J. Math. Kyoto Univ. (JMKYAZ)
45-1 (2005), 145–158

Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities

By

Reika Fukuizumi and Masahito Ohta

1. Introduction

In this paper we study the nonlinear Schrödinger equations

$$i\partial_t u = -\Delta u - g(x, |u|^2)u, \quad (t, x) \in \mathbb{R}^{1+n}. \tag{1.1}$$

When $g(x, |u|^2) = V(x)|u|^{p-1}$, equation (1.1) can model beam propagation in an inhomogeneous medium where $V(x)$ is proportional to the electron density ([18]). Akhmediev [1], Jones [14] and Grillakis, Shatah and Strauss [12] studied the existence and stability of solitary waves of (1.1) for the case where $g(x, |u|^2)$ describes three layered media where the outside two are nonlinear and the sandwiched one is linear. Also, Merle [19] investigated the existence and nonexistence of blowup solutions of (1.1) for certain types of inhomogeneities in case that $g(x, |u|^2) = V(x)|u|^{4/n}$. In this paper, we consider the case $g(x, |u|^2) = V(x)|u|^{p-1}$ with the following type of $V(x)$, assuming that $n \geq 3$, $0 < b < 2$ and $1 < p < 1 + (4 - 2b)/(n - 2)$.

(V1) $V(x) \geq 0, \quad V(x) \not\equiv 0, \quad V(x) \in C^2(\mathbb{R}^n, \mathbb{R})$,

(V2) There exist $C > 0$ and $a > \{(n+2) - (n-2)p\}/2 > b$ such that

$$\left| x^\alpha \partial_x^\alpha \left(V(x) - \frac{1}{|x|^b} \right) \right| \leq \frac{C}{|x|^a}$$

for $|x| \geq 1$ and $|\alpha| \leq 2$.

The main purpose in this paper is to show that under the above assumptions on $V(x)$, the standing wave solution of (1.1) is unstable for $p > 1 + (4 - 2b)/n$ and sufficiently small frequency.

By a standing wave, we mean a solution of (1.1) of the form

$$u_\omega(t, x) = e^{i\omega t}\phi_\omega(x),$$

Received June 1, 2004

where $\omega > 0$, and $\phi_\omega(x)$ is a ground state of the following stationary problem

$$\begin{cases} -\Delta\phi + \omega\phi - V(x)|\phi|^{p-1}\phi = 0, & x \in \mathbb{R}^n, \\ \phi \in H^1(\mathbb{R}^n), \quad \phi \not\equiv 0. \end{cases} \tag{1.2}$$

We recall previous results. Several authors have been studying the problem of stability and instability of standing waves for (1.1) (see, e.g., [2, 5, 6, 7, 8, 9, 10, 12, 17, 22, 25, 26]). First, we consider the case $V(x) \equiv 1$, namely,

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{1.3}$$

where $1 < p < \infty$ if $n = 1, 2$, and $1 < p < 1 + 4/(n-2)$ if $n \geq 3$.

For $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$\begin{cases} -\Delta\psi + \omega\psi - |\psi|^{p-1}\psi = 0, & x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \quad \psi \not\equiv 0. \end{cases} \tag{1.4}$$

(see Strauss [23] and Berestycki and Lions [3] for the existence, and Kwong [15] for the uniqueness). It is known that a positive solution of (1.4) is a ground state. In [5] Cazenave and Lions proved that if $p < 1 + 4/n$ then the standing wave solution $e^{i\omega t}\psi_\omega(x)$ is stable for any $\omega > 0$. On the other hand, it is shown that if $p \geq 1 + 4/n$ then the standing wave solution $e^{i\omega t}\psi_\omega(x)$ is unstable for any $\omega > 0$ (see Berestycki and Cazenave [2] for $p > 1+4/n$, and Weinstein [25] for $p = 1 + 4/n$).

We define the energy functional E and the charge Q on $H^1(\mathbb{R}^n)$ by

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} V(x)|v(x)|^{p+1}dx, \quad Q(v) := \frac{1}{2}\|v\|_2^2.$$

We remark that by the assumptions $(V1)$ and $(V2)$, the functional E is well-defined on $H^1(\mathbb{R}^n)$.

The time local well-posedness for the Cauchy problem to (1.1) in $H^1(\mathbb{R}^n)$, the conservation of energy and $L^2(\mathbb{R}^n)$-norm, and the virial identity hold (see, e.g., Theorem 4.4.6 and Section 6.5 of Cazenave [4]). That is, we have the following proposition.

Proposition 1.1. *For any $u_0 \in H^1(\mathbb{R}^n)$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u(t) \in C([0,T], H^1(\mathbb{R}^n))$ of (1.1) with $u(0) = u_0$ satisfying*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0,T].$$

In addition, if $u_0 \in H^1(\mathbb{R}^n)$ satisfies $|x|u_0 \in L^2(\mathbb{R}^n)$, then the virial identity

$$\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8P(u(t))$$

holds for $t \in [0, T]$, *where*

$$P(v) := \|\nabla v\|_2^2 - \frac{n(p-1)}{2(p+1)} \int_{\mathbb{R}^n} V(x)|v(x)|^{p+1} dx + \frac{1}{p+1} \int_{\mathbb{R}^n} x \cdot \nabla V(x)|v(x)|^{p+1} dx. \tag{1.5}$$

Before we state our theorem, we give some precise definitions.

Definition 1.1. For $\omega > 0$, we define two functionals on $H^1(\mathbb{R}^n)$:

$$S_\omega(v) := E(v) + \omega Q(v) \qquad \text{(action)},$$
$$I_\omega(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 - \int_{\mathbb{R}^n} V(x)|v(x)|^{p+1} dx.$$

Let $\mathcal{G}_\omega$ be the set of all non-negative minimizers for

$$\inf\{S_\omega(v) : \ v \in H^1(\mathbb{R}^n) \setminus \{0\}, \ I_\omega(v) = 0\}. \tag{1.6}$$

The existence of non-negative minimizers for (1.6) is proved by the standard variational argument since $V(x)$ vanishes as $|x| \to \infty$ (see Stuart [24]). In Section 3, we prove the following lemma for the sake of completeness.

Lemma 1.1. *Let* $n \geq 3$ *and* $1 < p < 1 + 4/(n-2)$. *Assume* $(V1)$ *and* $\lim_{|x|\to\infty} V(x) = 0$. *Then* $\mathcal{G}_\omega$ *is not empty for* $\omega > 0$.

Remark 1. (i) We note that

$$I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v\rangle, \quad P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1},$$

where $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.
(ii) Let $\phi_\omega \in \mathcal{G}_\omega$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(\phi_\omega) = \Lambda I'_\omega(\phi_\omega)$. Thus, we have $\langle S'_\omega(\phi_\omega), \phi_\omega\rangle = \Lambda\langle I'_\omega(\phi_\omega), \phi_\omega\rangle$. Since $\langle S'_\omega(\phi_\omega), \phi_\omega\rangle = I_\omega(\phi_\omega) = 0$ and $\langle I'_\omega(\phi_\omega), \phi_\omega\rangle = -(p-1)\int V(x)|\phi_\omega|^{p+1} < 0$, we have $\Lambda = 0$. Namely, ϕ_ω satisfies (1.2). Moreover, for any $v \in H^1(\mathbb{R}^n) \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $\mathcal{G}_\omega$, we have $S_\omega(\phi_\omega) \leq S_\omega(v)$. Namely, $\phi_\omega \in \mathcal{G}_\omega$ is a ground state (minimal action solution) of (1.2) in $H^1(\mathbb{R}^n)$. It is easy to see that a ground state of (1.2) in $H^1(\mathbb{R}^n)$ is a minimizer of (1.6).

The stability and the instability in this paper are formulated as follows.

Definition 1.2. For $\phi_\omega \in \mathcal{G}_\omega$ and $\delta > 0$, we put

$$U_\delta(\phi_\omega) := \left\{ v \in H^1(\mathbb{R}^n) : \ \inf_{\theta\in\mathbb{R}} \|v - e^{i\theta}\phi_\omega\|_{H^1} < \delta \right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable in $H^1(\mathbb{R}^n)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in U_\delta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in $H^1(\mathbb{R}^n)$.

The following theorem is our main result in this paper.

Theorem 1.1. *Let $n \geq 3$ and $1+(4-2b)/n < p < 1+(4-2b)/(n-2)$. Assume $(V1)$ and $(V2)$. Let $\phi_\omega \in \mathcal{G}_\omega$. Then, there exists $\omega_* > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is unstable in $H^1(\mathbb{R}^n)$ for any $\omega \in (0, \omega_*)$.*

For the proof of Theorem 1.1, we use the virial identity and the following sufficient condition for instability, which is a modification of Theorem 3 in Ohta [20] (see also [9, 11, 17, 22]).

Proposition 1.2. *Let $n \geq 3$ and $1 < p < 1 + 4/(n-2)$. Assume $(V1)$ and $\lim_{|x|\to\infty} V(x) = 0$. Let $\phi_\omega \in \mathcal{G}_\omega$. If*

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0, \tag{1.7}$$

then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is unstable in $H^1(\mathbb{R}^n)$. Here, $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.

Since $\|v^\lambda\|_2^2 = \|v\|_2^2$ for any $\lambda > 0$, (1.7) implies that $\phi_\omega(x)$ is not a local minimizer of E on $\{v \in H^1(\mathbb{R}^n) : \ \|v\|_2 = \|\phi_\omega\|_2\}$.

Remark 2. We do not require $p < 1 + (4-2b)/(n-2)$ with $0 < b < 2$, but $p < 1 + 4/(n-2)$ in Propositions 1, 2 and Lemma 1.1.

Grillakis, Shatah and Strauss [12, 13] gave an almost sufficient and necessary condition for the stability and instability of stationary states for the Hamiltonian systems under certain assumptions. By the abstract theory in Grillakis, Shatah and Strauss [12, 13], under some assumptions on the spectrum of linearized operators, $e^{i\omega_0 t}\phi_{\omega_0}(x)$ is stable (resp. unstable) if the function $\|\phi_\omega\|_2^2$ is strictly increasing (resp. decreasing) at $\omega = \omega_0$. In the papers of Shatah [21], Shatah and Strauss [22], they used the variational characterization of ground states instead of assumptions on the spectrum of linearized operators. In the case $V(x) \equiv 1$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, it is easy to check the increase and decrease of $\|\psi_\omega\|_2^2$. However, it seems difficult to check this property of $\|\phi_\omega\|_2^2$ for $V(x) \not\equiv$ constant in general.

By applying another sufficient condition as in Proposition 1.2, we may avoid such difficulty. However still, it is not easy to verify condition (1.7) directly. Therefore, we first study a limiting problem. We investigate the rescaling limit of $\phi_\omega(x)$ as $\omega \to 0$. We show that as $\omega \to 0$, the rescaled function $\tilde{\phi}_\omega(x)$ defined by $\phi_\omega(x) = \omega^{(2-b)/2(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x)$ tends to the unique positive radial solution $\psi_{1,b}(x)$ of (1.2) with $\omega = 1$ and $V(x) = |x|^{-b}$. From known stability properties of $\psi_{1,b}(x)$, we are able to prove (1.7) in the limit. For that reason, in Section 2, we review and summarize the properties of standing wave solution for the case where $V(x) = |x|^{-b}$ in (1.1). In Section 3, we verify the convergence property of the rescaled function $\tilde{\phi}_\omega(x)$, using its variational characterization. In Section 4, we check the condition (1.7) and we prove Theorem 1.1.

2. Case $V(x) = |x|^{-b}$

Let $n \geq 3$ and $0 < b < 2$. Stability and instability of the standing wave solution for (1.1) with $V(x) = |x|^{-b}$ follows from the method of Shatah [21], Shatah and Strauss [22].

Let $1 < p < 1 + (4 - 2b)/(n - 2)$. For any $\omega > 0$ there exists a unique positive radial solution $\psi_{\omega,b}(x) \in H^1(\mathbb{R}^n)$ of

$$-\Delta\psi + \omega\psi - \frac{1}{|x|^b}|\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^n. \tag{2.1}$$

See Stuart [24] and Remark 3 below for existence. The positivity of solutions follows from the maximum principle. Radial symmetry of solutions was showed by Li and Ni [16] and Yanagida [27] proved the uniqueness.

The unique positive solution $\psi_{\omega,b}(x)$ is a minimizer of

$$d_b(\omega) := \inf\{S_{\omega,b}(v) : \ v \in H^1(\mathbb{R}^n) \setminus \{0\}, \ I_{\omega,b}(v) = 0\}, \tag{2.2}$$

where

$$S_{\omega,b}(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} \frac{1}{|x|^b}|v(x)|^{p+1}dx,$$
$$I_{\omega,b}(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 - \int_{\mathbb{R}^n} \frac{1}{|x|^b}|v(x)|^{p+1}dx.$$

We apply the method of [21, 22] to the present case using the variational characterization $d_b(\omega)$ and we check the sufficient condition for stability $d_b''(\omega) > 0$ in [21] and instability $d_b''(\omega) < 0$ in [22]. Since $\psi_{\omega,b}(x)$ is a solution of $S'_{\omega,b}(v) = 0$, we have $d_b'(\omega) = Q(\psi_{\omega,b})$. In this case, by the scaling $\psi_{\omega,b}(x) = \omega^{(2-b)/2(p-1)}\psi_{1,b}(\sqrt{\omega}x)$, we have

$$2Q(\psi_{\omega,b}) = \|\psi_{\omega,b}\|_2^2 = \omega^{(2-b)/2(p-1)-n/2}\|\psi_{1,b}\|_2^2.$$

Therefore, for any $\omega > 0$, the standing wave solution is stable if $1 < p < 1 + (4 - 2b)/n$, and unstable if $1 + (4 - 2b)/n < p < 1 + (4 - 2b)/(n - 2)$. We have also blow-up instability for the case $p \geq 1 + (4 - 2b)/n$ following Weinstein [25] and Berestycki and Cazenave [2].

3. Convergence property of variational problems

First, we briefly explain the proof of Lemma 1.1 for the completeness. We know that the problem (1.6) is equivalent to the minimizing problem

$$\inf\{\|\nabla v\|_2^2 + \omega\|v\|_2^2 : \ v \in H^1(\mathbb{R}^n) \setminus \{0\}, \ I_\omega(v) = 0\},$$

and also equivalent to

$$d_V(\omega) := \inf\{\|\nabla v\|_2^2 + \omega\|v\|_2^2 : \ v \in H^1(\mathbb{R}^n), \ \int_{\mathbb{R}^n} V(x)|v(x)|^{p+1}dx = 1\}, \tag{3.1}$$

by (V1) (See Proposition 4.1 of [9]).

Proof of Lemma 1.1. Let $\{v_j\} \subset H^1(\mathbb{R}^n)$ be a minimizing sequence for the problem (3.1). Then, the sequence $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$. Thus, there exists a subsequence (still denoted by $\{v_j\}$) and $v_0 \in H^1(\mathbb{R}^n)$ such that $v_j \to v_0$ weakly in $H^1(\mathbb{R}^n)$. Here we put

$$\varphi(u) := \int_{\mathbb{R}^n} V(x)|u(x)|^{p+1}dx,$$

and we show that $\varphi(v_j) \to \varphi(v_0)$ as $j \to \infty$. Since $\lim_{|x|\to\infty} V(x) = 0$, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $V(x) \le \varepsilon$ for $|x| > C(\varepsilon)$. For $C(\varepsilon) \equiv C > 0$, we define $B(C) := \{x \in \mathbb{R}^n : \ |x| \le C\}$. By the compactness of the Sobolev embeddings on bounded domains, we have that $\|v_j - v_0\|_{L^{p+1}(B(C))} \to 0$ as $j \to \infty$ for $1 \le p < 1+4/(n-2)$. This also means that $\||v_j|^p - |v_0|^p\|_{L^{p+1/p}(B(C))} \to 0$ as $j \to \infty$. By (V1), we have

$$\begin{aligned}
&\left|\int_{|x|\le C} (V(x)|v_j(x)|^{p+1} - V(x)|v_0(x)|^{p+1})dx\right| \\
&\le \left|\int_{|x|\le C} V(x)(|v_j(x)|^p - |v_0(x)|^p)|v_j(x)|dx\right| \\
&\qquad\qquad + \left|\int_{|x|\le C} V(x)|v_0(x)|^p(|v_j(x)| - |v_0(x)|)dx\right| \\
&\le M\||v_j|^p - |v_0|^p\|^{p/(p+1)}_{L^{p+1/p}(B(C))}\|v_j\|_{p+1} + M\|v_0\|^{p/(p+1)}_{p+1}\|v_j - v_0\|^{1/(p+1)}_{L^{p+1}(B(C))},
\end{aligned}$$

where $M = \sup_{x\in B(C)} V(x)$. For the part $|x| > C$,

$$\left|\int_{|x|>C} (V(x)|v_j(x)|^{p+1} - V(x)|v_0(x)|^{p+1})dx\right| \le \varepsilon(\|v_j\|^{p+1}_{p+1} + \|v_0\|^{p+1}_{p+1}).$$

Accordingly, we have

$$\begin{aligned}
|\varphi(v_j) - \varphi(v_0)| &= \left|\int_{|x|\le C} (V(x)|v_j(x)|^{p+1} - V(x)|v_0(x)|^{p+1})dx\right| \\
&\quad + \left|\int_{|x|>C} (V(x)|v_j(x)|^{p+1} - V(x)|v_0(x)|^{p+1})dx\right| \\
&\le \varepsilon(\|v_j\|^{p+1}_{p+1} + \|v_0\|^{p+1}_{p+1}) + M\||v_j|^p - |v_0|^p\|^{p/(p+1)}_{L^{p+1/p}(B(C))}\|v_j\|_{p+1} \\
&\quad + M\|v_0\|^{p/(p+1)}_{p+1}\|v_j - v_0\|^{1/(p+1)}_{L^{p+1}(B(C))} \to 0, \quad j \to \infty
\end{aligned}$$

for $1 \le p < 1 + 4/(n-2)$ since v_j is bounded in $L^{p+1}(\mathbb{R}^n)$.

It follows from the above argument that

$$\int_{\mathbb{R}^n} V(x)|v_0(x)|^{p+1}dx = \lim_{j\to\infty}\int_{\mathbb{R}^n} V(x)|v_j(x)|^{p+1}dx = 1.$$

By the definition of (3.1), we have

$$d_V(\omega) \le \|\nabla v_0\|_2^2 + \omega\|v_0\|_2^2 \le \liminf_{j\to\infty}(\|\nabla v_j\|_2^2 + \omega\|v_j\|_2^2) = d_V(\omega).$$

Namely, v_0 is a minimizer and $v_j \to v_0$ strongly in $H^1(\mathbb{R}^n)$ as $j \to \infty$. □

Remark 3. This proof is valid for the case $V(x) = |x|^{-b}$ with $0 < b < 2$. However, we have to assume $p < 1 + (4-2b)/(n-2)$ so that we can have $|x|^{-b} \in L^\theta(B(C))$ where $\theta = 2n/\{(n+2)-(n-2)p\}$. Actually we use the fact that $|v_j|^{p+1}$ converges to $|v_0|^{p+1}$ weakly in $L^{2n/(n-2)(p+1)}(\mathbb{R}^n)$ which follows from $v_j \to v_0$ weakly in $L^{2n/(n-2)}(\mathbb{R}^n)$. The exponent θ is the conjugate relation with $2n/(n-2)(p+1)$.

Now, we shall prove a certain convergence property of $\phi_\omega \in \mathcal{G}_\omega$ as $\omega \to 0$. We rescale $\phi_\omega \in \mathcal{G}_\omega$ as follows:

$$\phi_\omega(x) = \omega^{(2-b)/\{2(p-1)\}}\tilde{\phi_\omega}(\sqrt{\omega}x), \quad \omega > 0. \tag{3.2}$$

Then, the rescaled function $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\tilde{\phi_\omega} + \tilde{\phi_\omega} = \omega^{-b/2}V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}|^{p-1}\tilde{\phi_\omega}, \quad x \in \mathbb{R}^n. \tag{3.3}$$

The main claim in this section is the following.

Proposition 3.1. *Let $n \ge 3$, $0 < b < 2$ and $1 < p < 1+(4-2b)/(n-2)$. Assume $(V1)$ and $(V2)$. Let $\phi_\omega \in \mathcal{G}_\omega$, $\tilde{\phi_\omega}(x)$ be the rescaled function and $\psi_{1,b}(x)$ be the unique positive radial solution of* (2.1) *with $\omega = 1$ in $H^1(\mathbb{R}^n)$. Then, we have*

$$\lim_{\omega\to 0}\|\tilde{\phi_\omega} - \psi_{1,b}\|_{H^1} = 0.$$

To prove this proposition, we consider the following functionals.

$$\tilde{I_\omega}(v) := \|\nabla v\|_2^2 + \|v\|_2^2 - \omega^{-b/2}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|v(x)|^{p+1}dx,$$
$$I_{1,b}(v) := \|\nabla v\|_2^2 + \|v\|_2^2 - \int_{\mathbb{R}^n}\frac{1}{|x|^b}|v(x)|^{p+1}dx,$$

where $0 < b < 2$.

Lemma 3.1. *Let $n \ge 3$, $0 < b < 2$ and $1 < p < 1 + (4-2b)/(n-2)$. Assume $(V1)$ and $(V2)$. Let $\phi_\omega \in \mathcal{G}_\omega$, $\tilde{\phi_\omega}(x)$ be the rescaled function and $\psi_{1,b}(x)$ be the unique positive radial solution of* (2.1) *with $\omega = 1$ in $H^1(\mathbb{R}^n)$. Then, we have*

$$\text{(i)} \quad \lim_{\omega\to 0}\|\tilde{\phi_\omega}\|_{H^1}^2 = \|\psi_{1,b}\|_{H^1}^2, \qquad \text{(ii)} \quad \lim_{\omega\to 0} I_{1,b}(\tilde{\phi_\omega}) = 0.$$

Proof. First of all, we remark that $\tilde{\phi_\omega}(x)$ is a minimizer of

$$\inf\{\|v\|_{H^1}^2 :\ v \in H^1(\mathbb{R}^n) \setminus \{0\},\ \tilde{I_\omega}(v) \le 0\},$$

and $\psi_{1,b}(x)$ is a minimizer of

$$\inf\{\|v\|_{H^1}^2 :\ v \in H^1(\mathbb{R}^n) \setminus \{0\},\ I_{1,b}(v) \le 0\}. \tag{3.4}$$

In order to prove (i), we show that for any $\mu > 1$, there exists $\omega(\mu) > 0$ such that

$$\tilde{I_\omega}(\mu\psi_{1,b}) < 0 \tag{3.5}$$

and

$$I_{1,b}(\mu\tilde{\phi_\omega}) < 0 \tag{3.6}$$

hold for any $\omega \in (0, \omega(\mu))$. If this is true, then the above variational characterizations of $\tilde{\phi_\omega}(x)$ and $\psi_{1,b}(x)$ yield that

$$\frac{1}{\mu^2}\|\psi_{1,b}\|_{H^1}^2 \le \|\tilde{\phi_\omega}\|_{H^1}^2 \le \mu^2\|\psi_{1,b}\|_{H^1}^2, \quad \omega \in (0, \omega(\mu)).$$

Since $\mu > 1$ is arbitrary, we conclude (i). First, we show (3.5). We put $V_\omega(x) := \omega^{-b/2}V(x/\sqrt{\omega})$ and $V_0(x) := |x|^{-b}$. From $I_{1,b}(\psi_{1,b}) = 0$, we have

$$\mu^{-2}\tilde{I_\omega}(\mu\psi_{1,b}) = -(\mu^{p-1} - 1)\|\psi_{1,b}\|_{H^1}^2 + \mu^{p-1}\int_{\mathbb{R}^n}(V_0(x) - V_\omega(x))|\psi_{1,b}(x)|^{p+1}dx.$$

Since

$$\lim_{\omega\to 0}\int_{\mathbb{R}^n}(V_0(x) - V_\omega(x))|\psi_{1,b}(x)|^{p+1}dx = 0$$

for any $\mu > 1$, there exists $\omega_1(\mu) > 0$ such that $\tilde{I_\omega}(\mu\psi_{1,b}) < 0$ for any $\omega \in (0, \omega_1(\mu))$. Namely, we have

$$\|\tilde{\phi_\omega}\|_{H^1}^2 \le \mu^2\|\psi_{1,b}\|_{H^1}^2 \tag{3.7}$$

for any $\omega \in (0, \omega_1(\mu))$. Indeed, we have

$$\int_{\mathbb{R}^n}(V_\omega(x) - V_0(x))|\psi_{1,b}(x)|^{p+1}dx \le \omega^{(n/2 - b\theta/2)/\theta}\|V - V_0\|_{L^\theta}\|\psi_{1,b}\|_{2n/(n-2)}^{p+1},$$

where $\theta = 2n/\{(n+2) - (n-2)p\}$ and $n/2 - b\theta/2 > 0$.

Next, we prove (3.6). Similarly to above, using $\tilde{I_\omega}(\tilde{\phi_\omega}) = 0$, we have

$$\begin{aligned} \mu^{-2}I_{1,b}(\mu\tilde{\phi_\omega}) &= \|\tilde{\phi_\omega}\|_{H^1}^2 - \mu^{p-1}\int_{\mathbb{R}^n}V_0(x)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ &= -(\mu^{p-1} - 1)\|\tilde{\phi_\omega}\|_{H^1}^2 + \mu^{p-1}\int_{\mathbb{R}^n}(V_\omega(x) - V_0(x))|\tilde{\phi_\omega}(x)|^{p+1}dx. \end{aligned} \tag{3.8}$$

We also have

$$\int_{\mathbb{R}^n} (V_\omega(x) - V_0(x))|\tilde{\phi_\omega}(x)|^{p+1}dx \le \omega^{(n/2-b\theta/2)/\theta}\|V - V_0\|_{L^\theta}\|\tilde{\phi_\omega}\|_{2n/(n-2)}^{p+1}.$$

Therefore, by Sobolev embedding,

$$\begin{aligned}(3.8) &\le -(\mu^{p-1} - 1)\|\tilde{\phi_\omega}\|_{H^1}^2 + \mu^{p-1}\omega^{(n/2-b\theta/2)/\theta}\|V - V_0\|_{L^\theta}\|\tilde{\phi_\omega}\|_{2n/(n-2)}^{p+1} \\ &\le -(\mu^{p-1} - 1)\|\nabla\tilde{\phi_\omega}\|_2^2 + C\mu^{p-1}\omega^{(n/2-b\theta/2)/\theta}\|V - V_0\|_{L^\theta}\|\nabla\tilde{\phi_\omega}\|_2^{p+1}.\end{aligned}$$

Taking $\mu = 2$ in (3.7), we have $\|\nabla\tilde{\phi_\omega}\|_2^{p-1} \le 2^{p-1}\|\psi_{1,b}\|_{H^1}^{p-1}$ for any $\omega \in (0, \omega_1(2))$. Accordingly,

$$(3.8) \le -\{(\mu^{p-1} - 1) - C\mu^{p-1}\omega^{(n/2-b\theta/2)/\theta}\|V - V_0\|_{L^\theta}\|\psi_{1,b}\|_{H^1}^{p-1}\}\|\nabla\tilde{\phi_\omega}\|_2^2,$$

for any $\omega \in (0, \omega_1(2))$. Thus, for any $\mu > 1$ there exists $\omega_2(\mu) \in (0, \omega_1(2))$ such that $I_{1,b}(\mu\tilde{\phi_\omega}) < 0$ for any $\omega \in (0, \omega_2(\mu))$.

(ii) follows from the same proof as Lemma 2.1 of [9]. □

Finally, we are in position to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\phi_\omega \in \mathcal{G}_\omega$. By (V1), $\phi_\omega(x)$ is positive in $\mathbb{R}^n$. By Lemma 3.1, for any $\{\omega_j\} \to 0$, $\{\tilde{\phi_{\omega_j}}\}$ is a minimizing sequence of (3.4). As we mentioned at the beginning of this section, it follows from a similar proof to Proposition 4.1 of [9] that (3.4) is equivalent to

$$(3.9) \qquad \inf\left\{\|v\|_{H^1}^2 :\ v \in H^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \frac{1}{|x|^b}|v(x)|^{p+1}dx = 1\right\}.$$

Thus, by the proof of Lemma 1.1, we obtain a minimum of (3.9) to which a subsequence of $\{\tilde{\phi_{\omega_j}}\}$ converges. It follows from uniqueness result by Yanagida [27] that such minimum is a unique solution of (2.1), namely $\psi_{1,b}(x)$. □

4. Orbital instability

In this section we check the sufficient condition for instability (1.7) in Proposition 1.2. By simple computations, we have

$$\begin{aligned}E(v^\lambda) &= \frac{\lambda^2}{2}\|\nabla v\|_2^2 - \frac{\lambda^{n(p-1)/2}}{p+1}\int_{\mathbb{R}^n} V\left(\frac{x}{\lambda}\right)|v(x)|^{p+1}dx, \\ \partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} &= \|\nabla\phi_\omega\|_2^2 - \frac{n(p-1)}{2(p+1)}\left\{\frac{n(p-1)}{2} - 1\right\}\int_{\mathbb{R}^n} V(x)|\phi_\omega(x)|^{p+1}dx \\ &\quad + \frac{n(p-1) - 2}{p+1}\int_{\mathbb{R}^n} x\cdot\nabla V(x)|\phi_\omega(x)|^{p+1}dx \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^n}\sum_{j,k=1}^n x_j x_k \partial_j\partial_k V(x)|\phi_\omega(x)|^{p+1}dx.\end{aligned}$$

Since $P(\phi_\omega) = \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$ (see (1.5) and Remark 1), we have

$$\begin{aligned}\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = {} & \frac{n(p-1)}{2(p+1)}\left\{2 - \frac{n(p-1)}{2}\right\}\int_{\mathbb{R}^n} V(x)|\phi_\omega(x)|^{p+1}dx \\ & + \frac{n(p-1)-3}{p+1}\int_{\mathbb{R}^n} x\cdot\nabla V(x)|\phi_\omega(x)|^{p+1}dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^n}\sum_{j,k=1}^n x_j x_k \partial_j\partial_k V(x)|\phi_\omega(x)|^{p+1}dx.\end{aligned} \tag{4.1}$$

Here, we rescale $\phi_\omega(x)$ as in (3.3) and we have

$$\begin{aligned}(4.1) = \omega^\gamma \Bigg[& \frac{n(p-1)}{2(p+1)}\left\{2 - \frac{n(p-1)}{2}\right\}\int_{\mathbb{R}^n} \omega^{-b/2} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ & + \frac{n(p-1)-3}{p+1}\int_{\mathbb{R}^n} \omega^{-b/2}\frac{x}{\sqrt{\omega}}\cdot\nabla V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^n} \omega^{-b/2}\sum_{j,k=1}^n \frac{x_j}{\sqrt{\omega}}\frac{x_k}{\sqrt{\omega}}\partial_j\partial_k V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx\Bigg],\end{aligned}$$

where $\gamma = (2-b)(p+1)/2(p-1) - n/2 + b/2 > 0$. Therefore, it suffices to show the following.

Lemma 4.1. *There exists $\omega_3 > 0$ such that $K_\omega < 0$ for any $\omega \in (0, \omega_3)$, where*

$$\begin{aligned}K_\omega := {} & \frac{n(p-1)}{2(p+1)}\left\{2 - \frac{n(p-1)}{2}\right\}\int_{\mathbb{R}^n} \omega^{-b/2} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ & + \frac{n(p-1)-3}{p+1}\int_{\mathbb{R}^n} \omega^{-b/2}\frac{x}{\sqrt{\omega}}\cdot\nabla V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^n} \omega^{-b/2}\sum_{j,k=1}^n \frac{x_j}{\sqrt{\omega}}\frac{x_k}{\sqrt{\omega}}\partial_j\partial_k V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx.\end{aligned}$$

We need the following lemma to prove Lemma 4.1.

Lemma 4.2. *Let $n \geq 3$, $0 < b < 2$ and $1 < p < 1 + (4-2b)/(n-2)$. Assume $(V1)$ and $(V2)$. Let $\phi_\omega \in \mathcal{G}_\omega$. Then the followings hold.*

(i) $$\lim_{\omega\to 0}\omega^{-b/2}\int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx = \int_{\mathbb{R}^n}\frac{1}{|x|^b}|\psi_{1,b}(x)|^{p+1}dx,$$

(ii) $$\lim_{\omega\to 0}\omega^{-b/2}\int_{\mathbb{R}^n} \frac{x}{\sqrt{\omega}}\cdot\nabla V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx = -\int_{\mathbb{R}^n}\frac{b}{|x|^b}|\psi_{1,b}(x)|^{p+1}dx,$$

(iii) $$\begin{aligned}& \lim_{\omega\to 0}\omega^{-b/2}\int_{\mathbb{R}^n}\sum_{j,k=1}^n \frac{x_j}{\sqrt{\omega}}\frac{x_k}{\sqrt{\omega}}\partial_j\partial_k V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi_\omega}(x)|^{p+1}dx \\ & \quad = b(b+1)\int_{\mathbb{R}^n}\frac{1}{|x|^b}|\psi_{1,b}(x)|^{p+1}dx.\end{aligned}$$

Proof. We put $V_0(x) := |x|^{-b}$. Then $V_0(x)$ satisfies

$$\omega^{-b/2} V_0\left(\frac{x}{\sqrt{\omega}}\right) = V_0(x), \quad \omega^{-b/2} \frac{x}{\sqrt{\omega}} \cdot \nabla V_0\left(\frac{x}{\sqrt{\omega}}\right) = -bV_0(x),$$
$$\omega^{-b/2} \sum_{j,k=1}^{n} \frac{x_j}{\sqrt{\omega}} \frac{x_k}{\sqrt{\omega}} \partial_j \partial_k V_0\left(\frac{x}{\sqrt{\omega}}\right) = b(b+1)V_0(x).$$

Since $\tilde{\phi_\omega} \to \psi_{1,b}$ strongly in $H^1(\mathbb{R}^n)$ as $\omega \to 0$, we see that $\tilde{\phi_\omega} \to \psi_{1,b}$ strongly in $L^{2n/n-2}(\mathbb{R}^n)$, so that $\tilde{\phi_\omega}^{p+1} \to \psi_{1,b}^{p+1}$ strongly in $L^{2n/\{(n-2)(p+1)\}}(\mathbb{R}^n)$. Therefore, it is enough for (i), (ii) and (iii) if we prove

$$\lim_{\omega\to 0} \left\| \omega^{-b/2} V\left(\frac{x}{\sqrt{\omega}}\right) - V_0(x) \right\|_{L^\theta} = 0, \tag{4.2}$$
$$\lim_{\omega\to 0} \left\| \omega^{-b/2} \frac{x}{\sqrt{\omega}} \cdot \nabla V\left(\frac{x}{\sqrt{\omega}}\right) - (-bV_0(x)) \right\|_{L^\theta} = 0, \tag{4.3}$$
$$\lim_{\omega\to 0} \left\| \omega^{-b/2} \sum_{j,k=1}^{n} \frac{x_j}{\sqrt{\omega}} \frac{x_k}{\sqrt{\omega}} \partial_j \partial_k V\left(\frac{x}{\sqrt{\omega}}\right) - b(b+1)V_0(x) \right\|_{L^\theta} = 0, \tag{4.4}$$

where $\theta = 2n/\{(n+2)-(n-2)p\}$. Indeed,

$$\int_{\mathbb{R}^n} \left| \omega^{-b/2} V\left(\frac{x}{\sqrt{\omega}}\right) - V_0(x) \right|^\theta dx = \int_{\mathbb{R}^n} \left| \omega^{-b/2} V\left(\frac{x}{\sqrt{\omega}}\right) - \omega^{-b/2} V_0\left(\frac{x}{\sqrt{\omega}}\right) \right|^\theta dx$$
$$= \omega^{-b\theta/2+n/2} \int_{\mathbb{R}^n} |V(x) - V_0(x)|^\theta dx.$$

On the other hand, by the assumptions (V1) and (V2),
we see that $\int_{\mathbb{R}^n} |V(x) - V_0(x)|^\theta dx$ is finite and independent of ω. Indeed,

$$\int_{\mathbb{R}^n} |V(x) - V_0(x)|^\theta dx = \int_{|x|\le 1} |V(x) - V_0(x)|^\theta dx + \int_{|x|\ge 1} |V(x) - V_0(x)|^\theta dx$$
$$\le \int_{|x|\le 1} |V(x)|^\theta dx + C\int_0^1 r^{-b\theta+n-1} dr + C\int_1^\infty r^{-\alpha\theta+n-1} dr < \infty$$

if $p < 1+(4-2b)/(n-2)$. Also, we have $-b\theta/2+n/2 > 0$ if $p < 1+(4-2b)/(n-2)$, which concludes (4.2). (4.3) and (4.4) follow from the same reason. □

Proof of Lemma 4.1. By Lemma 4.2, we have

$$\lim_{\omega\to 0} 4(p+1)K_\omega = -\{n(p-1)+2b\}\{n(p-1)+2(b-2)\} \int_{\mathbb{R}^n} \frac{1}{|x|^b} |\psi_{1,b}|^{p+1} dx.$$

Therefore, $\lim_{\omega\to 0} K_\omega < 0$ since $p > 1+(4-2b)/n$, which implies that Lemma 4.1 holds. □

Proof of Proposition 1.2 is similar to that of Proposition 1.1 of [9], except for a point that we have used the constraint $\|v\|_{p+1} = \|\phi_\omega\|_{p+1}$ in Lemma 3.1 of [9]. However, we may instead apply the constraint $\|v\|_{H^1} = \|\phi_\omega\|_{H^1}$ for our present case.

Acknowledgement. The authors would like to express their deep gratitude to Professor Yoshio Tsutsumi for his helpful advice and suggestions.

Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
e-mail: reika@math.sci.hokudai.ac.jp

Department of Mathematics
Faculty of Science
Saitama University
Saitama 338-8570, Japan
e-mail: mohta@rimath.saitama-u.ac.jp

References

[1] N. N. Akhmediev, *Novel class of nonlinear surfaces waves: asymmetric modes in a symmetric layered structure*, Sov. Phys. JETP **56** (1982), 299–303.

[2] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires*, C. R. Acad. Sci. Paris. **293** (1981), 489–492.

[3] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I–Existence of a ground state*, Arch. Ration. Mech. Anal. **82** (1983), 313–346.

[4] T. Cazenave, *"Semilinear Schrödinger equations,"* Courant Lecture Notes in Math. **10**, New York University, New York, 2003.

[5] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[6] A. Comech and D. Pelinovsky, *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure Appl. Math. **56** (2003), 1565–1607.

[7] G. Fibich and X. P. Wang, *Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*, Physica D. **175** (2003), 96–108.

[8] R. Fukuizumi, *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*, Discrete Contin. Dynam. Systems **7** (2001), 525–544.

[9] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with potentials*, Differential and Integral Equations. **16** (2003), 691–706.

[10] ———, *Stability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations. **16** (2003), 111–128.

[11] J. M. Gonçalves Ribeiro, *Instability of symmetric stationary states for some nonlinear Schrödinger equations with an external magnetic field*, Ann. Inst. H. Poincaré. Phys. Théor. **54** (1991), 403–433.

[12] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry* I, J. Funct. Anal. **74** (1987), 160–197.

[13] ———, *Stability theory of solitary waves in the presence of symmetry* II, J. Funct. Anal. **94** (1990), 308–348.

[14] C. K. R. T. Jones, *Instability of standing waves for non-linear Schrödinger-type equations*, Ergodic Theory Dynam. Systems **8*** (1988), 119–138.

[15] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u - u^p = 0$ *in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. **105** (1989), 234–266.

[16] Y. Li and W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Comm. Partial Differential Equations **18** (1991), 1043–1054.

[17] Y. Liu, X.-P. Wang and K. Wang, *Instability of standing waves of the Schrödinger equation with inhomogeneous nonlinearity*, preprint.

[18] C. S. Liu and V. K. Tripathi, *Laser guiding in an axially nonuniform plasma channel*, Phys. Plasmas **1** (1994), 3100–3103.

[19] F. Merle, *Nonexistence of minimal blow-up solutions of equations* $iu_t = -\Delta u - k(x)|u|^{4/N}u$ *in* $\mathbb{R}^n$, Ann. Inst. H. Poincaré Phys. Théor. **64** (1996), 33–85.

[20] M. Ohta, *Instability of standing waves for the generalized Davey-Stewartson system*, Ann. Inst. Henri Poincaré. Phys. Théor. **62** (1995), 69–80.

[21] J. Shatah, *Stable standing waves for nonlinear Klein-Gordon equations*, Comm. Math. Phys. **91** (1983), 313–327.

[22] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), 173–190.

[23] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[24] C. A. Stuart, *Bifurcation for Dirichlet problems without eigenvalues*, Proc. London Math. Soc. **45** (1982), 169–192.

[25] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), 567–576.

[26] ———, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

[27] E. Yanagida, *Uniqueness of positive radial solutions of* $\Delta u + g(r)u + h(r)u^p = 0$ *in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. **115** (1991), 257–274.