Generalized obstacle problem

By

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Abstract

In their 1995 paper [3], Fehlmann and Gardiner posed an extremal problem, which will be called an obstacle problem throughout the present paper, for quadratic differentials on Riemann surfaces. A compact set E in a Riemann surface S of finite type is called an obstacle if each component of E is relatively contractible in S and if $S \setminus E$ is connected. For a given obstacle E and a symmetric integrable quadratic differential $\varphi \neq 0$ on S, the obstacle problem is to find a conformal embedding g of $S \setminus E$ into another Riemann surface R of the same type as S and a symmetric quadratic differential ψ on R so that the following three conditions hold: (i) the borders and punctures are preserved under the mapping g, (ii) the pull-back $g^* \psi$ gives the same heights vector as that of φ, and (iii) the norm $\|\psi\|_{L^1(R)}$ is maximal among those embeddings. Fehlmann and Gardiner asserted existence and uniqueness of a solution to the obstacle problem when E consists only of finitely many components. It seems, however, that the uniqueness assertion is not correct in their form. In the present paper, we extend the existence theorem and give a correction to the uniqueness assertion in the general case. As an application we provide a slit mapping theorem for an open Riemann surface of finite genus.

1. Introduction

Since early 20th century, through the work of Koebe, de Possel and others, it has turned out that a solution to an extremal problem often gives a conformal mapping with distinguished properties, such as parallel slit mappings. In this regard, it is important to find a good extremal problem for the investigation of complex analysis. Fehlmann and Gardiner [3] studied such an extremal problem, which will be called the obstacle problem (see also [5, Chap.13]). In this paper, we extend their results to the case where an obstacle may have uncountably many connected components. In order to formulate the problem, we give some explanations for necessary notions.

For a holomorphic quadratic differential $\varphi = \varphi(z)dz^2$ on a Riemann surface S, we define the L^1-norm $\|\varphi\|_{L^1(S)}$ of it by

$$\|\varphi\|_{L^1(S)} = \iint_S |\varphi(z)|dx\,dy, \quad z = x + iy.$$

If $\|\varphi\|_{L^1(S)} < \infty$, the quadratic differential φ is called integrable. We denote by $A(S)$ the set of integrable holomorphic quadratic differentials on S.

Suppose that S is a hyperbolic Riemann surface. Then there is a holomorphic universal covering projection P of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto S. The group

$$\Gamma = \{\gamma \in \text{Aut}\mathbb{D} : P \circ \gamma = P\}$$

is called a Fuchsian model of S. When Γ is of the second kind, namely, when the limit set $\Lambda(\Gamma)$ of Γ does not coincide with $\partial\mathbb{D}$, the Riemann surface $S^{\text{d}} = (\bar{\mathbb{C}} \setminus \Lambda(\Gamma))/\Gamma$ is called the (Schottky) *double* of S, and $\partial S = (\partial\mathbb{D} \setminus \Lambda(\Gamma))/\Gamma$ is called the *border* of S. When Γ is of the first kind, we set $S^{\text{d}} = S$ and $\partial S = \emptyset$.

Let S be a Riemann surface of finite type (κ, m, l), that is, S is obtained by removing mutually disjoint l topological compact disks and m points from a compact Riemann surface of genus κ. When $l = 0$, we say that S is of finite analytic type (κ, m). When $l > 0$, the double S^{d} is of finite analytic type $(2\kappa + l - 1, 2m)$. We assume that the number $6\kappa - 6 + 3l + 2m$ is positive in the sequel so that $\dim_{\mathbb{C}} A(S^{\text{d}}) > 0$.

A quadratic differential $\varphi = \varphi(z)dz^2$ in $A(S)$ is called *symmetric* relative to ∂S if there is a quadratic differential $\varphi^{\text{d}} \in A(S^{\text{d}})$ with $\varphi^{\text{d}}|_S = \varphi$ for which $\overline{j^*\varphi^{\text{d}}} = \varphi^{\text{d}}$ holds. Here $j : S^{\text{d}} \to S^{\text{d}}$ is the canonical anti-conformal involution of S^{d} induced by the conjugation $z \mapsto 1/\bar{z}$, in other words, the lift $\tilde{\varphi}$ of φ^{d} to $\bar{\mathbb{C}} \setminus \Lambda(\Gamma)$ is symmetric relative to $\partial\mathbb{D} : \overline{\tilde{\varphi}(1/\bar{z})} \cdot z^{-4} = \tilde{\varphi}(z)$. We denote by $A_{\text{s}}(S)$ the set of quadratic differentials $\varphi \in A(S)$ symmetric relative to ∂S. Note that $\dim_{\mathbb{R}} A_{\text{s}}(S) = 6\kappa - 6 + 3l + 2m$ if S is of finite type (κ, m, l) with $l > 0$.

For a Riemann surface S of finite analytic type let $\mathfrak{S}(S)$ be the set of simple closed curves γ on S which are homotopic neither to a point in S nor to a puncture of S, and let $\mathfrak{S}[S]$ be the set of the free homotopy classes $[\gamma]$ of $\gamma \in \mathfrak{S}(S)$.

For a given $\varphi \in A(S) \setminus \{0\}$ and $\gamma \in \mathfrak{S}(S)$, we set

$$\text{height}_\varphi(\gamma) := \int_\gamma \left|\text{Im}\left(\sqrt{\varphi(z)}dz\right)\right|$$

and

$$\text{height}_\varphi[\gamma] := \inf_\beta \text{height}_\varphi(\beta),$$

where the infimum is taken over all closed curves $\beta \in \mathfrak{S}(S)$ freely homotopic to γ.

We are now ready to state the obstacle problem in the sense of Fehlmann and Gardiner [3]. They thought of a "simply connected" compact subset with finitely many connected components as an obstacle. Here we will include a set

with uncountably many components as an obstacle. In this paper we say that a subset E of a Riemann surface S is *relatively contractible* in S if there is a continuous map $H : E \times [0,1] \to S$ such that $H(p,1) = p$ for all $p \in E$ and that $H(p,0) = p_0 \in S$ is independent of $p \in E$.

Definition 1.1 (allowable subset)**.** A subset E of a Riemann surface S is said to be *allowable* if E is compact and relatively contractible in S and if $S \setminus E$ is connected. For an element $\varphi \in A(S)$, if each component of an allowable subset E is either a horizontal arc (resp. vertical arc) of φ or the union of finitely many horizontal arcs (resp. vertical arcs) and critical points of φ, then E is called an *allowable horizontal slit* (resp. *allowable vertical slit*) with respect to φ.

Note that an allowable set E may have uncountably many components.

Let S be a Riemann surface of finite type and that $E \subset S$ be allowable. A conformal embedding $g : S \setminus E \to S_g$ will be called *type-preserving* (with respect to S) if S_g is a Riemann surface of the same type as S and if g maps the border and the punctures of S onto the border and the punctures of S_g, respectively. Then the mapping g naturally extends to a conformal map g^{d} of $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ into the double S_g^{d} of S_g. We remark that $E^{\mathrm{d}} = E \cup j(E)$ and $E_g = S_g \setminus g(S \setminus E)$ are both allowable (see Lemma 2.2). For further properties of an allowable set, see Section 2. Then g^{d} induces an isomorphism ι_g of the fundamental group $\pi_1(S^{\mathrm{d}})$ of S^{d} onto that of S_g^{d} (see Lemma 2.3). Note that the isomorphism ι_g induces naturally the bijection

$$\iota_g : \mathfrak{S}[S^{\mathrm{d}}] \to \mathfrak{S}[S_g^{\mathrm{d}}].$$

Definition 1.2 (obstacle problem)**.** Let E be an allowable subset of a Riemann surface S of finite type and $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. The *obstacle problem for* (S, E, φ) is to find a triple (g, S_g, ψ), where g is a type-preserving conformal embedding of $S \setminus E$ with respect to S into another Riemann surface S_g and $\psi \in A_{\mathrm{s}}(S_g)$, with the following properties:
  (i) $\mathrm{height}_{\psi^{\mathrm{d}}} \circ \iota_g = \mathrm{height}_{\varphi^{\mathrm{d}}}$ on $\mathfrak{S}[S^{\mathrm{d}}]$ and
  (ii) the norm $\|\psi\|_{L^1(S_g)}$ is maximal among those triples satisfying (i).

Our first main theorem is stated as in the following, which is a generalization of a result of Fehlmann and Gardiner [3].

Theorem 1.3 (existence)**.** *Suppose that S is a Riemann surface of finite type and that $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. Let E be an allowable subset of S. Then there is a solution to the obstacle problem for (S, E, φ). For every solution (g, S_g, ψ), $E_g = S_g \setminus g(S \setminus E)$ is an allowable horizontal slit with respect to ψ, and E_g is of zero area.*

In [3], Fehlmann and Gardiner showed the above result under the additional assumption that E consists of finitely many components. They also asserted in the paper that the solution (g, S_g, ψ) is unique in the sense that, if (u, S_u, θ) is also a solution for (S, E, φ), then $g \circ u^{-1}$ extends to a conformal

map w of S_u onto S_g and $w_* \theta = \psi$ holds. The uniqueness, however, does not necessarily hold in this sense (see [9] for a counterexample).

In this paper we show a uniqueness result for the obstacle problem in the following form.

Theorem 1.4 (uniqueness). *Under the same hypotheses as in Theorem 1.3, the extremal differential $\psi \in A(S_g) \setminus \{0\}$ is uniquely determined. Namely, if a triple (u, S_u, θ) is also a solution for (S, E, φ), then θ satisfies*

$$\theta = (\psi \circ w)(w')^2 \ on \ u(S \setminus E),$$

where $w = g \circ u^{-1}$.

The present paper is organized as follows. Section 2 will be devoted to preparations including basic properties of allowable sets and some facts about quadratic differentials. Theorems 1.3 and 1.4 will be proved in Sections 3 and 4, respectively. In Section 5, as an application, we give a slit mapping theorem for an open Riemann surface of finite genus.

We remark that, if we consider the extremal problem of finding a triple (g, S_g, ψ) satisfying (i) in Definition 1.2 and

(ii)' the norm $\|\psi\|_{L^1(S_g)}$ is minimal among those triples satisfying (i), then the similar result can be obtained just by replacing the term "horizontal" by "vertical". That is, there is a solution of the extremal problem and for every solution (g, S_g, ψ), E_g is an allowable vertical slit with respect to ψ, and E_g is of zero area.

2.   Preliminaries

We collect here basic properties of allowable subsets of a Riemann surface. First we give another characterization of relatively contractible set in S.

Lemma 2.1. *Let E be a compact subset of a Riemann surface S. Then E is relatively contractible in S if and only if E is contained in a compact topological disk in S.*

Proof. The "if" part is trivial. We show the "only if" part when S is hyperbolic. The non-hyperbolic cases can be treated similarly. Suppose that E is relatively contractible in S . Then there is a continuous map $H : E \times [0,1] \to S$ such that $H(p,1) = p$ for all $p \in E$ and that $H(p,0) = p_0$ is independent of $p \in E$. Let P be the universal covering projection of $\mathbb{D}$ onto S such that $P(0) = p_0$ and let Γ be the covering transformation group of P. Let $\tilde{H} : E \times [0,1] \to \mathbb{D}$ be a lift of the homotopy H via P such that $\tilde{H}(p,0) = 0$ for all $p \in E$. The set $\tilde{E} = \{\tilde{H}(p,1) : p \in E\}$ is compact. For every $\tilde{p} \in \tilde{E}$ we can take a positive number $r_{\tilde{p}}$ with the property that the closed disk $V_{\tilde{p}}$ centered at $\tilde{p}$ with radius $r_{\tilde{p}}$ satisfies

$$\mathrm{dist}(A(V_{\tilde{p}}), \tilde{E}) \geq 2r_{\tilde{p}} \ \text{for every} \ A \in \Gamma, A \neq \mathrm{id}.$$

By compactness of $\tilde{E}$ we may assume that there exist finitely many points $\tilde{p}_1, \ldots, \tilde{p}_k \in \tilde{E}$ so that $\tilde{E}$ is contained in the interior of the closed set $\tilde{V} = \cup_{i=1}^{k} V_{\tilde{p}_i}$. We may assume that by replacing radii, if necessary, $V_{\tilde{p}_i}$ is not tangent to $V_{\tilde{p}_j}$ for every $i \neq j$. We can see that $A(\tilde{V}) \cap \tilde{V} = \emptyset$ for every $A \in \Gamma, A \neq \mathrm{id}$. Indeed, if $A(\tilde{V}) \cap \tilde{V} \neq \emptyset$ for some $A \in \Gamma, A \neq \mathrm{id}$, there exist $i, j \in \{1, \ldots, k\}$ and $\tilde{q}_i \in V_{\tilde{p}_i}$ and $\tilde{q}_j \in V_{\tilde{p}_j}$ such that $A(\tilde{q}_i) = \tilde{q}_j$. Because of $\mathrm{dist}(A(V_{\tilde{p}_i}), \tilde{E}) \geq 2r_{\tilde{p}_i}$, we have $r_{\tilde{p}_j} \geq \mathrm{dist}(\tilde{q}_j, \tilde{p}_j) \geq \mathrm{dist}(\tilde{q}_j, \tilde{E}) \geq \mathrm{dist}(A(V_{\tilde{p}_i}), \tilde{E}) \geq 2r_{\tilde{p}_i}$ holds. On the other hand, by considering A^{-1}, we have $r_{\tilde{p}_i} \geq 2r_{\tilde{p}_j}$. This is a contradiction. Hence $A(\tilde{V}) \cap \tilde{V} = \emptyset$ for every $A \in \Gamma, A \neq \mathrm{id}$. Let $\tilde{W}$ be a polynomially convex hull of $\tilde{V}$, that is $\tilde{W} \subset \mathbb{D}$ is the union of $\tilde{V}$ and all compact components of $\mathbb{D} \setminus \tilde{V}$. Then $\tilde{W}$ is a disjoint union of finitely many compact topological disks whose boundaries are contained in $\tilde{V}$. Then we can see that for every components $\tilde{W}_1, \tilde{W}_2$ of $\tilde{W}$ and for every $A \in \Gamma, A \neq \mathrm{id}$, the intersection $A(\tilde{W}_1) \cap \tilde{W}_2$ is empty or coincident with either $A(\tilde{W}_1)$ or $\tilde{W}_2$, because by the above argument $A(\partial \tilde{W}_1) \cap \partial \tilde{W}_2 = \phi$ holds. Therefore the projection $P(\tilde{W})$ also consists of mutually disjoint compact topological disks in S whose union contains E in its interior. Consequently, by joining these disks with suitable canals we obtain a compact topological disk in S containing E. $\qquad \square$

By using the above lemma, we can also show the following.

Lemma 2.2. *Let E be an allowable subset of a Riemann surface S of finite type. Then $E^{\mathrm{d}} = E \cup j(E)$ is an allowable subset of the double S^{d}. Also, for any type-preserving conformal embedding $g : S \setminus E \to S_g$ with respect to S the set $E_g = S_g \setminus g(S \setminus E)$ is allowable in S_g.*

Proof. By joining a compact topological disk Δ containing E and its reflection in ∂S with a suitable canal, we obtain a compact topological disk Δ^{d} containing E^{d}. Thus E^{d} is allowable by Lemma 2.1. We now show the latter assertion. We may take a compact topological disk Δ so that E is contained in the interior of Δ. If S is of finite type (κ, m, l), then $S \setminus \Delta$ and $g(S \setminus \Delta)$ are of type $(\kappa, m, l + 1)$. Then the set $\Delta_g = S_g \setminus g(S \setminus \Delta) \subset S_g$ is of finite type, say, (κ', m', l'). Since S_g is reconstructed from $g(S \setminus \Delta)$ and Δ_g by gluing along a single Jordan curve, $(\kappa, m, l) = (\kappa, m, l + 1) + (\kappa', m', l') - (0, 0, 2)$, i.e., $(\kappa', m', l') = (0, 0, 1)$. Then the set Δ_g which contains E_g in the interior is a compact topological disk in S_g. Thus the latter assertion has been proved. $\quad \square$

Lemma 2.3. *Let E be an allowable subset of a Riemann surface S of finite type. Then the induced homomorphism $\sigma : \pi_1(S \setminus E) \to \pi_1(S)$ by the inclusion mapping $S \setminus E \hookrightarrow S$ is surjective. Furthermore, let $g : S \setminus E \to S_g$ be a type-preserving conformal embedding with respect to S. Then the induced isomorphism $(g)_* : \pi_1(S \setminus E) \to \pi_1(S_g \setminus E_g)$ decends to an isomorphism $\overline{(g)_*} : \pi_1(S) \to \pi_1(S_g)$ so that the following diagram commutes, where $\sigma_g : \pi_1(S_g \setminus$*

$E_g) \to \pi_1(S_g)$ *is the induced homomorphism*:

$$
\begin{array}{ccc}
\pi_1(S \setminus E) & \xrightarrow{(g)_*} & \pi_1(S_g \setminus E_g) \\
\sigma \downarrow & & \sigma_g \downarrow \\
\pi_1(S) & \xrightarrow{\overline{(g)_*}} & \pi_1(S_g).
\end{array}
$$

Proof. By Lemma 2.2 there is a compact topological disk $V \subset S$ bounded by an analytic curve and containing E in its interior V°. Let γ be an arbitrary closed curve in S. We will find a closed curve γ' in $S \setminus V^\circ$ freely homotopic to γ in S. We may assume that γ is smooth and intersects ∂V transversally at finitely many points. If $\gamma \cap \partial V = \emptyset$, we can take γ' as γ itself. If $\gamma \cap \partial V \neq \emptyset$, γ can be devided into closed subarcs $\gamma_1, \delta_1, \gamma_2, \delta_2, \ldots, \gamma_k, \delta_k$ in such a way that $\gamma = \gamma_1 \delta_1 \gamma_2 \delta_2 \cdots \gamma_k \delta_k$, $\gamma_j \subset S \setminus V^\circ$ and $\delta_j \subset V$ for $j = 1, 2, \ldots, k$. Let δ'_j be a subarc of ∂V sharing end points with δ_j. Set $\gamma' = \gamma_1 \delta'_1 \gamma_2 \delta'_2 \cdots \gamma_k \delta'_k$. Since $\delta_j \sim \delta'_j$ in V, $\gamma' \sim \gamma$ in S. Thus we can see that the homomorphism $\sigma : \pi_1(S \setminus E) \to \pi_1(S)$ is surjective.

Next define $\overline{(g)_*} : \pi_1(S) \to \pi_1(S_g)$ by $\overline{(g)_*}[\gamma] = [g(\gamma')]$, where γ' is the curve constructed above. By construction, $\overline{(g)_*}$ is well-defined, homomorphic and makes the diagram in the lemma commute. Considering $g^{-1} : S_g \setminus E_g \to S \setminus E$, we see that $\overline{(g)_*}$ is isomorphic. $\qquad \square$

From Lemmas 2.2 and 2.3 we can see that for every type-preserving conformal embedding $g : S \setminus E \to S_g$ with respect to S, the natural isomorphism $(g^d)_* : \pi_1(S^d \setminus E^d) \to \pi_1(S_g^d \setminus E_g^d)$ induces an isomorphism $\overline{(g^d)_*} : \pi_1(S^d) \to \pi_1(S_g^d)$. We denote $\overline{(g^d)_*}$ by ι_g.

Here we recall a definition of the Teichmüller space $T(S)$ of a Riemann surface S of finite analytic type (see [6] for the compact case). Let (R, ι) be a marked Riemann surface of the same type as S, that is, a pair of a Riemann surface R of the same type as S and an orientation-preserving isomorphism ι of the fundamental group $\pi_1(S)$ onto $\pi_1(R)$. (More rigorously, we should consider the fundamental group with base point. Though we do not refer to the base point to avoid complexity, the reader can formulate it in an obvious way.) Two pairs (R_1, ι_1) and (R_2, ι_2) are called (Teichmüller) equivalent if there exists a conformal mapping u of R_1 onto R_2 such that $(u)_* \circ \iota_1 = \iota_2$. The set of such equivalent classes $[R, \iota]$ is called the Teichmüller space of S and denoted by $T(S)$. Every point in $T(S)$ is represented as $[R, f_*]$ by a (smooth) quasiconformal map $f : S \to R$. For the existence of such a quasiconformal map, see [7].

We next recall heights mappings (cf. [4, §11.7], [5, §12.6]). Let $|dv|$ be a measured foliation on a Riemann surface S of finite analytic type, namely, there are finitely many singularities $p_1, \ldots, p_t$ in $\hat{S}$, where $\hat{S}$ is the completion of S and an open cover $\{U_j\}$ of $S \setminus \{p_1, \ldots, p_t\}$ and real-valued continuous functions v_j on U_j with locally L^2 derivatives in such a way that $v_j = \pm v_k + \text{const.}$ on $U_j \cap U_k$ and $|dv|$ behaves around each p_s like the pull-back of $|\mathrm{Im}\,(z^{n/2}dz)|$ under a quasiconformal map with certain restriction on the integer n.

Let φ be a non-zero integrable holomorphic quadratic differential on a Riemann surface S of finite analytic type. We say that a curve $\gamma \in \mathfrak{S}(S)$ is *φ-polygonal* if γ is a union of finitely many horizontal and vertical arcs of φ. For a measured foliation $|dv|$ on S, by abuse of language, we define the height of γ relative to $|dv|$ by

$$\operatorname{height}_v(\gamma) = \int_\gamma |dv|.$$

Note that the height is defined for almost every φ-polygonal curves in $\mathfrak{S}(S)$. We denote by $\operatorname{height}_v[\gamma]$ the essential infimum of $\operatorname{height}_v(\gamma')$ over $\gamma' \in [\gamma]$. Measured foliations $|dv_1|$ and $|dv_2|$ on S are called measure equivalent if $\operatorname{height}_{v_1}[\gamma] = \operatorname{height}_{v_2}[\gamma]$ holds for all $[\gamma] \in \mathfrak{S}[S]$. Let $\mathfrak{MF}(S)$ be the set of measure equivalence classes of measured foliations on S. Every measure equivalence class $[|dv|] \in \mathfrak{MF}(S)$ induces the real-valued function

$$\operatorname{height}_v : \mathfrak{S}[S] \to \mathbb{R} \quad ([\gamma] \mapsto \operatorname{height}_v[\gamma]).$$

In this way, we obtain an embedding

$$\mathfrak{MF}(S) \to \mathbb{R}^{\mathfrak{S}[S]} \quad ([|dv|] \mapsto \operatorname{height}_v).$$

Then the product topology of $\mathbb{R}^{\mathfrak{S}[S]}$ induces a topology of $\mathfrak{MF}(S)$. It is known that the mapping

$$\Psi : A(S) \setminus \{0\} \to \mathfrak{MF}(S) \quad (\varphi \mapsto [|\operatorname{Im}(\sqrt{\varphi(z)}dz)|])$$

is a homeomorphism (see, for example, [5, p. 227]).

For a given $\varphi \in A(S) \setminus \{0\}$ and a local homeomorphism $f : S \to R$, the measured foliation $|dv| = |\operatorname{Im}(\sqrt{\varphi(z)}dz)|$ on S induces a measured foliation $f_*(|dv|) = |d(v \circ f^{-1})|$ on R. The relation

$$\operatorname{height}_\varphi[\gamma] = \operatorname{height}_{f_*(|dv|)}[f(\gamma)], \quad [\gamma] \in \mathfrak{S}[S],$$

implies that the measure equivalence class $[f_*(|dv|)] \in \mathfrak{MF}(R)$ depends only on the Teichmüller equivalence class $\tau = [R, f_*]$. Then we obtain the unique holomorphic quadratic differential $\tau_*\varphi \in A(R) \setminus \{0\}$ as $\Psi^{-1}[f_*(|dv|)]$, namely,

$$\operatorname{height}_{\tau_*\varphi}[\gamma] = \operatorname{height}_{f_*(|dv|)}[\gamma]$$
$$= \operatorname{height}_\varphi[f^{-1}(\gamma)]$$

for every $[\gamma] \in \mathfrak{S}[R]$. The mapping $\tau_* : A(S) \setminus \{0\} \to A(R) \setminus \{0\}$ is called the heights mapping.

The following variant of the second minimal norm property will play an important role in this note. The assertion can be seen by analyzing the proof of Theorem 9 in [4, p. 54].

Proposition 2.4. *Let S be a Riemann surface of finite analytic type. Suppose that $\varphi \in A(S)$ and ψ is an integrable (not necessarily holomorphic) quadratic differential on S such that*

$$\operatorname{height}_\varphi[\gamma] \leq \operatorname{height}_\psi(\gamma)$$

for almost every φ-polygonal curve $\gamma \in \mathfrak{S}(S)$. Then

$$\|\varphi\|_{L^1(S)} \leq \iint_S \left| \sqrt{\varphi} \sqrt{\psi} \right|$$

and, in particular, $\|\varphi\|_{L^1(S)} \leq \|\psi\|_{L^1(S)}$. Moreover $\|\varphi\|_{L^1(S)} = \|\psi\|_{L^1(S)}$ only if $\varphi = \psi$ a.e.

Now we can show

Lemma 2.5. *Let E be an allowable subset of a Riemann surface S of finite type and $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. Suppose that $g : S \setminus E \to S_g$ is a type-preserving conformal embedding with respect to S. Then there exists the unique $\psi \in A_{\mathrm{s}}(S_g)$ satisfying the condition* (i) *in Definition* 1.2.

Proof.   We can take $\psi = \tau_*(\varphi^{\mathrm{d}})\big|_{S_g}$, where $\tau = [S_g^{\mathrm{d}}, \iota_g] \in T(S^{\mathrm{d}})$ and τ_* is the heights mapping. Then we have only to show that the quadratic differential $\tau_*(\varphi^{\mathrm{d}})$ is symmetric relative to ∂S_g. Indeed,

$$\begin{aligned}
\mathrm{height}_{\overline{j_g^* \tau_* \varphi^{\mathrm{d}}}}[\gamma] &= \mathrm{height}_{\tau_* \varphi^{\mathrm{d}}}[j_g(\gamma)] \\
&= \mathrm{height}_{\varphi^{\mathrm{d}}}[\iota_g^{-1} \circ j_g(\gamma)] \\
&= \mathrm{height}_{\varphi^{\mathrm{d}}}[j \circ \iota_g^{-1}(\gamma)] \\
&= \mathrm{height}_{\varphi^{\mathrm{d}}}[\iota_g^{-1}(\gamma)] \\
&= \mathrm{height}_{\tau_* \varphi^{\mathrm{d}}}[\gamma],
\end{aligned}$$

for every $[\gamma] \in \mathfrak{S}[S_g^{\mathrm{d}}]$, where $j_g : S_g^{\mathrm{d}} \to S_g^{\mathrm{d}}$ is the canonical anticonformal involution of S_g^{d}, and Proposition 2.4 implies

$$\overline{j_g^* \tau_* \varphi^{\mathrm{d}}} = \tau_* \varphi^{\mathrm{d}},$$

that is, $\tau_* \varphi^{\mathrm{d}}$ is symmetric relative to ∂S_g.

The uniqueness of ψ follows from injectivity of τ_*. $\square$

The quadratic differential ψ in Lemma 2.5 will be denoted by φ_g in the sequel for a type-preserving conformal embedding $g : S \setminus E \to S_g$ with respect to S. We also write

$$M_g = \|\varphi_g\|_{L^1(S_g)}.$$

Following [3], we now introduce a few lemmas. Let S' be a subdomain of a Riemann surface S. For a simple closed curve β on S' we denote by $\lambda([\beta]_S, S')$ the extremal length of the family of all closed curves on S' freely homotopic to β in S. (For the definition of the extremal length, see [2].) We remark that if subdomains S' and S'' of S satisfy $S' \subset S''$ then

$$\lambda([\beta]_S, S'') \leq \lambda([\beta]_S, S')$$

follows. For a point $\tau = [R, \iota]$ in $T(S)$, let $K_0(\tau)$ be the dilatation of the unique extremal quasiconformal map from S onto R which induces the isomorphism $\iota : \pi_1(S) \to \pi_1(R)$. In other words, $\log K_0(\tau)$ is the Teichmüller distance between τ and the base point $[S, \mathrm{id}]$.

Lemma 2.6 ([5, p. 247]). *Let S be a Riemann surface of finite analytic type. Then there exists a positive constant c and finitely many simple closed curves $\beta_1, \ldots, \beta_N$ on S such that the inequality*

$$K_0(\tau) \leq c \max_{1 \leq l \leq N} \lambda(\iota[\beta_l]_R, R)$$

holds for every point $\tau = [R, \iota]$ in $T(S)$.

Lemma 2.7 ([4, p. 218]). *Let S be a Riemann surface of finite analytic type and $\varphi \in A(S) \setminus \{0\}$. Then for every $\tau \in T(S)$*

$$K_0(\tau)^{-1} \|\varphi\|_{L^1(S)} \leq \|\tau_* \varphi\|_{L^1(S_\tau)} \leq K_0(\tau) \|\varphi\|_{L^1(S)}.$$

Let E be an allowable subset of a Riemann surface S of finite type. Let $\mathfrak{F}(S, E)$ denote the set of pairs (g, S_g), where $g : S \setminus E \to S_g$ is a type-preserving conformal embedding with respect to S. Set $\mathfrak{T}(S, E) = \{[S_g^{\mathrm{d}}, \iota_g] \in T(S^{\mathrm{d}}) : (g, S_g) \in \mathfrak{F}(S, E)\}$.

Lemma 2.8. *Let E be an allowable subset of a Riemann surface S of finite type and $\varphi \in A_{\mathrm{s}}(S) \setminus \{0\}$. Then, the set $\mathfrak{T}(S, E)$ is relatively compact in $T(S^{\mathrm{d}})$ and there is a positive constant c_0 such that*

$$c_0^{-1} \|\varphi\|_{L^1(S)} \leq M_h \leq c_0 \|\varphi\|_{L^1(S)}$$

for every $(h, S_h) \in \mathfrak{F}(S, E)$.

Proof. For the double S^{d} we take a family of curves $\beta_1, \ldots, \beta_N$ and a positive constant c as in Lemma 2.6. We may assume that each β_k is contained in $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$. For every $(h, S_h) \in \mathfrak{F}(S, E)$ set $\tau = [S_h^{\mathrm{d}}, \iota_h] \in T(S^{\mathrm{d}})$. Lemma 2.7, together with $\|\varphi^{\mathrm{d}}\|_{L^1(S^{\mathrm{d}})} = 2\|\varphi\|_{L^1(S)}$ and $\|\varphi_h^{\mathrm{d}}\|_{L^1(S_h^{\mathrm{d}})} = 2\|\varphi_h\|_{L^1(S_h)}$, implies

$$\frac{\|\varphi\|_{L^1(S)}}{K_0(\tau)} \leq \|\varphi_h\|_{L^1(S_h)} \leq K_0(\tau) \|\varphi\|_{L^1(S)}.$$

Moreover $h^{\mathrm{d}}(S^{\mathrm{d}} \setminus E^{\mathrm{d}}) \subset S_h^{\mathrm{d}}$ and Lemma 2.6 imply

$$\begin{aligned}
K_0(\tau) &\leq c \max_{1 \leq l \leq N} \lambda(\iota_h[\beta_l]_{S_h^{\mathrm{d}}}, S_h^{\mathrm{d}}) \\
&= c \max_{1 \leq l \leq N} \lambda([h^{\mathrm{d}}(\beta_l)]_{S_h^{\mathrm{d}}}, S_h^{\mathrm{d}}) \\
&\leq c \max_{1 \leq l \leq N} \lambda([h^{\mathrm{d}}(\beta_l)]_{S_h^{\mathrm{d}}}, h^{\mathrm{d}}(S^{\mathrm{d}} \setminus E^{\mathrm{d}})) \\
&= c \max_{1 \leq l \leq N} \lambda([\beta_l]_{S^{\mathrm{d}}}, S^{\mathrm{d}} \setminus E^{\mathrm{d}}) < \infty.
\end{aligned}$$

The last constant is independent of $(h, S_h) \in \mathfrak{F}(S, E)$. This means that $\mathfrak{T}(S, E)$ is relatively compact in $T(S^{\mathrm{d}})$. Moreover, letting

$$c_0 = c \max_{1 \leq l \leq N} \lambda([\beta_l]_{S^{\mathrm{d}}}, S^{\mathrm{d}} \setminus E^{\mathrm{d}}),$$

we have

$$c_0^{-1}\|\varphi\|_{L^1(S)} \le M_h = \|\varphi_h\|_{L^1(S_h)} \le c_0\|\varphi\|_{L^1(S)}$$

for every $(h, S_h) \in \mathfrak{F}(S, E)$. □

3. Proof of Theorem 1.3

Take a sequence $\{(g_n, S_{g_n})\}_n \subset \mathfrak{F}(S, E)$ such that

$$\lim_{n\to\infty} M_{g_n} = \sup_{(h,S_h)\in\mathfrak{F}(S,E)} M_h.$$

Set $\tau_n = [S^{\mathrm{d}}_{g_n}, \iota_{g_n}] \in T(S^{\mathrm{d}})$. Let Γ and Γ_n $(n \in \mathbb{N})$ be the normalized Fuchsian models ([6, p. 59]) of S and S_{g_n}, respectively. Since $\{[S^{\mathrm{d}}_{g_n}, \iota_{g_n}]\}_n$ is relatively compact in $T(S^{\mathrm{d}})$ by Lemma 2.8, we may assume that the sequence $[S^{\mathrm{d}}_{g_n}, \iota_{g_n}]$ is convergent in $T(S^{\mathrm{d}})$. Then the isomorphisms $\chi_n : \Gamma \to \Gamma_n$ induced by ι_{g_n} algebraically converge to an isomorphism $\chi_\infty : \Gamma \to \Gamma_\infty$. We denote by S_∞ the quotient surface $\mathbb{D}/\Gamma_\infty$. Let G be a Fuchsian model of $S \setminus E$ and $\rho : G \to \Gamma$ be a surjective homomorphism induced by the inclusion mapping $S \setminus E \hookrightarrow S$. For each $n \in \mathbb{N}$, set

$$\rho_n = \chi_n \circ \rho : G \to \Gamma_n.$$

Since χ_n converge to χ_∞, the homomorphisms ρ_n converge to the surjective homomorphism

$$\rho_\infty = \chi_\infty \circ \rho : G \to \Gamma_\infty.$$

Let $\tilde{g}_n : \mathbb{D} \to \mathbb{D}$ $(n \in \mathbb{N})$ be a lift of $g_n : S \setminus E \to S_g$ associated with ρ_n, e.g.,

$$\tilde{g}_n \circ A = \rho_n(A) \circ \tilde{g}_n$$

for every $A \in G$. Since $\{\tilde{g}_n\}_n$ is normal, there is a subsequence which converges to a holomorphic map $\tilde{g}$ on $\mathbb{D}$ uniformly on any compact subset of $\mathbb{D}$. Then we obtain

$$\tilde{g} \circ A = \rho_\infty(A) \circ \tilde{g}$$

for every $A \in G$. If $\tilde{g}$ is a constant c, then Γ_∞ has the common fixed point c, and thus Γ_∞ is a cyclic group, which is not the case. So $\tilde{g}$ is non-constant and thus a holomorphic map into $\mathbb{D}$. Therefore $\tilde{g}$ descends to an injective holomorphic map g of $S \setminus E$ into S_∞ such that the homomorphism $(g)_*$ corresponds to ρ_∞. We denote S_∞ by S_g. Then it is easy to see that $(g, S_g) \in \mathfrak{F}(S, E)$.

Let $\hat{\Gamma}$, $\hat{\Gamma}_n$ $(n \in \mathbb{N})$ and $\hat{\Gamma}_\infty$ be the normalized Fuchsian models of the compact Riemann surfaces $\hat{S}$, $\hat{S}_{g_n}$ and $\hat{S}_g$, which are the completion of the doubles S^{d}, $S^{\mathrm{d}}_{g_n}$ and S^{d}_g, respectively. (In the case when $\hat{S}$ is not hyperbolic, the proof will be simpler.) Let f_n be a quasiconformal map of S^{d} onto $S^{\mathrm{d}}_{g_n}$ which represents $[S^{\mathrm{d}}_{g_n}, \iota_{g_n}] \in T(S^{\mathrm{d}})$ and let f be a quasiconformal map of S^{d} onto S^{d}_g representing the element $[S^{\mathrm{d}}_g, \iota_g] \in T(S^{\mathrm{d}})$. Then f_n and f extend to quasiconformal mappings of $\hat{S}$ onto $\hat{S}_{g_n}$ and $\hat{S}_g$, respectively. Let $\hat{f}_n$ and $\hat{f} : \mathbb{D} \to \mathbb{D}$ be the normalized lifts of f_n and f to the universal covering space $\mathbb{D}$ over $\hat{S}_{g_n}$ and $\hat{S}_g$, respectively (see the following commutative diagram).

$$\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\hat{f}} & \mathbb{D} \\
\downarrow & & \downarrow \\
\hat{S} & \xrightarrow{f} & \hat{S}_g
\end{array}$$

Let $\hat{\Omega}$ be a relatively compact Dirichlet fundamental domain of $\hat{\Gamma}$ with base point at the origin. Set $\hat{\Omega}_n = \hat{f}_n(\hat{\Omega})$ ($n \in \mathbb{N}$) and $\hat{\Omega}_\infty = \hat{f}(\hat{\Omega})$. Then $\hat{\Omega}_n$ and $\hat{\Omega}_\infty$ are fundamental domains of $\hat{\Gamma}_n$ and $\hat{\Gamma}_\infty$, respectively. Let $\hat{\varphi}_{g_n}$ be a lift of $\varphi_{g_n}^{\mathrm{d}}$ to the universal covering space $\mathbb{D}$ over $\hat{S}_{g_n}$. For each $n \in \mathbb{N}$, $\hat{\varphi}_{g_n}$ is a meromorphic quadratic differential on $\mathbb{D}$ with at most simple poles. Let D be the subdomain of $\mathbb{D}$ which is obtained from $\mathbb{D}$ by removing the points corresponding to the punctures of S_g^{d}.

Lemma 3.1.   *The family $\{\hat{\varphi}_{g_n}\}_n$ is normal on D.*

Proof.   It is sufficient to show that $\{\|\hat{\varphi}_{g_n}\|_{L^1(K)}\}_n$ is bounded for any compact subset K of $\mathbb{D}$. Let

$$\{A \in \hat{\Gamma}_\infty : A(\hat{\Omega}_\infty \cup \partial\hat{\Omega}_\infty) \cap K \neq \emptyset\} = \{A_1, \dots, A_{m_0}\}.$$

Then, the euclidean distance d_∞ between K and $(\cup_{i=1}^{m_0} A_i(\hat{\Omega}_\infty))^c$ is positive. Since we can assume that $\hat{f}_n$ converge to $\hat{f}$ uniformly on any compact subset of $\mathbb{D}$, for each A_i ($1 \leq i \leq m_0$) the Möbius transformations

$$A_{i,n} = (\hat{f}_n \circ \hat{f}^{-1}) \circ A_i \circ (\hat{f}_n \circ \hat{f}^{-1})^{-1} \in \hat{\Gamma}_n$$

converge to A_i as $n \to \infty$. Then the distances d_n between K and $(\cup_{i=1}^{m_0} A_{i,n}(\hat{\Omega}_n))^c$ converge to d_∞. Hence $d_n > 0$ for sufficiently large n. This implies

$$K \subset \bigcup_{i=1}^{m_0} A_{i,n}(\hat{\Omega}_n \cup \partial\hat{\Omega}_n).$$

Then

$$\#\{A \in \hat{\Gamma}_n : A(\hat{\Omega}_n \cup \partial\hat{\Omega}_n) \cap K \neq \emptyset\} \leq m_0$$

for sufficiently large n. This result together with boundedness of $\{M_{g_n}\}_n$ (Lemma 2.8) implies that $\{\|\hat{\varphi}_{g_n}\|_{L^1(K)}\}_n$ is bounded. $\qquad\square$

Lemma 3.2. $M_g = \sup\{M_h : (h, S_h) \in \mathfrak{F}(S, E)\}$.

Proof.   By Lemma 3.1, there is a subsequence of $\{\hat{\varphi}_{g_n}\}_n$ which converges uniformly on any compact subset of D to a holomorphic function $\hat{\varphi}_\infty$. For each $n \in \mathbb{N}$, $\hat{\varphi}_{g_n}$ is $\hat{\Gamma}_n$-invariant, that is,

$$(\hat{\varphi}_{g_n} \circ A_n)(A_n')^2 = \hat{\varphi}_{g_n}$$

for every $A_n \in \hat{\Gamma}_n$. Then we see that $\hat{\varphi}_\infty$ is $\hat{\Gamma}_\infty$-invariant, so we can project $\hat{\varphi}_\infty$ to a holomorphic quadratic differential φ_∞^* on S_g^{d}. Moreover for any compact subset K of D,

$$\|\hat{\varphi}_\infty\|_{L^1(K)} = \lim_{n \to \infty} \|\hat{\varphi}_{g_n}\|_{L^1(K)} \leq 2c_0 \|\varphi\|_{L^1(S)}$$

holds, where c_0 is the positive number obtained in Lemma 2.8. Hence we can see that $\hat{\varphi}_\infty$ is integrable on $\hat{\Omega}_\infty$.

Next we show that the integrable holomorphic quadratic differential φ_∞^* on S_g^{d} is symmetric relative to ∂S_g. Let $\Lambda_n(n \in \mathbb{N})$ and Λ_∞ be the limit sets of Γ_n and Γ_∞, respectively. Let $\tilde{\varphi}_{g_n}(n \in \mathbb{N})$ be the lift of $\varphi_{g_n}^{\mathrm{d}}$ to the covering space $\bar{\mathbb{C}} \setminus \Lambda_n$ over $S_{g_n}^{\mathrm{d}}$ and let $\tilde{\varphi}_\infty$ be the lift of φ_∞^* to $\bar{\mathbb{C}} \setminus \Lambda_\infty$. By assumption, for each $n \in \mathbb{N}$, $\tilde{\varphi}_{g_n}$ satisfies

$$\overline{\tilde{\varphi}_{g_n}(1/\bar{z})} \cdot z^{-4} = \tilde{\varphi}_{g_n}(z) \quad \text{on } \bar{\mathbb{C}} \setminus \Lambda_n.$$

Since $\tilde{\varphi}_{g_n}$ converge to $\tilde{\varphi}_\infty$ uniformly on any compact subset of $\bar{\mathbb{C}} \setminus \Lambda_\infty$, we have

$$\overline{\tilde{\varphi}_\infty(1/\bar{z})} \cdot z^{-4} = \tilde{\varphi}_\infty(z) \quad \text{on } \bar{\mathbb{C}} \setminus \Lambda_\infty.$$

Hence $\tilde{\varphi}_\infty$ is symmetric relative to $\partial\mathbb{D}$, that is, the projection φ_∞^* of $\tilde{\varphi}_\infty$ to S_g^{d} is symmetric relative to ∂S_g. We denote by φ_∞ the restriction of φ_∞^* to S_g.

Now we show

$$\lim_{n \to \infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} = \|\varphi_\infty\|_{L^1(S_g)}.$$

Let $p_1, \ldots, p_k$ be all the points in $\hat{\Omega}_\infty \cup \partial\hat{\Omega}_\infty$ corresponding to the punctures of S_g^{d}. We may assume that all $p_1, \ldots, p_k$ are points of $\hat{\Omega}_\infty$. Set

$$p_{i,n} = \hat{f}_n \circ \hat{f}^{-1}(p_i) \quad (n \in \mathbb{N}, \ 1 \leq i \leq k).$$

Then $p_{i,n} \to p_i$ as $n \to \infty$.

For sufficiently small $\varepsilon > 0$, let $V_i \subset \hat{\Omega}_\infty$ $(1 \leq i \leq k)$ be the open disk of radius ε centered at p_i such that $V_i \cap V_j = \phi$ if $i \neq j$. Since holomorphic functions $(z - p_{i,n})\hat{\varphi}_{g_n}(z)$ $(n \in \mathbb{N})$ on V_i converge to $(z - p_i)\hat{\varphi}_\infty(z)$ uniformly on V_i, the residues $c_{i,n} = \mathrm{Res}(\hat{\varphi}_{g_n}, p_{i,n})$ converge to $c_i = \mathrm{Res}(\hat{\varphi}_\infty, p_i)$ as $n \to \infty$. Moreover we can also see that holomorphic functions

$$\hat{\varphi}_{g_n}(z) - \frac{c_{i,n}}{z - p_{i,n}}$$

converge to

$$\hat{\varphi}_\infty(z) - \frac{c_i}{z - p_i}$$

uniformly on V_i. We take the union $\bigcup_{n \geq n_0}(\hat{\Omega}_n \cup \partial\hat{\Omega}_n)$ for sufficiently large $n_0 \in \mathbb{N}$. Then $\hat{\varphi}_{g_n}$ converge to $\hat{\varphi}_\infty$ uniformly on the compact set

$$\left(\bigcup_{n \geq n_0}(\hat{\Omega}_n \cup \partial\hat{\Omega}_n) \bigcup (\hat{\Omega}_\infty \cup \partial\hat{\Omega}_\infty) \right) \setminus \bigcup_{i=1}^{k} V_i.$$

Set

$$C = \max_{1 \le i \le k} |c_i|.$$

Then we have

$$\limsup_{n \to \infty} \left| \|\hat{\varphi}_{g_n}\|_{L^1(\hat{\Omega}_n)} - \|\hat{\varphi}_\infty\|_{L^1(\hat{\Omega}_\infty)} \right|$$

$$\le \limsup \sum_{i=1}^k \left(\left\| \frac{c_i}{z - p_i} \right\|_{L^1(V_i)} + \left\| \frac{c_{i,n}}{z - p_{i,n}} \right\|_{L^1(V_i)} \right)$$

$$+ \limsup \sum_{i=1}^k \left\| \left(\hat{\varphi}_{g_n} - \frac{c_{i,n}}{z - p_{i,n}} \right) - \left(\hat{\varphi}_\infty - \frac{c_i}{z - p_i} \right) \right\|_{L^1(V_i)}$$

$$+ \limsup \left| \|\hat{\varphi}_{g_n}\|_{L^1(\hat{\Omega}_n \setminus \cup_{i=1}^k V_i)} - \|\hat{\varphi}_\infty\|_{L^1(\hat{\Omega}_\infty \setminus \cup_{i=1}^k V_i)} \right|$$

$$\le Ck4\pi\varepsilon.$$

Thus

$$\lim_{n \to \infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} = \|\varphi_\infty\|_{L^1(S_g)},$$

and we obtain $\|\varphi_\infty\|_{L^1(S_g)} = \sup_{(h, S_h) \in \mathfrak{F}(S, E)} M_h$. Moreover, by Lemma 2.8, we have

$$\|\varphi_\infty\|_{L^1(S_g)} = \lim_{n \to \infty} \|\varphi_{g_n}\|_{L^1(S_{g_n})} \ge c_0^{-1} \|\varphi\|_{L^1(S)}.$$

Therefore $\varphi_\infty \in A_s(S_g) \setminus \{0\}$.

Next we show $\varphi_\infty = \varphi_g$ on S_g. Take any curve $\gamma \in \mathfrak{S}(S_g^d)$ and sufficiently small $\varepsilon > 0$. Set $\gamma_n = f_n \circ f^{-1}(\gamma)$ $(n \in \mathbb{N})$. Assume that $A_\infty \in \hat{\Gamma}_\infty$ corresponds to the homotopy class $[\gamma]$ on $\hat{S}_g$. Then

$$A_n = \left(\hat{f}_n \circ \hat{f}^{-1} \right) \circ A_\infty \circ \left(\hat{f}_n \circ f^{-1} \right)^{-1}$$

corresponds to the homotopy class $[\gamma_n]$ on $\hat{S}_{g_n}$.

Let $\alpha_n \in \mathfrak{S}(S_{g_n}^d)$ be a $\varphi_{g_n}^d$-geodesic curve freely homotopic to γ_n on $S_{g_n}^d$, that is, α_n satisfies $[\alpha_n] = [\gamma_n]$ on $S_{g_n}^d$ and $\ell_{\varphi_{g_n}^d}[\gamma_n] = \ell_{\varphi_{g_n}^d}(\alpha_n)$, where $\ell_{\varphi_{g_n}^d}(\alpha_n)$ is the length

$$\int_{\alpha_n} \left| \sqrt{\varphi_{g_n}^d(z)} dz \right|$$

of α_n associated with $\varphi_{g_n}^d$ and $\ell_{\varphi_{g_n}^d}[\gamma_n]$ is that of homotopy class $[\gamma_n]$ defined by

$$\inf_\beta \ell_{\varphi_{g_n}^d}(\beta),$$

where the infimum is taken over all closed curves β freely homotopic to γ_n on $S_{g_n}^d$.

We remark that the geodesic α_n may pass through punctures of $S_{g_n}^{\mathrm{d}}$. Therefore, we should interpret the condition $[\alpha_n] = [\gamma_n]$ on $S_{g_n}^{\mathrm{d}}$ in an appropriate way.

It is known [11, Theorem 24.1] that each α_n satisfies $\mathrm{height}_{\varphi_{g_n}^{\mathrm{d}}}[\gamma_n] = \mathrm{height}_{\varphi_{g_n}^{\mathrm{d}}}(\alpha_n)$. Let $\hat{\alpha}_n$ $(n \in \mathbb{N})$ be a lift of α_n to the universal covering space $\mathbb{D}$ over $\hat{S}_{g_n}$ such that $\hat{\alpha}_n$ is a closed arc starting from a point on $\partial\hat{\Omega}_n$. We parametrize each $\hat{\alpha}_n$ by the $\hat{\varphi}_{g_n}$-length s and we set $l(n) = \ell_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n)$. Then $\{l(n)\}_n$ is bounded because the set $\{[S_{g_n}^{\mathrm{d}}, \iota_{g_n}]\}_n$ is relatively compact in $T(S^{\mathrm{d}})$ by Lemma 2.8. Thus we may assume that $l(n)$ converge to a positive constant l_∞. For each $n \in \mathbb{N}$ by re-parametrizing $\hat{\alpha}_n$ by $\hat{\alpha}_n(l(n)s)$, we may further assume that for every $n \in \mathbb{N}$, $\hat{\alpha}_n$ is defined on $[0,1]$. The curve α_n $(n \in \mathbb{N})$ is analytic except for finitely many points where $\varphi_{g_n}^{\mathrm{d}}$ vanishes. Since the number of punctures of $S_{g_n}^{\mathrm{d}}$ are the same for every $n \in \mathbb{N}$, the number of zeros of $\varphi_{g_n}^{\mathrm{d}}$ is uniformly bounded for $n \in \mathbb{N}$. And $\{l(n)\}_n$ is also bounded. Hence the number of singularities of $\hat{\alpha}_n$ are uniformly bounded. Then we can assume that there exists an integer N_0, and real numbers $t_1, t_2, \ldots, t_{N_0} \in [0,1]$ $(t_1 \le t_2 \le \cdots \le t_{N_0})$ such that the number of critical points of $\hat{\alpha}_n$ is N_0 for all $n \in \mathbb{N}$ and for each $i = 1, \ldots, N_0$, the i-th critical point $t_{n,i}$ of $\hat{\alpha}_n$ converge to t_i as $n \to \infty$. We can easily see that the family $\{\hat{\alpha}_n\}_{n \in \mathbb{N}}$ is equicontinuous. Then, together with uniform boundedness of $\{\hat{\alpha}_n\}_{n \in \mathbb{N}}$, we may assume that $\hat{\alpha}_n$ uniformly converge to a continuous function $\hat{\alpha}_\infty$ on $[0,1]$. The image $\hat{\alpha}_\infty([0,1])$ is a lift of a closed curve on $\hat{S}_g$.

Now we can show that $\mathrm{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n)$ converge to $\mathrm{height}_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)$. Let s_0 be any point on $(0,1) \setminus \{t_1, \ldots, t_{N_0}\}$ such that $\hat{\varphi}_\infty(\hat{\alpha}_\infty(s_0)) \ne 0$ and $\hat{\alpha}_\infty(s_0) \in D$. We can take a neighborhood U of s_0 in $(0,1)$ such that for every $n \in \mathbb{N}$, $\hat{\varphi}_{g_n}(\hat{\alpha}_n(s)) \ne 0$ on $U \cup \partial U$ and $\hat{\alpha}_n(U \cup \partial U)$ does not contain any points corresponding to punctures of $S_{g_n}^{\mathrm{d}}$. Let $\hat{\zeta}_n$ $(n \in \mathbb{N})$ and $\hat{\zeta}_\infty$ be the natural parameters near $\hat{\alpha}_n(s_0)$ and $\hat{\alpha}_\infty(s_0)$, which are represented by

$$\hat{\zeta}_n = \int_{\hat{\alpha}_n(s_0)}^{z} \sqrt{\hat{\varphi}_{g_n}(z)}\,dz$$

and

$$\hat{\zeta}_\infty = \int_{\hat{\alpha}_\infty(s_0)}^{z} \sqrt{\hat{\varphi}_\infty(z)}\,dz,$$

respectively. For every $n \in \mathbb{N}$, α_n is geodesic relative to $\varphi_{g_n}^{\mathrm{d}}$-length, so we see that

$$\frac{d\hat{\zeta}_n}{|d\hat{\zeta}_n|} = e^{i\theta_n} \text{ on } U$$

for a constant $\theta_n \in [0, 2\pi)$. For any subsequence $\{\theta_{n_k}\}_k$ which is converging to

some $\theta_\infty \in [0, 2\pi]$

$$
\begin{aligned}
\frac{d\hat{\alpha}_{n_k}}{ds} &= \frac{l(n_k)d\hat{\alpha}_{n_k}}{|d\hat{\zeta}_{n_k}|} \\
&= \frac{l(n_k)d\hat{\alpha}_{n_k}}{d\hat{\zeta}_{n_k}} \cdot \frac{d\hat{\zeta}_{n_k}}{|d\hat{\zeta}_{n_k}|} \\
&= \frac{l(n_k)}{\sqrt{\hat{\varphi}_{g_{n_k}}(\hat{\alpha}_{n_k})}} \cdot e^{i\theta_{n_k}} \\
&\xrightarrow{k\to\infty} \frac{l_\infty}{\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty)}} \cdot e^{i\theta_\infty}.
\end{aligned}
$$

The convergence is uniform on U, so we can see that $\hat{\alpha}_\infty$ is of the class C^1 on U and

$$
\frac{d\hat{\alpha}_\infty}{ds} = \lim_{k\to\infty} \frac{d\hat{\alpha}_{n_k}}{ds} = \frac{l_\infty}{\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty)}} \cdot e^{i\theta_\infty}.
$$

Since the argument θ_∞ is independent of the choice of subsequence, the sequence $\{\theta_n\}_n$ itself is converging and we obtain

$$
\frac{d\hat{\alpha}_\infty}{ds} = \lim_{n\to\infty} \frac{d\hat{\alpha}_n}{ds} \quad \text{on } U.
$$

Consequently the curve $\hat{\alpha}_\infty$ satisfies

$$
\frac{d\hat{\alpha}_\infty}{ds} = \lim_{n\to\infty} \frac{d\hat{\alpha}_n}{ds} \quad \text{on } (0,1)
$$

except for finitely many points. Then Lebesgue's convergence theorem yields

$$
\lim_{n\to\infty} \text{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n) = \text{height}_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)
$$

because

$$
\left| \text{Im}\left(\sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n)} \frac{d\hat{\alpha}_n}{ds} \right) \right| \leq \left| \sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n)} \frac{d\hat{\alpha}_n}{ds} \right| = l(n) \leq \sup_{n\in\mathbb{N}} l(n) < \infty
$$

for every $n \in \mathbb{N}$ and

$$
\lim_{n\to\infty} \left| \text{Im}\left(\sqrt{\hat{\varphi}_{g_n}(\hat{\alpha}_n(s))} \frac{d\hat{\alpha}_n}{ds} \right) \right| = \left| \text{Im}\left(\sqrt{\hat{\varphi}_\infty(\hat{\alpha}_\infty(s))} \frac{d\hat{\alpha}_\infty}{ds} \right) \right| \quad \text{a.e. on } [0,1].
$$

If there exists $s_1 \in [0,1]$ such that $\hat{\alpha}_\infty(s_1)$ corresponds to a puncture of S_g^{d}, we modify the curve $\hat{\alpha}_\infty$ locally in a neighborhood of s_1 to a curve $\hat{\alpha}'_\infty$ so that the projection on S_g^{d} is homotopic to γ on S_g^{d}. It is possible to choose $\hat{\alpha}'_\infty$ so that $\ell_{\hat{\varphi}_\infty}(\hat{\alpha}'_\infty)$ is as close to $\ell_{\hat{\varphi}_\infty}(\hat{\alpha}_\infty)$ as we want. Therefore, for any $\varepsilon > 0$,

there is a curve $\hat{\alpha}'_{\infty}$ so that

$$\begin{aligned}
\text{height}_{\varphi^{\mathrm{d}}_{\infty}}[\gamma] &\leq \text{height}_{\hat{\varphi}_{\infty}}(\hat{\alpha}'_{\infty}) \\
&< \text{height}_{\hat{\varphi}_{\infty}}(\hat{\alpha}_{\infty}) + \varepsilon \\
&= \lim_{n\to\infty} \text{height}_{\hat{\varphi}_{g_n}}(\hat{\alpha}_n) + \varepsilon \\
&= \lim_{n\to\infty} \text{height}_{\varphi^{\mathrm{d}}_{g_n}}[\gamma_n] + \varepsilon \\
&= \text{height}_{\varphi^{\mathrm{d}}_g}[\gamma] + \varepsilon.
\end{aligned}$$

Then we have the inequality $\text{height}_{\varphi^{\mathrm{d}}_{\infty}}[\gamma] \leq \text{height}_{\varphi^{\mathrm{d}}_g}[\gamma]$. In particular, by Proposition 2.4,

$$\|\varphi^{\mathrm{d}}_{\infty}\|_{L^1(S^{\mathrm{d}}_g)} \leq \|\varphi^{\mathrm{d}}_g\|_{L^1(S^{\mathrm{d}}_g)}.$$

On the other hand, from maximality of $\|\varphi_{\infty}\|_{L^1(S_g)}$ in $\{M_h; (h, S_h) \in \mathfrak{F}(S, E)\}$ we have

$$\|\varphi^{\mathrm{d}}_g\|_{L^1(S^{\mathrm{d}}_g)} \leq \|\varphi^{\mathrm{d}}_{\infty}\|_{L^1(S^{\mathrm{d}}_g)}.$$

Therefore $\|\varphi^{\mathrm{d}}_{\infty}\|_{L^1(S^{\mathrm{d}}_g)} = \|\varphi^{\mathrm{d}}_g\|_{L^1(S^{\mathrm{d}}_g)}$ holds, and by Proposition 2.4, we have

$$\varphi^{\mathrm{d}}_{\infty} = \varphi^{\mathrm{d}}_g \text{ on } S^{\mathrm{d}}_g.$$

$\square$

Lemma 3.3. $E_g = S_g \setminus g(S \setminus E)$ *has measure zero.*

Proof. If E_g has positive measure, we can find a Beltrami differential ν on S^{d}_g so that $\nu = 0$ on $S^{\mathrm{d}}_g \setminus E^{\mathrm{d}}_g$ and $\iint_{S^{\mathrm{d}}_g} \nu\varphi^{\mathrm{d}}_g > 0$.

For sufficiently small $t > 0$, let $f^{t\nu}$ be a quasiconformal map of S^{d}_g onto the double $S^{\mathrm{d}}_{t\nu}$ of a Riemann surface $S_{t\nu}$ with Beltrami coefficient $t\nu$. Let $\tau(t) = [S^{\mathrm{d}}_{t\nu}, (f^{t\nu})_*] \in T(S^{\mathrm{d}}_g)$. For $\varphi_g \in A_{\mathrm{s}}(S_g) \setminus \{0\}$ we set $\varphi_{t\nu} = \tau(t)_*\varphi^{\mathrm{d}}_g|_{S_{t\nu}} \in A_{\mathrm{s}}(S_{t\nu}) \setminus \{0\}$ whose heights on $S^{\mathrm{d}}_{t\nu}$ are equal to those of φ^{d}_g on S^{d}_g. By the variational formula [4, p. 217], we have

$$\log\|\varphi^{\mathrm{d}}_{t\nu}\|_{L^1(S^{\mathrm{d}}_{t\nu})} = \log\|\varphi^{\mathrm{d}}_g\|_{L^1(S^{\mathrm{d}}_g)} + \frac{2}{\|\varphi^{\mathrm{d}}_g\|_{L^1(S^{\mathrm{d}}_g)}}\text{Re}\iint_{S^{\mathrm{d}}_g} t\nu\varphi^{\mathrm{d}}_g \, dxdy + o(\|t\nu\|_{\infty}).$$

Then we can see

$$\|\varphi_{t\nu}\|_{L^1(S_{t\nu})} > \|\varphi_g\|_{L^1(S_g)}$$

for sufficiently small $t > 0$. On the other hand, $(f^{t\nu} \circ g, S_{t\nu}) \in \mathfrak{F}(S, E)$ for small $t > 0$, and the maximality of $\|\varphi_g\|_{L^1(S_g)}$ yields the inequality

$$\|\varphi_{t\nu}\|_{L^1(S_{t\nu})} \leq \|\varphi_g\|_{L^1(S_g)},$$

which is a contradiction. $\square$

Lemma 3.4. *A component of E_g is either*

(i) *a horizontal arc of φ_g or,*

(ii) *a connected union of finitely many horizontal arcs and critical points of φ_g.*

In particular, E_g is an allowable horizontal slit with respect to φ_g.

Proof. Fix any component J of E_g. We consider the obstacle problem for (S_g, J, φ_g). Then there is a solution $(h, S_h) \in \mathfrak{F}(S_g, J)$ and $\varphi_h \in A_s(S_h) \setminus \{0\}$ satisfying

$$\|\varphi_h\|_{L^1(S_h)} = \sup\{\|\varphi_f\|_{L^1(S_f)} : (f, S_f) \in \mathfrak{F}(S_g, J)\}.$$

From the conclusion of Fehlmann and Gardiner [3, Theorem 1] we know that $S_g \setminus h(S_g \setminus J)$ is a possibly branched arc on horizontal trajectories of φ_h. Since $(h \circ g, S_h) \in \mathfrak{F}(S, E)$, we have $\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_g\|_{L^1(S_g)}$. On the other hand, $(\mathrm{id}, S_g) \in \mathfrak{F}(S_g, J)$ yields the inequality $\|\varphi_g\|_{L^1(S_g)} \leq \|\varphi_h\|_{L^1(S_h)}$. Thus $(\mathrm{id}, S_g) \in \mathfrak{F}(S_g, J)$ attains the maximum. By Theorem 2 in [3], we conclude that the component J satisfies either (i) or (ii) in the lemma. $\qquad\square$

We have proved Theorem 1.3 completely.

4. Proof of Theorem 1.4

Let $E_n \subset S$ ($n \in \mathbb{N}$) be an allowable subset of S such that each E_n is a disjoint union of finitely many closed analytic disks and satisfying $E_n \supset E_{n+1} \supset \cdots$ and $\cap_{n=1}^{\infty} E_n = E$. Set $U_n = S \setminus E_n$ ($n \in \mathbb{N}$). (Consider a regular exhaustion $\{R_n\}_{n \in \mathbb{N}}$ of the Riemann surface $\hat{S} \setminus E^d$ and set $U_n = S \cap R_n$ and $E_n = S \setminus U_n$ for sufficiently large n's; cf. [1, p. 144].) First we show that we can obtain a solution $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ of the obstacle problem for (S, E, φ) from the family of solutions $(h_n, S_{h_n}) \in \mathfrak{F}(S, E_n)$ for (S, E_n, φ) ($n \in \mathbb{N}$).

For each $n \in \mathbb{N}$, by considering the obstacle problem for (S, E_n, φ), we obtain a solution $(h_n, S_{h_n}) \in \mathfrak{F}(S, E_n)$. Let $\tau_n = [S_{h_n}^d, \iota_{h_n}] \in T(S^d)$. We can see in a similar manner to Section 2 that the set $\{\tau_n\}_n \subset T(S^d)$ is relatively compact in $T(S^d)$. In fact, we may assume that the curve family $\beta_1, \ldots, \beta_N$ in Lemma 2.6 is contained in $S^d \setminus U_1^d$. Then the dilatation $K_0(\tau_n)$ of the extremal quasiconformal map of the Teichmüller class τ_n satisfies

$$K_0(\tau_n) \leq c \max_{1 \leq l \leq N} \lambda([\beta_l]_{U_1^d}, U_1^d) =: c_1$$

for every $n \in \mathbb{N}$. Since the constant c_1 is independent of $n \in \mathbb{N}$, $\{\tau_n\}_n$ is relatively compact in $T(S^d)$. We can also see

$$c_1^{-1} \|\varphi\|_{L^1(S)} \leq \|\varphi_{h_n}\|_{L^1(S_{h_n})} \leq c_1 \|\varphi\|_{L^1(S)}$$

for every $n \in \mathbb{N}$. By the same argument as in the proof of Theorem 1.3, we obtain a subsequence $\{\tau_{n_k}\}_k$ and a point $\tau_\infty = [S_{h_\infty}, \iota_{h_\infty}] \in T(S^d)$ associated with the element $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ such that τ_{n_k} converge to τ_∞ in $T(S^d)$,

and the lift of $h_{n_k}^{\mathrm{d}}$ to the covering space $\mathbb{D}$ over $S^{\mathrm{d}} \setminus E^{\mathrm{d}}$ converge to that of h_∞^{d} uniformly on any compact subset of $\mathbb{D}$. For brevity, we renumber n_k by k. For every $(h, S_h) \in \mathfrak{F}(S, E)$ and every $n \in \mathbb{N}$, by considering the restriction of h to U_n, we have

$$\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_{h_n}\|_{L^1(S_{h_n})}.$$

Since $\|\varphi_{h_n}\|_{L^1(S_{h_n})}$ converge to $\|\varphi_{h_\infty}\|_{L^1(S_{h_\infty})}$, we have

$$\|\varphi_h\|_{L^1(S_h)} \leq \|\varphi_{h_\infty}\|_{L^1(S_{h_\infty})}$$

for every $(h, S_h) \in \mathfrak{F}(S, E)$. Hence, $(h_\infty, S_{h_\infty}) \in \mathfrak{F}(S, E)$ is extremal for (S, E, φ);

$$M_{h_\infty} = \sup_{(h, S_h) \in \mathfrak{F}(S, E)} M_h.$$

Next we assume that an element $(u, S_u) \in \mathfrak{F}(S, E)$ also attains the supremum. Set

$$w_n = h_n \circ u^{-1}, \qquad \psi_n = (\varphi_{h_n} \circ w_n)(w_n')^2 \ (n \in \mathbb{N})$$
$$w_\infty = h_\infty \circ u^{-1}, \qquad \psi_\infty = (\varphi_{h_\infty} \circ w_\infty)(w_\infty')^2.$$

We show that $\varphi_u = \psi_\infty$ on $u(S \setminus E)$. For the sake of convenience, we extend ψ_n to S_u so that $\psi_n = 0$ on $S_u \setminus u(U_n)$ and similarly, $\psi_\infty = 0$ on $S_u \setminus u(S \setminus E)$. Then $\psi_n \ (n \in \mathbb{N})$ and ψ_∞ are integrable quadratic differentials on S_u.

Fix any point $p_0 \in S_u^{\mathrm{d}}$ with $\varphi_u^{\mathrm{d}}(p_0) \neq 0$. Let $\zeta_u = \xi_u + i\eta_u$ be the natural parameter of φ_u^{d}, which is defined about p_0 by

$$\zeta_u = \int_{z_0}^{z} \sqrt{\varphi_u^{\mathrm{d}}(z)} dz,$$

where z is a local chart near p_0 and $z(p_0) = z_0$. Let N_{p_0} be a sufficiently small closed neighborhood of p_0 such that N_{p_0} is mapped by ζ_u onto a square centered at p_0 and each side of which is parallel to the axis. Since ψ_n^{d} is integrable on S_u^{d}, the height relative to ψ_n^{d} of almost every vertical segment of φ_u^{d} on S_u^{d}, particularly on N_{p_0}, is well defined for each $n \in \mathbb{N} \cup \{\infty\}$.

We denote by $\mathcal{V}_{p_0}$ the set of such vertical segments of φ_u^{d} in N_{p_0}.

For each $\beta = \beta(\xi_u) \in \mathcal{V}_{p_0}$ we set $\beta_n = \beta_n(\xi_u) = \beta(\xi_u) \cap u^{\mathrm{d}}(U_n^{\mathrm{d}}) \ (n \in \mathbb{N})$. Fix any $\varepsilon > 0$ and any 1-dimensional measurable set $A \subset N_{p_0}$ on a horizontal trajectory of φ_u^{d}. Since φ^{d} and φ_u^{d} are integrable on S^{d} and S_u^{d}, respectively, by Fubini's theorem and Schwarz's inequality, for every $n, m \in \mathbb{N} \ (n \geq m)$ we have

$$
\begin{aligned}
\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_m)| d\xi_u \quad &\leq \quad \int_A \int_{\beta_n \setminus \beta_m} |\sqrt{\psi_n^{\mathrm{d}}(\zeta_u)}| d\eta_u \, d\xi_u \\
&= \quad \iint_{V(n,m,A)} |\sqrt{\psi_n^{\mathrm{d}}(\zeta_u)}| d\xi_u \, d\eta_u \\
&\leq \quad \|\psi_n^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \cdot \|\varphi_u^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \\
&\leq \quad \left(c_1 \|\varphi^{\mathrm{d}}\|_{L^1(S^{\mathrm{d}})}\right)^{\frac{1}{2}} \cdot \|\varphi_u^{\mathrm{d}}\|_{L^1(V(n,m,A))}^{\frac{1}{2}} \\
&\xrightarrow{m \to \infty} \quad 0,
\end{aligned}
$$

where $V(n, m, A) = \{p \in N_{p_0} \cap u^{\mathrm{d}}(U_n^{\mathrm{d}} \setminus U_m^{\mathrm{d}}) \mid \operatorname{Re} \zeta_u(p) \in A\}$. Then there exists a natural number N_1 such that for every $n \geq N_1$

$$\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1})| d\xi_u < \varepsilon.$$

Furthermore we can see that

$$\int_A |\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1})| d\xi_u \to 0 \quad (n \to \infty),$$

because ψ_n^{d} uniformly converge to ψ_∞^{d} on $N_{p_0} \cap u^{\mathrm{d}}(U_{N_1}^{\mathrm{d}})$. Then, by Fatou's lemma, we have

$$\int_A \liminf_{n \to \infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) d\xi_u$$

$$\leq \liminf_{n \to \infty} \int_A \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) d\xi_u$$

$$\leq \liminf_{n \to \infty} \left\{ \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) \right) d\xi_u \right.$$

$$\left. + \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) \right) d\xi_u + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) d\xi_u \right\}$$

$$= \liminf_{n \to \infty} \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) - \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) \right) d\xi_u$$

$$+ \lim_{n \to \infty} \int_A \left(\mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta_{N_1}) - \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) \right) d\xi_u + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta_{N_1}) d\xi_u$$

$$\leq \varepsilon + \int_A \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta) d\xi_u.$$

Therefore almost every $\beta \in \mathcal{V}_{p_0}$ satisfies

$$\liminf_{n \to \infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\beta) \leq \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\beta).$$

The same holds for almost every horizontal and vertical segments on S_u^{d}. Then we can see that almost every φ_u^{d}-polygonal curve γ, which is a union of finitely many horizontal and vertical segments, satisfies

$$\liminf_{n \to \infty} \mathrm{height}_{\psi_n^{\mathrm{d}}}(\gamma) \leq \mathrm{height}_{\psi_\infty^{\mathrm{d}}}(\gamma).$$

Moreover, for every $n \in \mathbb{N}$, $S_{h_n}^{\mathrm{d}} \setminus w_n^{\mathrm{d}}(u^{\mathrm{d}}(S^{\mathrm{d}} \setminus E_n^{\mathrm{d}}))$ is an allowable horizontal slit with respect to $\varphi_{h_n}^{\mathrm{d}}$, which consists of finitely many components. Then for almost every φ_u^{d}-polygonal curve γ we can obtain from $w_n^{\mathrm{d}}(\gamma)$ a simple closed curve $\tilde{w}_n^{\mathrm{d}}(\gamma)$ by supplying finitely many horizontal segments of $\varphi_{h_n}^{\mathrm{d}}$. Hence we have

$$\mathrm{height}_{\varphi_u^{\mathrm{d}}}[\gamma] = \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}[\tilde{w}_n^{\mathrm{d}}(\gamma)]$$

$$\leq \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}\left(\tilde{w}_n^{\mathrm{d}}(\gamma) \right)$$

$$= \mathrm{height}_{\varphi_{h_n}^{\mathrm{d}}}\left(w_n^{\mathrm{d}}(\gamma) \right)$$

$$= \mathrm{height}_{\psi_n^{\mathrm{d}}}(\gamma)$$

for every $n \in \mathbb{N}$. Therefore $\text{height}_{\varphi_u^{\mathrm{d}}}[\gamma] \leq \liminf_{n \to \infty} \text{height}_{\psi_n^{\mathrm{d}}}(\gamma)$ holds, and we obtain $\text{height}_{\varphi_u^{\mathrm{d}}}[\gamma] \leq \text{height}_{\psi_\infty^{\mathrm{d}}}(\gamma)$ and, in particular, $\|\varphi_u^{\mathrm{d}}\|_{L^1(S_u^{\mathrm{d}})} \leq \|\psi_\infty^{\mathrm{d}}\|_{L^1(S_u^{\mathrm{d}})}$. Then the maximality of $M_u = \|\varphi_u\|_{L^1(S_u)}$ implies, together with Proposition 2.4,

$$\varphi_u^{\mathrm{d}} = \psi_\infty^{\mathrm{d}} \text{ on } u^{\mathrm{d}}(S^{\mathrm{d}} \setminus E^{\mathrm{d}}).$$

Consequently we have

$$\varphi_u = (\varphi_{h_\infty} \circ w_\infty)(w_\infty')^2 \text{ on } u(S \setminus E).$$

We have proved Theorem 1.4. □

5. Slit mapping theorem of open Riemann surface of finite genus

Let R be an open Riemann surface of finite genus. As was observed by Bochner, R can be embedded into a compact Riemann surface S of the same genus. Let w be such a conformal embedding of R into S. We note the following fact.

Lemma 5.1. $E = S \setminus w(R)$ *is allowable in* S.

Proof.   Let κ be the genus of R. We can take a family $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_\kappa, \beta_\kappa\}$ of simple closed curves on R with common base point at $p_0 \in R$ such that $R \setminus \cup_{i=1}^\kappa (\alpha_i \cup \beta_i)$ is connected. Then $D = S \setminus \cup_{i=1}^\kappa (w(\alpha_i) \cup w(\beta_i))$ is a simply connected subdomain of S. Since E is a compact subset of D, E is relatively contractible in D, and hence, in S. Since $S \setminus E = w(R)$ is connected, E is allowable. □

Let $\mathfrak{M}$ be the family of pairs (g, S_g), where g is a conformal map of R into a compact Riemann surface S_g of the same genus. For every $(g, S_g) \in \mathfrak{M}$, $g \circ w^{-1} : S \setminus E \hookrightarrow S_g$ induces an isomorphism of $\pi_1(S)$ onto $\pi_1(S_g)$ by Lemma 2.3. We denote by $\tau = [S_g, (g \circ w^{-1})_*]$ the Teichmüller class in $T(S)$. Then for this $\tau \in T(S)$ and for $\varphi \in A(S) \setminus \{0\}$ we obtain the unique element $\varphi_g \in A(S_g) \setminus \{0\}$ whose heights on S_g are equal to those of φ on S. Consider the extremal problem to find an element $(g, S_g) \in \mathfrak{M}$ maximizing $\|\varphi_g\|_{L^1(S_g)}$. By Theorem 1.3, we obtain an extremal element $(g, S_g) \in \mathfrak{M}$. The set $S_g \setminus g(R)$ is of zero area and each component of $S_g \setminus g(R)$ is either

(i) a horizontal arc of φ_g or,

(ii) a connected union of finitely many horizontal arcs and critical points of φ_g.

Thus we have shown the following

Corollary 5.2.   *Let* R *be an open Riemann surface of finite genus. Then there exist a compact Riemann surface* S *of the same genus and a conformal embedding* $g : R \to S$ *and a holomorphic quadratic differential* ψ *on* S *such that* $S \setminus g(R)$ *has measure zero and each component of* $S \setminus g(R)$ *is a possibly branched arc on horizontal trajectories of* ψ.

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References

[1] L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton, New Jersey Princeton University Press, 1960.

[2] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand Reinhold, Princeton, 1966.

[3] R. Fehlmann and F. P. Gardiner, *Extremal Problem for Quadratic Differentials*, Michigan Math. J. **42** (1995), 573–591.

[4] F. P. Gardiner, *Teichmüller Theory and Quadratic differentials*, Wiley Interscience, 1987.

[5] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Amer. Math. Soc., 2000.

[6] Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Space*, Springer-Verlag, Tokyo, 1992.

[7] C. Maclachlan and W. J. Harvey, *On mapping-class groups and Teichmüller Spase*, Proc. London Math. Soc. (3) **30** (1975), 496–512.

[8] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, Berlin Heidelberg, 1970.

[9] R. Sasai, *Non-uniqueness of obstacle problem on finite Riemann surface*, preprint.

[10] M. Shiba, *The Riemann-Hurwitz relation, parallel slit covering map, and continuation of an open Riemann surface of finite genus*, Hiroshima Math. J. **14** (1984), 371–399.

[11] K. Strebel, *Quadratic Differentials*, Springer-Verlag, Berlin Heidelberg, New York, Tokyo, 1984.