J. Math. Kyoto Univ. (JMKYAZ)
46-1 (2006), 167–178

Solvability of a third-order two-point boundary value problem in Banach Spaces

By

Shihuang Hong

Abstract

In this paper we discuss the two-point boundary value problem of third order ordinary differential inclusions in ordered Banach Spaces. By using a fixed point theorem due to Hong [8], we establish the existence of solutions the mentioned above problems with discontinuous right hand side.

1. Introduction

Many techniques arose in the studies of boundary value problems of third order ordinary differential equations, for instance, differential inequality [1], topological transversality [2], the shooting method [3], the lower and upper solutions method [4], [5], analysis comparable to that of classical equations[6], the Lyapunov-Schmidt procedure and the continuum theory for O-epi maps [7] and so on. A common hypothesis has been used in the above cited works, that is, the function f is assumed to be continuous. However, it is common knowledge that many problems considered in the engineering and technology lead to nonlinear ordinary differential equations of which are not supposed the functions to be continuous. So, it is significant to study the boundary value problems with discontinuous functions.

In this paper, we are concerned with the third-order two point boundary value problem of third order differential inclusions in order Banach spaces

$$\begin{cases} -u'''(t) \in F(t, u(t), u'(t)), \quad 0 \leq t \leq 1, \\ u(0) = u'(0) = u'(1) = \theta, \end{cases} \tag{1.1}$$

where θ denotes zero element of E, $F : [0,1] \times E \times E \to 2^E$ with E a Banach space. Our purpose is to establish existence of solutions for the problem (1.1) via a fixed point theorem due to Hong [8].

Very recently, under the conditions of F being continuous and single valued real number function, Feng and Liu [5] used the upper and lower solutions method to prove some existence results of solutions to (1.1). The considerations

Received October 10, 2005

of this paper differ from them in the sense that we do not assume that both of lower and upper solutions of (1.1) exist in advance, besides, we omit the continuity assumption for the map F in ordered Banach spaces.

2. Preliminaries

In this section, we introduce some definitions, notations and preliminaries facts from multivalued analysis (for example, see [9]) which are used throughout this paper.

Let $(E, |\cdot|)$ be a Banach space with a partially order "$\leq$" introduced by a cone P of E, that is, $x \leq y$ if and only if $y - x \in P$, $x < y$ if and only if $x \leq y$ and $x \neq y$. Throughout this paper we always denote with θ the zero element of E.

Take $x_0, y_0 \in E$ and let $K = \{x \in E : x_0 \leq x\}$, $K_1 = \{x \in E : x \leq y_0\}$ be given ordered sets of E. The ordered interval of E be written as $[u, v] = \{x \in E : u \leq x \leq v\}$.

$C([0,1], E)$ is a Banach space consisting of all continuous functions from $[0,1]$ into E with the norm $\|x\| = \sup\{|x(t)| : 0 \leq t \leq 1\}$. For any $x, y \in C([0,1], E)$, define $x \leq y$ if and only if $x(t) \leq y(t)$ for each $t \in [0,1]$, $x < y$ if and only if $x \leq y$ and there exists some $t \in [0,1]$ such that $x(t) \neq y(t)$. $C^k([0,1], E)$ stands for the Banach space consisting of all functions $x(t)$ belonging to $C([0,1], E)$ and $x^{(k)}(t)$ existing and continuous, where, $k = 1, 2, 3$.

For $0 < p < \infty$, let $L^p([a,b], E)$ denote the Banach space of measurable functions $x : [a,b] \to E$ which are Bochner integrable with norm

$$\|x\|_p = \left(\int_a^b |x(t)|^p dt \right)^{1/p}.$$

The partial order in $L^p([a,b], E)$ is defined as $x \leq y$ iff $x(t) \leq y(t)$ a.e. for $t \in [a,b]$.

For $D \subset L^p([0,1], E)$, we denote $D(t) = \{x(t) : x \in D\}$ and $\int_0^t D(s)ds = \{\int_0^t x(s)ds : x \in D\}$ with $t \in [0,1]$. We always denote with $\stackrel{w}{\rightarrow}$ the weak convergence, with $\lim(w)$ the weak limit and with $cl(B)$ the closure of the set B.

For two subsets M, N of E we write $M \leq N$ if

$$\forall x \in M, \exists y \in N \text{ such that } x \leq y.$$

A multivalued operator $T : M \subset E \to 2^E \setminus \{\emptyset\}$ is said to be increasing upwards if $u, v \in M$ with $u \leq v$ and $x \in Tu$ imply that there exists $y \in Tv$ such that $x \leq y$. T is increasing downwards if $u, v \in M$, $u \leq v$ and $y \in Tv$ imply an existence of $x \in Tu$ such that $x \leq y$. If T is increasing upwards and downwards we say that T is increasing.

The multivalued operator $T : E \to 2^E$ is said to be measurable if for each $x \in E$ the distance between x and Tx is a measurable function.

The multivalued operator T has a fixed point if there is $x \in E$ such that $x \in Tx$. Let the multivalued operator $T : [0,1] \times E \to 2^E$. For each $x \in$

$C([0,1],E)$, the set of L^1-selections $S_{T,x}$ of T is defined by

$$S_{T,x} = \{f_x \in L^1([0,1],E) : f_x(t) \in T(t,x(t)) \text{ a.e. for } t \in [0,1]\}.$$

This may be empty. It is nonempty if and only if the function $y : [0,1] \to R$ defined by

$$y(t) = \inf\{|v| : v \in T(t,y(t))\}$$

belongs to $L^1([0,1],R)$ (see [10]).

Throughout this paper we always assume that the multivalued map T has nonempty, weakly closed values and L^1-selections $S_{T,x}$ is nonempty.

At the end of this section we give he following lemmas which are crucial in the proof of our main theorems.

Lemma 2.1 ([8]). *If the operator $A : K \to 2^E$ satisfies the following hypotheses:*

(H1) *A is increasing upwards and Ax is totally ordered subset for any $x \in E$.*

(H2) *If $C = \{x_n\} \subset K$ is countable and totally ordered subset and*

$$C \subset cl\left(\{x_1\} \cup A(C)\right),$$

then C is weakly relatively compact.

(H3) $\{x_0\} \leq Ax_0$.

Then A has at least one fixed point.

Lemma 2.2 ([8]). *Suppose that the conditions* (H2) *and* (H3) *of Lemma 1 are satisfied. If the following condition is satisfied:*

(H'1) *A is increasing and Ax is totally ordered subset for any $x \in E$.*

(H4) *$Ay_0 \leq \{y_0\}$ with $x_0 < y_0$.*

Then A has minimal and maximal fixed points $x_*, x^* \in [x_0, y_0]$.

Lemma 2.3 ([11]). *Let $p \in (0,\infty)$. Suppose that $M \subset L^p([0,1],E)$ is countable and exists some $v \in L^p([0,1],R_+)$ with $|u(t)| \leq v(t)$ a.e. on $[0,1]$ for all $u \in M$. If $M(t)$ is relatively compact in E for a.e. $t \in [0,1]$, then M is weakly relatively compact in $L^p([0,1],E)$.*

Lemma 2.4 ([12]). *Let $D = \{x_n\} \subset L^1([0,1],E)$. If there exists $v \in L^1([0,1],R)$ such that $|x_n(t)| \leq v(t)$ a.e. on $[0,1]$, then $\gamma(D(t)) \in L^1([0,1],R_+)$ and*

$$\gamma\left(\left\{\int_0^t x_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \gamma(D(s))ds.$$

Here γ is Kuratowskii's measure of noncompactness on E.

Let $X = C([0,1],E)$, $X_+ = \{x \in X : x(t) \geq \theta, 0 \leq t \leq 1\}$, then X_+ is a cone of X. Similarly [5], we can prove the following lemma.

Lemma 2.5. *If $z \in C^2([0,1], E)$ satisfies*

$$\begin{cases} z''(t) \geq \theta, & 0 \leq t \leq 1, \\ z(0) \leq \theta, z(1) \leq \theta, \end{cases}$$

then $z \leq \theta$.

Proof. Let X_+^* denote the dual cone of X_+, and for any $g \in X_+^*$, let $h(t) = g(z(t))$. Then we have $h \in C^2([0,1], R)$ and $h'(t) = g(z'(t))$, $h''(t) = g(z''(t))$ for any $t \in [0,1]$. In view of the above inequality we have

$$\begin{cases} h''(t) \geq 0, & 0 \leq t \leq 1, \\ h(0) \leq 0, h(1) \leq 0. \end{cases}$$

We now prove $h(t) \leq 0$ for all $t \in [0,1]$. From the above inequality it follows that $h(t)$ is a convex function on $[0,1]$. This implies that

$$h(t) = h(0(1-t) + 1t) \leq (1-t)h(0) + th(1) \leq 0, \quad \forall t \in [0,1].$$

Note that $g \in X_+^*$ is arbitrary, therefore we obtain $z(t) \leq \theta$ for all $t \in [0,1]$. This proof is completed. □

3. Main results

Let us define that a function $u \in C^2([0,1], E)$ is called the solution of problem (1.1) if $u^{(2)}$ is abstractly continuous and u satisfies (1.1) on $[0,1]$.

Let $v(t) = u'(t)$, $H(t, v(t)) = F\left(t, \int_0^t v(s)ds, v(t)\right)$, then (1.1) is equivalent to

$$\begin{cases} -v''(t) \in H(t, v(t)), & 0 \leq t \leq 1, \\ v(0) = v(1) = \theta. \end{cases} \tag{3.1}$$

Define a single valued operator $L : \Omega \subset X \to X$ and a multivalued operator $N : X \to 2^X$ as follows:

$$(Lv)(t) = -v''(t),$$
$$(Nv)(t) = \{u : u(t) = g_v(t) \text{ for } g_v \in S_{H,v}\},$$

where $\Omega = \{v \in X : v'' \in X, v(0) = v(1) = \theta\}$ and $t \in [0,1]$.

By the definition of L and N, equation(3.1) is equivalent to the following operator inclusion:

$$Lv \in Nv. \tag{3.2}$$

In order to apply Lemma 1 to prove that (3.2) has solutions, we impose the following hypothesis on map F.

(h1) $F(t, u, v)$ is a totally ordered subset in E for each $u, v \in E$ and $t \in [0,1]$. In addition, for each given $t \in [0,1]$ F is increasing upwards, i.e., for any $u_i, v_i \in E$ $(i = 1, 2)$ with $u_1 \leq u_2, v_1 \leq v_2$, we have

$$F(t, u_1, v_1) \leq F(t, u_2, v_2).$$

(h2) There exists $z_0 \in C^3([0,1], E)$ such that

$$\{-z_0'''(t)\} \leq F(t, z_0(t), z_0'(t)), \quad 0 < t < 1,$$
$$z_0(0) = \theta,\ z_0'(0) \leq \theta, z_0'(1) \leq \theta.$$

(h3) There exists $w_0 \in C^3([0,1], E)$ such that

$$F(t, w_0(t), w_0'(t)) \leq \{-w_0'''(t)\}, \quad 0 < t < 1,$$
$$w_0(0) = \theta,\ w_0'(0) \geq \theta, w_0'(1) \geq \theta.$$

(h4) $\sup\{|f(t)| : f(t) \in F(t, u, v)\} \leq b(t)$ *a.e. on* J for all $u, v \in E$. Here $b(t) \in L^1([0,1], R_+)$.

(h5) For any $t \in [0,1]$, $u_i, v_i \in C^2([0,1], E)$ $(i = 1,2)$, there exists a Carath*é*odory function $\phi : [0,1] \times R_+ \to R_+$ satisfying that $\phi(t, \cdot)$ is non-decreasing for fixed $t \in [0,1]$ and for all $f_i(t) \in F(t, u_i(t), v_i(t))$ $(i = 1,2)$ and almost every $t \in J$,

$$|f_1(t) - f_2(t)| \leq \phi(t, \max\{|u_1(t) - u_2(t)|, |v_1(t) - v_2(t)|\}).$$

In addition $\rho(t) = 0$ for every $t \in [0,1]$ is the unique solution in $L^1([0,1], R_+)$ to the inequality

$$\rho(t) \leq 2 \int_0^1 G(t,s)\phi(s, \rho(s))ds \quad \textit{a.e. on } [0,1].$$

with

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Now we can state and prove our main results.

Theorem 3.1. *Assume that the operator F satisfies conditions* (h1), (h2), (h4) *and* (h5), *then there exists a solution to* (1.1).

Proof. For every $\eta \in X$, from the well known results of ordinary differential equations it follows that the boundary value problem

$$\begin{cases} Lv(t) = -v''(t) = \eta, & 0 \leq t \leq 1, \\ v(0) = v(1) = 0. \end{cases}$$

has an unique solution v satisfying

$$v(t) = (L^{-1}\eta)(t) = \int_0^1 G(t,s)\eta(s)ds \tag{3.3}$$

with $G(t,s)$ given in (h5). Let $A = L^{-1}N$, then

$$Ax = \{L^{-1}u : u \in Nx\}.$$

It is obvious that the existence of solutions to inclusion (3.2) is equivalent to the existence of fixed points of the operator A. In order to prove that A has fixed points, we seek to apply Lemma 2.1. It is sufficient to show that A has nonempty, weakly closed values and the conditions (H1)-(H3) are satisfied. First, Let $\eta \in X, \{\eta_n\} \subset X, \eta_n \to \eta$ and $v = L^{-1}\eta, v_n = L^{-1}\eta_n$, then

$$v(t) = \int_0^1 G(t,s)\eta(s)ds, \quad v_n(t) = \int_0^1 G(t,s)\eta_n(s)ds.$$

Note that $\max\limits_{0\le t\le 1} \int_0^1 G(t,s)ds = \frac{1}{8}$, we have

$$\begin{aligned}\|v_n - v\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)(\eta_n(s)-\eta(s))ds\right| \\ &\le \max_{0\le t\le 1}\int_0^1 G(t,s)\|\eta_n(s)-\eta(s)\|ds \le \frac{1}{8}\|\eta_n - \eta\|.\end{aligned}$$

This indicates $v_n \to v$, i.e., $L^{-1} : E \to E$ is continuous. From the fact that N has nonempty and weakly closed values (because F has nonempty and weakly closed values), together with the continuity of L^{-1}, it follows that A has nonempty and weakly closed values.

Next, we show that the condition (H1) holds. To validate that A is increasing upwards, we take $x, y \in X$ with $x \le y$. Thus, for any $u \in Ax$, take $\nu \in Nx$ satisfying $u = L^{-1}\nu$, in view of assumption (h1), there exists $\xi \in Ny$ such that $\nu \le \xi$. Let $v = L^{-1}\xi$, then $v \in Ay$ and $Lu \le Lv$. Let $p(t) = u(t) - v(t)$, then $p''(t) \ge \theta$ for any $t \in [0,1]$. In view of Lemma 2.5, we have $u \le v$. This implies that $A : X \to 2^X$ is increasing upwards.

Now we prove that Ax is totally ordered subset for any $x \in E$. In fact, for any $u, v \in Ax$, there exist $x, y \in Nx$ such that $u = L^{-1}x, v = L^{-1}y$. (h1) guarantees that Nx is totally ordered. Without loss of generality, we can assume $x \le y$. Hence, $Lu \le Lv$. Again applying Lemma 2.5, it can be see that $u \le v$. This shows that Ax is totally ordered. Consequently, (H1) is satisfied.

Third, we shall prove that the condition (H3) holds. Let $x_0 = z_0'$, we claim $\{x_0\} \le Ax_0$. Indeed, by the definition of z_0, there exists $\beta \in Nx_0$ such that

$$-x_0''(t) \le \beta(t) \text{ for } 0 \le t \le 1, x_0(0) \le \theta, x_0(1) \le \theta.$$

Denote $y_0 = L^{-1}\beta$, thus $y_0 \in Ax_0$ and $Ly_0 = \beta$, that is

$$-y_0''(t) = \beta(t) \text{ for } 0 \le t \le 1, y_0(0) = y_0(1) = \theta.$$

Let $p(t) = x_0(t) - y_0(t)$. The above two expressions imply

$$-p''(t) \le \theta \text{ for } 0 \le t \le 1, p(0) \le \theta, p(1) \le \theta.$$

In virtue of Lemma 2.5, we get $p(t) \le \theta$, hence, $x_0 \le y_0$.

Finally, we cheek condition (H2) in Lemma 2.1. In order to do this, we consider any countable totally ordered subset $C = \{x_n : n \ge 1\} \subset K$ such that $C \subset cl(\{x_1\} \cup A(C))$. We shall prove that C is weakly relatively compact. Since C is countable, we can find a countable set $V = \{v_n : n \geq 1\} \subset A(C)$ with $C \subset cl(\{x_1\} \cup V)$. There exists $x_n \in C$ such that $v_n(t) \in (Ax_n)(t)$, i.e., there exists $u \in Nx_n$, $f_{x_n} \in S_{H,x_n}$ such that

$$v_n(t) = (L^{-1}u)(t) = \int_0^1 G(t,s) f_{x_n}(s)\, ds.$$

By means of the assumption (h4) we have

$$|v_n(t)| \leq \int_0^1 G(t,s)\, |f_{x_n}(s)|\, ds \leq \int_0^1 G(t,s) b(s) ds.$$

Obviously, $\int_0^1 G(t,s)b(s)ds =: c(t) \in L^1([0,1], R_+)$ and we obtain

$$|v_n(t)| \leq c(t). \tag{3.4}$$

From $C \subset cl(\{x_1\} \cup V)$ it follows that the left hand of (3.4) is also true with any $x_n \in C$ instead of v_n.

Next, let us take $x_n, x_m \in C$ and for any $f_n \in F(t, \int_0^t x_n(s)ds,, x_n(t))$, $f_m \in F(t, \int_0^t x_m(s)ds,, x_m(t))$, from the assumption (h5), we have

$$|f_n(t) - f_m(t)| \leq \phi(t, \max\{|y_n(t) - y_m(t)|, |x_n(t) - x_m(t)|\}). \tag{3.5}$$

where $y_k(t) = \int_0^t x_k(s)ds$ $(k = n, m)$. Note that $|y_n(t) - y_m(t)| \leq \int_0^t |x_n(s) - x_m(s)|ds$ for each $t \in [0,1]$. Load this into (3.5), we get

$$\begin{aligned} |f_n(t) - f_m(t)| &\leq \phi\left(t, \max\left\{\int_0^t |x_n(s) - x_m(s)|ds, |x_n(t) - x_m(t)|\right\}\right) \\ &\leq \phi\left(t, \max\left\{\int_0^t \operatorname{diam}(C(s))ds, \operatorname{diam}(C(t))\right\}\right). \end{aligned}$$

This yields

$$\operatorname{diam}\left(F\left(t, \int_0^t C(s)ds, C(t)\right)\right) \leq \phi\left(t, \max\left\{\int_0^t \operatorname{diam}(C(s))ds, \operatorname{diam}(C(t))\right\}\right).$$

Fixed $t \in [0,1]$, for any given $\varepsilon > 0$, there exists a finite number of subsets $D_1(t), D_2(t), \ldots, D_l(t)$ of $C(t)$ such that

$$C(t) \subset \bigcup_{i=1}^{l} D_i(t), \quad \operatorname{diam} D_i(t) \leq \gamma(C(t)) + \varepsilon.$$

Since

$$F\left(t, \int_0^t C(s)ds, C(t)\right) \subset \bigcup_{i=1}^{l} F\left(t, \int_0^t D_i(s)ds, D_i(t)\right).$$

By the monotonicity of $\phi(s,\cdot)$ we have

$$\begin{aligned}
&\operatorname{diam}\left(F\left(t,\int_0^t D_i(s)ds, D_i(t)\right)\right)\\
&\quad\le \phi\left(t,\max\left\{\int_0^t \operatorname{diam}(D_i(s))ds, \operatorname{diam}(D_i(t))\right\}\right)\\
&\quad\le \phi\left(t,\max\left\{\int_0^t [\gamma(C(s))+\varepsilon]ds, \gamma(C(t))+\varepsilon\right\}\right) \quad (i=1,2,\dots,l).
\end{aligned}$$

This yields

$$\gamma\left(F\left(t,\int_0^t C(s)ds, C(t)\right)\right) \le \phi\left(t,\max\left\{\int_0^t [\gamma(C(s))+\varepsilon]ds, \gamma(C(t))+\varepsilon\right\}\right).$$

Letting $\varepsilon \to 0$, we have

$$\gamma\left(F\left(t,\int_0^t C(s)ds, C(t)\right)\right) \le \phi\left(t,\max\left\{\int_0^t \gamma(C(s))ds, \gamma(C(t))\right\}\right).$$

If $\gamma(C(t)) \le \int_0^t \gamma(C(s))ds$, then $\gamma(C(t)) = 0$ by Gronwall inequality. This implies that $C(t)$ is relatively compact on $[0,1]$. In virtue of Lemma 2.3, we obtain that C is weakly relatively compact. If $\gamma(C(t)) > \int_0^t \gamma(C(s))ds$, then

$$\max\left\{\int_0^t \gamma(C(s))ds, \gamma(C(t))\right\} = \gamma(C(t)).$$

Hence,

$$\gamma\left(F\left(t,\int_0^t C(s)ds, C(t)\right)\right) \le \phi\left(t,\gamma(C(t))\right). \tag{3.6}$$

For any $v_n(t) \in V(t)$, there exists $u \in Nx_n$, $f_{x_n} \in S_{H,x_n}$ with $x_n \in C$ such that

$$v_n(t) = (L^{-1}u)(t) = \int_0^1 G(t,s)f_{x_n}(s)\,ds.$$

By this expression and Lemma 2.4 we have

$$\begin{aligned}
\gamma(V(t)) &= \gamma\left(\left\{\int_0^1 G(t,s)f_{x_n}(s)\,ds : n \ge 1\right\}\right)\\
&\le 2\int_0^1 G(t,s)\gamma(\{f_{x_n}(s) : n \ge 1\})ds
\end{aligned}$$

for every $t \in [0,1]$. While by means of (3.6) we get

$$\gamma(C(t)) \le \gamma(V(t)) \le 2\int_0^1 G(t,s)\phi(s,\gamma(C(s)))ds.$$

By virtue of the character of ϕ, we obtain

$$\gamma(C(t)) = 0$$

for any $t \in [0,1]$. This implies that $C(t)$ is relatively compact for almost every $t \in J$. In view of Lemma 2.3, we obtain that C is weakly relatively compact. Consequently, (H2) holds.

To sum up, Lemma 2.1 guarantees that the operator A has a fixed point which is clearly a solutions of (3.2). This proof is completed. □

Theorem 3.2. *Assume that the operator F satisfies conditions* (h1), (h2) *and* (h4) *and the following condition*

(h'5) *For any $t \in [0,1]$, $u_i, v_i \in C^2([0,1], E)$ $(i = 1,2)$, there exists a function $\phi : R_+ \to R_+$ satisfying that $\phi(\cdot)$ is nondecreasing, $\phi(r+) \leq r(\forall r > 0)$ and for all $f_i(t) \in F(t, u_i(t), v_i(t))(i = 1,2)$ and almost every $t \in J$,*

$$|f_1(t) - f_2(t)| \leq \phi(\max\{\|u_1(t) - u_2(t)\|, \|v_1(t) - v_2(t)\|\}).$$

then there exists a solution to (1.1) *in K.*

Proof. We only want to prove that C is weakly relatively compact. Suppose, on the contrary, $\gamma(C) > 0$. In the same way as the proof of Theorem 3.1, we may obtain

$$\gamma(C(t)) \leq \gamma(V(t)) \leq 2\int_0^1 G(t,s)\phi(\gamma(C)+)ds.$$

This implies that

$$\gamma(C) \leq 2\int_0^1 G(t,s)\phi(\gamma(C)+)ds < \phi(\gamma(C)+) \leq \gamma(C),$$

a contradiction. Hence, $\gamma(C) = 0$, i.e., C is relatively compact. This proof is completed. □

The next result is dual to that of Theorem 3.1 (Theorem 3.2).

Theorem 3.3. *Assume that the operator F satisfies conditions* (h3), (h4) *and* (h5) ((h'5)) *and the following condition*

- *$F(t,u,v)$ is a totally ordered subset in E for each $u, v \in E$ and $t \in [0,1]$.*

In addition, for each given $t \in [0,1]$ F is increasing downwards, then there exists a solution to (1.1) *in K_1.*

In the light of Lemma 2.2, we can obtain the existence theorem of maximal and minimal solutions to (1.1):

Theorem 3.4. *Let the conditions* (h2)–(h5)((h'5)) *hold with $x_0 = z_0' < w_0' = y_0$. Moreover, if*

(h) *$F(t, u, v)$ is a totally ordered subset in E for each $u, v \in E$ and $t \in [0, 1]$. In addition, for each given $t \in [0, 1]$ F is increasing.*
Then (1.1) *has maximal and minimal solutions on* $[x_0, y_0]$.

In what follows, we consider existence of multiple solutions for (1.1)

Theorem 3.5. *Let the conditions* (h), (h2)–(h5)((h'5)) *hold with* $x_0 = z_0' > w_0' = y_0$. *Then* (1.1) *has at least two solutions.*

Proof. In view of Theorem 3.1 (Theorem 3.2) and Theorem 3.3 we immediately get that problem (1.1) has solutions $x^* \in K$ and $y^* \in K_1$. From our assumptions it follows $K \cap K_1 = \emptyset$, which yields $x^* \neq y^*$. Therefore problem (1.1) has two solutions. This proof is completed. □

4. An example

Consider the two point boundary value problem of infinite system for nonlinear scalar third order differential inclusions

$$\left\{ \begin{array}{l} -u_j'''(t) \in F_j(t, u(t), u'(t)), \quad 0 \le t \le 1, \\ u_j(0) = u_j'(0) = u_j'(1) = \theta, \end{array} \right. \tag{4.1}$$

where $F_j = [\min\{f_j, g_j\}, \max\{f_j, g_j\}]$ with

$$f_j(t, u(t), v(t)) = \frac{t+1}{j^2} + \frac{1}{32(j+1)^2}[\sin v_{j+1}(t) + \cos u_j(t)],$$
$$g_j(t, u(t), v(t)) = \frac{t+1}{j^2} + \frac{1}{32(j+1)^2}[\cos v_{j+1}(t) + \sin u_j(t)].$$

Here, $j = 1, 2, \ldots$. Let $E = l^1 = \{u_1, u_2, \ldots, u_j, \ldots) : \sum_{j=1}^{\infty} |u_j| < \infty\}$ with norm $|u| = \sum_{j=1}^{\infty} |u_j|$, cone $P = \{u \in l^1 : u_j \ge 0, j = 1, 2, \ldots\}$, $f = (f_1, f_2, \ldots, f_j, \ldots)$ and multivalued operator $F(t, u(t), u'(t))$ stand for the totally ordered subset of infinite rectangle $F_1(t, u(t), u'(t)) \times F_2(t, u(t), u'(t)) \times \cdots \times F_j(t, u(t), u'(t)) \times \cdots$. Clearly, F has nonempty, weakly closed values and the hypothesis (h) is satisfied. *Conclusion.* If (4.1) is regarded as a boundary value problem of form (1.1), then it admits a solution in P. Accordingly, (4.1) admits at least two solutions.

Proof. We shall show that all conditions of Theorem 3.5 are satisfied. Let $z_0 = \theta = (0, 0, \ldots)$, $w_0(t) = (-t^3, \ldots, -\frac{t^3}{j^2}, \ldots)$ and $b(t) = \frac{\pi^2 t}{6} + \frac{\pi^2}{3} - 1$, then $z_0' > w_0'$ and (h2), (h3) and (h4) hold. Let $\phi(x) = x$, then for any $t \in [0, 1]$, $u_1, u_2, v_1, v_2 \in X$ and any $h_1 \in F(t, u_1, v_1), h_2 \in F(t, u_2, v_2)$, denote $h_1 = (f_1^1, f_2^1, \ldots), h_2 = (f_1^2, f_2^2, \ldots)$ with $f_j^1 \in F_j(t, u_1, v_1), f_j^2 \in F_j(t, u_2, v_2)$ for

$j = 1, 2, \ldots$, we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &= \sum_{j=1}^{\infty} |f_j^1(t, u_1(t), v_1(t)) - f_j^2(t, u_2(t), v_2(t))| \\
&\le \frac{1}{32} \sum_{j=1}^{\infty} 16|v_{j+1}^1(t) - v_{j+1}^2(t)| + \sum_{j=1}^{\infty} 16|u_j^1(t) - u_j^2(t)| \\
&\le \frac{1}{32}(16|v_1(t) - v_2(t)| + 16|u_1(t) - u_2(t)|) \\
&\le \max\{|v_1(t) - v_2(t)|, |u_1(t) - u_2(t)|\} \\
&\le \max\{\|v_1 - v_2\|, \|u_1 - u_2\|\} = \phi(\max\{\|v_1 - v_2\|, \|u_1 - u_2\|\}),
\end{aligned}$$

where $w_i = (w_1^i, w_2^i, \ldots)$ with $i = 1, 2$ and $w = u, v$. Consequently, (h'5) is satisfied. □

Department of Mathematics
Hangzhou Dianzi University
Hangzhou 310018, People's Republic of China
e-mail: hongshh@hotmail.com

References

[1] L. K. Jackson, *Existence and uniqueness of solutions of boundary value problems for third order differntial equations*, J. Differntial Equations, **13** (1973), 432–437.

[2] D. J. O'Regan, *Topological transversality: Application to third-order boundary value problem*, SIAM J. Math. Anal. **19** (1987), 630–541.

[3] W. C. Troy, *Solution of third order differntial equations relevant to draining and coating flows*, SIAM J. Math. Anal. **24** (1993), 166–171.

[4] A. Cabada, *The method of lower and upper solutions for second, third, fourth and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994), 302–320.

[5] Y. Feng and S. Liu, *Solvability of a third-order two-point boundary value problem*, Appl. Math. Lett. **18** (2005), 1034–1040.

[6] F. Bernis and L. A. Peleter, *Two problems from draining flows involving third-order ordinary differntial equation*, SIAM J. Math. Anal. **27** (1996), 515–527.

[7] M. Ruyun, *Multiplicity results for a third order boundary value problem at resonance*, Nonlinear Anal. **32** (1998), 493–500.

[8] S. H. Hong, *Fixed points of discontinuous multivalued increasing operators in Banach spaces with applications*, J. Math. Anal. Appl. **282** (2003), 151–162.

[9] K. Deimling, Multivalued differential equations, Walter de Gruyter, Berlin-New York, 1992.

[10] N. S. Papageorgiou, *Boundary value problems for evolution inclusions*, Comment. Math. Univ. Carolin. **29** (1988), 355–363.

[11] J. Diestel, W. M. Ruess and W. Schachemayer, *Weak compactness in* $L^1(\mu, X)$, Proc. Amer. Math. Soc. **118** (1993), 447–453.

[12] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983), 1351–1371.