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Spectra of deranged Cantor set by weak local dimensions

By

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Abstract

We decompose the most generalized Cantor set into a spectral class using weak lower (upper) local dimension. Each member of the spectral class is related to a quasi-self-similar measure, so the information of its Hausdorff (packing) dimension can be obtained. In the end, we give an example of the Cantor set having countable members composing the spectral class.

1. Introduction

Many authors ([9], [11]) studied the multi-fractals of an irregular set in Euclidean space using some measure. In particular, they used a self-similar measure to analyze a self-similar Cantor set. The self-similar Cantor set is decomposed into a spectral class from the measure and its lower (upper) local dimensions. Using the strong law of large numbers, we can relate a member of the spectral class from the local dimensions of the self-similar measure with a distribution set ([7], [10]), which means that the spectral class by the self-similar measure and its local dimensions is in fact the union of the distribution sets. So a self-similar Cantor set has a spectral class of distribution sets. When we consider a deranged Cantor set which is the most generalized Cantor set, its spectral class by a measure and its local dimensions is hard to analyze and so is to get the information of dimensions of the members of the spectral class. Recently we ([2]) attempted such trial to find a spectral class using a quasi-local dimension, which we call a weak local dimension which is a dimension of a perturbed Cantor set ([1]) in local sense. We note that we got a spectral class of a deranged Cantor set using weak local dimensions while Olsen or Falconer did a spectral class of a self-similar set using a self-similar measure and its local dimensions. In our case, we just considered only a weak local dimension, a united concept of measure and local dimension like the distribution set. In [2], we positively conjectured that in a spectral class of the deranged Cantor set weak local dimension is related to the local dimension of a natural

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measure with respect to the weak local dimension. In this paper, we show that such conjecture is right. We note that the natural measure is a quasi-self-similar measure in the sense that it is a self-similar measure on a self-similar Cantor set. From the relationship of the quasi-self-similar measure and the subset composing the spectral class by weak local dimension, we obtain some information of the dimensions of the member of the spectral class. As a result, we have an interesting fact that a perturbed Cantor set ([1]) which is regular in the sense that its Hausdorff and packing dimensions coincide has a natural measure which has an exact dimension, and a non-regular perturbed Cantor set has two natural measures which have a lower exact dimension and an upper exact dimension respectively without the assumption of Cutler ([8]) of positive exact lower dimensional Hausdorff measure or positive exact upper dimensional packing measure. We will prove it using weak local measures ([2]).

2. Preliminaries

We recall the definition of the deranged Cantor set ([2]). Let $I_\phi = [0,1]$. Then we obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ by deleting the middle open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \ldots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid = c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$,where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^{\infty} E_n$ a deranged Cantor set. If $x \in I_\tau$ where $\tau \in \{1,2\}^n$, then $c_n(x)$ denotes I_τ for each $n = 0, 1, 2, \ldots$.

We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbf{N}}$ such that $\bigcap_{k=0}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \ldots, i_k$ where $\sigma = i_1, i_2, \ldots, i_k, i_{k+1}, \ldots$). Hereafter, we use $\sigma \in \{1,2\}^{\mathbf{N}}$ and $x \in F$ as the same identity freely.

We ([2]) recall the local Hausdorff dimension $f(\sigma)$ of σ in F

$$f(\sigma) = \inf\{s > 0 : h^s(\sigma) = 0\} = \sup\{s > 0 : h^s(\sigma) = \infty\}$$

where the s-dimensional local Hausdorff measure or the s-dimensional weak lower local measure of σ

$$h^s(\sigma) = \liminf_{k\to\infty}(c_1^s + c_2^s)(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s)(c_{\sigma|2,1}^s + c_{\sigma|2,2}^s)\cdots(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s),$$

and dually the local packing dimension $g(\sigma)$ of σ in F

$$g(\sigma) = \inf\{s > 0 : q^s(\sigma) = 0\} = \sup\{s > 0 : q^s(\sigma) = \infty\}$$

where the s-dimensional local packing measure or the s-dimensional weak upper local measure of σ

$$q^s(\sigma) = \limsup_{k\to\infty}(c_1^s + c_2^s)(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s)(c_{\sigma|2,1}^s + c_{\sigma|2,2}^s)\cdots(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s).$$

We call the local Hausdorff (packing) dimension of σ in F as the weak lower (upper) local dimension of σ in F compared with a lower (upper) local dimension of σ in F with respect to some mass distribution.

We recall the s-dimensional Hausdorff measure of F:

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} \mid U_n \mid^s \colon \{U_n\}_{n=1}^{\infty}$ is a δ-cover of $F\}$, and the Hausdorff dimension ([9]) of F:

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\} (= \inf\{s > 0 : H^s(F) = 0\}).$$

Also we recall the s-dimensional packing measure of F:

$$p^s(F) = \inf\left\{\sum_{n=1}^{\infty} P^s(F_n) : \bigcup_{n=1}^{\infty} F_n = F\right\},$$

where $P^s(E) = \lim_{\delta\to 0} P^s_\delta(E)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^{\infty} \mid U_n \mid^s \colon \{U_n\}_{n=1}^{\infty}$ is a δ-packing of E $\}$, and the packing dimension ([9]) of F:

$$\dim_p(F) = \sup\{s > 0 : p^s(F) = \infty\} (= \inf\{s > 0 : p^s(F) = 0\}).$$

We note that a deranged Cantor set satisfying $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$, for each $n = 0,1,2,\ldots$ is called a perturbed Cantor set ([1]).

We recall the lower and upper local dimension of a Borel probability measure μ at x are given by $\underline{\dim}_{loc}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$ and $\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$ where $B_r(x)$ is the closed ball with center x and radius $r > 0$ ([9]). We also recall that a measure μ has exact lower (upper) dimension s if $\underline{\dim}_{loc}\mu(x) = s$ ($\overline{\dim}_{loc}\mu(x) = s$) for μ-almost all x ([9]).

We are now ready to study the ratio geometry of the deranged Cantor set.

3. Main results

In this section, F means a deranged Cantor set determined by $\{c_\tau\}$ with $\tau \in \{1,2\}^n$ where $n = 1,2,\ldots$. Hereafter we only consider a deranged Cantor set whose contraction ratios $\{c_\tau\}$ and gap ratios $\{d_\tau (= 1 - (c_{\tau,1} + c_{\tau,2}))\}$ are uniformly bounded away from 0.

Lemma 3.1. *Given a Borel probability measure μ on F, for all $x \in F$,*

$$\liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = \liminf_{n\to\infty} \frac{\log \mu(c_n(x))}{\log |c_n(x)|}$$

and

$$\limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = \limsup_{n\to\infty} \frac{\log \mu(c_n(x))}{\log |c_n(x)|}.$$

Proof. It is obvious from the uniform boundedness of $\{c_\tau\}$ and $\{d_\tau\}$ away from 0. □

Theorem 3.1. *Let μ_s be the Borel probability measure on F satisfying*

$$\mu_s(I_\tau) = \frac{|I_\tau|^s}{(c_1^s + c_2^s)(c_{i_1,1}^s + c_{i_1,2}^s)\dots(c_{i_1,i_2,\dots,i_{n-1},1}^s + c_{i_1,i_2,\dots,i_{n-1},2}^s)}$$

for each $\tau = i_1, i_2, \dots, i_{n-1}, i_n$, where $i_j \in \{1,2\}$ for $1 \le j \le n$ and $n \in \mathbf{N}$.

We have for $s > 0$

(1) *if $h^s(\sigma) > 0$, then $\underline{\dim}_{loc}\mu_s(x) \ge s$,*
(2) *if $q^s(\sigma) > 0$, then $\overline{\dim}_{loc}\mu_s(x) \ge s$,*
(3) *if $h^s(\sigma) < \infty$, then $\underline{\dim}_{loc}\mu_s(x) \le s$,*
(4) *if $q^s(\sigma) < \infty$, then $\overline{\dim}_{loc}\mu_s(x) \le s$.*

Proof. If $h^s(\sigma) > 0$, then $\prod_{k=0}^{n-1}(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s) \ge A$ for all $n \in \mathbf{N}$ and some $A > 0$. Then we have

$$\liminf_{n\to\infty} \frac{\log \mu_s(c_n(x))}{\log |c_n(x)|} = s - \limsup_{n\to\infty} \frac{\log \prod_{k=0}^{n-1}(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s)}{\log |c_n(x)|} \ge s.$$

Therefore (1) follows from Lemma 3.1. The similar arguments give (2).

If $h^s(\sigma) < \infty$, then $\prod_{k=0}^{n-1}(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s) \le B$ for infinitely many $n \in \mathbf{N}$ and some $B < \infty$. Then we have

$$\liminf_{n\to\infty} \frac{\log \mu_s(c_n(x))}{\log |c_n(x)|} = s - \limsup_{n\to\infty} \frac{\log \prod_{k=0}^{n-1}(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s)}{\log |c_n(x)|} \le s.$$

Therefore (3) follows from Lemma 3.1. The similar arguments give (4). ☐

Remark 1. The Borel probability measure in the above Theorem is called *a quasi-self-similar measure* ([5]) on F since it turns out to be a self-similar measure on F if F is a self-similar Cantor set.

Lemma 3.2. *Fix $x \in F$. Then $\underline{\dim}_{loc}\mu_s(x)$ is a continuous function for $s > 0$. Similarly $\overline{\dim}_{loc}\mu_s(x)$ is a continuous function for $s > 0$.*

Proof. Fix $x = \sigma \in \{1,2\}^{\mathbf{N}}$. Let $\delta_n(s) = \frac{\log \prod_{k=0}^{n-1}(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s)}{\log |c_n(x)|}$. Clearly for each $s > 0$, $\{\delta_n(s)\}$ are bounded for all $n \in \mathbf{N}$. We note that contraction ratios are uniformly bounded away from 0, which means that there exist B_1 and B_2 such that $0 < B_1 \le c_{\sigma|k-1,1}, c_{\sigma|k-1,2} \le B_2 < 1$ for all $k \in \mathbf{N}$. From the mean value theorem we easily see that $|\frac{c_{\sigma|k-1,1}^s + c_{\sigma|k-1,2}^s}{c_{\sigma|k-1,1}^t + c_{\sigma|k-1,2}^t} - 1| \le \frac{|\log B_1|}{B_1}|s-t|$ for all $k \in \mathbf{N}$. Hence

$$|\delta_n(s) - \delta_n(t)| \le \frac{K|s-t|}{|\log B_2|}$$

for all $n \in \mathbf{N}$ where $0 < K < \infty$ which is from B_1 and independent of n. Putting $\frac{K}{|\log B_2|} = C$, we have $|\delta_n(s) - \delta_n(t)| \le C|s-t|$ all $n \in \mathbf{N}$. Writing $\delta(s) = \limsup_{n\to\infty} \delta_n(s)$ for every $s > 0$, we only need to show that $\delta(s)$ is continuous for $s > 0$. Fix $s > 0$ and suppose that $\lim_{t\to s} \delta(t) \ne \delta(s)$.

Then there is $\epsilon > 0$ and a sequence $\{t_m\}$ of positive real numbers such that $t_m \to s$ satisfying $\delta(t_m) > \delta(s) + \epsilon$ or $\delta(t_m) < \delta(s) - \epsilon$. Consider m satisfying $C|t_m - s| < \frac{\epsilon}{3}$. Then $|\delta_n(t_m) - \delta_n(s)| < \frac{\epsilon}{3}$ for all $n \in \mathbf{N}$.

Suppose that $\delta(t_m) > \delta(s) + \epsilon$. There is a sequence $\{m_k\}$ of natural numbers such that $\delta_{m_k}(t_m) \to \delta(t_m)$ and $|\delta_{m_k}(t_m) - \delta_{m_k}(s)| < \frac{\epsilon}{3}$ for all m_k. We have a contradiction since $\limsup_{k\to\infty} \delta_{m_k}(s) \geq \delta(s) + \frac{2\epsilon}{3}$.

Now assume that $\delta(t_m) < \delta(s) - \epsilon$. There is a natural number N_m such that $\delta_n(t_m) < \delta(s) - \epsilon$ for all $n \geq N_m$ and $|\delta_n(t_m) - \delta_n(s)| < \frac{\epsilon}{3}$ for such n. We have a contradiction since $\limsup_{n\to\infty} \delta_n(s) \leq \delta(s) - \frac{2\epsilon}{3}$. Similarly $\liminf_{n\to\infty} \delta_n(s)$ is also a continuous function for s. □

Theorem 3.2. *$\underline{\dim}_{loc}\mu_{f(\sigma)}(x) = f(\sigma)$ and $\overline{\dim}_{loc}\mu_{g(\sigma)}(x) = g(\sigma)$ for every $\sigma \in \{1,2\}^{\mathbf{N}}$.*

Proof. If $s < f(\sigma)$, then $h^s(\sigma) > 0$. By Theorem 3.1, $\underline{\dim}_{loc}\mu_s(x) \geq s$. If $s > f(\sigma)$, then $h^s(\sigma) < \infty$. By Theorem 3.1, $\underline{\dim}_{loc}\mu_s(x) \leq s$. It follows from the intermediate value theorem since $\underline{\dim}_{loc}\mu_s(x)$ is a continuous function for s for fixed $x \in F$ by the above Lemma. Similar arguments hold for g. □

Now we ([2]) can think of a multifractal structure E_s, G_s on F using weak local dimensions,

$$E_s = \{\sigma \in F : f(\sigma) = s\},$$

$$G_s = \{\sigma \in F : g(\sigma) = s\}.$$

Then F is classified as $F = \bigcup_{0<s<1} E_s$ and $F = \bigcup_{0<s<1} G_s$. From the above Theorem, we get the relation between $E_s(G_s)$ and the set having lower (upper) local dimension s of μ_s.

Corollary 3.1. *$E_s = \{x \in F : \underline{\dim}_{loc}\mu_s(x) = s\}$ and $G_s = \{x \in F : \overline{\dim}_{loc}\mu_s(x) = s\}$ for every $s \in (0,1)$.*

Proof. It is immediate from the above Theorem. □

Corollary 3.2. *Let F be a perturbed Cantor set. Then there exist s_1 and s_2 such that $f(\sigma) = s_1$ and $g(\sigma) = s_2$ for all $\sigma \in \{1,2\}^{\mathbf{N}}$. Further μ_{s_1} has exact lower dimension s_1 which is the Hausdorff dimension of F, and μ_{s_2} has exact upper dimension s_2 which is the packing dimension of F.*

Proof. It is immediate from the definitions. □

To get informations of the dimensions of E ($\subset \mathbf{R}$) we need the following Proposition.

Proposition 3.1 ([9]). *Let $E \subset \mathbf{R}$ be a Borel set and let μ be a finite measure.*

(a) *If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$ then $\dim_H(E) \geq s$.*
(b) *If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$ then $\dim_H(E) \leq s$.*
(c) *If $\overline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$ then $\dim_p(E) \geq s$.*
(d) *If $\overline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$ then $\dim_p(E) \leq s$.*

Theorem 3.3.

$$\inf_{\sigma\in\{1,2\}^{\mathbf{N}}} f(\sigma) \le \dim_H(F) \le \sup_{\sigma\in\{1,2\}^{\mathbf{N}}} f(\sigma)$$

and

$$\inf_{\sigma\in\{1,2\}^{\mathbf{N}}} g(\sigma) \le \dim_p(F) \le \sup_{\sigma\in\{1,2\}^{\mathbf{N}}} g(\sigma).$$

Proof. If $s < \inf_{\sigma\in\{1,2\}^{\mathbf{N}}} f(\sigma)$, then $s < f(\sigma)$ for all $\sigma \in \{1,2\}^{\mathbf{N}}$. By Theorem 3.1, $\underline{\dim}_{loc}\mu_s(x) \ge s$ for all $x \in F$. Since $\mu_s(F) = 1 > 0$, by the above Proposition, $\dim_H(F) \ge s$. If $s > \sup_{\sigma\in\{1,2\}^{\mathbf{N}}} f(\sigma)$, then $s > f(\sigma)$ for all $\sigma \in \{1,2\}^{\mathbf{N}}$. By Theorem 3.1, $\underline{\dim}_{loc}\mu_s(x) \le s$ for all $x \in F$, which gives $\dim_H(F) \le s$ by the above Proposition. Similar arguments hold for packing case. □

We have a better estimation of dimensions of a deranged Cantor set from the followings.

Theorem 3.4. *If* $\mu_s(\{x : f(\sigma) \ge s\}) > 0$ *for some* $s > 0$ *then* $\dim_H(\{x : f(\sigma) \ge s\}) \ge s$. *Similarly if* $\mu_s(\{x : g(\sigma) \ge s\}) > 0$ *for some* $s > 0$ *then* $\dim_p(\{x : g(\sigma) \ge s\}) \ge s$.

Proof. By Theorem 3.2, $\underline{\dim}_{loc}\mu_s(x) \ge s$ for $f(\sigma) = s$ since $\underline{\dim}_{loc}\mu_s(x) = s$. By Theorem 3.1, if $f(\sigma) > s$ then $\underline{\dim}_{loc}\mu_s(x) \ge s$. Hence $\dim_H(\{x : f(\sigma) \ge s\}) \ge s$ by the above Proposition. Similarly it holds for packing case. □

Corollary 3.3.

$$\dim_H(F) \ge \sup\{s > 0 : \mu_s(\{x : f(\sigma) \ge s\}) > 0\},$$

and

$$\dim_p(F) \ge \sup\{s > 0 : \mu_s(\{x : g(\sigma) \ge s\}) > 0\}.$$

Proof. It is immediate from the above Theorem. □

Remark 2. Perturbed Cantor set has a measure which has exact lower dimension of its Hausdorff dimension and exact upper dimension of its packing dimension without Cutler's assumption ([8]) of positive exact lower dimensional Hausdorff measure or positive exact upper dimensional packing measure. A regular perturbed Cantor set has a measure having an exact dimension of its Hausdorff and packing dimension (cf. [4]).

Example 3.1. Consider a deranged Cantor set with $c_{1,\tau,1} = a_{n+2}$ and $c_{1,\tau,2} = b_{n+2}$ and $c_{2,1,\tau,1} = a'_{n+3}$ and $c_{2,1,\tau,2} = b'_{n+3}$ and $c_{2,2,1,\tau,1} = a''_{n+4}$ and $c_{2,2,1,\tau,2} = b''_{n+4}, \ldots,$ for all $\tau \in \{1,2\}^n$, for each $n = 0, 1, 2, \ldots$. Then we have at most countable disjoint non-empty E_s whose Hausdorff dimension is s and G_s whose packing dimension is s. Clearly $\dim_H(F) = \sup\{s : E_s \ne \phi\}$ and

$\dim_p(F) = \sup\{s : G_s \neq \phi\}$ from Theorem 3.3 and Corollary 3.3. We note that if there is only one E_{s_1}, which means $E_{s_1} = F$, then the Hausdorff dimension of F is s_1. Similarly if there is only one G_{s_2}, which means $G_{s_2} = F$, then the packing dimension of F is s_2. But such an example of a deranged Cantor set is quite different from the perturbed Cantor set.

Remark 3. $f(\sigma) = \liminf_{n\to\infty} y_{\sigma|n}$ and $g(\sigma) = \limsup_{n\to\infty} y_{\sigma|n}$ where

$$\prod_{i=0}^{n}(c_{\sigma|i,1}^{y_{\sigma|n}} + c_{\sigma|i,2}^{y_{\sigma|n}}) = 1 \text{ ([6]).}$$

Remark 4. If a deranged Cantor set is given, naturally all the uncountable elements have their own weak lower(upper) local dimensions. If $\mu_s(E_s) > 0$ ($\mu_s(G_s) > 0$), then $\dim_H(E_s) = s$ ($\dim_p(G_s) = s$). However if $\mu_s(E_s) = 0$ ($\mu_s(G_s) = 0$), then we get no information of its dimension except for the fact that $\dim_H(E_s) \le s$ ($\dim_p(G_s) \le s$).

Remark 5. We conjecture that the theorems above hold for the deranged Cantor set without the uniform boundedness conditions of contraction ratios $\{c_\tau\}$ away from 0 (cf. [3]).

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