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# Differential equations for Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$

By

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**Abstract**

We construct a system of non-linear differential equations from the uniformizing differential equations of an orbifold attached to certain Hilbert modular surface. Generic solutions of this system can be given by the logarithmic derivatives of Hilbert modular forms.

## 1. Introduction

The theory of modular forms has a long history. It relates to many branches of mathematics. In this paper, we shall study modular forms from an analytic view points. Namely we shall construct a holonomic system of nonlinear differential equations which characterize symmetric Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$.

It is known that modular forms of one complex variable satisfy algebraic ordinary differential equations. Algebraic differential equations are convenient to study analytic properties of their solutions. Therefore, it is an interesting problem to find algebraic differential equations satisfied by modular forms. The first successful attempt was done by Jacobi, who gave a differential equation for theta constants of genus one ([5]).

For certain modular forms, their logarithmic derivatives satisfy a system of differential equations. Halphen first found such a system ([2]) and his method was analyzed and generalized by several authors ([9], [3], [7]). Note that Halphen's systems is equivalent to Jacobi's equation.

For modular forms of several variables, M. Sato showed that logarithmic derivatives of theta constants of genus two satisfy a holonomic system of partial differential equations ([12], [8]).

In the present paper we shall construct a holonomic system of partial differential equations satisfied by logarithmic derivatives of Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$. There are several approaches to find such holonomic systems. Sato obtained his results by using differential relations of theta constants and the heat equation satisfied by theta functions. Here, modular forms are regarded as functions on the moduli spaces of certain geometric objects. The second

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approach is the one to use differential operators preserving modular properties and the algebraic structure of the graded ring of modular forms.

In the present paper we shall use another method. An important feature of modular forms is that they are obtained as the inverse functions of solutions of certain linear differential equations. We shall use such differential equations to obtain a holonomic system which characterizes modular forms. This method was initiated by Jacobi. In the present paper we shall use the uniformizing differential equations of Hilbert modular orbifold obtained by Sasaki and Yoshida ([11]). From the differential equations we shall deduce the holonomic system satisfied by the logarithmic derivatives of Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$ (Theorem 4.1). The holonomic system characterizes the Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$. Namely, generic solutions of the holonomic system are given by logarithmic derivatives of the Hilbert modular forms (for details, see Theorem 5.1 below).

Let us describe briefly the content of the present paper. In Section 2 we shall discuss the structure of the ring of Hilbert modular forms for a certain subgroup $\Gamma$ of the Hilbert modular group $SL(2,\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of $\mathbb{Q}(\sqrt{2})$. It is an important fact due to Hirzebruch that the subring consisting of the symmetric Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$ is isomorphic to the polynomial ring of three variables. In Section 3 we shall recall the result due to Sasaki and Yoshida ([11]). They described the inverse map of $\mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\mathbb{H}/\langle\Gamma,\tau\rangle$ by a system of linear differential equations of two variables with rank four, where $\mathbb{H}$ is the upper half plane in the complex plane and $\tau$ is the involution of interchanging the factors of $\mathbb{H}\times\mathbb{H}$.

Section 4 is the main part of the present paper. We shall construct a system of nonlinear differential equations for logarithmic derivatives of Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$ by using Sasaki-Yoshida's equations (Theorem 4.1). In Section 5 we shall show that generic solutions of our system of differential equations will be given in terms of the logarithmic derivatives of symmetric Hilbert modular forms of $\mathbb{Q}(\sqrt{2})$ (Theorem 5.1). Also we shall describe degenerate solutions of the system.

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## 2. Ring structure of Hilbert modular forms

Let $K$ be the real quadratic field $\mathbb{Q}(\sqrt{2})$ and $\mathcal{O}$ be the ring of integers in $K$. We consider the principal congruence subgroup $\Gamma(2)$ of $SL(2,\mathcal{O})$ for the ideal (2) in $\mathcal{O}$:

$$\Gamma(2)=\left\{\gamma\in SL(2,\mathcal{O});\gamma\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\mod (2)\right\}.$$

The group $SL(2,\mathcal{O})/\Gamma(2)$ is an extension of the symmetric group $S_4$ by the group of order 2 which is the center of $SL(2,\mathcal{O})/\Gamma(2)$. The fundamental unit of $\mathcal{O}$ is $\varepsilon_0=1+\sqrt{2}$. Then the non-trivial element in the center is represented by the matrix

$$\begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} = D_{\varepsilon_0}.$$

Let $\Gamma$ be the subgroup of $SL(2, \mathcal{O})$ obtained by extending $\Gamma(2)$ by $D_{\varepsilon_0}$. Then $SL(2, \mathcal{O})/\Gamma$ is isomorphic to $S_4$. Let $\mathbb{H}$ be the upper half plane in $\mathbb{C}$. There are two embedding of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ into $\mathbb{R}$:

$$\iota : a \mapsto a, \ \iota' : a \mapsto a'.$$

Then $\Gamma$ acts properly discontinuously on $\mathbb{H} \times \mathbb{H}$ in the following manner: for $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\gamma(z_1, z_2) = (\gamma z_1, \gamma' z_2) = \left( \frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

Now let us define Hilbert modular forms for $\Gamma$.

**Definition 2.1.** A holomorphic function $f$ on $\mathbb{H} \times \mathbb{H}$ is a Hilbert modular form of weight $k(\in \mathbb{N})$ for $\Gamma$, if, for any $\gamma \in \Gamma$, $f$ satisfies

$$f(\gamma z_1, \gamma' z_2) = (cz_1 + d)^k (c'z_2 + d')^k f(z_1, z_2).$$

F. Hirzebruch [4] determined the ring of Hilbert modular forms for $\Gamma$ by studying the Hilbert modular surface attached to $\Gamma$.

**Theorem 2.1.** *The ring of modular forms for the group* $\Gamma$ *is isomorphic to*

$$\mathbb{C}[x_1, x_2, x_3, x_4, c]/(x_1 + x_2 + x_3 + x_4, c^2 - C),$$

*where* $C = x_1x_2x_3x_4(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)$, $x_i$ $(i = 1, \dots, 4)$ *is of weight* 1 *and* $c$ *is of weight* 5. *Moreover* $x_i$*'s are symmetric modular forms, i.e.* $x_i(z_1, z_2) = x_i(z_2, z_1)$.

Put

$$y_1 = \frac{1}{2}(x_4 - x_3), y_2 = \frac{1}{2}(x_2 - x_1), y_3 = \frac{1}{2}(x_4 + x_3).$$

Then we can define the holomorphic mapping from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{P}^2(\mathbb{C})$:

$$\pi : (z_1, z_2) \mapsto (y_1(z_1, z_2) : y_2(z_1, z_2) : y_3(z_1, z_2)).$$

This mapping $\pi$ is factored through $\mathbb{H} \times \mathbb{H}/\langle \Gamma, \tau \rangle$ and gives an isomorphism between $\mathbb{H} \times \mathbb{H}/\langle \Gamma, \tau \rangle$ and $\mathbb{P}^2(\mathbb{C}) \setminus \{6 \text{ points}\}$, where $\tau$ is the natural involution which interchanges the factors of $\mathbb{H} \times \mathbb{H}$. Let $x = y_1/y_3$ and $y = y_2/y_3$ be an inhomogeneous coordinate of $\mathbb{P}^2(\mathbb{C})$. The branch loci of $\pi$ is given by $D = (1 - x^2)(1 - y^2)(1 - x^2y^2)(2 - x^2 - y^2)$ in $\mathbb{C}^2$ and the ramification index is equal to two. The above six points are exactly six multiple points of $D$. The projective plane $\mathbb{P}^2(\mathbb{C})$ equipped with the ramification locus $D$ and the index 2 is called the Hilbert modular orbifold $M$ associated with $\langle \Gamma, \tau \rangle$.

## 3. Uniformizing equation for the Hilbert modular orbifold $M$

Naturally $\mathbb{H} \times \mathbb{H}$ is a domain of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which can be considered as a quadratic surface in $\mathbb{P}^3(\mathbb{C})$:

$$\iota : \quad (z_1, z_2) \mapsto (1 : z_1 : z_2 : z_1 z_2).$$

The conformal structure on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ as a hypersurface in $\mathbb{P}^3(\mathbb{C})$, pulled back by the inverse map of $\pi$ on the Hilbert modular orbifold $M$, can be described by the linear differential equation of the form:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = l \dfrac{\partial^2 u}{\partial x \partial y} + a \dfrac{\partial u}{\partial x} + b \dfrac{\partial u}{\partial y} + pu \\ \dfrac{\partial^2 u}{\partial y^2} = m \dfrac{\partial^2 u}{\partial x \partial y} + c \dfrac{\partial u}{\partial x} + d \dfrac{\partial u}{\partial y} + qu. \end{cases} \tag{3.1}$$

The equation (3.1) must satisfy integrability conditions and the dimension of its solution space is of four. Moreover there is a quadratic relation between any four linearly independent solutions. That is to say, the linear differential equation (3.1) has quadric property. Conversely, if we take four linearly independent solutions of (3.1) suitably, the map

$$\varphi : \quad (x, y) \mapsto (u_1 : u_2 : u_3 : u_4) = (1 : z_1 : z_2 : z_1 z_2)$$

gives the inverse map of $\pi$. In what follows, we fix linearly independent solutions as this. Coefficients of (3.1) were determined by Sasaki and Yoshida [11] after R. Kobayashi and I. Naruki [6] derived the conformal structure on $M$ induced from $\iota \circ \pi^{-1}$:

$$l = -\frac{2 - y^2 - x^2 y^2}{xy(1 - x^2)}, \tag{3.2}$$

$$m = -\frac{2 - x^2 - x^2 y^2}{xy(1 - y^2)}, \tag{3.3}$$

$$\begin{aligned} a = &-\frac{3}{2}\frac{\partial}{\partial x} \log \frac{(1 - x^2 y^2)(2 - x^2 - y^2)}{1 - x^2} \\ &+ \frac{l}{2}\frac{\partial}{\partial y} \log \frac{(1 - x^2 y^2)^2 (2 - x^2 - y^2)^2}{(1 - y^2)^2 (2 - y^2 - x^2 y^2)}, \end{aligned} \tag{3.4}$$

$$b = \frac{l}{2}\frac{\partial}{\partial x} \log \frac{(2 - y^2 - x^2 y^2)(1 - x^2 y^2)(2 - x^2 - y^2)}{(1 - x^2)^2}, \tag{3.5}$$

$$c = \frac{m}{2}\frac{\partial}{\partial y} \log \frac{(2 - x^2 - x^2 y^2)(1 - x^2 y^2)(2 - x^2 - y^2)}{(1 - y^2)^2}, \tag{3.6}$$

$$d = -\frac{3}{2}\frac{\partial}{\partial y} \log \frac{(1 - x^2 y^2)(2 - x^2 - y^2)}{1 - y^2} \tag{3.7}$$

$$+\frac{m}{2}\frac{\partial}{\partial x}\log\frac{(1-x^2y^2)^2(2-x^2-y^2)^2}{(1-x^2)^2(2-x^2-x^2y^2)},$$

$$p=\frac{-2(x^2-y^2)}{(1-x^2)^2(1-y^2)}, \tag{3.8}$$

$$q=\frac{-2(y^2-x^2)}{(1-x^2)(1-y^2)^2}. \tag{3.9}$$

For notational simplicity, we denote by $f_x$ (resp. $f_y$) the partial derivative of a function $f$ with respect to $x$ (resp. $y$). Put

$$e^{2\theta}=\det(u,u_x,u_y,u_{xy})=\begin{vmatrix} u_1 & u_2 & u_3 & u_4\\ u_{1,x} & u_{2,x} & u_{3,x} & u_{4,x}\\ u_{1,y} & u_{2,y} & u_{3,y} & u_{4,y}\\ u_{1,xy} & u_{2,xy} & u_{3,xy} & u_{4,xy}\end{vmatrix}$$

which is called a normalization factor of (3.1). In [11], the normalization factor is taken as

$$e^{2\theta}=(1-lm)^{-\frac{7}{2}}(xy)^{-6}. \tag{3.10}$$

**Proposition 3.1.** *When the differential equation* (3.1) *has quadric property, coefficients $a$, $b$, $c$ and $d$ can be written by $l$, $m$, and $\theta$ in the following form*:

$$\begin{cases} b=\dfrac{l}{2}\left(\dfrac{l_x}{l}-\dfrac{3}{4}\xi_x-\theta_x\right)\\ c=\dfrac{m}{2}\left(\dfrac{m_y}{m}-\dfrac{3}{4}\xi_y-\theta_y\right),\end{cases} \tag{3.11}$$

$$\begin{cases} a=\dfrac{1}{4}\xi_x+\theta_x-\dfrac{l}{2}\left(\dfrac{l_y}{l}-\dfrac{1}{4}\xi_y+\theta_y\right)\\ d=\dfrac{1}{4}\xi_y+\theta_y-\dfrac{m}{2}\left(\dfrac{m_x}{m}-\dfrac{1}{4}\xi_x+\theta_x\right),\end{cases} \tag{3.12}$$

*where* $\xi=\log(1-lm)$.

*Proof.* See [10]. □

## 4. Construction of differential equations for Hilbert modular forms

First, we need to determine the differential equation of the form (3.1) which the modular form $y_3$ satisfies. Put $w_1(x,y)=y_3(z_1(x,y),z_2(x,y))$, $w_2(x,y)=z_1w_1$, $w_3(x,y)=z_2w_1$ and $w_4(x,y)=z_1z_2w_1$, where $z_1(x,y)=u_2(x,y)/u_1(x,y)$ and $z_2=u_3/u_1$, then $w_i(x,y)$'s are multi-valued functions on $\mathbb{P}^2(\mathbb{C})\setminus D$. Let us observe the behaviors of $w_i$'s under analytic continuations. Let $\pi_1(\mathbb{P}^2\setminus D)$ be the fundamental group of $\mathbb{P}^2\setminus D$, then $(w_1:w_2:w_3:w_4)$ gives a projective monodromy representation of $\pi_1(\mathbb{P}^2\setminus D)$. For $\alpha\in\pi_1(\mathbb{P}^2\setminus D)$

and the fixed base point $(x, y)$, there exists $\gamma_\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ corresponding to $\alpha$ such that

$$z_1(\alpha(x,y)) = \frac{az_1(x,y)+b}{cz_1(x,y)+d},$$
$$z_2(\alpha(x,y)) = \frac{a'z_2(x,y)+b'}{c'z_2(x,y)+d'},$$

where, for a holomorphic function $f$ around the point $(x, y)$, $f(\alpha(x, y))$ denotes the analytic continuation of $f$ along $\alpha$. Hence, for $w_i$'s, we have

$$\begin{aligned} w_1(\alpha(x,y)) &= (cz_1+d)(c'z_2+d')w_1(z_1,z_2) \\ &= dd'w_1 + cd'w_2 + dc'w_3 + cc'w_4, \\ w_2(\alpha(x,y)) &= (az_1+b)(c'z_2+d')w_1(z_1,z_2) \\ &= bd'w_1 + ad'w_2 + bc'w_3 + ac'w_4, \\ w_3(\alpha(x,y)) &= (cz_1+d)(a'z_2+b')w_1(z_1,z_2) \\ &= db'w_1 + da'w_2 + cb'w_3 + ca'w_4, \\ w_4(\alpha(x,y)) &= (az_1+b)(a'z_2+b')w_1(z_1,z_2) \\ &= bb'w_1 + ab'w_2 + ba'w_3 + aa'w_4. \end{aligned}$$

Therefore the projective monodromy representation of $\pi_1(\mathbb{P}^2 \setminus D)$ given by $w_i$'s is just the same as one given by $u_i$'s. The following lemma is checked by direct calculations.

**Lemma 4.1.** *In* (3.1)*, perform a change of the unknown $u$ by multiplying a factor $e^\rho$. Then the coefficients of the transformed equation, which are denoted by the same letter with primes, are given in the following*

$$l' = l, \quad m' = m, \tag{4.1}$$
$$a' = a + 2\rho_x - l\rho_y, \quad c' = c - m\rho_y, \tag{4.2}$$
$$b' = b - l\rho_x, \quad d' = d + 2\rho_y - m\rho_x, \tag{4.3}$$
$$p' = p - a\rho_x - b\rho_y + (\rho_{xx} - \rho_x^2) - l(\rho_{xy} - \rho_x\rho_y), \tag{4.4}$$
$$q' = q - c\rho_x - d\rho_y + (\rho_{yy} - \rho_y^2) - m(\rho_{xy} - \rho_x\rho_y), \tag{4.5}$$
$$e^{2\theta'} = e^{4\rho+2\theta}. \tag{4.6}$$

Applying this lemma to our case, we have

**Proposition 4.1.** *Put $e^\rho = w_1/u_1$, then $w_1$, $w_2 = z_1w_1$, $w_3 = z_2w_1$ and $w_4 = z_1z_2w_1$ are solutions of the differential equation of type* (3.1) *with coefficients $l', m', \ldots, e^{2\theta'}$ given in the above lemma.*

Therefore in order to determine coefficients $l', m', \ldots, q'$, we have to determine only the normalization factor

$$e^{2\theta'} = \det(w, w_x, w_y, w_{xy}).$$

Computing directly, we have

$$e^{2\theta'} = \begin{vmatrix} w_1 & z_1w_1 & z_2w_1 & z_1z_2w_1 \\ w_{1,x} & (z_1w_1)_x & (z_2w_1)_x & (z_1z_2w_1)_x \\ w_{1,y} & (z_1w_1)_y & (z_2w_1)_y & (z_1z_2w_1)_y \\ w_{1,xy} & (z_1w_1)_{xy} & (z_2w_1)_{xy} & (z_1z_2w_1)_{xy} \end{vmatrix} = STw_1^4, \tag{4.7}$$

where

$$S = z_{1,x}z_{2,y} + z_{2,x}z_{1,y} \tag{4.8}$$

and

$$T = z_{1,x}z_{2,y} - z_{2,x}z_{1,y}. \tag{4.9}$$

On the other hand, the differential equation which have $w_1$, $w_2$, $w_3$ and $w_4$ as a set of independent solutions are also given in terms of determinants:

$$\begin{vmatrix} w & w_1 & w_2 & w_3 & w_4 \\ w_x & w_{1,x} & w_{2,x} & w_{3,x} & w_{4,x} \\ w_y & w_{1,y} & w_{2,y} & w_{3,y} & w_{4,y} \\ w_{xy} & w_{1,xy} & w_{2,xy} & w_{3,xy} & w_{4,xy} \\ w_{xx} & w_{1,xx} & w_{2,xx} & w_{3,xx} & w_{4,xx} \end{vmatrix} = 0, \tag{4.10}$$

$$\begin{vmatrix} w & w_1 & w_2 & w_3 & w_4 \\ w_x & w_{1,x} & w_{2,x} & w_{3,x} & w_{4,x} \\ w_y & w_{1,y} & w_{2,y} & w_{3,y} & w_{4,y} \\ w_{xy} & w_{1,xy} & w_{2,xy} & w_{3,xy} & w_{4,xy} \\ w_{yy} & w_{1,yy} & w_{2,yy} & w_{3,yy} & w_{4,yy} \end{vmatrix} = 0. \tag{4.11}$$

Therefore we have

$$l' = e^{-2\theta'} \det(w, w_x, w_y, w_{xx}), \tag{4.12}$$

$$m' = e^{-2\theta'} \det(w, w_x, w_y, w_{yy}), \tag{4.13}$$

$$p' = -e^{-2\theta'} \det(w_x, w_y, w_{xy}, w_{xx}), \tag{4.14}$$

$$q' = -e^{-2\theta'} \det(w_x, w_y, w_{xy}, w_{yy}), \tag{4.15}$$

$$a' = e^{-2\theta'} \det(w, w_y, w_{xy}, w_{xx}), \tag{4.16}$$

$$b' = -e^{-2\theta'} \det(w, w_x, w_{xy}, w_{xx}), \tag{4.17}$$

$$c' = e^{-2\theta'} \det(w, w_y, w_{xy}, w_{yy}), \tag{4.18}$$

$$d' = -e^{-2\theta'} \det(w, w_x, w_{xy}, w_{yy}). \tag{4.19}$$

From (4.12) and (4.13), we have

$$l' = 2z_{1,x}z_{2,x}/S \tag{4.20}$$

and

$$m' = 2z_{1,y}z_{2,y}/S. \tag{4.21}$$

Therefore we can write $S$ as

$$(1 - l'm')S^2 = T^2. \tag{4.22}$$

The following lemma is used to calculate the normalization factor $e^{2\theta'}$.

**Lemma 4.2.** *Let $T^{-1}$ be the inverse of $T$, i.e.,*

$$T^{-1}(z_1, z_2) = \begin{vmatrix} \dfrac{\partial x}{\partial z_1} & \dfrac{\partial x}{\partial z_2} \\ \dfrac{\partial y}{\partial z_1} & \dfrac{\partial y}{\partial z_2} \end{vmatrix}.$$

*Then $w_1^4/T^{-2}$ is a symmetric Hilbert modular function. In particular, $w_1^4/T^{-2}$ can be regarded as a meromorphic function on $\mathbb{P}^2(\mathbb{C})$.*

*Proof.* First, we shall check behaviors of $T^{-1}$ under actions of $\Gamma$. For any $\gamma \in \Gamma$, we can check easily

$$T^{-1}(\gamma z_1, \gamma' z_2) = (cz_1 + d)^2(c' z_2 + d')^2 T^{-1}(z_1, z_2).$$

Hence $w_1^4/T^{-2}$ is a meromorphic function on $\mathbb{H} \times \mathbb{H}$ and $\Gamma$-invariant. Moreover, since $T^{-1}$ is a skew-symmetric function, $w_1^4/T^{-2}$ is a symmetric function. Therefore $w_1^4/T^{-2}$ gives a meromorphic function on $\mathbb{H} \times \mathbb{H}/\langle \Gamma, \tau \rangle \cong \mathbb{P}^2(\mathbb{C}) \setminus \{6 \text{ points}\}$, which extends to $\mathbb{P}^2(\mathbb{C})$ automatically. $\square$

**Proposition 4.2.** *Normalization factor $e^{2\theta'}$ is equal to*

$$xy(x^2 - 1)^{-1/2}(y^2 - 1)^{-1/2}(x^2y^2 - 1)^{-3/2}(x^2 + y^2 - 2)^{-3/2}. \tag{4.23}$$

*Proof.* From (4.7) and (4.22), we have $e^{2\theta'} = T^2 w_1^4/\sqrt{1 - l'm'}$. Here $l'$ and $m'$ are the same as $l$ and $m$ by Lemma 4.1. Hence $1 - l'm' = 2(1 - x^2y^2)(x^2 + y^2 - 2)x^{-2}y^{-2}(1 - x^2)^{-1}(1 - y^2)^{-1}$. Note that $T$ is a Jacobian of $\varphi$ whose ramification index are equal to two on each component of $D = (1 - x^2)(1 - y^2)(1 - x^2y^2)(2 - x^2 - y^2)$. Hence $T^2 w_1^4$ is equal to $(x^2 - 1)^{-1}(y^2 - 1)^{-1}(x^2y^2 - 1)^{-1}(x^2 + y^2 - 2)^{-1}$ with some constant multiple. However difference of a constant multiple is not essential. Therefore we obtain the desired equality. $\square$

We can determine rests of coefficients $p'$, $q'$ and $e^{4\rho}$:

$$\begin{aligned} p' &= -\frac{1}{x^2 - 1}, \qquad q' = -\frac{1}{y^2 - 1}, \\ e^{4\rho} &= (x^2 - 1)^{-4}(y^2 - 1)^{-4}(x^2y^2 - 1)^2(x^2 + y^2 - 2)^2. \end{aligned}$$

Now we can deduce differential relations for logarithmic derivatives of a Hilbert modular form $w_1(z_1, z_2)$.

**Lemma 4.3.** *Put*

$$A_1(z_1, z_2) = \frac{\partial}{\partial z_1} \log w_1(z_1, z_2), \quad A_2(z_1, z_2) = \frac{\partial}{\partial z_2} \log w_1(z_1, z_2). \tag{4.24}$$

*Then we have the following differential relations for* $A_i$ $(i = 1, 2)$:

$$\frac{\partial A_i}{\partial z_i} - A_i^2 = p' \left(\frac{\partial x}{\partial z_i}\right)^2 + q' \left(\frac{\partial y}{\partial z_i}\right)^2, \tag{4.25}$$

*and*

$$A_2(z_1, z_2) = A_1(z_2, z_1). \tag{4.26}$$

*Proof.* We substitute $w_2 = z_1 w_1$, $w_3 = z_2 w_1$, $w_4 = z_1 z_2 w_1$ in (4.14). Then we have

$$p' = (z_{1,x})^2 \frac{T}{S} \left(\frac{\partial A_1}{\partial z_1} - A_1^2\right) - (z_{2,x})^2 \frac{T}{S} \left(\frac{\partial A_2}{\partial z_2} - A_2^2\right). \tag{4.27}$$

In the similar way, we have

$$q' = -(z_{1,y})^2 \frac{T}{S} \left(\frac{\partial A_1}{\partial z_1} - A_1^2\right) + (z_{2,y})^2 \frac{T}{S} \left(\frac{\partial A_2}{\partial z_2} - A_2^2\right) \tag{4.28}$$

from (4.15). These are equivalent to

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \frac{T}{S} \begin{pmatrix} z_{1,x}^2 & -z_{2,x}^2 \\ -z_{1,y}^2 & z_{2,y}^2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial A_1}{\partial z_1} - A_1^2 \\ \dfrac{\partial A_2}{\partial z_2} - A_2^2 \end{pmatrix}. \tag{4.29}$$

Therefore we obtain

$$\begin{pmatrix} \dfrac{\partial A_1}{\partial z_1} - A_1^2 \\ \dfrac{\partial A_2}{\partial z_2} - A_2^2 \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial x}{\partial z_1}\right)^2 & \left(\dfrac{\partial y}{\partial z_1}\right)^2 \\ \left(\dfrac{\partial x}{\partial z_2}\right)^2 & \left(\dfrac{\partial y}{\partial z_2}\right)^2 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}. \tag{4.30}$$

As for the equality (4.26), it is obvious from that $w_1(z_1, z_2) = w_1(z_2, z_1)$. □

We introduce

$$X_i(z_1, z_2) = \frac{\partial}{\partial z_i} \log(x - 1) = \frac{1}{x-1} \frac{\partial x}{\partial z_i}, \tag{4.31}$$

$$Y_i(z_1, z_2) = \frac{\partial}{\partial z_i} \log(x + 1) = \frac{1}{x+1} \frac{\partial x}{\partial z_i}, \tag{4.32}$$

$$Z_i(z_1, z_2) = \frac{\partial}{\partial z_i} \log(y - 1) = \frac{1}{y-1} \frac{\partial y}{\partial z_i}, \tag{4.33}$$

$$W_i(z_1, z_2) = \frac{\partial}{\partial z_i} \log(y + 1) = \frac{1}{y+1} \frac{\partial y}{\partial z_i}, \tag{4.34}$$

for $i = 1, 2$. Then (4.25) can be written as

$$\frac{\partial A_i}{\partial z_i} = A_i^2 - X_i Y_i - Z_i W_i. \tag{4.35}$$

Moreover, since differentials by $z_1$ and $z_2$ are commutative, there exists a relation

$$\frac{\partial}{\partial z_2}A_1(z_1,z_2)=\frac{\partial}{\partial z_1}A_2(z_1,z_2). \tag{4.36}$$

Next we need to derive differentiations of $X_i$ and $Y_i$ by $z_1$ and $z_2$.

**Lemma 4.4.** *Second order derivatives of $z_1$ and $z_2$ by $x$ and $y$ are given by*

$$z_{1,xy}=-\frac{1}{2}\frac{S}{T}(z_{1,y}(l_y+l\frac{S_y}{S})-z_{1,x}(m_x+m\frac{S_x}{S})), \tag{4.37}$$

$$z_{2,xy}=\frac{1}{2}\frac{S}{T}(z_{2,y}(l_y+l\frac{S_y}{S})-z_{2,x}(m_x+m\frac{S_x}{S})), \tag{4.38}$$

$$z_{1,xx}=z_{1,x}a'+z_{1,y}b'-2z_{1,x}\frac{w_{1,x}}{w_1}+lz_{1,x}\frac{w_{1,y}}{w_1}+lz_{1,y}\frac{w_{1,x}}{w_1}+lz_{1,xy}, \tag{4.39}$$

$$z_{2,xx}=z_{2,x}a'+z_{2,y}b'-2z_{2,x}\frac{w_{1,x}}{w_1}+lz_{2,x}\frac{w_{1,y}}{w_1}+lz_{2,y}\frac{w_{1,x}}{w_1}+lz_{2,xy}, \tag{4.40}$$

$$z_{1,yy}=z_{1,x}c'+z_{1,y}d'-2z_{1,y}\frac{w_{1,y}}{w_1}+mz_{1,x}\frac{w_{1,y}}{w_1}+mz_{1,y}\frac{w_{1,x}}{w_1}+mz_{1,xy}, \tag{4.41}$$

$$z_{2,yy}=z_{2,x}c'+z_{2,y}d'-2z_{2,y}\frac{w_{1,y}}{w_1}+mz_{2,x}\frac{w_{1,y}}{w_1}+mz_{2,y}\frac{w_{1,x}}{w_1}+mz_{2,xy}. \tag{4.42}$$

*Proof.* First, differentiate $l=2z_{1,x}z_{2,x}/S$ by $y$. Then we have

$$Sl_y+lS_y=2z_{1,x}z_{2,xy}+2z_{2,x}z_{1,xy}.$$

In the similar way, we have

$$Sm_x+mS_x=2z_{1,y}z_{2,xy}+2z_{2,y}z_{1,xy}.$$

Therefore we obtain (4.37) and (4.38). From (4.16), (4.17), (4.18) and (4.19), we have

$$a'=-\frac{1}{T}\left(z_{1,y}z_{2,xx}-2T\frac{w_{1,x}}{w_1}-lz_{1,y}z_{2,xy}+lT\frac{w_{1,y}}{w_1}-z_{2,y}z_{1,xx}+lz_{2,y}z_{1,xy}\right), \tag{4.43}$$

$$b'=\frac{1}{T}\left(z_{1,x}z_{2,xx}-lz_{1,x}z_{2,xy}-lT\frac{w_{1,x}}{w_1}-z_{2,x}z_{1,xx}+lz_{2,x}z_{1,xy}\right), \tag{4.44}$$

$$c'=-\frac{1}{T}\left(z_{1,y}z_{2,yy}-mz_{1,y}z_{2,xy}+mT\frac{w_{1,y}}{w_1}-z_{2,y}z_{1,yy}+mz_{2,y}z_{1,xy}\right), \tag{4.45}$$

$$d'=\frac{1}{T}\left(z_{1,x}z_{2,yy}+2T\frac{w_{1,y}}{w_1}-mz_{1,x}z_{2,xy}-mT\frac{w_{1,x}}{w_1}-z_{2,x}z_{1,yy}+mz_{2,x}z_{1,xy}\right). \tag{4.46}$$

Therefore we obtain (4.39) and (4.40) (resp. (4.41) and (4.42)) from (4.43) and (4.44) (resp. (4.45) and (4.46)). □

**Lemma 4.5.** *The derivatives* $\frac{\partial x}{\partial z_i}$ *and* $\frac{\partial y}{\partial z_i}$ *satisfy the following relations*:

$$-2xy(y^2-1)\frac{\partial x}{\partial z_1}\frac{\partial x}{\partial z_2} = (2-x^2-x^2y^2)\left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2}+\frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right), \tag{4.47}$$

$$-2xy(x^2-1)\frac{\partial y}{\partial z_1}\frac{\partial y}{\partial z_2} = (2-y^2-x^2y^2)\left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2}+\frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right). \tag{4.48}$$

*Proof.* From (4.8) and (4.20), we have the equality

$$(z_{1,x}z_{2,y}+z_{2,x}z_{1,y})l = 2z_{1,x}z_{2,x}. \tag{4.49}$$

From (3.2), we obtain the equality (4.47). The second equality (4.48) can be obtained in the similar way. □

**Remark 1.** These two relations (4.47) and (4.48) are equivalent to

(4.50)
$$\begin{aligned}&2xy(x^2-1)(y^2-1)\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_1}\\ =&-(y^2-1)(2-y^2-x^2y^2)\left(\frac{\partial x}{\partial z_1}\right)^2-(x^2-1)(2-x^2-x^2y^2)\left(\frac{\partial y}{\partial z_1}\right)^2,\end{aligned}$$

(4.51)
$$\begin{aligned}&2xy(x^2-1)(y^2-1)\frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_2}\\ =&-(y^2-1)(2-y^2-x^2y^2)\left(\frac{\partial x}{\partial z_2}\right)^2-(x^2-1)(2-x^2-x^2y^2)\left(\frac{\partial y}{\partial z_2}\right)^2.\end{aligned}$$

The discriminant of these quadratic relations are equal to

$$-8D = -8(x^2-1)(y^2-1)(x^2y^2-1)(x^2+y^2-2). \tag{4.52}$$

Therefore, we have

$$\mathbb{C}\left(A_1,A_2,x,y,\frac{\partial x}{\partial z_1},\frac{\partial y}{\partial z_1},\frac{\partial x}{\partial z_2},\frac{\partial y}{\partial z_2}\right) = \mathbb{C}\left(A_1,A_2,x,y,\frac{\partial x}{\partial z_1},\frac{\partial x}{\partial z_2}\right)(\sqrt{D}),$$

which is a quadratic extension field of $\mathbb{C}(A_1,A_2,x,y,\frac{\partial x}{\partial z_1},\frac{\partial x}{\partial z_2})$.

Using Lemma 4.4 and Lemma 4.5, we obtain the following differential relations.

**Proposition 4.3.** *Second order derivatives of $x$ and $y$ by $z_1$ and $z_2$ are* *given by*

$$\frac{\partial^2 x}{\partial z_i^2} = 2\frac{\partial x}{\partial z_i}A_i + \left(\frac{\partial x}{\partial z_i}\right)^2 \frac{3x}{x^2-1} + \left(\frac{\partial y}{\partial z_i}\right)^2 \frac{x}{y^2-1}, \tag{4.53}$$

$$\frac{\partial^2 y}{\partial z_i^2} = 2\frac{\partial y}{\partial z_i}A_i + \left(\frac{\partial x}{\partial z_i}\right)^2 \frac{y}{x^2-1} + \left(\frac{\partial y}{\partial z_i}\right)^2 \frac{3y}{y^2-1}, \tag{4.54}$$

$$\frac{\partial^2 x}{\partial z_1 \partial z_2} = \frac{\partial x}{\partial z_1}\frac{\partial x}{\partial z_2}\frac{x}{x^2-1} + \frac{1}{2}\left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2} + \frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right)\frac{y}{y^2-1}, \tag{4.55}$$

$$\frac{\partial^2 y}{\partial z_1 \partial z_2} = \frac{\partial y}{\partial z_1}\frac{\partial y}{\partial z_2}\frac{y}{y^2-1} + \frac{1}{2}\left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2} + \frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right)\frac{x}{x^2-1}. \tag{4.56}$$

*Proof.* We shall prove (4.53) and (4.55) and rests can be deduced in the similar way. First,

$$\begin{aligned}
\frac{\partial^2 x}{\partial z_1^2} &= \frac{\partial x}{\partial z_1}\frac{\partial}{\partial x}\left(\frac{z_{2,y}}{T}\right) + \frac{\partial y}{\partial z_1}\frac{\partial}{\partial y}\left(\frac{z_{2,y}}{T}\right) \\
&= \frac{\partial x}{\partial z_1}\frac{1}{T}\left(z_{2,xy} - z_{2,y}\frac{T_x}{T}\right) + \frac{\partial y}{\partial z_1}\frac{1}{T}\left(z_{2,yy} - z_{2,y}\frac{T_y}{T}\right) \\
&= \frac{1}{T^2}\left(z_{2,y}z_{2,xy} - z_{2,y}^2\frac{T_x}{T} - z_{2,x}z_{2,yy} + z_{2,x}z_{2,y}\frac{T_y}{T}\right).
\end{aligned}$$

From (4.38) and (4.42),

$$\begin{aligned}
\frac{\partial^2 x}{\partial z_1^2} = \frac{1}{T^2}\Bigg[&(z_{2,x})^2\left(-c' - m\frac{w_{1,y}}{w_1}\right) + (z_{2,y})^2\left(\frac{1}{2}\left(l_y + l\frac{S_y}{S}\right) - \frac{T_x}{T}\right) \\
&+ z_{2,x}z_{2,y}\left(-\frac{1}{2}\left(m_x + m\frac{S_x}{S}\right) + \frac{T_y}{T} - d' + 2\frac{w_{1,y}}{w_1} - m\frac{w_{1,x}}{w_1}\right)\Bigg].
\end{aligned}$$

From (4.7) and (4.22), we have

$$\frac{T_x}{T} = \theta_x - 2\frac{w_{1,x}}{w_1} + \frac{1}{4}\xi_x, \quad \frac{S_x}{S} = \frac{T_x}{T} - \frac{1}{2}\xi_x. \tag{4.57}$$

Substituting (3.11), (3.12) and (4.57), we obtain

$$\begin{aligned}
\frac{\partial^2 x}{\partial z_1^2} &= \left(\frac{\partial x}{\partial z_1}\right)^2\left(2\frac{w_{1,x}}{w_1} + \frac{1}{2}l_y + \frac{1}{2}l\theta_y - \frac{1}{8}l\xi_y - \theta_x - \frac{1}{4}\xi_x\right) \\
&\quad + 2\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_1}\frac{w_{1,y}}{w_1} + \left(\frac{\partial y}{\partial z_1}\right)^2\left(-\frac{m_y}{2} + \frac{3}{8}m\xi_y + \frac{m}{2}\theta_y\right) \\
&= 2\frac{\partial x}{\partial z_1}A_1 + \left(\frac{\partial x}{\partial z_1}\right)^2\frac{3x}{x^2-1} + \left(\frac{\partial y}{\partial z_1}\right)^2\frac{x}{y^2-1}.
\end{aligned}$$

As for (4.55),

$$\begin{aligned}
\frac{\partial^2 x}{\partial z_1 \partial z_2} &= \frac{\partial x}{\partial z_2}\frac{\partial}{\partial x}\left(\frac{z_{2,y}}{T}\right) + \frac{\partial y}{\partial z_2}\frac{\partial}{\partial y}\left(\frac{z_{2,y}}{T}\right) \\
&= \frac{1}{T^2}\left(-z_{1,y}\left(z_{2,xy} - z_{2,y}\frac{T_x}{T}\right) + z_{1,x}\left(z_{2,yy} - z_{2,y}\frac{T_y}{T}\right)\right) \\
&= \frac{S}{T^2}\left(-\frac{1}{2}m_x + \frac{1}{4}m\xi_x + \frac{1}{4}lm_y + \frac{1}{4}ml_y - \frac{1}{4}lm\xi_y\right) \\
&= \left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2} + \frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right)\left(\frac{1}{2}\frac{2x^2y^2 - y^2 + x^2 - 2}{y(y^2-1)(x^2-1)}\right).
\end{aligned}$$

Using (4.47), we have the equality (4.55). □

We can obtain the system of differential equations which $A_i$, $X_i$, $Y_i$, $Z_i$ and $W_i$ satisfy.

**Theorem 4.1.** *Functions defined by* (4.24), (4.31), (4.32), (4.33) *and* (4.34)

$$\begin{aligned}
A_i(z_1, z_2) &= \frac{\partial}{\partial z_i}\log w_1(z_1, z_2), \\
X_i(z_1, z_2) &= \frac{\partial}{\partial z_i}\log(x-1) = \frac{1}{x-1}\frac{\partial x}{\partial z_i}, \\
Y_i(z_1, z_2) &= \frac{\partial}{\partial z_i}\log(x+1) = \frac{1}{x+1}\frac{\partial x}{\partial z_i}, \\
Z_i(z_1, z_2) &= \frac{\partial}{\partial z_i}\log(y-1) = \frac{1}{y-1}\frac{\partial y}{\partial z_i}, \\
W_i(z_1, z_2) &= \frac{\partial}{\partial z_i}\log(y+1) = \frac{1}{y+1}\frac{\partial y}{\partial z_i}
\end{aligned}$$

*satisfy the following system of differential equations* (*we call this system HMS*):

$$\frac{\partial A_i}{\partial z_i} = A_i^2 - X_iY_i - Z_iW_i, \tag{4.58}$$

$$\frac{\partial A_2}{\partial z_1} = \frac{\partial A_1}{\partial z_2} = -\frac{1}{3}X_1Y_2 - \frac{1}{3}Z_1W_2 - \frac{1}{6}S_1T_2 - \frac{1}{6}S_2T_1, \tag{4.59}$$

$$\frac{\partial X_i}{\partial z_i} = 2X_iA_i + \frac{3}{2}X_iY_i + \frac{1}{2}X_i^2 + \frac{1}{2}Z_iW_i + \frac{1}{2}\frac{X_iZ_iW_i}{Y_i}, \tag{4.60}$$

$$\begin{aligned}
\frac{\partial X_1}{\partial z_2} &= \frac{\partial X_2}{\partial z_1} \\
&= -\frac{1}{2}X_1X_2 + \frac{1}{2}X_1Y_2 + \frac{1}{4}X_1(Z_2 + W_2) + \frac{1}{4}X_2(Z_1 + W_1),
\end{aligned}\tag{4.61}$$

$$\frac{\partial Y_i}{\partial z_i} = 2Y_iA_i + \frac{3}{2}X_iY_i + \frac{1}{2}Y_i^2 + \frac{1}{2}Z_iW_i + \frac{1}{2}\frac{Y_iZ_iW_i}{X_i}, \tag{4.62}$$

$$\frac{\partial Y_1}{\partial z_2} = \frac{\partial Y_2}{\partial z_1} \tag{4.63}$$

$$= -\frac{1}{2}Y_1Y_2 + \frac{1}{2}X_1Y_2 + \frac{1}{4}Y_1(Z_2 + W_2) + \frac{1}{4}Y_2(Z_1 + W_1),$$

$$\frac{\partial Z_i}{\partial z_i} = 2Z_iA_i + \frac{3}{2}Z_iW_i + \frac{1}{2}Z_i^2 + \frac{1}{2}X_iY_i + \frac{1}{2}\frac{Z_iX_iY_i}{W_i}, \tag{4.64}$$

$$\frac{\partial Z_1}{\partial z_2} = \frac{\partial Z_2}{\partial z_1} \tag{4.65}$$

$$= -\frac{1}{2}Z_1Z_2 + \frac{1}{2}Z_1W_2 + \frac{1}{4}Z_1(X_2 + Y_2) + \frac{1}{4}Z_2(X_1 + Y_1),$$

$$\frac{\partial W_i}{\partial z_i} = 2W_iA_i + \frac{3}{2}W_iZ_i + \frac{1}{2}W_i^2 + \frac{1}{2}Y_iX_i + \frac{1}{2}\frac{W_iY_iX_i}{Z_i}, \tag{4.66}$$

$$\frac{\partial W_1}{\partial z_2} = \frac{\partial W_2}{\partial z_1} \tag{4.67}$$

$$= -\frac{1}{2}W_1W_2 + \frac{1}{2}Z_1W_2 + \frac{1}{4}W_1(X_2 + Y_2) + \frac{1}{4}W_2(X_1 + Y_1),$$

$$X_1Y_2 - X_2Y_1 = 0, \tag{4.68}$$

$$Z_1W_2 - Z_2W_1 = 0, \tag{4.69}$$

$$3S_1(Z_2 + W_2) + 3S_2(Z_1 + W_1) = 4(2X_1Y_2 - Z_1W_2 + S_1T_2 + S_2T_1), \tag{4.70}$$

$$3T_1(X_2 + Y_2) + 3T_2(X_1 + Y_1) = 4(-X_1Y_2 + 2Z_1W_2 + S_1T_2 + S_2T_1), \tag{4.71}$$

*where* $i = 1, 2$ *and* $S_i = 2X_iY_i/(X_i + Y_i)$, $T_i = 2Z_iW_i/(Z_i + W_i)$.

*Proof.* We have already derived the equality (4.58). Equalities (4.60)–(4.67) can be derived from Proposition 4.3. The algebraic relation (4.68) is nothing but

$$X_1/Y_1 = X_2/Y_2 = (x + 1)/(x - 1) \tag{4.72}$$

and we obtain (4.69) in the same way. Another two relations (4.70) and (4.71) are given by rewriting (4.47) and (4.48) in Lemma 4.5. Derivation of the differential relation (4.59) is not so direct. Since differentiations of the Hilbert modular function $x(z_1, z_2)$ by $z_1$ and $z_2$ must be commutative, we have

$$\frac{\partial^3 x}{\partial z_1^2 \partial z_2} = \frac{\partial}{\partial z_2}\left(2\frac{\partial x}{\partial z_1}A_1 + \frac{3x}{x^2 - 1}\left(\frac{\partial x}{\partial z_1}\right)^2 + \frac{x}{y^2 - 1}\left(\frac{\partial y}{\partial z_1}\right)^2\right) \tag{4.73}$$

$$= \frac{\partial}{\partial z_1}\left(\frac{x}{x^2 - 1}\frac{\partial x}{\partial z_1}\frac{\partial x}{\partial z_2} + \frac{y}{2(y^2 - 1)}\left(\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2} + \frac{\partial x}{\partial z_2}\frac{\partial y}{\partial z_1}\right)\right)$$

from (4.53) and (4.55). This relation is not trivial and gives the equation which contain $\frac{\partial A_1}{\partial z_2}$. That equation is equal to (4.59). Also from (4.54) and (4.56), we obtain the same equation. □

**Theorem 4.2.** *Whole differential and algebraic equations in HMS are compatible with each others, and the equations* $(4.68 - 4.71)$ *are algebraically independent. Particularly, HMS is essentially a nonlinear differential system of sixth order.*

*Proof.* We can check it by direct calculations. $\square$

## 5. Initial value problems for HMS

In this section, we shall give generic solutions of HMS. This will be done by constructing the solution for initial conditions at any given points. First, we shall prove that the differential system HMS has the following remarkable properties.

**Proposition 5.1.** *Given a set of solutions* $\{F_i(z_1, z_2)\}$ $(F = A, X, Y, Z, W$ *and* $i = 1, 2)$ *of HMS,* $\{F_i^T(z_1, z_2)\}$ *are also solutions of HMS, where*

$$F_1^T(z_1, z_2) = F_2(z_2, z_1), \tag{5.1}$$

$$F_2^T(z_1, z_2) = F_1(z_2, z_1). \tag{5.2}$$

*Moreover, for any* $\gamma = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, *put*

$$A_i^\gamma(z_1, z_2) = \frac{1}{(c_i z_i + d_i)^2} A_i\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right) - \frac{c_i}{c_i z_i + d_i}, \tag{5.3}$$

$$G_i^\gamma(z_1, z_2) = \frac{1}{(c_i z_i + d_i)^2} G_i\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right), \tag{5.4}$$

*where* $G = X, Y, Z, W$. *Then* $\{F_i^\gamma(z_1, z_2)\}$ $(F = A, X, Y, Z, W$ *and* $i = 1, 2)$ *are also solutions of HMS.*

*Proof.* The first assertion of the proposition is obvious. The second part is proved by direct calculations. For example,

$$\begin{aligned}
\frac{\partial}{\partial z_i} A_i^\gamma(z_1, z_2) =& \frac{-2c_i}{(c_i z_i + d_i)^2} A_i(\gamma(z_1, z_2)) \\
&+ \frac{1}{(c_i z_i + d_i)^4} \frac{\partial A_i}{\partial z_i}(\gamma(z_1, z_2)) + \frac{c_i^2}{(c_i z_i + d_i)^2} \\
=& \frac{-2c_i}{(c_i z_i + d_i)^2} A_i + \frac{c_i^2}{(c_i z_i + d_i)^2} \\
&+ \frac{1}{(c_i z_i + d_i)^4}(A_i^2 - X_i Y_i - Z_i W_i).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(A_i^\gamma)^2 - X_i^\gamma Y_i^\gamma - Z_i^\gamma W_i^\gamma \\
=& \frac{1}{(c_i z_i + d_i)^4} A_i^2 - \frac{2c_i}{(c_i z_i + d_i)^2} A_i + \frac{c_i^2}{(c_i z_i + d_i)^2} \\
&- \frac{1}{(c_i z_i + d_i)^4} X_i Y_i - \frac{1}{(c_i z_i + d_i)^4} Z_i W_i.
\end{aligned}$$

□

We can construct solutions with six parameters of HMS from the particular solution in the previous section by using Proposition 5.1. We shall prove that these solutions are generic solutions of HMS. We consider initial value problems of HMS for generic initial conditions.

**Theorem 5.1.** *Take complex numbers* $A_i^0$, $X_i^0$, $Y_i^0$, $Z_i^0$, $W_i^0$, $(i = 1, 2)$ *satisfying the algebraic relations* (4.68), (4.69), (4.70) *and* (4.71). *We assume that*

$$x^0 y^0 \left((x^0)^2 - 1\right) \left((y^0)^2 - 1\right) \left((x^0)^2 (y^0)^2 - 1\right) \left((x^0)^2 + (y^0)^2 - 2\right) \neq 0 \tag{5.5}$$

*and*

$$x_1^0 x_2^0 y_1^0 y_2^0 (x_1^0 y_2^0 - x_2^0 y_1^0) \neq 0, \tag{5.6}$$

*where* $x^0 = (X_1^0 + Y_1^0)/(X_1^0 - Y_1^0)$, $y^0 = (Z_1^0 + W_1^0)/(Z_1^0 - W_1^0)$, $x_1^0 = 2X_1^0 Y_1^0/(X_1^0 - Y_1^0)$, $x_2^0 = 2X_2^0 Y_2^0/(X_2^0 - Y_2^0)$, $y_1^0 = 2Z_1^0 W_1^0/(Z_1^0 - W_1^0)$ *and* $y_2^0 = 2Z_2^0 W_2^0/(Z_2^0 - W_2^0)$. *Then, for any* $(z_1^0, z_2^0) \in \mathbb{C} \times \mathbb{C}$, *there exists*

$$\gamma = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$$

*such that the set of solutions given by transformations in Proposition* 5.1 *from our special solution in Theorem* 4.1 *satisfies the initial conditions*

$$\begin{aligned} A_i^\gamma(z_1^0, z_2^0) &= A_i^0 \\ X_i^\gamma(z_1^0, z_2^0) &= X_i^0 \\ Y_i^\gamma(z_1^0, z_2^0) &= Y_i^0 \\ Z_i^\gamma(z_1^0, z_2^0) &= Z_i^0 \\ W_i^\gamma(z_1^0, z_2^0) &= W_i^0 \end{aligned}$$

*or*

$$\begin{aligned} A_i^{\gamma \circ T}(z_1^0, z_2^0) &= (A_i^\gamma)^T (z_1^0, z_2^0) = A_i^0 \\ X_i^{\gamma \circ T}(z_1^0, z_2^0) &= (X_i^\gamma)^T (z_1^0, z_2^0) = X_i^0 \\ Y_i^{\gamma \circ T}(z_1^0, z_2^0) &= (Y_i^\gamma)^T (z_1^0, z_2^0) = Y_i^0 \\ Z_i^{\gamma \circ T}(z_1^0, z_2^0) &= (Z_i^\gamma)^T (z_1^0, z_2^0) = Z_i^0 \\ W_i^{\gamma \circ T}(z_1^0, z_2^0) &= (W_i^\gamma)^T (z_1^0, z_2^0) = W_i^0. \end{aligned}$$

*Proof.* Under the assumptions (5.5) and (5.6), there are relations

$$\begin{aligned} &2x^0 y^0 (x^{02} - 1)(y^{02} - 1) x_i^0 y_i^0 \\ &= - (y^{02} - 1)(2 - y^{02} - x^{02} y^{02}) x_i^{02} - (x^{02} - 1)(2 - x^{02} - x^{02} y^{02}) y_i^{02}, \end{aligned} \tag{5.7}$$

for $i = 1$ and 2. Take $(\overline{z_1}, \overline{z_2}) \in \mathbb{H} \times \mathbb{H}$ satisfying $x^0 = x(\overline{z_1}, \overline{z_2})$ and $y^0 = y(\overline{z_1}, \overline{z_2})$. From (5.7) and Remark 1, we have

$$\frac{y_1^0}{x_1^0} = \frac{\partial y}{\partial z_1}(\overline{z_1}, \overline{z_2}) \Big/ \frac{\partial x}{\partial z_1}(\overline{z_1}, \overline{z_2}) \tag{5.8}$$

or

$$\frac{y_1^0}{x_1^0} = \frac{\partial y}{\partial z_1}(\overline{z_2}, \overline{z_1}) \Big/ \frac{\partial x}{\partial z_1}(\overline{z_2}, \overline{z_1}) = \frac{\partial y}{\partial z_2}(\overline{z_1}, \overline{z_2}) \Big/ \frac{\partial x}{\partial z_2}(\overline{z_1}, \overline{z_2}). \tag{5.9}$$

1. The case of (5.8)

By the assumption $y_1^0/x_1^0 \neq y_2^0/x_2^0$, we have

$$\frac{y_2^0}{x_2^0} = \frac{\partial y}{\partial z_2}(\overline{z_1}, \overline{z_2}) \Big/ \frac{\partial x}{\partial z_2}(\overline{z_1}, \overline{z_2}). \tag{5.10}$$

Take $\gamma = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ which satisfies conditions

$$\overline{z_i} = \frac{a_i z_i^0 + b_i}{c_i z_i^0 + d_i}, \tag{5.11}$$

$$A_i^0 (c_i z_i^0 + d_i)^2 + c_i (c_i z_i^0 + d_i) - \frac{\partial w_1}{\partial z_i}(\overline{z_1}, \overline{z_2}) / w_1(\overline{z_1}, \overline{z_2}) = 0, \tag{5.12}$$

and

$$(c_i z_i^0 + d_i)^2 = \frac{1}{x_i^0} \frac{\partial x}{\partial z_i}(\overline{z_1}, \overline{z_2}), \tag{5.13}$$

for $i = 1$ and 2. Then for this $\gamma$, we have

$$\begin{aligned} A_i^\gamma(z_1^0, z_2^0) &= A_i^0 \\ X_i^\gamma(z_1^0, z_2^0) &= X_i^0 \\ Y_i^\gamma(z_1^0, z_2^0) &= Y_i^0 \\ Z_i^\gamma(z_1^0, z_2^0) &= Z_i^0 \\ W_i^\gamma(z_1^0, z_2^0) &= W_i^0. \end{aligned}$$

2. The case of (5.9)

In the same as the previous case, we have

$$\frac{y_2^0}{x_2^0} = \frac{\partial y}{\partial z_1}(\overline{z_1}, \overline{z_2}) \Big/ \frac{\partial x}{\partial z_1}(\overline{z_1}, \overline{z_2}). \tag{5.14}$$

Take $\gamma = \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ as

$$\overline{z_1} = \frac{a_1 z_2^0 + b_1}{c_1 z_2^0 + d_1}, \quad \overline{z_2} = \frac{a_2 z_1^0 + b_2}{c_2 z_1^0 + d_2}, \tag{5.15}$$

$$(c_1 z_2^0 + d_1)^2 = \frac{1}{x_2^0} \frac{\partial x}{\partial z_1}(\overline{z_1}, \overline{z_2}), \quad (c_2 z_1^0 + d_2)^2 = \frac{1}{x_1^0} \frac{\partial x}{\partial z_2}(\overline{z_1}, \overline{z_2}), \tag{5.16}$$

$$A_2^0 (c_1 z_2^0 + d_1)^2 + c_1 (c_1 z_2^0 + d_1) - \frac{\partial w_1}{\partial z_1}(\overline{z_1}, \overline{z_2}) / w_1(\overline{z_1}, \overline{z_2}) = 0, \tag{5.17}$$

and

$$A_1^0 (c_2 z_1^0 + d_2)^2 + c_2 (c_2 z_1^0 + d_2) - \frac{\partial w_1}{\partial z_2}(\overline{z_1}, \overline{z_2}) / w_1(\overline{z_1}, \overline{z_2}) = 0. \tag{5.18}$$

Then for this $\gamma$, we have

$$\begin{aligned} A_i^{\gamma \circ T}(z_1^0, z_2^0) &= A_i^0 \\ X_i^{\gamma \circ T}(z_1^0, z_2^0) &= X_i^0 \\ Y_i^{\gamma \circ T}(z_1^0, z_2^0) &= Y_i^0 \\ Z_i^{\gamma \circ T}(z_1^0, z_2^0) &= Z_i^0 \\ W_i^{\gamma \circ T}(z_1^0, z_2^0) &= W_i^0. \end{aligned}$$

□

Finally, we shall give reductions and special solutions of HMS. First, we reduce HMS to an ordinary differential equation by restricting to the diagonal part $z_1 = z_2 = t$. In HMS, we assume that $A_1(t,t) = A_2(t,t)$, $X_1(t,t) = X_2(t,t)$, $Y_1(t,t) = Y_2(t,t)$, $Z_1(t,t) = Z_2(t,t)$, $W_1(t,t) = W_2(t,t)$ and that $X_i(t,t) = Y_i(t,t) = 0$, then we have an ordinary differential system:

$$\begin{cases} \dfrac{d}{dt} A(t) = A(t)^2 - Z(t)W(t) \\ \dfrac{d}{dt} Z(t) = 2Z(t)A(t) + 2Z(t)W(t) \\ \dfrac{d}{dt} W(t) = 2W(t)A(t) + 2Z(t)W(t). \end{cases} \tag{5.19}$$

The system (5.19) is changed to

$$\begin{cases} \omega_1' = \omega_1\omega_2 + \omega_1\omega_3 - \omega_2\omega_3 \\ \omega_2' = \omega_1\omega_2 + \omega_2\omega_3 - \omega_1\omega_3 \\ \omega_3' = \omega_1\omega_3 + \omega_2\omega_3 - \omega_1\omega_2 \end{cases} \tag{5.20}$$

by the transformation

$$\omega_1(t) = A(t), \tag{5.21}$$

$$\omega_2(t) = Z(t) + A(t), \tag{5.22}$$

$$\omega_3(t) = W(t) + A(t). \tag{5.23}$$

This system was solved by Halphen in terms of elliptic modular forms ([2]).

Next, we assume that $X_i(z_1, z_2) = Y_i(z_1, z_2)$ and $Z_i(z_1, z_2) = W_i(z_1, z_2)$. Then HMS is reduced to

$$\frac{\partial A_i}{\partial z_i} = A_i^2 - X_i^2 - Z_i^2, \tag{5.24}$$

$$\frac{\partial A_2}{\partial z_1} = \frac{\partial A_1}{\partial z_2} = -X_1 X_2, \tag{5.25}$$

$$\frac{\partial X_i}{\partial z_i} = 2X_i A_i + 2X_i^2 + Z_i^2, \tag{5.26}$$

$$\frac{\partial X_2}{\partial z_1} = \frac{\partial X_1}{\partial z_2} = X_1 X_2, \tag{5.27}$$

$$\frac{\partial Z_i}{\partial z_i} = 2Z_i A_i + 2Z_i^2 + X_i^2, \tag{5.28}$$

$$\frac{\partial Z_2}{\partial z_1} = \frac{\partial Z_1}{\partial z_2} = Z_1 Z_2, \tag{5.29}$$

$$X_1 X_2 = Z_1 Z_2 = \frac{1}{2}(X_1 Z_2 + X_2 Z_1). \tag{5.30}$$

We can solve this system (5.24)–(5.30) directly. Put $P_i = X_i + A_i$ and $Q_i = Y_i + A_i$. Then the system is changed to

$$\frac{\partial A_i}{\partial z_i} = -A_i^2 + 2(P_i + Q_i)A_i - P_i^2 - Q_i^2, \tag{5.31}$$

$$\frac{\partial A_2}{\partial z_1} = \frac{\partial A_1}{\partial z_2} = -(P_1 - A_1)(P_2 - A_2), \tag{5.32}$$

$$\frac{\partial P_i}{\partial z_i} = P_i^2, \tag{5.33}$$

$$\frac{\partial P_2}{\partial z_1} = \frac{\partial P_1}{\partial z_2} = 0, \tag{5.34}$$

$$\frac{\partial Q_i}{\partial z_i} = Q_i^2, \tag{5.35}$$

$$\frac{\partial Q_2}{\partial z_1} = \frac{\partial Q_1}{\partial z_2} = 0, \tag{5.36}$$

and algebraic equation (5.30) demands $P_i = Q_i$ $(i = 1, 2)$ or $P_1 = Q_1 = A_1$. Then we have the following solutions:

1. $A_i = P_i = Q_i = 0$ $(i = 1, 2)$,
2. $A_1 = 0, A_2 = \dfrac{1}{z_2}, P_i = Q_i = 0$,
3. $A_i = \dfrac{1}{z_1 + z_2}, P_i = Q_i = 0$,
4. $A_1 = 0, A_2 = \dfrac{\omega}{z_2}, P_1 = Q_1 = 0, P_2 = -\dfrac{1}{z_2}, Q_2 = 0$ $(\omega = e^{2\pi i/3})$,
5. $A_1 = 0, A_2 = \dfrac{\omega^2}{z_2}, P_1 = Q_1 = 0, P_2 = -\dfrac{1}{z_2}, Q_2 = 0$,

6. $A_1 = 0, A_2 = \dfrac{\omega + \omega^2 z_2^{\omega^2-\omega}}{z_2(1 + z_2^{\omega^2-\omega})}, P_1 = Q_1 = 0, P_2 = -\dfrac{1}{z_2}, Q_2 = 0.$

The general solutions of the system (5.24)–(5.30) are obtained by the transformations in the Proposition 5.1.

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