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Remarks on long range scattering for nonlinear Schrödinger equations with Stark effects

By

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Abstract

In this paper, the global existence and asymptotic behavior in time of solutions for the nonlinear Schrödinger equation with the Stark effect in one or two space dimensions are studied. The nonlinearity is cubic and quadratic in one and two dimensional cases, respectively, and it is a summation of a gauge invariant term and non-gauge invariant terms. This nonlinearity is critical between the short range scattering and the long range one. A modified wave operator to this equation is constructed for small final states. Its domain is a certain small ball in $H^2 \cap \mathcal{F}H^2$, where $\mathcal{F}$ is the Fourier transform.

1. Introduction

We study the global existence and large time behavior of solutions for the nonlinear Schrödinger equation with the Stark effect in one or two space dimensions:

$$i\partial_t u = -\frac{1}{2}\Delta u + (E\cdot x)u + \widetilde{F}_n(u), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

where $n = 1, 2$ and u is a complex valued unknown function of (t, x). Here $\widetilde{F}_n(u)$ and $E \cdot x$ are a nonlinearity and a linear potential, respectively. The nonlinearity is given by

$$\begin{gathered} \widetilde{F}_n(u) = G_n(u) + \widetilde{N}_n(u), \\ G_n(u) = \lambda_0 |u|^{2/n} u, \\ \widetilde{N}_1(u) = \lambda_1 u^3 + \lambda_2 \bar{u}^3, \quad \text{when } n = 1, \\ \widetilde{N}_2(u) = \lambda_1 u^2 + \lambda_2 \bar{u}^2 + \lambda_3 u\bar{u}, \quad \text{when } n = 2, \end{gathered} \tag{1.2}$$

where $\lambda_0 \in \mathbb{R}$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $E \in \mathbb{R}^n \setminus \{0\}$. We remark that the cubic nonlinearity $u\bar{u}^2$ is excluded in one dimensional case. $\widetilde{F}_n$ is a summation of the gauge invariant nonlinearity $G_n(u)$ and the non-gauge invariant one $\widetilde{N}_n(u)$,

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and it is a critical power nonlinearity between the short range case and the long range one in n space dimensions $(n = 1, 2)$. The potential $E \cdot x$ is called the Stark potential with a constant electric field E. In this paper, we prove the existence of modified wave operators to the equation (1.1) for small final states, and extend our previous results [10]. The domains of these modified wave operators are $H^2 \cap H^{0,2}$, which are larger than those in [10]. Namely, the assumptions on final states are weakened. The method to estimate the non-gauge invariant terms is essentially different from that in [10] (the detail will be mentioned below).

The theory of scattering for the ordinary nonlinear Schrödinger equations with critical power nonlinearities was studied, e.g., in [4, 5, 6, 7, 8, 9, 10, 11, 12]. We recall the result in [10] on the long range scattering for the equation (1.1). Let $U(t)$ be the free Schrödinger group, that is,

$$U(t) = e^{it\Delta/2}.$$

The Schrödinger operator $-(1/2)\Delta + E\cdot x$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. H_E denotes the self-adjoint realization of that operator defined on $C_0^\infty(\mathbb{R}^n)$ and we define the unitary group U_E generated by H_E:

$$U_E(t) = e^{-itH_E}.$$

$\widetilde{F}_n(u)$ is a critical power nonlinearity between the short range scattering and the long range one. In [10], the existence of the modified wave operator $\widetilde{W}_+$ was shown. The modified wave operator $\widetilde{W}_+$ for the equation (1.1) is defined as follows. Let ϕ be a final state. Modifying the solution $U_E(t)\phi$ for the linear Schrödinger equation with the Stark potential, we construct a suitable modified free dynamics A, which depends on ϕ, and we show the existence of a unique solution u for the equation (1.1) which approaches A in L^2 as $t \to \infty$. The mapping

$$\widetilde{W}_+ : \phi \mapsto u(0)$$

is called a modified wave operator. The domain of the modified wave operator in [10] is $H^2 \cap H^{0,2}$ and $H^2 \cap H^{0,3}$ in one and two space dimensions, respectively. First we reduce our problem to the equation

$$i\partial_t v = -\frac{1}{2}\Delta v + F_n(t, v), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3}$$

where $n = 1, 2$,

$$\begin{aligned} F_n(t, v) &= G_n(v) + N_n(t, v),\\ N_1(t, v) &= \lambda_1 v^3 e^{-2i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 \bar{v}^3 e^{4i(tE\cdot x - t^3|E|^2/3)},\\ N_2(t, v) &= \lambda_1 v^2 e^{-i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 \bar{v}^2 e^{3i(tE\cdot x - t^3|E|^2/3)}\\ &\quad + \lambda_3 v\bar{v} e^{i(tE\cdot x - t^3|E|^2/3)}, \end{aligned}$$

$G_n(v)$ is defined by (1.2). By a suitable change of variables (see Proposition 2.1 below), our problem is equivalent to constructing modified wave operators for the equation (1.3). In [10], we constructed a modified free dynamics of the form $B = v_a + v_1$, where

$$\begin{aligned} v_a(t,x) =& (U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi)(x) \\ =& \frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}, \end{aligned}$$

$S(t,x) = \lambda_0|\hat{\phi}(x)|^{2/n}\log t$ and v_1 is a faster decaying term mentioned below. This modified free dynamics v_a was introduced by Ozawa [8] for the ordinary nonlinear Schrödinger equation with a nonlinearity $\lambda|u|^2u$ in one space dimension. In order to overcome difficulties caused by the gauge invariant nonlinearity $G_n(v)$ which is a long range interaction (see Barab [1]), we introduced the principal term v_a of the asymptotics B with a phase shift so that $\mathcal{L}v_a - G_n(v_a)$ decays faster than $G(v_a)$, where $\mathcal{L} = i\partial_t + (1/2)\Delta$. In order to treat the non-gauge invariant nonlinearity $N_n(t,v)$, we construct a second correcting term v_1 such that v_1 and $\mathcal{L}v_1 - N_n(t,v_a)$ decay faster than v_a and $N_n(t,v_a)$, respectively (more precisely, the L^2-norms of $\mathcal{L}v_a - G_n(v_a)$ and $\mathcal{L}v_1 - N_n(t,v_a)$ are integrable over the interval $[1,\infty)$, while those of $G(v_a)$ and $N_n(t,v_a)$ are not). So we see that $\mathcal{L}B - F_n(t,B) = (\mathcal{L}v_a - G_n(v_a)) + (\mathcal{L}v_1 - N_n(t,v_a)) +$ (faster decaying terms) and its L^2-norm is integrable over the interval $[1,\infty)$. By the Cook-Kuroda method, we obtained a unique solution v of the equation (1.3) which approaches the profile B. Since v_1 decays faster than v_a, the solution v approaches the modified free dynamics v_a.

In this paper, we prove the existence of modified wave operators for the equation (1.1), which is equivalent to that for the equation (1.3), without constructing a second correcting term such as the function v_1 in [10] mentioned above. The domain of the modified wave operator in this paper is slightly larger than that in [10]. As in [10], we introduce the principal term v_a in order to overcome difficulties caused by the gauge invariant nonlinearity $G_n(v)$ (see Lemma 3.2). In order to treat the non-gauge invariant nonlinearity $N_n(t,v)$, we show that

$$\left\|\int_t^\infty U(t-s)N_n(s,v_a(s))\,ds\right\|_{L_x^2},$$

which appears in the associated integral equation, is integrable over the interval $[1,\infty)$. More precisely, we prove that it decays at a suitable rate in time (see Lemma 3.3). Hence we see that

$$\left\|\int_t^\infty U(t-s)(\mathcal{L}v_a(s) - F_n(s,v_a(s)))\,ds\right\|_{L_x^2}$$

decays suitably in time and we can directly construct a unique solution u which approaches the asymptotic profile v_a. The method in this paper is more precise than that in [10], because we do not construct an approximation in order to overcome the effect from $N_n(t,v)$. This is the reason why we can extend the domain of the modified wave operator.

Before stating our main results, we introduce several notations.

Notation. We denote the Schwartz space on $\mathbb{R}^n$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^n$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi}\, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

and put $H^s = H^{s,0}$.

C denotes a constant and so forth. They may differ from line to line, when it does not cause any confusion.

Our result is as follows.

Theorem 1.1. *Let $n = 1$ or 2. Assume that $\phi \in H^2 \cap H^{0,2}$ and that $\|\phi\|_{H^2 \cap H^{0,2}}$ is sufficiently small. Then the equation (1.1) has a unique solution u satisfying*

$$u \in C([0,\infty); L^2),$$
$$\sup_{t\geq 1}(t^d \|u(t) - U_E(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi\|_{L^2}) < \infty,$$
$$\sup_{t\geq 1}\left[t^d \left(\int_t^\infty \|U(s)(U_E(-s)u(s) - e^{-i|\cdot|^2/2s}e^{-iS(s,-i\nabla)}\phi)\|_{Y_n}^4\, ds\right)^{1/4}\right] < \infty,$$

where

$$S(t,x) = \lambda_0|\phi(x)|^{2/n}\log t \tag{1.4}$$

and d is a constant satisfying $n/4 < d < 1$, $Y_1 = L_x^\infty$ and $Y_2 = L_x^4$.

Furthermore the modified wave operator $\widetilde{W}_+ \colon \phi \mapsto u(0)$ is well-defined.

A similar result holds for negative time.

Remark 1.1. Since the multiplication operator $e^{-i|\cdot|^2/2t}$ converges the identity strongly in L^2 as $t \to \infty$, the solution obtained in Theorem 1.1 approaches $U_E(t)e^{-iS(t,-i\nabla)}\phi$ in L^2. Noting the phase correction S depends only on the gauge invariant nonlinearity $G_n(u)$, we see that the contribution of the non-gauge invariant term $\widetilde{N}_n(u)$ is a short range interaction, that is, it is negligible as $t \to \infty$, under our assumptions. We also note that the assumption $\phi \in H^2$ is needed only if $\widetilde{N}_n(u) \neq 0$ (see Lemma 3.3 below).

Remark 1.2. If we consider the asymptotic behavior of solutions to the Cauchy problem for the equation (1.1) with initial data $u(0,x) = \phi_0(x)$, $x \in \mathbb{R}^n$, then we see from Theorem 1.1 that for any initial data ϕ_0 belonging to the range of the modified wave operator $\widetilde{W}_+$, there exists a unique global solution $u \in C([0,\infty); L^2)$ of the Cauchy problem for the equation (1.1) which has the modified free profile $U_E(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi$. More precisely, u satisfies the asymptotic formula of Theorem 1.1. However it is not clear how to describe the initial data belonging to the range of the operator $\widetilde{W}_+$.

The outline of this paper is as follows. In Section 2, we reduce the scattering problem for the equation (1.1) to that of the equation (1.3), and we solve the Cauchy problem at infinite initial time for the equation (1.3) under suitable decay and approximate conditions on v_a. In Section 3, we show that the asymptotics v_a satisfies the assumptions of this Cauchy problem at infinite initial time and prove Theorem 1.1.

2. The Cauchy Problem at Infinite Initial Time

First we reduce the scattering problem for the equation (1.1) to that of the non-autonomous nonlinear Schrödinger equation (1.3) without a potential as mentioned in Section 1. By a direct calculation, we obtain the following relation between a solution to the equation (1.1) and that to the equation (1.3). The following proposition is not essentially new but almost well-known (see Cycon, Froese, Kirsch and Simon [3]).

Proposition 2.1. *If v solves the equation* (1.3)*, then*

$$u(t,x) = v\left(t, x + \frac{t^2}{2}E\right) e^{-i(tE\cdot x + t^3|E|^2/6)}$$

solves the equation (1.1).

Conversely, if u solves the equation (1.1)*, then*

$$v(t,x) = u\left(t, x - \frac{t^2}{2}E\right) e^{i(tE\cdot x - t^3|E|^2/3)}$$

solves the equation (1.3).

Remark 2.1. Recently, the above change of variables has been applied to the nonlinear Schrödinger equation with the Stark effects and the gauge invariant nonlinearity by Carles and Nakamura [2].

According to Proposition 2.1, Theorem 1.1 is an immediate consequence of Proposition 2.2 below.

Proposition 2.2. *Assume that ϕ satisfies all the assumptions of Theo-*

rem 1.1. *Then there exists a unique solution* v *for the equation* (1.3) *satisfying*

$$v \in C([0,\infty);L^2),$$
$$\sup_{t\ge 1}\left(t^d\|v(t)-U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi\|_{L^2}\right)<\infty,$$
$$\sup_{t\ge 1}\left[t^d\left(\int_t^\infty \|v(s)-U(s)e^{-i|\cdot|^2/2s}e^{-iS(s,-i\nabla)}\phi\|_{Y_n}\,ds\right)^{1/4}\right]<\infty,$$

where S *is defined by* (1.4), d *is a constant satisfying* $n/4<d<1$, $Y_1=L^\infty_x$ *and* $Y_2=L^4_x$.

A similar result holds for negative time.

In what follows, we shall prove Proposition 2.2.

Let $n=1,2$, and let v_a be a given asymptotic profile of the equation (1.3), namely an approximate solution for that equation as $t\to\infty$. We introduce the following function:

$$R=\mathcal{L}v_a-F_n(t,v_a), \tag{2.1}$$

where

$$\mathcal{L}=i\partial_t+\frac{1}{2}\Delta.$$

The function R is difference between the left hand sides and the right hand ones in the equation (1.3) substituted $v=v_a$.

We can prove the following proposition (see Propositions 3.4 and 3.5 in [10]).

Proposition 2.3. *Assume that there exists a constant* $\eta'>0$ *such that*

$$\|v_a(t)\|_{L^2}\le \eta',$$
$$\|v_a(t)\|_{L^\infty}\le \eta'(1+t)^{-n/2},$$
$$\left\|\int_t^\infty U(t-s)R(s)\,ds\right\|_{L^2_x}+\left\|\int_\tau^\infty U(\tau-s)R(s)\,ds\right\|_{L^4_\tau((t,\infty);Y_n)}\le \eta'(1+t)^{-d},$$

for $t\ge 0$, *where* $Y_1=L^\infty_x$ *and* $Y_2=L^4_x$, *and assume that* $\eta'>0$ *is sufficiently small. Then there exists a unique solution* v *for the equation* (1.3) *satisfying*

$$v\in C([0,\infty);L^2),$$
$$\sup_{t\ge1}\left(t^d\|v(t)-v_a(t)\|_{L^2}\right)<\infty,$$
$$\sup_{t\ge1}\left[t^d\left(\int_t^\infty\|v(s)-v_a(s)\|^4_{Y_n}\,ds\right)^{1/4}\right]<\infty,$$

where d *is a constant satisfying* $n/4<d<1$, $Y_1=L^\infty_x$ *and* $Y_2=L^4_x$.

A similar result holds for negative time.

3. Remainder Estimates and Proof of Theorem 1.1

In this section, we prove Proposition 2.2 to obtain Theorem 1.1.

First we introduce the Strichartz estimate for the free Schrödinger equation obtained by Yajima [13]. We define the linear operator

$$(\Gamma h)(t) = \int_t^\infty U(t-s)h(s)\,ds,$$

where h is a function of (t,x).

Lemma 3.1. *Let n denote the space dimension, and let (q,r) and $(\tilde{q},\tilde{r})$ be pairs of positive numbers satisfying $2/q = n(1/2-1/r)$, $2 < q \le \infty$, $2/\tilde{q} = n(1/2-1/\tilde{r})$ and $2 < \tilde{q} \le \infty$. Then Γ is a bounded operator from $L_t^{\tilde{q}'}((T_0,\infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$ into $L_t^q((T_0,\infty); L_x^r(\mathbb{R}^n))$ with norm uniformly bounded with respect to T_0, where $(\tilde{q}',\tilde{r}')$ is a pair of positive numbers satisfying $1/\tilde{q} + 1/\tilde{q}' = 1$ and $1/\tilde{r} + 1/\tilde{r}' = 1$. Furthermore, if $h \in L_t^{\tilde{q}'}((T_0,\infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$, then $\Gamma h \in C_t([T_0,\infty); L_x^2(\mathbb{R}^n))$.*

Let

$$\begin{aligned} v_a(t,x) =&(U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi)(x) \\ =&\frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}, \end{aligned} \tag{3.1}$$

where S is defined by (1.4). This modified free dynamics was introduced by Ozawa [8] for the ordinary nonlinear Schrödinger equation with a nonlinearity $\lambda|u|^2u$ in one space dimension. In order to prove Proposition 2.2, we show that v_a satisfies the assumptions in Proposition 2.3. It is sufficient to show only the estimates

$$\|v_a(t)\|_{L^2} \le \eta', \tag{3.2}$$

$$\|v_a(t)\|_{L^\infty} \le \eta' t^{-n/2}, \tag{3.3}$$

$$\begin{aligned} &\left\|\int_t^\infty U(t-s)R(s)\,ds\right\|_{L_x^2} \\ &\quad + \left\|\int_s^\infty U(s-\tau)R(\tau)\,d\tau\right\|_{L_s^4((t,\infty);Y_n)} \le \eta' t^{-d}, \end{aligned} \tag{3.4}$$

where R is defined by (2.1). In fact, in order to avoid a singularity at $t=0$, multiplying a cut off function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t) = 0$ if $t \le 1/2$ and $\theta(t) = 1$ if $t \ge 3/4$ to v_a, we easily see from the estimates (3.2)–(3.4) that the resulting function satisfies the assumptions in Proposition 2.3.

First we consider the gauge invariant nonlinearity $G_n(u)$.

Lemma 3.2. *There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\|v_a(t)\|_{L^2} = \|\phi\|_{L^2},$$
$$\|v_a(t)\|_{L^\infty} \leq C\|\phi\|_{L^1} t^{-n/2},$$
$$\|\mathcal{L}v_a(t) - G_n(v_a(t))\|_{L^2} \leq C(\|\phi\|_{H^{0,2}} + \|\phi\|_{H^{0,2}}^3)\frac{(\log t)^2}{t^2}.$$

Since we can prove this lemma in the same way as Lemma 2.2 in [10], we omit the proof.

We next consider the non-gauge invariant and non-autonomous nonlinearity $N_n(t,u)$. In order to obtain the estimate (3.4), we need the following lemma.

Lemma 3.3. *Assume that $\|\phi\|_{H^2 \cap H^{0,2}} \leq 1$. Then, there exists a constant $C > 0$ such that for $t \geq 1$,*

$$\left\| \int_t^\infty U(t-s) N_n(s, v_a(s))\, ds \right\|_{L^2_x} + \left\| \int_s^\infty U(s-\tau) N_n(\tau, v_a(\tau))\, d\tau \right\|_{L^4_s((t,\infty);Y_n)} \leq C\|\phi\|_{H^2\cap H^{0,2}} t^{-d},$$

where $0 < d < 1$.

Proof. It is sufficient to prove for a single power nonlinearity of the form

$$N_n(t,v) = \lambda v^l \bar{v}^m e^{-i(\alpha-1)(tE\cdot x - t^3|E|^2/3)},$$

where $\lambda \in \mathbb{C}$,

$$(l,m) = (3,0) \text{ or } (0,3), \text{ when } n = 1,$$
$$(l,m) = (2,0),\ (1,1) \text{ or } (0,2) \text{ when } n = 2,$$
$$\alpha = l - m.$$

Note that $l + m = 1 + 2/n$ and $\alpha \neq \pm 1$. Then

$$\begin{aligned} &N_n(t, v_a) \\ &= \frac{1}{t^{1+n/2}} P\left(\frac{x}{t}\right) e^{i\alpha\theta_1(t,x)} e^{i(\alpha-1)(\theta_2(t,x)+\theta_3(t))} \\ &= \frac{1}{i(\alpha-1)|E|^2} \frac{1}{t^{3+n/2}} P\left(\frac{x}{t}\right) e^{i\alpha\theta_1(t,x)} e^{i(\alpha-1)\theta_2(t,x)} \partial_t(e^{i(\alpha-1)\theta_3(t)}), \end{aligned} \tag{3.5}$$

where

$$P(x) = i^{-\alpha n/2} \widehat{\phi}(x)^l \overline{\widehat{\phi}(x)}^m,$$
$$\theta_1(t,x) = \frac{|x|^2}{2t} - S\left(t, \frac{x}{t}\right), \quad \theta_2(t,x) = -tE\cdot x, \quad \theta_3(t) = \frac{t^3|E|^2}{3}.$$

We calculate the integrand $U(-s)N_n(s, v_a(s))$:

$$
\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
=&\partial_s\left[U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right]\\
&+\frac{i}{2}U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\\
&+iU(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}\\
&+\frac{i}{2}U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\Delta\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}\\
&-U(-s)\left\{\partial_s\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\right)e^{i(\alpha-1)\theta_3(s)}\right\}.
\end{aligned}
$$

Noting the relation

$$
\Delta\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)=i(\alpha-1)e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)}),
$$

we have

$$
\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
=&\partial_s\left[U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right]\\
&+\frac{i}{2}U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\\
&+iU(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}\\
&-\frac{\alpha-1}{2}U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
&-U(-s)\left\{\partial_s\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\right)e^{i(\alpha-1)\theta_3(s)}\right\}.
\end{aligned}
$$

Since $\alpha\neq-1$, we have

$$
\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
=&\frac{2}{\alpha+1}\partial_s\left[U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right]\\
&+\frac{i}{\alpha+1}U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\\
&+\frac{2i}{\alpha+1}U(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}
\end{aligned}
$$

$$- \frac{2}{\alpha+1} U(-s) \left\{ \partial_s \left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)\theta_2(s,x)} \right) e^{i(\alpha-1)\theta_3(s)} \right\}.$$

By the identity (3.5), the above identity is equivalent to

$$\begin{aligned} &U(-s)N_n(s, v_a(s)) \\ (3.6) \qquad &= \frac{1}{i(\alpha-1)|E|^2} \left(\partial_s (U(-s)I_1(s)) + \sum_{j=2}^{4} U(-s)I_j(s) \right), \end{aligned}$$

where

$$\begin{aligned} I_1(s) &= \frac{2}{\alpha+1} \left\{ \frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))} \right\}, \\ I_2(s) &= \frac{i}{\alpha+1} \left\{ \Delta \left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} \right) e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))} \right\}, \\ I_3(s) &= \frac{2i}{\alpha+1} \left\{ \nabla \left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} \right) \cdot \nabla \left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))} \right) \right\}, \\ I_4(s) &= -\frac{2}{\alpha+1} \left\{ \partial_s \left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)\theta_2(s,x)} \right) e^{i(\alpha-1)\theta_3(s)} \right\}. \end{aligned}$$

Integrating the identity (3.6) over the interval (t, ∞) and applying $U(t)$ to the resulting equality, we have

$$\begin{aligned} &\int_t^\infty U(t-s)N_n(s, v_a(s))\, ds \\ (3.7) \qquad &= \frac{1}{i(\alpha-1)|E|^2} \left(-I_1(t) + \sum_{j=2}^{4} \int_t^\infty U(t-s)I_j(s)\, ds \right). \end{aligned}$$

By the definitions of I_1, I_2, I_3 and I_4, we have

$$\begin{aligned} \|I_1(t)\|_{L^2} &\le Ct^{-3} \|\hat{\phi}\|_{L^2} \|\hat{\phi}\|_{L^\infty}^{2/n}, \\ \|I_1(t)\|_{L^\infty} &\le Ct^{-7/2} \|\hat{\phi}\|_{L^\infty}^3, \quad \text{when } n = 1, \\ \|I_1(t)\|_{L^4} &\le Ct^{-4} \|\hat{\phi}\|_{L^8}^2, \quad \text{when } n = 2, \\ \|I_2(s)\|_{L^2} &\le Cs^{-3} (\log s)^2 \|\phi\|_{H^2 \cap H^{0,2}}, \\ \|I_3(s)\|_{L^2} &\le Cs^{-2} (\log s) \|\phi\|_{H^2 \cap H^{0,2}}, \\ \|I_4(s)\|_{L^2} &\le Cs^{-2} (\log s) \|\phi\|_{H^2 \cap H^{0,2}}. \end{aligned}$$

We have used Hölder's inequality, the Sobolev embedding and the assumption $\|\phi\|_{H^2 \cap H^{0,2}} \le 1$. We note that the L^2-norms of I_2, I_3 and I_4 are integrable over the interval (t, ∞). Applying the above inequalities and Lemma 3.1 to the identity (3.7), we obtain this lemma. □

Proof of Theorem 1.1. Assume all the assumptions in Theorem 1.1. Let v_a be the function defined by (3.1). According to Proposition 2.3, as mentioned before, it is sufficient to show the estimates (3.2) through (3.4). The estimates (3.2) and (3.3) immediately follow from the definition of v_a. We prove the estimate (3.4). Since

$$R = \mathcal{L}v_a - G_n(v_a) - N_n(t, v_a),$$

by Lemmas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}
&\left\| \int_t^\infty U(t-s)R(s)\,ds \right\|_{L_x^2} + \left\| \int_s^\infty U(s-\tau)R(\tau)\,ds \right\|_{L_s^4((t,\infty);Y_n)} \\
&\le C \int_t^\infty \|\mathcal{L}v_a(s) - G_n(v_a(s))\|_{L^2}\,ds \\
&\quad + \left\| \int_t^\infty U(t-s)N_n(s, v_a(s))\,ds \right\|_{L_x^2} \\
&\quad + \left\| \int_s^\infty U(s-\tau)N_n(\tau, v_a(\tau))\,d\tau \right\|_{L_s^4((t,\infty);Y_n)} \\
&\le C\|\phi\|_{H^2\cap H^{0,2}} t^{-d},
\end{aligned}$$

where $n/4 < d < 1$ appearing in the assumption of Theorem 1.1. Taking $\eta' = C\|\phi\|_{H^2\cap H^{0,2}}$, we see that the condition (3.4) is satisfied. According to Proposition 2.3, this completes the proof of Theorem 1.1. □

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