_{T(u)} T(u) - K(u) \nabla_{T(u)} T(u) = 3k'(u) \nabla_{T(u)} N(u), \tag{3.1}$$

where $K(u) = \frac{k''(u)}{k(u)} + \tau^2(u)$.

Proof. Suppose that α is general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$. Then from (2.6), we have

(3.2)
$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) = (k''(u)+k(u)\tau(u))N(u)+(2k'(u)\tau(u)+k(u)\tau'(u))B(u).$$

Now, since α is general helix with respect to the Frenet Frame

$$\frac{k(u)}{\tau(u)} = \text{const}$$

and this upon the derivation gives rise to

$$k'(u)\tau(u) = k(u)\tau'(u). \tag{3.3}$$

If we substitute the equations (3.3),

$$N(u) = \frac{1}{k(u)}\nabla_{T(u)}T(u), \tag{3.4}$$

and

$$B(u) = \frac{1}{\tau(u)}\nabla_{T(u)}N(u) \tag{3.5}$$

in (3.2), we obtain (3.1).

Conversely let us assume that the equation (3.1) holds. We show that the curve α is a general helix. Differentiating covariantly (3.4) we obtain

$$\nabla_{T(u)}N(u) = -\frac{k'(u)}{k^2(u)}\nabla_{T(u)}T(u) + \frac{1}{k(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) \tag{3.6}$$

and so

$$\begin{aligned}\nabla_{T(u)}\nabla_{T(u)}N(u) = {} & \left(-\frac{k'(u)}{k^2(u)}\right)'\nabla_{T(u)}T(u) - 2\frac{k'(u)}{k^2(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) \\ & + \frac{1}{k(u)}\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u).\end{aligned} \tag{3.7}$$

If we use (3.1) in (3.7) and make some calculations, we have

$$\begin{aligned}\nabla_{T(u)}\nabla_{T(u)}N(u) = {} & \left[\left(-\frac{k'(u)}{k^2(u)}\right)' + \frac{K(u)}{k(u)}\right]\nabla_{T(u)}T(u) - 2\frac{k'(u)}{k^2(u)}N(u) \\ & + \frac{k'(u)\tau(u)}{k(u)}B(u).\end{aligned} \tag{3.8}$$

Also we obtain

$$\nabla_{T(u)}\nabla_{T(u)}N(u) = \tau^2(u)N(u) + \tau'(u)B(u) \tag{3.9}$$

since (3.8) and (3.9) are equal, routine calculations show that α is a general helix. $\square$

Theorem 3.2. *Let α be a curve of ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$ if and only if*

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) - K(u)\nabla_{T(u)}T(u) = 3\lambda\tau'(u)\nabla_{T(u)}N(u), \tag{3.10}$$

where $K(u) = \frac{k''(u)}{k(u)} + \tau^2(u)$ and $\lambda = \frac{k(u)}{\tau(u)} = \mathrm{const}$.

Proof. It is similar to the proof of Theorem 3.1. □

Corollary 3.1. *Let α be a curve of ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a circular helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$ if and only if*

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) = \tau^2(u)\nabla_{T(u)}T(u). \tag{3.11}$$

Proof. From the hypotesis of corollary 3.1 and since α is a circular helix, we can show easily (3.11). □

Theorem 3.3. *If α be a curve of ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$, then*

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) - \tilde{K}(u)\nabla_{T(u)}B(u) = 3k'(u)\nabla_{T(u)}N(u) \tag{3.12}$$

where $\tilde{K}(u) = \frac{k''(u)}{\tau(u)} + k(u)\tau(u)$.

Proof. Suppose that α is a general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$. Then from (3.2) and (3.3)

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) = (k''(u) + k(u)\tau^2(u))N(u) + (3k'(u)\tau(u))B(u). \tag{3.13}$$

If we substitute the equations

$$N(u) = \frac{1}{\tau(u)}\nabla_{T(u)}B(u) \tag{3.14}$$

and (3.5) in (3.13), we obtain (3.12). □

Theorem 3.4. *If α be a curve of ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a general helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$, then*

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) - \tilde{K}(u)\nabla_{T(u)}B(u) = 3\lambda\tau'(u)\nabla_{T(u)}N(u), \tag{3.15}$$

where $\tilde{K}(u) = \frac{k''(u)}{\tau(u)} + k(u)\tau(u)$ and $\lambda = \frac{k(u)}{\tau(u)} = \mathrm{const}$.

Proof. It is similar to the proof of Theorem 3.3. □

Corollary 3.2. *Let α be a curve of ruled surface of type* I *or* II *in pseudo-Galilean space G_3^1. α is a circular helix with respect to the Frenet frame $\{T(u), N(u), B(u)\}$ if and only if*

$$\nabla_{T(u)}\nabla_{T(u)}\nabla_{T(u)}T(u) = k(u)\tau(u)\nabla_{T(u)}B(u). \tag{3.16}$$

Proof. From the hypotesis of Corollary 3.2 and since α is a circular helix, we can show easily (3.16). □

4. Example

In this section we give a example helix of ruled surfaces in 3-dimensional pseudo-Galilean space G_3^1 which it can be parametrized by

$$x(u,v) = \left(u, a\mathrm{sh}\frac{u}{p}, a\mathrm{ch}\frac{u}{p}\right) + v\left(1, \frac{1}{b}a\mathrm{ch}\frac{u}{p}, \frac{1}{b}\mathrm{sh}\frac{u}{p}\right).$$

This is a ruled surfaces of type I which is obtained by revolving and simultaneously moving along the axis x with the constant speed a straight line $x = bv$, $y = v$, $z = a$ (see [2]). Its curvature and torsion are defined as the following, respectively;

$$k(u) = \frac{1}{p} = \text{const.}, \quad \tau(u) = -\frac{b}{p}\text{const.}$$

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