J. Math. Kyoto Univ. (JMKYAZ)
45-1 (2005), 159–182

Remarks on degree 4 projective curves

By

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Abstract

In this paper we characterize the degree 4 multiple lines with generic embedding dimension 3 and among them the ones with very degenerate hyperplane section, and the ones which contain a degree 3 planar subcurve. Using that characterization, we prove that the degree 4 curves containing a planar subcurve of degree 3 are the general element of an irreducible component of the Hilbert scheme. Moreover, we show that all the multiple lines we consider belong to the same connected component of the corresponding Hilbert scheme.

1. Introduction

A classical method to study curves embedded in projective spaces is to consider their general hyperplane section. For example, bounds on the postulation of the general hyperplane section give bounds on the cohomology (see [7] for curves in $\mathbb{P}^3$ and [2] for curves in $\mathbb{P}^n, n \geq 3$) as well as bounds on the arithmetic genus of the curve (see [2] for locally Cohen-Macaulay curves or [5] for smooth curves).

Conversely, the lifting problem for curves consists in obtaining information on the curve assuming analogous properties of its general hyperplane section. In particular, in [3], the authors studied the existence of planar subcurves of a curve C under the assumption that the general hyperplane section $C \cap H$ of C contains a large linear subscheme. One of the main result of [3] is that the planar subcurve exists and it has the expected degree if the degree r of the linear subscheme exceeds $(d+3)/2$ where $d = \deg(C)$ is the degree of C. For $r = d-1$ and $d \geq 5$ their result implies the equivalence between "$C \cap H$ contains a linear subscheme of degree $d-1$" and "C contains a degree $d-1$ planar subcurve". This equivalence was used in [14] to prove that the curves satisfying the previous hypothesis, called curves with very degenerate hyperplane section, are the general element of an irreducible component of the corresponding Hilbert scheme, whatever the genus of the curve is.

One of the aim of this paper is to prove an analogous result for curves of degree 4. The main problem in proving it is that in this case there exist curves

2000 *Mathematics Subject Classification*(*s*). Primary 14H45; Secondary 14C05.
Received July 20, 2004
Revised January 31, 2005

with very degenerate hyperplane section which do not contain planar subcurves of degree 3, as proved in [3]. Those exceptional curves are multiple lines of degree 4 and generic embedding dimension 3. So, we were forced to study those multiple lines in great detail. After stating notation and preliminaries in Section 2, in Section 3 we characterize the possible general hyperplane section of such a curve, we compute its Cohen-Macaulay filtration, a classical tool to study multiple lines, and we show how to construct such a curve. Moreover, we obtain some numerical information on the curve in terms of the data of the filtration. In Section 4, we characterize the multiple lines which have very degenerate hyperplane section, and among those, the ones which contain a degree 3 planar subcurve.

From their description, it follows that there are a lot of different algebraic families parameterizing those multiple lines. In Section 5, we describe those algebraic families, and we address the natural problem of understanding if all those families belong to the same connected component of the corresponding Hilbert scheme. Of course, we consider two more algebraic families, the degree 4 ropes supported on a line and the curves containing a degree 3 planar subcurve, studied respectively in [11] and [14], which appear in projective spaces of dimension at least 4. The problem of the connectedness of the Hilbert scheme parameterizing locally Cohen-Macaulay curves, up to now, was considered only for curves in $\mathbb{P}^3$, and the results obtained by various authors suggest that the answer to that problem should be positive. See, for example, [12] and [13] for the connectedness of Hilbert schemes of degree 3 and 4 curves, respectively, in $\mathbb{P}^3$, of whatever genus. In proving that all the families we consider belong to the same connected component we study the deformation of one parameter flat families, following an approach analogous to the one in [12].

In last Section 6, we prove that the degree 4 curves containing a planar subcurve of degree 3 fill an irreducible component of the Hilbert scheme, giving an affirmative answer to our initial question.

Both the authors would like to warmly thank Silvio Greco for suggesting them the study of such interesting curves. Furthermore, the second author profited of a stay at the Department of Mathematics of the University of Messina during the preparation of this paper, and he would like to thank that institution for the warm hospitality.

2. Preliminaries and notation

In this section, we collect the notation we' ll use throughout the paper and the known results we need to develop the subject.

2.1. Notation

K is an algebraically closed field of characteristic zero, $R = K[x_0, \dots, x_n]$ is the polynomial ring in $n+1$ unknowns of degree 1, and $\mathbb{P}^n = \text{Proj}(R)$ is the n-dimensional projective space. $\mathbb{P}^n$ is our ambient space. The structure sheaf of a closed scheme $X \subset \mathbb{P}^n$ is $\mathcal{O}_X$ while its ideal sheaf is $\mathcal{I}_X$. The saturated ideal I_X defining a closed scheme X is equal to $I_X = H^0_*(\mathbb{P}^n, \mathcal{I}_X) =$

$\oplus_{j\in\mathbb{Z}}H^0(\mathbb{P}^n,\ \mathcal{I}_X(j))$. The dimension of the degree j homogeneous piece of I_X is $h^0(\mathcal{I}_X(j)) = \dim_K[I_X]_j$, while the Hilbert function of X is defined as $h_X(j) = \dim_K[R/I_X]_j = \binom{n+j}{j} - h^0(\mathcal{I}_X(j))$.

A curve $C \subset \mathbb{P}^n$ is a locally Cohen-Macaulay closed scheme of pure dimension 1, i.e., C has no embedded or isolated zero-dimensional component. Two very important discrete invariants of a projective curve are the degree $\deg(C)$ and the arithmetic genus g_C. Another important object to be studied for a curve C is the Hartshorne-Rao module defined as $H^1_*(\mathbb{P}^n, \mathcal{I}_C) = \oplus_{j\in\mathbb{Z}}H^1(\mathbb{P}^n, \mathcal{I}_C(j))$. It is a finite length R-module. The Rao function of C is $h^1(\mathcal{I}_C(j)) = \dim_K H^1(\mathbb{P}^n, \mathcal{I}_C(j))$, as K-vector space.

2.2. Multiple lines

In this subsection, we adapt some results by Banica and Forster [1] to multiplicity structures on lines. For a survey on the construction of Cohen-Macaulay scheme structures on a smooth variety, see [6].

Definition 2.1. If L is a line, a multiple line C supported on L is a curve C whose ideal sheaf satisfies $\mathcal{I}_L^k \subseteq \mathcal{I}_C \subseteq \mathcal{I}_L$, for some integer k.

A multiple line C admits a filtration via multiple lines with the same support and smaller degrees which can be constructed as follows. Let $L^{(i)}$ be the curve defined by the ideal sheaf $\mathcal{I}_L^i$. Let C_i denote the subscheme obtained by removing embedded points from $C \cap L^{(i)}$. C_i is the largest Cohen-Macaulay subscheme contained in $C \cap L^{(i)}$, and it is uniquely determined. If k is the smallest integer such that $\mathcal{I}_L^k \subseteq \mathcal{I}_C$, then

$$L = C_1 \subset C_2 \subset \cdots \subset C_k = C$$

is the Cohen-Macaulay filtration of C.

The sheaves $\mathcal{L}_j = \mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}}$ are associated to the filtration and they are $\mathcal{O}_L$-modules. In fact, for each $i, j \geq 1$ such that $1 \leq i + j \leq k$ it holds $\mathcal{I}_{C_i}\mathcal{I}_{C_j} \subseteq \mathcal{I}_{C_{i+j}}$. Moreover, they are locally free, and so free because the support is a line. All those sheaves are related by the short exact sequences

$$0 \to \mathcal{I}_{C_{i+1}} \longrightarrow \mathcal{I}_{C_i} \longrightarrow \mathcal{L}_i \to 0. \tag{2.1}$$

2.3. Ropes supported on lines

In this subsection, we collect some results from [10] and [11] on the multiple lines which have a Cohen-Macaulay filtration as short as possible.

Definition 2.2. A rope Z supported on a line is a multiple line that satisfies $\mathcal{I}_L^2 \subset \mathcal{I}_Z \subset \mathcal{I}_L$.

In [10, Theorem 2.4, Remark 2.5], the authors proved that every rope Z of degree d with $2 \leq d \leq n-1$ is uniquely determined by fixing the supporting line L and an exact sequence

$$0 \to \bigoplus_{j=1}^{n-d} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-1}(-1) \xrightarrow{\varphi_A} \bigoplus_{i=0}^{d-2} \mathcal{O}_L(\alpha_i - 1) \to 0. \tag{2.2}$$

The arithmetic genus of Z is $g_Z = -\sum_{i=0}^{d-2} \alpha_i = -\sum_{j=1}^{n-d} \beta_j$. Furthermore, if $I_L = (x_0, \ldots, x_{n-2})$ and $S = K[x_{n-1}, x_n]$, the saturated ideal of Z is generated by $I_Z = (I_L^2, [x_0, \ldots, x_{n-2}]B)$, where B is a matrix which represents the map φ_B and does not drop rank in codimension 1, i.e., the ideal $I_{n-d}(\varphi_B) \subset S$ generated by the maximal minors of B contains a regular sequence of length 2, or equivalently, the map $\varphi_B \otimes K_P$ is injective for each closed point $P \in L$, where K_P is the residual field of the local ring $\mathcal{O}_{L,P}$. Of course, the entries of B are in S. The Hartshorne-Rao module of Z is an S-module because it is annihilated by I_L, and it is isomorphic to coker(φ_A) as S-module. The Rao function of Z is

$$h^1(\mathcal{I}_Z(j)) = \sum_{i=0}^{d-2} \binom{j+\alpha_i}{1} - (n-1)\binom{j}{1} + \sum_{i=1}^{n-d} \binom{j-\beta_i}{1}. \tag{2.3}$$

Definition 2.3. Given a rope Z of degree d and genus g, the sequence $\alpha = (\alpha_0, \ldots, \alpha_{d-2})$ with $\alpha_0 \leq \cdots \leq \alpha_{d-2}$ is called the right-type of Z. Analogously, if $\beta_1 \leq \cdots \leq \beta_{n-d}$, the sequence $\beta = (\beta_1, \ldots, \beta_{n-d})$ is called the left-type of Z.

The ropes of degree d and right-type α form an irreducible family into the corresponding Hilbert scheme parameterized by the closure $\mathcal{R}_{n,d,\alpha}$ of $j(V_\alpha)$ where V_α is the product of the affine space $\mathbb{A}^{2(n-1)}$ isomorphic to the open subset $W_{0,\ldots,n-2}$ of the Grassmannian Grass$(1,n)$ of lines in $\mathbb{P}^n$ and the open subset of the affine space parameterizing the homogeneous matrices giving surjective maps $S^{n-1}(-1) \to \oplus_{i=0}^{d-2} S(\alpha_i - 1)$ which do not drop rank in codimension 1, and $j : V_\alpha \to \mathrm{Hilb}_{d,g}(\mathbb{P}^n)$ is the natural map.

The dimension of $\mathcal{R}_{n,d,\alpha}$ [11, Proposition 6.7] is equal to

$$\dim \mathcal{R}_{n,d,\alpha} = (n-1)(d+1-g) - \dim \mathrm{Aut}\left(\bigoplus_{i=0}^{d-2} \mathcal{O}_L(\alpha_i - 1)\right). \tag{2.4}$$

All these families in the Hilbert scheme lie into the closure of one of them, the one with "balanced" right type. Let $H_{d,g}(\mathbb{P}^n)$ be the closure into the Hilbert scheme of their union, where we consider as Hilbert scheme only the one parameterizing the locally Cohen-Macaulay closed schemes.

Under some hypotheses on the genus, $H_{d,g}(\mathbb{P}^n)$ is an irreducible component of the Hilbert scheme Hilb(n,d,g) parameterizing the curves in $\mathbb{P}^n$ of degree d and genus g, and this component is generically smooth.

2.4. Curves with very degenerate hyperplane section

In [9], the author proved that the general hyperplane section of a non-degenerate projective curve spans a linear space of dimension at least 2.

Definition 2.4. If the general hyperplane section of a degree d curve contains a degree $d-1$ subscheme spanning a line, we say that C has very degenerate hyperplane section.

If the degree of a curve with the worst general hyperplane section is at least 5, then such a curve contains a planar subcurve of degree one less. The curves of degree d containing a planar subcurve of degree $d-1$ mostly consist of a planar curve of degree $d-2$ and a degree 2 rope supported on a line contained in the same plane. Those curves were studied in [14]. In particular, the saturated ideal I_C of such a curve C is generated by

$$I_C = (I_L I_D, [x_0, \ldots, x_{n-3}, x_{n-2}f]B) \tag{2.5}$$

where $I_L = (x_0, \ldots, x_{n-2})$ defines the line support of the rope, $I_D = (x_0, \ldots, x_{n-3}, x_{n-2}f)$ defines the union of the plane curve of degree $d-2$ and the line L, and B is a matrix with entries in S which represents a map φ_B that fits into the short exact sequence

$$0 \to \bigoplus_{i=1}^{n-2} \mathcal{O}_L(-b_i - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-2}(-1) \oplus \mathcal{O}_L(-d+1) \xrightarrow{\varphi_A} \mathcal{O}_L(a-1) \to 0. \tag{2.6}$$

Furthermore, the arithmetic genus of C is $g = \binom{d-2}{2} - a$, and the Hartshorne-Rao module of C is an S-module isomorphic to $\operatorname{coker}(\varphi_A)$ [14, Theorems 2.1, 3.2, Remark 3.4, Proposition 3.7].

The curves with very degenerate hyperplane section of degree d and genus g form an irreducible family parameterized by a smooth, quasi-projective variety $\mathcal{V}_{n,d,g}$ of dimension

$$\dim \mathcal{V}_{n,d,g} = n \left[\binom{d-2}{2} + 4 - g \right] + 3(d-4) + g \tag{2.7}$$

defined analogously to the case of ropes (see [14, Section 7]). In [14], Theorems 7.8, and 7.11, it is proved that those curves are the general element of an irreducible component of the corresponding Hilbert scheme if $d \geq 5$.

2.5. Generic embedding dimension of curves

For the definitions and results of this subsection we refer to [3].

Definition 2.5. If $(A, \mathfrak{M}, k)$ is a local ring, the embedding dimension of A is the integer $\operatorname{emdim}(A) = \dim_k(\mathfrak{M}/\mathfrak{M}^2)$. If X is a scheme and x is a point of X, the embedding dimension of X at x is equal to $\operatorname{emdim}_x(X) = \operatorname{emdim}(\mathcal{O}_{X,x})$. If X is irreducible with generic point η, the generic embedding dimension of X is the integer $\operatorname{ged}(X) = \operatorname{emdim}_\eta(X) + \dim X$.

The generic embedding dimension of an irreducible scheme X is related to the embedding dimension of the closed points of X. In fact, it holds [3, Lemma 2.2].

Lemma 2.1. *Let X be an irreducible algebraic K-scheme. Then*

(i) *there is a non-empty open set $U \subseteq X$ such that* $\operatorname{ged}(X) = \operatorname{emdim}_x(X)$ *for all closed points $x \in U$;*

(ii) $\mathrm{ged}(X) = \min\{\mathrm{emdim}_x(X), x \in X, x \text{ closed}\}$.

Moreover, the embedding dimension of a curve C at a point x is related to the embedding dimension of $C \cap H$ at $x \in C \cap H$. In fact, we have (see, [3, Remark 2.3 (i)])

Lemma 2.2. *Let C be an irreducible curve, H a general hyperplane for C and $x \in C \cap H$. Then,* $\mathrm{emdim}_x(C) = 1 + \mathrm{emdim}_x(C \cap H)$.

A classical method to study curves consists in studying their general hyperplane sections and then in lifting the information to the curves themselves. In particular, in [3], the authors address the problem of finding a planar subcurve if the general hyperplane section contains a large subscheme spanning a line. Among other results, they proved

Proposition 2.1. *Assume* $\mathrm{char}(K) = 0$. *Let $C \subseteq \mathbb{P}^n (n \geq 4)$ be a non degenerate curve of degree $d \geq 4$ such that for every general hyperplane H the section $C \cap H$ contains a subscheme of degree $d-1$ spanning a line. Then C contains a planar subcurve E of degree $d-1$, with the following exception: C is a multiple line with* $\deg(C) = 4, \mathrm{ged}(C) = 3$.

Now, we give two examples to show that such curves can contain a planar degree 3 subcurve, but this is not always the case, and so they are really exceptions to Proposition 2.1.

Example 2.1 ([3, Remark 4.13]). Let $C \subset \mathbb{P}^4 = \mathrm{Proj}(K[x,y,z,t,w])$ be the multiple line defined by the ideal

$$I_C = (y^2 - 4xz, 2xzt + yzw, yzt + 2z^2w, 2x^2t + xyw, xyt + 2xzw,$$
$$xt^2 + ytw + zw^2, z^3, yz^2, xz^2, xyz, x^3, x^2y, x^2z).$$

C has degree 4, genus -5 and $\mathrm{ged}(C) = 3$. Furthermore, as explained in [3, Remark 4.13], C does not contain a degree 3 planar subcurve.

Example 2.2. Let $C \subset \mathbb{P}^4$ be the multiple line defined by the ideal

$$I_C = \left(x^2, xy, y^2, xz, yz, z^4, (x, y, z^3) \begin{pmatrix} w^6 & 0 \\ t^6 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

As before, C has degree 4 and genus -5. Moreover, $H = V(t-w)$ is a general hyperplane for C and $C \cap H|H$ is defined by $(x+y, y^2, yz, z^3)$. By Lemma 2.2, $\mathrm{ged}(C) = 3$, because $\mathrm{emdim}(C \cap H) = 2$.

C contains a degree 3 planar subcurve, because $I_C \subset (x, y, z^3)$. However, it is easy to prove that C is a flat limit of a one parameter family contained in $\mathcal{V}_{4,4,-5}$. To this end, we consider the family $\{C_a\}_{a \in \mathbb{A}^1}$ defined by the ideal

$$I_a = \left((x,y,z)(x,y,z^2(z^2+atw)), (x,y,z(z^2+atw)) \begin{pmatrix} w^6 & 0 \\ t^6 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

When $a \to 0$, we have that $I_a \to I_C$, and if $a \neq 0$, I_a defines a curve $C_a \in \mathcal{V}_{4,4,-5}$.

3. Degree 4 multiple lines with ged = 3

In this section we construct all the multiple lines of degree 4 and generic embedding dimension 3 by giving the Cohen-Macaulay filtration of such curves. Moreover, we bound the genus and the Rao function of such curves in terms of the data of the filtration.

To study multiple lines of degree 4 and generic embedding dimension 3, we start from the possible general hyperplane section of such a curve. Hence, let H be a hyperplane, general for C. The scheme $C \cap H|H$ is a degree 4 zero-dimensional scheme not contained in a line. Then, its Hilbert function can be either

$$h_1(j) = \begin{cases} 1 & \text{if} \quad j = 0 \\ 4 & \text{if} \quad j \geq 1 \end{cases}$$

or

$$h_2(j) = \begin{cases} 1 & \text{if} \quad j = 0 \\ 3 & \text{if} \quad j = 1 \\ 4 & \text{if} \quad j \geq 2. \end{cases}$$

Now, we consider the two cases, one at a time. To start with, we consider the function h_1.

Lemma 3.1. *Let $C \subset \mathbb{P}^n$ be a multiple line of degree 4 and assume that the Hilbert function of its general hyperplane section $C \cap H|H$ is h_1. Then* $\mathrm{ged}(C) = 3$ *if, and only if, as subscheme of $\mathbb{P}^3 = \mathrm{Proj}(K[x, y, z, t])$, the ideal $I_{C\cap H|H}$ is generated by $x^2 - a_1 t\ell, xy - a_2 t\ell, y^2 - a_3 t\ell, xz - a_4 t\ell, yz - a_5 t\ell, z^2 - a_6 t\ell$ for some non zero $\ell \in [K[x, y, z]]_1$ and $a_i \in K$ not all of them equal to zero.*

Proof. Because of the Hilbert function h_1 of the general hyperplane section $C \cap H|H$ there exist a $\mathbb{P}^3 \subset H$ such that $C \cap H \subset \mathbb{P}^3$. Assume that $\mathbb{P}^3 = \mathrm{Proj}(K[x, y, z, t])$ and that $C \cap H$ is supported on $P = (0 : 0 : 0 : 1)$. Because of the Hilbert function, $I_{C\cap H|\mathbb{P}^3}$ is generated by 6 quadrics, and $I_P^k \subset I_{C\cap H|\mathbb{P}^3} \subset I_P$. Then, there exist 6 linear forms $\ell_1, \dots, \ell_6 \in K[x, y, z]$ such that $I_{C\cap H|\mathbb{P}^3}$ is generated by $x^2 - t\ell_1, xy - t\ell_2, y^2 - t\ell_3, xz - t\ell_4, yz - t\ell_5, z^2 - t\ell_6$.

If $\mathrm{ged}(C) = 3$, then we can assume that $\mathrm{emdim}_P(C \cap H) = 2$, and then $\dim_K \mathcal{L}(\ell_1, \dots, \ell_6) = 1$, because only one indeterminate can be computed in terms of the others. Hence, $\ell_i = a_i \ell$ for some non zero $\ell \in K[x, y, z]$ and for some $a_i \in K$ not all of them equal to zero.

Conversely, if $I_{C\cap H|\mathbb{P}^3}$ is generated by the quadrics $x^2 - a_1 t\ell, \dots, z^2 - a_6 t\ell$ then the embedding dimension of $C \cap H$ at P is equal to 2 and so $\mathrm{ged}(C) = 3$. $\square$

If the Hilbert function of the general hyperplane section of C is h_2, we have the following lemma, that we state in more general hypotheses.

Lemma 3.2. *Let $C \subset \mathbb{P}^n, n \geq 4$, be a multiple line of degree $d \geq 4$ with very degenerate hyperplane section. Then,* $\mathrm{ged}(C) = 3$ *if, and only if, $C \cap H|H$ is defined by the ideal $(y_1, \dots, y_{n-3}, y_{n-2}^2, y_{n-2}y_{n-1}, y_{n-1}^{d-1})$ up to suitably choosing the coordinate system in $H = \mathrm{Proj}(K[y_1, \dots, y_n])$.*

If C has degree 4, it is equivalent to assume that C has very degenerate hyperplane section, or to assume that the Hilbert function of the general hyperplane section of C is h_2.

Proof. The linear span $\langle C \cap H\rangle$ of $C \cap H$ has dimension 2, and so we can work in $\mathbb{P}^2 = \mathrm{Proj}(K[x, y, z])$. Assume that $(C \cap H)_{red} = (0 : 0 : 1) = P$, and assume that the line $L = V(x)$ contains the linear subscheme of $C \cap H$ of degree $d - 1$. Then, we have the following exact sequence

$$0 \to \mathcal{I}_P(-1) \longrightarrow \mathcal{I}_{C\cap H|\mathbb{P}^2} \longrightarrow \mathcal{I}_{C\cap H\cap L|L} \to 0$$

where the first map is the multiplication by x. Then,

$$I_{C\cap H|\mathbb{P}^2} = (x^2, xy, y^{d-1} + axz^{d-2})$$

for some $a \in K$. It is straightforward that the embedding dimension of $C \cap H|H$ is 2 if, and only if, $a = 0$, and so the claim follows by Lemma 2.2. □

Now, we are ready to construct the Cohen-Macaulay filtration of the multiple lines we are interested in.

Proposition 3.1. *Let $C \subset \mathbb{P}^n$ be a degree 4 multiple line supported on a line L. If $\mathrm{ged}(C) = 3$ then there exists a rope Z of degree 3 supported on L such that the Cohen-Macaulay filtration of C is $L \subset Z \subset C$.*

Proof. To construct the Cohen-Macaulay filtration of C we have to intersect C with the various infinitesimal neighborhoods $L^{(i)}$ of L. The first one to be considered is $L^{(2)}$, defined by the ideal I_L^2. Let H be a hyperplane general both for C and for $L^{(2)}$. $C \cap L^{(2)} \cap H$ is defined, in H, by the ideal $\left(I_C + I_L^2 + I_H/I_H\right)^{sat}$. But it holds

Claim 1. $\left(\frac{I_C+I_L^2+I_H}{I_H}\right)^{sat} = \left(\left(\frac{I_C+I_H}{I_H}\right)^{sat} + \left(\frac{I_L^2+I_H}{I_H}\right)^{sat}\right)^{sat}.$

Before proving the claim, we show how the statement follows from the claim.

By Lemma 3.1 and Lemma 3.2, $(I_C + I_H/I_H)^{sat}$ is defined either by $y_1, \dots, y_{n-4}, y_{n-3}^2 - a_1 y_n\ell, \dots, y_{n-1}^2 - a_6 y_n\ell$, or by $y_1, \dots, y_{n-3}, y_{n-2}^2, y_{n-2}y_{n-1}, y_{n-1}^3$, while it is easy to show that $(I_L^2 + I_H/I_H)^{sat}$ is defined by $(y_1, \dots, y_{n-1})^2$. Then, the general hyperplane section of $C \cap L^{(2)}$ is defined either by $y_1, \dots, y_{n-4}, \ell, y_{n-2}^2, y_{n-2}y_{n-1}, y_{n-1}^2$ if $\ell = y_{n-3} + \dots$, or by $y_1, \dots, y_{n-3}, y_{n-2}^2, y_{n-2}y_{n-1}, y_{n-1}^2$. So $\deg(C \cap L^{(2)}) = 3$. Let Z be the largest locally Cohen-Macaulay scheme contained in $C \cap L^{(2)}$. The previous argument shows that $\deg(Z) = 3$, because we can always choose H in such a way that it does not contain any embedded point of $C \cap L^{(2)}$. Hence, the Cohen-Macaulay filtration of C is $L \subset Z \subset C$, and furthermore, $C \subset L^{(3)}$. $\square$

Proof of the Claim. It is trivial that

$$\frac{I_C + I_L^2 + I_H}{I_H} \subset \left(\frac{I_C + I_H}{I_H}\right)^{sat} + \left(\frac{I_L^2 + I_H}{I_H}\right)^{sat}$$

and so the first inclusion follows by taking the saturation of both sides.

To prove the inverse inclusion, at first we notice that $I_L^2 + I_H/I_H$ is saturated because $L^{(2)}$ is an arithmetically Cohen-Macaulay scheme.

Now, if $F \in (I_C + I_H/I_H)^{sat} + (I_L^2 + I_H/I_H)$, there exists $G \in I_L^2 + I_H/I_H$ such that $F - G \in (I_C + I_H/I_H)^{sat}$. Hence, there exist integers $m_1, \dots, m_n$ such that $(F - G)y_i^{m_i} \in (I_C + I_H/I_H)$, for each $i = 1, \dots, n$. This is equivalent to say that $Fy_i^{m_i} \in I_C + I_L^2 + I_H/I_H$, i.e., $F \in (I_C + I_L^2 + I_H/I_H)^{sat}$. By taking the saturation of both the ideals, we get the claim. $\square$

Now, we want to emphasize the data we need to construct such a degree 4 multiple line, because of the properties of the Cohen-Macaulay filtration.

Corollary 3.1. *Let $C \subset \mathbb{P}^n$ be a degree 4 multiple line supported on L with generic embedding dimension 3. Then there exist two short exact sequences of $\mathcal{O}_L$-modules*

$$0 \to \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_L^{n-1}(-1) \xrightarrow{\varphi_A} \mathcal{F}_1 = \begin{array}{c} \mathcal{O}_L(\alpha_0 - 1) \\ \oplus \\ \mathcal{O}_L(\alpha_1 - 1) \end{array} \to 0 \tag{3.1}$$

and

$$0 \to \mathcal{F}_2 \longrightarrow \begin{array}{c} S_2(\mathcal{F}_1) \\ \oplus \\ \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-\beta_j - 1) \end{array} \xrightarrow{(\mu_1,\mu_2)} \mathcal{O}_L(\gamma) \to 0. \tag{3.2}$$

where $S_2(\mathcal{F}_1)$ is the second symmetric power of $\mathcal{F}_1$, and μ_1 is not the null map.

We denote μ the map $\mu = (\mu_1, \mu_2)$.

Proof. If C is a degree 4 multiple line supported on L with $\mathrm{ged}(C) = 3$ then its Cohen-Macaulay filtration is

$$L \subset Z \subset C$$

where Z is a degree 3 rope supported on L, by Proposition 3.1.

The rope Z is uniquely determined by the supporting line L and by the exact sequence (2.2) with $d = 3$. Hence, we get the first sequence of the two ones we need.

Because of the properties of the Cohen-Macaulay filtration of a multiple line, we have the inclusion $\mathcal{I}_L\mathcal{I}_Z \subset \mathcal{I}_C$, and that there exists a free $\mathcal{O}_L$-module $\mathcal{L}$ that fits into the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_Z \to \mathcal{L} \to 0.$$

By comparing the Euler characteristics of the three sheaves, we deduce that $\mathcal{L}$ has rank 1, and so there exists $\gamma \in \mathbb{Z}$ such that $\mathcal{L} = \mathcal{O}_L(\gamma)$. Moreover, if we factor out $\mathcal{I}_L\mathcal{I}_Z$ from the first two items of the previous exact sequence, we still get an exact sequence of $\mathcal{O}_L$-modules

$$0 \to \frac{\mathcal{I}_C}{\mathcal{I}_L\mathcal{I}_Z} \to \frac{\mathcal{I}_Z}{\mathcal{I}_L\mathcal{I}_Z} \to \mathcal{L} \to 0.$$

$\mathcal{I}_Z/\mathcal{I}_L\mathcal{I}_Z$ can be computed as $\operatorname{coker}(\varepsilon_2) \otimes \mathcal{O}_L$ where

$$\ldots \longrightarrow \mathcal{P}_2 \xrightarrow{\varepsilon_2} \mathcal{P}_1 \longrightarrow \mathcal{I}_Z \to 0$$

is a free resolution of $\mathcal{I}_Z$. Thanks to the explicit knowledge of ε_2 [11, Theorem 3.4], we have that $\mathcal{I}_Z/\mathcal{I}_L\mathcal{I}_Z \cong S_2(\mathcal{F}_1) \oplus \oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j)$, where $S_2(\mathcal{F}_1)$ is the second symmetric power of $\mathcal{F}_1 = \mathcal{O}_L(\alpha_0 - 1) \oplus \mathcal{O}_L(\alpha_1 - 1)$. If we set $\mathcal{I}_C/\mathcal{I}_L\mathcal{I}_Z \cong \mathcal{F}_2$ as $\mathcal{O}_L$-modules, we get the second exact sequence. The map μ_1 cannot be zero because otherwise $S_2(\mathcal{F}_1)$ would be a free addendum of $\mathcal{F}_2$. So $\mathcal{I}_C \supset \mathcal{I}_L^2$ and C would be a rope. □

Now, we can prove the converse of Proposition 3.1 and Corollary 3.1.

Theorem 3.1. *Let $L \subset \mathbb{P}^n$ be a line. Assume that the two short exact sequences (3.1) and (3.2) of $\mathcal{O}_L$-modules are given. Then, there exists a multiple line C supported on L, with $\deg(C) = 4$ and $\operatorname{ged}(C) = 3$ such that its associated sequences are the given ones.*

Proof. Let Z be the degree 3 rope supported on L and defined by the short exact sequence (3.1). Its defining ideal is $I_Z = (I_L^2, [I_L]B)$ by [10, Theorem 2.4], where $[I_L]$ is a row matrix whose entries are the generators of I_L in the ordering given by the basis of $\mathcal{O}_L^{n-1}(-1)$, and B represents the map φ_B.

By [11, Theorem 3.4], $\mathcal{I}_Z$ has a presentation given by

$$\mathcal{O}_{\mathbb{P}^n}^{\binom{n}{2}}(-2) \oplus \bigoplus_{j=1}^{n-3} \mathcal{O}_{\mathbb{P}^n}(-1-\beta_j) \longrightarrow \mathcal{I}_Z \to 0.$$

Then, $\mathcal{I}_Z \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_L \cong \mathcal{I}_Z/\mathcal{I}_L\mathcal{I}_Z$ has a presentation given by

$$\mathcal{O}_L^{\binom{n}{2}}(-2) \oplus \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j) \longrightarrow \mathcal{I}_Z/\mathcal{I}_L\mathcal{I}_Z \to 0.$$

Moreover, $\mathcal{O}_L^{\binom{n}{2}}(-2) = S_2(\mathcal{O}_L^{n-1}(-1))$, and so there exists a surjective map

$$\begin{matrix} \mathcal{O}_L^{\binom{n}{2}}(-2) \\ \oplus \\ \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j) \end{matrix} \xrightarrow{\phi_A} \begin{matrix} S_2(\mathcal{F}_1) \\ \oplus \\ \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j) \end{matrix}$$

where $\phi_A = \begin{pmatrix} S_2\varphi_A & 0 \\ 0 & id \end{pmatrix}$.

Let $\psi_D = \mu \circ \phi_A : \mathcal{O}_L^{\binom{n}{2}}(-2) \oplus \oplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j) \to \mathcal{O}_L(\gamma)$, and let ψ_E be the map which resolves it. Then we have the following exact sequence

$$0 \to \mathcal{G} \xrightarrow{\psi_E} \begin{matrix} \mathcal{O}_L^{\binom{n}{2}}(-2) \\ \oplus \\ \bigoplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j) \end{matrix} \xrightarrow{\psi_D} \mathcal{O}_L(\gamma) \to 0 \tag{3.3}$$

where $\mathcal{G}$ is a suitable $\mathcal{O}_L-$module of rank $\binom{n+1}{2} - 4$.

Let $[I_Z]$ be a row matrix whose entries correspond to the elements of a basis of $\mathcal{O}_L^{\binom{n}{2}}(-2) \oplus \oplus_{j=1}^{n-3} \mathcal{O}_L(-1-\beta_j)$ and let E be a matrix which represents the map ψ_E. We can construct the ideal $I_C = (I_L I_Z, [I_Z]E)$, and let $\mathcal{I}_C$ and $\mathcal{O}_C$ be the ideal sheaf and the structure sheaf of the corresponding closed scheme. Then, we have the following properties.

(a) $\mathcal{I}_C$ and $\mathcal{O}_C$ fit into the exact sequences

$$0 \to \mathcal{I}_C \to \mathcal{I}_Z \to \mathcal{O}_L(\gamma) \to 0 \tag{3.4}$$

and

$$0 \to \mathcal{O}_L(\gamma) \to \mathcal{O}_C \to \mathcal{O}_Z \to 0 \tag{3.5}$$

because the exact sequence (3.3) can be rewritten as

$$0 \to \frac{\mathcal{I}_C}{\mathcal{I}_L \mathcal{I}_Z} \to \frac{\mathcal{I}_Z}{\mathcal{I}_L \mathcal{I}_Z} \to \mathcal{O}_L(\gamma) \to 0$$

from which we get the exact sequence (3.4). The second exact sequence (3.5) is then straightforward.

By computing the Hilbert polynomials of the items of (3.5), we get that

$$p_C(z) = 4z + 1 - (g_z - \gamma - 1)$$

and so C has dimension 1 and degree 4.

(b) Taking the cohomology sequence associated to (3.4), we get the sequence

$$\cdots \to H^0(\mathcal{O}_L(j+\gamma)) \to H^1(\mathcal{I}_C(j)) \to H^1(\mathcal{I}_Z(j)) \to \dots$$

and so $H^1(\mathcal{I}_C(j)) = 0$ for $j \ll 0$, that is to say, C is a curve.

(c) $I_L^3 \subset I_C \subset I_L$ because $I_L^2 \subset I_Z \subset I_L$. Then, C is a multiple line supported on L, that admits a Cohen-Macaulay filtration given by $L \subset Z \subset C$, where Z is a degree 3 rope supported on L.

(d) From (3.4), we deduce the exact sequence

$$0 \to \mathcal{I}_{C\cap H|H} \to \mathcal{I}_{Z\cap H|H} \to \mathcal{O}_P(\gamma) \to 0$$

where H is a hyperplane general both for C and for Z, and $P = H \cap L$. Hence, we have

$$0 \to H^0(\mathcal{I}_{C\cap H|H}(j)) \to H^0(\mathcal{I}_{Z\cap H|H}(j)) \to H^0(\mathcal{O}_P(j+\gamma)) \to \dots$$

The ideal $I_{Z\cap H|H}$ is generated by $y_1, \dots, y_{n-3}, (y_{n-2}, y_{n-1})^2$ up to suitably choosing the coordinate system in $H = \operatorname{Proj}(K[y_1, \dots, y_n])$, while $h^0(\mathcal{O}_P(j + \gamma)) = 1$, for each $j \in \mathbb{Z}$.

For $j = 1$, there are two possible cases: the map $\delta : H^0(\mathcal{I}_{Z\cap H|H}(1)) \to H^0(\mathcal{O}_P(1+\gamma))$ is zero, or it is surjective.

Case 1: δ is the null map.

In this case, $H^0(\mathcal{I}_{C\cap H|H}(1)) = H^0(\mathcal{I}_{Z\cap H|H}(1)) = (y_1, \dots, y_{n-3})$. Then, the Hilbert function of $C \cap H|H$ is h_2 and C has very degenerate hyperplane section. Hence, we can lift two quadrics from $(I_{Z\cap H|H})_2$ with a common factor. Assume we lift y_{n-2}^2, and $y_{n-2}y_{n-1}$. Then, we can lift y_{n-1}^3 too, and so $I_{C\cap H|H}$ is the one computed in Lemma 3.2. Then, we get that $\operatorname{ged}(C) = 3$. Notice that lifting a different cubic form forces the scheme $C \cap H|H$ not to be supported only on $(0 : \dots : 0 : 1)$.

Case 2: δ is surjective.

In this case, $h^0(\mathcal{I}_C(1)) = n-4$. Without loss of generality, we can assume that $H^0(\mathcal{I}_C(1))$ is generated by $y_1, \dots, y_{n-4}$. Moreover, the Hilbert function of $\mathcal{I}_{C\cap H|H}$ is h_1, and we can lift 6 quadrics from $H^0(\mathcal{I}_Z(2))$ which have the shape $y_{n-3}^2 + a_1 y_{n-3}y_n, \dots, y_{n-1}^2 + a_6 y_{n-3}y_n$, because $I_{C\cap H|H}$ is a saturated ideal and because a different choice forces the Cohen-Macaulay filtration to be different. Hence, $\operatorname{emdim}_P(C \cap H) = 2$ and $\operatorname{ged}(C) = 3$.

Summarizing the results, we have that C is a multiple line supported on L with $\deg(C) = 4$ and $\operatorname{ged}(C) = 3$. Furthermore, the sequences associated to C according to Corollary 3.1 are the ones given as hypotheses. □

Now, we want to investigate the numerical data involved in the construction of such curves.

Corollary 3.2. *Let $C \subset \mathbb{P}^n$ be a multiple line supported on L, with $\deg(C) = 4$ and $\operatorname{ged}(C) = 3$. Then, $g_C = g_Z - \gamma - 1$, where Z is the degree 3 rope which appears in the Cohen-Macaulay filtration of C.*

Proof. See the proof of previous Theorem 3.1, where we computed the Hilbert polynomial of C. □

Corollary 3.3. *Let $C \subset \mathbb{P}^n$ be a multiple line as before. Then*

$$h^1(\mathcal{I}_C(j)) \leq \dim_K \left(\frac{S}{(D)}\right)_{j+\gamma} + h^1(\mathcal{I}_Z(j))$$

for every $j \in \mathbb{Z}$, where (D) is the ideal in S generated by the entries of a matrix D which represents the map ψ_D.

Proof. In the proof of previous Theorem 3.1, we considered the exact sequence (3.4) and the associated cohomology sequence. The result follows from it, because we can rewrite that sequence as

$$0 \to \left(\frac{S}{(D)}\right)_{j+\gamma} \to H^1(\mathcal{I}_C(j)) \to H^1(\mathcal{I}_Z(j))$$

where (D) is the ideal generated by the entries of the matrix D which represents the map ψ_D. $\square$

Remark 3.1. The dimension $\dim_K(S/(D))_{j+\gamma}$ can be computed from the sequence (3.3).

Corollary 3.4. *In the same hypotheses as before, if Z is an arithmetically Cohen-Macaulay curve, then $H^1_*(\mathcal{I}_C) \cong \dfrac{S}{(D)}(\gamma)$.*

Proof. If Z is arithmetically Cohen-Macaulay then $H^1_*(\mathcal{I}_Z) = 0$, and the claim follows from the previous Corollary. $\square$

Remark 3.2. Z is arithmetically Cohen-Macaulay if, and only if, $g_Z = 0$, or, equivalently, $\alpha_0 = \alpha_1 = \beta_1 = \cdots = \beta_{n-3} = 0$.

Now, we look for bounds for γ in terms of the other data.

Proposition 3.2. *Let $C \subset \mathbb{P}^n$ be a multiple line supported on L with* $\deg(C) = 4$, $\mathrm{ged}(C) = 3$, *and let Z be the degree 3 rope which appears in the Cohen-Macaulay filtration of C. Then $\gamma \geq 2\alpha_0 - 2$ and $g_Z \geq g_c - 1$.*

Proof. The genus of the degree 3 rope Z is $g_Z = -\alpha_0 - \alpha_1$ where $0 \leq \alpha_0 \leq \alpha_1$. Then, $g_Z \leq 0$. Assume that $\beta_1 \leq \cdots \leq \beta_{n-3}$.

To compute the bound for γ, we have to consider the existence of a surjective map μ as in the sequence (3.2), with $\mu_1 \neq 0$. This is possible if either there exists a retraction or in the image of μ there is a regular sequence of length 2.

We recall that the domain of μ is

$$\mathcal{O}_L(2\alpha_0-2)\oplus\mathcal{O}_L(\alpha_0+\alpha_1-2)\oplus\mathcal{O}_L(2\alpha_1-2)\oplus\mathcal{O}_L(-1-\beta_1)\oplus\cdots\oplus\mathcal{O}_L(-1-\beta_{n-3}),$$

and so the degrees of the entries of μ are $\gamma - 2\alpha_0 + 2 \geq \gamma - \alpha_0 - \alpha_1 + 2 \geq \gamma - 2\alpha_1 + 2, \gamma + 1 + \beta_1 \leq \cdots \leq \gamma + 1 + \beta_{n-3}$.

Hence, $\gamma \geq 2\alpha_0 - 2$. Moreover, $g_Z - g_C - 1 = \gamma \geq 2\alpha_0 - 2 \geq -2$, and the bound for g_Z follows. □

4. Characterization of some multiple lines

In this section, we want to characterize the multiple lines of degree 4 and generic embedding dimension 3 which have a very degenerate hyperplane section, or which contain a degree 3 planar subcurve. Of course, we characterize them in terms of the sequences (3.2) and (3.3).

At first, we characterize the multiple lines with very degenerate hyperplane section.

Theorem 4.1. *Let $C \subset \mathbb{P}^n$ be a multiple line supported on L with $\deg(C) = 4$ and $\mathrm{ged}(C) = 3$. C has very degenerate hyperplane section if, and only if, $\mu_2 : \oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j) \to \mathcal{O}_L(\gamma)$ is the null map.*

Proof. Assume that μ_2 is the null map. Then, the map ψ_D restricted to $\oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j)$ is zero, too, and so the free module $\mathcal{G}$ splits as $\mathcal{G} = \mathcal{G}' \oplus \oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j)$ and ψ_E restricted to $\oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j)$ is the identity map. To fix notation, the map ψ_E restricted to $\mathcal{G}'$ is represented by the matrix E'.

By Theorem 3.1, we have that $I_C = (I_L^3, [I_L^2]E', [I_L]B)$ defines a degree 4 multiple line supported on L with $\mathrm{ged}(C) = 3$.

Let $H = V(x_n - ax_{n-1})$ be a general hyperplane for C, for a suitable $a \in K$. Then, $I_{C\cap H|H} = (I_L^3, [I_L^2]E'_P, [I_L]B_P)^{sat}$ where E'_P (resp. B_P) is obtained from E' (resp. B) by setting $x_n = ax_{n-1}$, and the saturation is computed in $K[x_0, \ldots, x_{n-1}]$.

B does not drop rank in codimension 1, and so B_P has rank $n-3$, i.e., $[I_L]B_P = (x_{n-1}^{\beta_1}\ell_1, \ldots, x_{n-1}^{\beta_{n-3}}\ell_{n-3})$ where $\ell_1, \ldots, \ell_{n-3} \in (I_L)_1$ are linearly independent. Hence, the Hilbert function of $C \cap H|H$ is h_2 and C has very degenerate hyperplane section.

Conversely, assume that $\mu_2 : \oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j) \to \mathcal{O}_L(\gamma)$ is not the null map. Then, $\psi_{D|} : \oplus_{j=1}^{n-3}\mathcal{O}_L(-1-\beta_j) \to \mathcal{O}_L(\gamma)$ is represented by a matrix $D_2 = (d_1, \ldots, d_{n-3})$ which is non zero. Without loss of generality, assume that $d_{n-3}(P) \neq 0$ for some $P = (0 : \ldots : 0 : 1 : a) \in L$.

Let $H = V(x_n - ax_{n-1})$ be a hyperplane. We want to compute $I_{C\cap H|H}$. Let us consider the sequence (3.3). By the Hilbert-Burch Theorem [4, Theorem 20.15], the maximal minors of the matrix E are equal to the entries of the matrix $D = (D_1|D_2)$ which represents the map ψ_D, up to their sign. Of course, D_1 represents the map ψ_D restricted to $\mathcal{O}_L^{\binom{n}{2}}(-2)$.

We want to prove that $I_{C\cap H|H} = ([I_Z]_P E_P)^{sat}$, where the subscript P means that we substitute x_n with ax_{n-1} and the saturation is computed in $K[x_0, \ldots, x_{n-1}]$.

Let $E_\bullet$ be the last row of E and let $\tilde{E}$ be the submatrix of E obtained by erasing its last row. Then, $\det(\tilde{E}) = d_{n-3}$, if we assume we choose the signs in such a way that they agree. Moreover, let $[I_L]B = (F_1, \ldots, F_{n-3})$ be the generators of I_Z not in I_L^2. Then, in I_C there are the polynomials

$$[I_Z]E = [I_L^2, F_1, \dots, F_{n-4}]\tilde{E} + F_{n-3}E_\bullet.$$

If we multiply by $\mathrm{adj}(\tilde{E})$, classical adjoint matrix of $\tilde{E}$, we get that

$$\det(\tilde{E})[I_L^2, F_1, \dots, F_{n-4}]Id + F_{n-3}E_\bullet \mathrm{adj}(\tilde{E})$$

are polynomials in I_C.

It is a classical result that

$$E_\bullet \mathrm{adj}(\tilde{E}) = (\delta_1, \dots, \delta_{\binom{n}{2}}, d_1, \dots, d_{n-4}) =: \tilde{D},$$

where $D_1 = (\delta_1, \dots, \delta_{\binom{n}{2}})$, and so we can rewrite the previous polynomials as

$$d_{n-3}[I_L^2, F_1, \dots, F_{n-4}] + F_{n-3}\tilde{D}. \tag{4.1}$$

If we substitute $x_n = ax_{n-1}$ in $F_1, \dots, F_{n-3}$ and D, and we denote $F_i(P), i = 1, \dots, n-3$, and $D(P)$ the results, we get that $F_i(P) = x_{n-1}^{\beta_i}\ell_i$, for $i = 1, \dots, n-3$, where $\ell_1, \dots, \ell_{n-3} \in (I_L)_1$ are linearly independent, and $\delta_i(P) = p_i x_{n-1}^{\gamma+2}$, for $i = 1, \dots, \binom{n}{2}$, with $p_i \in K$, and $d_i(P) = q_i x_{n-1}^{\gamma+\beta_i+1}$ for $i = 1, \dots, n-3$, with $q_i \in K$, and $q_{n-3} \neq 0$. Hence, if we substitute as before $x_n = ax_{n-1}$ in the polynomials in (4.1), we get

$$q_{n-3}x_{n-1}^{\gamma+\beta_{n-3}+1}[I_L^2, x_{n-1}^{\beta_1}\ell_1, \dots, x_{n-1}^{\beta_{n-4}}\ell_{n-4}] + x_{n-1}^{\beta_{n-3}}\ell_{n-3}\tilde{D}(P) \in I_{C\cap H|H}.$$

By taking saturation, we have that

$$[I_L^2] + \ell_{n-3}\frac{x_{n-1}}{q_{n-3}}[p_1, \dots, p_{\binom{n}{2}}]$$

and

$$\ell_i + \frac{q_i}{q_{n-3}}\ell_{n-3} \qquad i = 1, \dots, n-4,$$

belong to $I_{C\cap H|H}$. Set J be the ideal generated by those polynomials.

Other generators of I_C are I_L^3 and $I_L(F_1, \dots, F_{n-3})$.

At first, we prove that $x_j F_i(P) = x_j x_{n-1}^{\beta_i}\ell_i \in J$.

Of course, the claim is equivalent to $x_j x_{n-1}^{\beta_i}\frac{q_i}{q_{n-3}}\ell_{n-3} \in J$.

$x_j\ell_{n-3} \in I_L^2$ and the coefficients are reordered in such a way that they are a syzygy of $(S_2\varphi_A)_P$. Hence, $x_j x_{n-1}^{\beta_i}\ell_i \in J$ for all i, j.

In the same way, we have that $I_L^3 \subset J$ and so $I_{C\cap H|H} = J^{sat}$. Moreover, the generators of J are a Grobner basis with respect to the reverse lexicographic order of the variables because their S-polynomials are in J for the same argument as before. Then, the initial ideal of J is saturated and so J is saturated, too. Hence, $d_{n-3}(P) \neq 0$ implies that the Hilbert function of $C \cap H|H$ is equal to 4 for $j \geq 1$, and so C does not have very degenerate hyperplane section, and the claim follows. $\square$

Now, we describe the Hartshorne-Rao module and the global sections of the structure sheaf of a multiple line with very degenerate hyperplane section.

Proposition 4.1. *Let $C \subset \mathbb{P}^n$ be a multiple line supported on a line L with* $\deg(C) = 4, \operatorname{ged}(C) = 3$, *and very degenerate hyperplane section. Then*
(a) $0 \to H^0_*(\mathcal{O}_L(\gamma)) \to H^0_*(\mathcal{O}_C) \to H^0_*(\mathcal{O}_Z) \to 0$ *is exact;*
(b) $0 \to \frac{S}{(D)}(\gamma) \to H^1_*(\mathcal{I}_C) \to H^1_*(\mathcal{I}_Z) \to 0$ *is exact.*

Proof. To prove (a), we recall some results on $H^0_*(\mathcal{O}_Z)$ from [11, Lemma 4.2, Remark 4.4, Theorem 4.5].

$H^0_*(\mathcal{O}_Z)$ is a free S-module, $S = K[x_{n-1}, x_n]$, with basis $1, Z_0, Z_1$, Z_i of degree $1 - \alpha_i, i = 0, 1$, where Z_0, Z_1 are defined by $[I_L] = (Z_0, Z_1)A$ in $\mathcal{O}_Z$.

In our hypotheses, the total ideal of the multiple line C is $I_C = (I_L^3, [I_L^2]E', [I_L]B)$. Then, in $\mathcal{O}_C$, $[I_L]B = 0$, and so we can define two sections z_0, z_1 of degrees $1 - \alpha_0, 1 - \alpha_1$, such that $[I_L] = (z_0, z_1)A$, because $AB = 0$. Of course, we have that 1 is a global section of $\mathcal{O}_C$ of degree 0, where 1 is the image of the unit in $R/I_C \to H^0_*(\mathcal{O}_C)$. Hence, the natural map $H^0_*(\mathcal{O}_C) \to H^0_*(\mathcal{O}_Z)$ induced by the inclusion of the ideals is surjective, because all the generators of $H^0(\mathcal{O}_Z)$ are in the image.

(b) follows from (a) and the proof of Corollary 3.3, because the surjection $H^0_*(\mathcal{O}_C) \to H^0_*(\mathcal{O}_Z)$ induces a surjective map $H^1_*(\mathcal{I}_C) \to H^1_*(\mathcal{I}_Z)$, being $H^1_*(\mathcal{I}_Z)$ generated by the images of Z_0 and Z_1. □

The second kind of multiple lines we want to characterize are the multiple lines containing a planar subcurve of degree 3.

Theorem 4.2. *Let $C \subset \mathbb{P}^n$ be a multiple line supported on a line L with* $\deg(C) = 4, \operatorname{ged}(C) = 3$, *and very degenerate hyperplane section. Then, C contains a planar subcurve of degree 3 if, and only if, an addendum splits off both from $\varphi_A : \mathcal{O}_L^{n-1}(-1) \to \mathcal{O}_L(\alpha_0 - 1) \oplus \mathcal{O}_L(\alpha_1 - 1)$ and from $\psi_D : \mathcal{O}_L^{\binom{n}{2}}(-2) \to \mathcal{O}_L(\gamma)$.*

Proof. The only degree 3 planar curves supported on L are generated by $\ell_0, \ldots, \ell_{n-3}, \ell_{n-2}^3$ where $\ell_0, \ldots, \ell_{n-2}$ generate I_L. Without loss of generality, we can assume that $\ell_i = x_i, i = 0, \ldots, n-2$.

If C satisfies the hypotheses, its saturated ideal is $I_C = (I_L^3, [I_L^2]E', [I_L]B)$, by Theorem 4.1.

Now, assume that C contains the planar subcurve D defined by $I_D = (x_0, .., x_{n-3}, x_{n-2}^3)$. Then, $[I_L]B \subset I_D$, and $[I_L^2]E' \subset I_D$. Hence, the last row of both B and E' is zero, and so the claim follows.

Conversely, if one addendum splits off both in φ_A and in ψ_D, then we can choose a basis in $\mathcal{O}_L^{n-1}(-1)$ in such a way that one row of B is zero, and so $I_Z = (I_L^2, [I_L]B) = (I_L^2, F_1, \ldots, F_{n-3})$ satisfies $F_i \in (x_0, \ldots, x_{n-3})$, without loss of generality. Moreover, either $\alpha_0 = 0$ or $\alpha_1 = 0$. Assume $\alpha_0 = 0$.

By hypothesis, the map $\psi_D = \mu \circ S_2\varphi_A$ has a retraction, and so $\gamma = -2$. Of course, the addendum which splits off is $\mathcal{O}_L(2\alpha_0 - 2) = \mathcal{O}_L(-2)$ and so $[I_L^2]E' \subset (x_0, \ldots, x_{n-3})$. Hence, $I_C \subset I_D$, and the claim follows. □

In this last case, we can describe more precisely the Hartshorne-Rao module.

Corollary 4.1. *In the same hypotheses as Theorem 4.2, the Hartshorne -Rao module of C is*

$$H^1_*(\mathcal{I}_C) \cong \left(\frac{S}{(A)}\right)(\alpha_1 - 1)$$

where (A) is the ideal generated by the entries of a matrix A which represents the map $\tilde{\varphi}_A : \mathcal{O}_L^{n-2}(-1) \to \mathcal{O}_L(\alpha_1 - 1)$ obtained from φ_A by canceling the extra addendum.

Proof. In our hypotheses, ψ_D has a retraction and so $S/(D) = 0$. Hence, by Proposition 4.1(b), $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_Z)$. By [10], Proposition 3.1, $H^1_*(\mathcal{I}_Z) \cong \operatorname{coker}(\varphi_A)$ and because of our hypotheses, we get the claim. $\square$

Remark 4.1. If the four-tuple line C contains a planar subcurve D of degree 3, then $g_C = 1 + g_Z$, because $\gamma = -2$, and Corollary 3.2.

5. A connectedness problem

In this section, we construct suitable parameter spaces for the multiple lines of degree 4 and generic embedding dimension 3. Because of the universal property of the Hilbert scheme, we can embed these parameter spaces into the corresponding Hilbert scheme parameterizing only locally Cohen-Macaulay schemes, and we can consider their closures. A natural problem is the following: is the union of those closed subschemes connected?

This problem was considered, up to now, only for curves in the projective three space. In that ambient, it is not known if the Hilbert scheme of locally Cohen-Macaulay schemes is connected, also if, in various papers, some authors gave partial results in that direction. Of course, we do not consider all the degree 4 curves of fixed genus, but only the multiple lines with generic embedding dimension 3, and so we don' t give a general answer for the degree 4 curves. To prove the connectedness, we consider one more family, the ropes of degree 4 and right genus, which appears in projective spaces of dimension higher than 3.

At first, we describe the parameter spaces.

A multiple line C supported on a line L, with $\deg(C) = 4$, and $\operatorname{ged}(C) = 3$, is known when we know the sequences (3.1) and (3.2). Then, we can parameterize them using the parameter space for the degree 3 ropes and a parameter space for the map μ. To specify the map μ, we need that the left-type β of the rope Z is fixed. Hence, we notice $\mathcal{R}_{n,3,\alpha}(\beta)$ the locally closed subset of $\mathcal{R}_{n,3,\alpha}$ where the left-type is β. It is locally closed because we fixed the Rao function of the rope.

Definition 5.1. Every multiple line C of degree 4, genus g_C and generic embedding dimension 3, which is filtered via a degree 3 rope of right-type α and left-type β corresponds to a closed point of an open subset of a $\mathbb{P}^N$-bundle over $\mathcal{R}_{n,3,\alpha}(\beta)$, where $g_Z = -\alpha_0 - \alpha_1$, $N = n(g_Z - g_C + 1) - 2g_Z + 2$, with fibers corresponding to surjective maps $\mu : S_2(\mathcal{O}_L(\alpha_0 - 1) \oplus \mathcal{O}_L(\alpha_1 - 1)) \oplus$

$\oplus_{j=1}^{n-3}\mathcal{O}_L(-\beta_j-1)\to\mathcal{O}_L(g_Z-g_C-1)$ which do not drop rank in codimension 1, modulo an automorphism of $\mathcal{O}_L(g_Z-g_C-1)$.

With abuse of notation, we denote with $\mathcal{L}_{\dots}$ both the parameter space and its embedding into the corresponding Hilbert scheme.

Remark 5.1. The maximal genus for non planar curves of degree 4 is $g_C=1$. If C is a multiple line with $\deg(C)=4$, $\mathrm{ged}(C)=3$ and $g_C=1$, then the degree 3 rope which appears in its filtration has genus $g_Z=0$, because of Proposition 3.2. Hence, there is only one choice for the right-type and for the left-type of Z, and so there is only one family of multiple lines to consider. In this case, the connectedness is trivial.

From now on, we assume that $g_C\le 0$.
At first, we consider the case $n\ge 5$.

Proposition 5.1. *Let $n\ge 5$. Every family $\mathcal{L}_{n,\alpha,\beta,g_C}$ such that there exists a surjective map μ_2 connects to $\mathcal{R}_{n,4,g_C}$.*

Proof. Set C a multiple line corresponding to a closed point of $\mathcal{L}_{a,\alpha,\beta,g_C}$ such that the associated map $\mu=(\mu_1,\mu_2)$ has μ_2 surjective. Here, we need $n\ge 5$ and some restrictions on the left- and right-type.

Set C_t the multiple line defined by the same degree 3 rope Z and by the map $\mu_t=(t\mu_1,\mu_2), t\in\mathbb{A}^1$. If $t\ne 0$, then C_t is a multiple line of degree 4, generic embedding dimension 3 and genus g_C. If $t=0$, C_0 is a degree 4 rope of genus g_C. In fact, $\mu_1=0$ implies that $I_L^2\subset I_{C_0}$ and the genus can be computed by resolving the map μ_2. Hence, we can connect the two families. $\square$

Remark 5.2. From a numerical point of view, there exists a surjective map μ_2 if $\gamma-2\alpha_0+2\le\gamma+1+\beta_{n-4}$, or if the right-type and the left-type are zero (i.e. $g_Z=0$) and $\gamma\ge -1$. Then, we have to consider the complementary cases, i.e. $\alpha_0=\beta_{n-4}=0$, $\alpha_1>0$.

Proposition 5.2. *Let $g_Z\le 0$. Set $\alpha'=(0,-g_Z),\beta'=(0,\dots,0,-g_Z)$, and $\alpha''=(0,1-g_Z)$, $\beta''=(0,\dots,0,1,-g_Z)$. Then, the families $\mathcal{L}'=\mathcal{L}_{n,\alpha',\beta',g_C}$ and $\mathcal{L}''=\mathcal{L}_{n,\alpha'',\beta'',g_C}$ are in the same connected component.*

Proof. We exhibit a flat family whose general member is in $\mathcal{L}''$ and whose special member is in $\mathcal{L}'$.

We choose Z_t, $t\in\mathbb{A}^1$, to be the degree 3 rope defined over $L=V(x_0,\dots,x_{n-2})$ by the matrix

$$A_t=\begin{pmatrix}0&\cdots&0&tx_n^{1-g_Z}&x_{n-1}x_n^{-g_Z}&x_{n-1}^{1-g_Z}&0\\0&\cdots&0&0&0&0&1\end{pmatrix}.$$

Then, the saturated ideal of Z_t is

$$I_{Z_t}=((I_L)^2,x_0,\dots,x_{n-6},x_{n-5}x_{n-1}-tx_{n-4}x_n,x_{n-4}x_{n-1}^{-g_Z}-x_{n-3}x_n^{-g_Z}).$$

To define a multiple line, we have to define the map μ_t. We choose

$$\mu_{1,t} = (0, 0, x_{n-1}^{g_Z - g_C + 1})$$

and

$$\mu_{2,t} = (0, \ldots, 0, x_n^{g_Z - g_C + 1}, 0)$$

for every $t \neq 0$. Notice that the only non zero entry of $\mu_{2,t}$ corresponds to the degree 2 generator of I_{Z_t}. If we compute the map ψ_{D_t} and the map ψ_{E_t} which resolves it, we get the following ideal that defines a multiple line C_t

$$I_{C_t} = (x_0, \ldots, x_{n-6}, x_{n-5}^2, x_{n-5}x_{n-4}, x_{n-4}^2, \ldots, x_{n-3}x_{n-2}, x_{n-2}^3,$$
$$x_{n-2}^2 x_n^{g_Z - g_C + 1} - x_{n-1}^{g_Z - g_C + 1}(x_{n-5}x_{n-1} - t x_{n-4}x_n), x_{n-4}x_{n-1}^{-g_Z} - x_{n-3}x_n^{-g_Z}).$$

If we let $t \to 0$, we get the ideal

$$I_0 = (x_0, \ldots, x_{n-6}, x_{n-5}^2, \ldots, x_{x-3}x_{x-2}, x_{n-2}^3,$$
$$x_{n-2}^2 x_n^{g_Z - g_C + 1} - x_{n-5}x_{n-1}^{g_Z - g_C + 2}, x_{n-4}x_{n-1}^{-g_Z} - x_{n-3}x_n^{-g_Z})$$

that defines a multiple line C_0 of degree 4, generic embedding dimension 3, whose filtration is $L \subset Z_0 \subset C_0$, where Z_0 is defined by

$$I_{Z_0} = ((I_L)^2, x_0, \ldots, x_{n-6}, x_{n-5}, x_{n-4}x_{n-1}^{-g_Z} - x_{n-3}x_n^{-g_Z})$$

and

$$\mu_{1,0} = (0, 0, x_{n-1}^{g_Z - g_C + 2})$$

while

$$\mu_{2,0} = (0, \ldots, 0, x_n^{g_Z - g_C + 1}, 0).$$

Hence, C_0 has genus g_C and the right-type of Z_0 is $(0, -g_Z)$, while its left-type is $(0, \ldots, 0, -g_Z)$, and so the claim holds. □

Now, we consider the case $n = 4$. In this case, the left-type of the rope is $-g_Z$ because $n - d = 1$. Moreover, in $\mathbb{P}^4$ the degree 4 ropes are $L^{(2)}$ and so we have degree 4 ropes only if $g_Z = 0$. Then, we have to follow a different strategy to get the claim.

Proposition 5.3. *Let $n = 4$, and let $g_C, g_Z \in \mathbb{Z}$ such that $g_C \leq 0, g_C - 1 \leq g_Z \leq 0$. Then the families $\mathcal{L}' = \mathcal{L}_{4,(\alpha_0,\alpha_1),-g_Z,g_C}$ and $\mathcal{L}'' = \mathcal{L}_{4,0,0,g_C}$ are in the same connected component, whatever $0 \leq \alpha_0 \leq \alpha_1$, with $\alpha_0 + \alpha_1 = -g_Z$.*

Proof. We construct a flat family of multiple lines whose general member belongs to $\mathcal{L}'$ and whose special member is in $\mathcal{L}''$.

Let $Z_t, t \in \mathbb{A}^1$ be the rope supported on the line $L = V(x, y, z) \subset \mathbb{P}^4 = \mathrm{Proj}(K[x, y, z, u, v])$ defined by the matrix

$$A_t = \begin{pmatrix} v^{\alpha_0} & t u^{\alpha_0} & 0 \\ 0 & v^{\alpha_1} & u^{\alpha_1} \end{pmatrix}.$$

Its saturated ideal is

$$I_{Z_t} = ((x,y,z)^2, t^2xu^{-g_Z} - tyu^{\alpha_1}v^{\alpha_0} + zv^{-g_Z}).$$

To define the multiple line we choose the map $\mu_t = (v^{-3\alpha_0-\alpha_1-g_C+1}, 0, 0, u^{-g_C})$. The exponent of v is non negative because of Proposition 3.2.

We can compute the ideal of C_t using the usual procedure, and we get

$$\begin{aligned} I_{C_t} = (&x^3, x^2y, xy^2, y^3, xz, yz, z^2, tx^2u^{\alpha_0} - xyv^{\alpha_0}, txyu^{\alpha_0} - y^2v^{\alpha_0}, \\ &y^2u^{-g_C-2\alpha_0} - t^2v^{-3\alpha_0-\alpha_1-g_C+1}(t^2xu^{-g_Z} - tyu^{\alpha_1}v^{\alpha_0} + zv^{-g_Z})). \end{aligned}$$

In I_{C_t} there are also the polynomials p such that $tp \in I_{C_t}$, because we flatten the family over $K[t]$. Then, we have to add two more forms

$$p_1 = xyu^{-g_C-\alpha_0} - t^3xu^{-g_Z}v^{-2\alpha_0-\alpha_1-g_C+1} + t^2yu^{\alpha_1}v^{g_Z-g_C+1} - tzv^{-\alpha_0-g_C+1}$$

obtained as $u^{-g_C-2\alpha_0}$ times the second last generator plus v^{α_0} times the last generator of I_{C_t}, and

$$p_2 = x^2u^{-g_C} - zv^{1-g_C} + tyu^{\alpha_1}v^{-\alpha_1-g_C+1} - t^2xu^{-g_Z}v^{g_Z-g_C+1}$$

obtained as $u^{-g_C-\alpha_0}$ times the third last generator plus v^{α_0} times p_1.

If we let $t \to 0$, and we take the saturation of the ideal we get, we obtain the ideal

$$I_{C_0} = (x^3, xy, y^2, xz, yz, z^2, x^2u^{-g_C} - zv^{1-g_C})$$

that defines a multiple line of degree 4, generic embedding dimension 3, genus g_C whose associated Cohen-Macaulay filtration is $L \subset Z_0 \subset C_0$, where Z_0 is

$$I_{Z_0} = ((x,y,z)^2, z)$$

and so its left-type is 0, while its right-type is $(0,0)$. □

Remark 5.3. If $g_C = 0$, we have two possible genera for the associated degree 3 rope, namely, $g_Z = 0$, or $g_Z = -1$. If $g_Z = 0$, the multiple lines either are arithmetically Cohen-Macaulay if their general hyperplane section is not very degenerate, or they have very degenerate hyperplane section and Hartshorne-Rao module isomorphic to K in degree 0. The arithmetically Cohen-Macaulay curves of degree 4 in $\mathbb{P}^4$ where studied in [8], and in [15]. In the first quoted paper, the authors proved that these curves fill an irreducible component of dimension 21, and the ropes are singular in that component. Then, the family $\mathcal{L}_{4,(0,0),0,0}$ is contained in the closure of that irreducible component, and so we find curves with very degenerate hyperplane section in that closure.

As last result of the section, we want to prove that the family of degree 4 curves containing a planar subcurve of degree 3 is in the same connected component as the multiple lines we studied in previous Sections 3 and 4.

Proposition 5.4. *$\mathcal{V}_{n,4,g_C}$ and $\mathcal{L}_{n,(0,1-g_C),(0,\dots,0,1-g_C),g_C}$ are in the same connected component.*

Proof. We construct a suitable flat family whose general member is a degree 4 curve of genus g_C containing a plane subcurve of degree 3, and whose special member is a degree 4 multiple line with generic embedding dimension 3, genus g_C, filtered via a degree 3 rope with right-type $(0, 1-g_C)$ and left-type $(0,\dots,0,1-g_C)$.

Let L be the line defined by $I_L = (x_0,\dots,x_{n-2})$, and let P_t the plane conic defined by the ideal $I_{P_t} = (x_0,\dots,x_{n-3}, x_{n-2}^2 + tx_{n-1}x_n)$, $t \in \mathbb{A}^1$. Set $D_t = L \cup P_t$ for $t \neq 0$. To define a curve with very degenerate hyperplane section we have to define a surjective map φ_A as in (2.6). We choose φ_A to be represented by $A = (0,\dots,0,x_n^{1-g_C}, x_{n-1}^{1-g_C}, 0)$ because $a = 1-g_C$ being $d = 4$ and $g = g_C$. Then, the matrix B which resolves A is

$$B = \begin{pmatrix} I & 0 & 0 \\ 0 & x_{n-1}^{1-g_C} & 0 \\ 0 & -x_n^{1-g_C} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The curve C_t we are constructing is then defined by

$$I_{C_t} = (I_L I_{D_t}, [I_{D_t}]B)$$

where $I_{D_t} = (x_0,\dots,x_{n-3}, x_{n-2}(x_{n-2}^2 + tx_{n-1}x_n))$.

When $t \to 0$ we get the ideal $I_{C_0} = (I_L I_{D_0}, [I_{D_0}]B) = (x_0,\dots,x_{n-5}, x_{n-4}^2, x_{n-4}x_{n-3}, x_{n-3}^2, x_{n-4}x_{n-2}, x_{n-3}x_{n-2}, x_{n-2}^3, x_{n-4}x_{n-1}^{1-g_C} - x_{n-3}x_n^{1-g_C})$.

It is evident that C_0 is a curve of degree 4, genus g_C, that $I_L^3 \subset I_{C_0} \subset I_L$, i.e., C_0 is a multiple line, and that its general hyperplane section is the one described in Lemma 3.2, i.e., $\mathrm{ged}(C_0) = 3$. The Cohen-Macaulay filtration of C_0 is $L \subset Z_0 \subset C_0$ where Z_0 is a degree 3 rope defined by

$$I_{Z_0} = ((I_L)^2, x_0,\dots,x_{n-5}, x_{n-4}x_{n-1}^{1-g_C} - x_{n-3}x_n^{1-g_C}),$$

and so its left-type is $(0,\dots,0,1-g_C)$. The syzygies of the associated matrix B_0 are

$$A_0 = \begin{pmatrix} 0 & \dots & 0 & x_n^{1-g_C} & x_{n-1}^{1-g_C} & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

and so the right-type of Z_0 is $(0, 1-g_C)$. As last remark, of course C_0 contains D_0 and so it contains a degree 3 plane subcurve. □

We can summarize the previous results in the statement

Theorem 5.1. *The families of degree 4 ropes of genus g_C, of curves with very degenerate hyperplane sections of degree 4 and genus g_C, and the families of degree 4 multiple lines with generic embedding dimension 3 and genus g_C are in the same connected component of the corresponding Hilbert scheme of locally Cohen-Macaulay curves.*

6. An openness problem

In this section, we want to prove that the family of curves of degree 4 containing a planar subcurve of degree 3 corresponds to an open subset of an irreducible component of the corresponding Hilbert scheme, whatever genus the curves have. An analogous result was proved in [14] for curves of degree $d \geq 5$. There, the main geometrical ingredient was the equivalence between the hypothesis on the curve and the fact that the curve has very degenerate hyperplane section. As showed in Section 4, for degree 4 curves the equivalence does not hold, and so we have to use a different argument.

At first, we characterize the curves containing a planar subcurve.

Proposition 6.1. *Let $C \subset \mathbb{P}^n$ be a non degenerate degree 4 curve of genus g_C. Assume that either $n \geq 5$ and $g_C \leq 0$, or $n = 4$ and $g_C \leq -1$. Then, the following are equivalent*

(a) *C contains a planar subcurve of degree 3;*

(b) *$h^1\mathcal{I}_C(g_C) \neq 0$.*

In our hypotheses, the curves we are considering are exactly the ones in $\mathcal{V}_{n,4,g_C}$.

Proof. In [14] there is a description of the curves containing a planar subcurve of degree one less, and in particular, there is a description of their Hartshorne-Rao module. It holds that, as $K[x_{n-1}, x_n]$-module, $H^1_*\mathcal{I}_C$ is isomorphic to $\operatorname{coker}(\varphi_A)$ as described in the exact sequence (2.6). Then, it is trivial that $h^1\mathcal{I}_C(g_C) = 1$.

Conversely, if C does not contain a degree 3 planar subcurve, we have two possibilities: either the general hyperplane section of C is non degenerate, or C is a multiple line with $\operatorname{ged}(C) = 3$ and very degenerate hyperplane section, but C does not contain a planar subcurve of degree 3.

In the first case, we use the Castelnuovo method to compute the Rao function of C in degree 0. At first, we get the following upper bound

$$h^2\mathcal{I}_C(0) \leq \sum_{t \geq 1} h^1\mathcal{I}_{C\cap H|H}(t).$$

The general hyperplane section of C is non degenerate and so $h^1\mathcal{I}_{C\cap H|H}(t) = 0$ for $t \geq 1$. Hence, $h^2\mathcal{I}_C(0) = 0$. Of course, $h^0\mathcal{I}_C(0) = 0$, and so we can compute $h^1\mathcal{I}_C(0)$. In fact, $h^0\mathcal{I}_C(0) - h^1\mathcal{I}_C(0) + h^2\mathcal{I}_C(0) = g_C$, that is to say, $h^1\mathcal{I}_C(0) = -g_C$. But the Rao function is strictly increasing in negative degrees and so $h^1\mathcal{I}_C(g_C) = 0$.

In the second case, let $L \subset Z \subset C$ be the Cohen-Macaulay filtration of the multiple line C where L is the line support of C and Z is a degree 3 rope. C does not contain a planar subcurve of degree 3 and so either φ_A has no retraction, or ψ_D has no retraction (see Theorem 4.2). Furthermore, by Proposition 4.1, for $j = g_C$,

$$h^1\mathcal{I}_C(g_C) = h^1\mathcal{I}_Z(g_C) + \dim_K(S/D)_{\gamma+g_C}.$$

We want to prove that both the addenda are zero.

By Corollary 3.2, $\gamma + g_C = g_Z - 1$, and so $\gamma + g_C \leq -1$ because $g_Z \leq 0$. Hence $\dim_K(S/D)_{\gamma+g_C} = 0$.

By formula (2.3), $h^1\mathcal{I}_Z(g_C) = \binom{\alpha_0+g_C}{1} + \binom{\alpha_1+g_C}{1}$. Hence, it is non zero only in the case $\alpha_0 = 0, \alpha_1 = 1 - g_C$ and $g_Z = -1 + g_C$. Because of the surjectivity of φ_A, it has a retraction. Moreover, the map μ_1 that defines the curve C is non zero and this forces the map ψ_D to have a retraction, too. Hence, C contains a degree 3 planar subcurve, and the claim follows. $\square$

Now, we can prove the main result of the section.

Theorem 6.1. *Let n, g be integers such that $g \leq 0$ if $n \geq 5$ or $g \leq -1$ if $n = 4$. Then, $\mathcal{V}_{n,4,g}$ is an open subset of an irreducible component of the corresponding Hilbert scheme.*

Proof. The proof is essentially the same as [14, Theorem 7.11], with the remark that there are no integral curves in $\mathbb{P}^n$ of degree 4 and genus g which satisfies our hypotheses. $\square$

Now, we consider the last case, namely $n = 4, g = 0$.

Proposition 6.2. *The Hilbert scheme parameterizing curves in $\mathbb{P}^4$ with Hilbert polynomial $p(z) = 4z + 1$ consists of two irreducible components of dimension* 21.

Proof. In this case, we are interested in curves of genus $g = 0$. Following Remark 5.3, if C has non degenerate general hyperplane section, then C is arithmetically Cohen-Macaulay and it lies in the component studied in [8] of dimension 21. The other component corresponds to the union of a planar curve of degree 3 and a line skew with the plane. Also this component has dimension 21. If C has very degenerate hyperplane section and is not arithmetically Cohen-Macaulay, then, either it contains a planar subcurve of degree 3 and so it lies in this second component, or it does not contain a degree 3 planar subcurve and so it lies in the closure of the component containing arithmetically Cohen-Macaulay curves (see Remark 5.3). These last curves are multiple lines with ged 3 by Proposition 2.1. $\square$

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