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On the separation of cohomology groups of increasing unions of $(1,1)$ convex-concave manifolds

By

Mihnea Colţoiu

Abstract

We construct a complex manifold X, $dimX \geq 3$, which is an increasing union of $(1,1)$ convex-concave open subsets having the same fixed convex boundary, and a holomorphic line bundle L on X, such that the cohomology group $H^1(X,L)$ is not separated.The manifold X is constructed as a proper modification of the $(1,1)$ convex-concave manifold $\mathbb{C}^k \setminus \{0\}$ at a discrete subset. It is also remarked that an increasing union of 1-concave manifolds has always separated cohomology (for locally free sheaves).

1. Introduction

If X is a complex manifold which is an increasing sequence of Stein open subsets $X = \bigcup_{n\in\mathbb{N}} X_n$, $X_n \subset X_{n+1}$, then X is not necessarily Stein [Fo]. In fact, in this case, the Steineness of X is equivalent to the separation of the cohomology group $H^1(X,\mathcal{O})$ [Ma], [Sil].

In this short note we consider a somehow dual situation, i.e. increasing sequences of $(1,1)$ convex-concave open sets $\{X_n\}_{n\in\mathbb{N}}$ such that they have the same fixed convex boundary. We denote $X = \bigcup_{n\in\mathbb{N}} X_n$ and let L be a holomorphic vector bundle over X. We assume that $k = dimX \geq 3$. As it is well-known (see e.g. [Ra], [A-G]) for each n the cohomology group $H^1(X_n,L)$ is of finite dimension. The aim of this short note is to show that, under the above considered situation, the cohomology of the union $H^1(X,L)$ may not be separated. More precisely one has:

Theorem 1.1. *For every integer $k \geq 3$ there exist*:

1. *a connected complex manifold X, $dimX = k$, which is an increasing union $X = \bigcup_{n\in\mathbb{N}} X_n$ of $(1,1)$ convex-concave open sets X_n and all X_n have the same fixed convex boundary*

2. *a holomorphic line bundle L on X*

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such that $H^1(X, L)$ *is not separated.*

Moreover X *can be chosen as a proper modification of the* $(1,1)$ *convex-concave manifold* $\mathbb{C}^k \setminus \{0\}$ *at a discrete set.*

2. Construction of the example

We recall that a complex manifold Y is said to be $(1,1)$ convex-concave [Ra] if there is a smooth function $\varphi : Y \to (0,\infty)$ such that $\{\varepsilon < \varphi < \alpha\} \subset\subset Y$, $\forall \varepsilon > 0$, $\forall \alpha > \varepsilon$, and φ is strongly plurisubharmonic outside a compact subset of Y. So $\varphi \to 0$ at the concave part of the boundary of Y and $\varphi \to \infty$ at the convex part of this boundary.

If $k = dimY \geq 3$ and F is a holomorphic vector bundle over a $(1,1)$ convex-concave manifold Y then all cohomology groups $H^i(Y,F)$ are separated if $1 \leq i \leq k-1$ and of finite dimension if $1 \leq i \leq k-2$ (see e.g. [Ra]). In particular $H^1(Y,F)$ is always of finite dimension for $k \geq 3$. In the case of dimension 2 the cohomology group $H^1(Y,F)$ is separated if the hole can be filled in and F can be extended to a holomorphic line bundle on the manifold obtained by filling in the hole (see [LT-Le2]). On $\mathbb{C}^2 \setminus \{0\}$, which is of course $(1,1)$ convex-concave, there are holomorphic line bundles F which cannot extend through $\{0\}$, therefore [Tra] $H^1(\mathbb{C}^2 \setminus \{0\}, F)$ is not separated.

Let now $X = \bigcup_{n\in\mathbb{N}} X_n$, $X_n \subset X_{n+1}$, be an increasing union of $(1,1)$ convex-concave open sets. We assume that all X_n have the same fixed convex boundary. This means the following: let $\varphi_n : X_n \to (0,\infty)$ be the functions describing the $(1,1)$ convexity-concavity of X_n. We assume that there is some $\alpha_0 > 0$ such that all sets $\{\varphi_n > \alpha_0\}$, $n \in \mathbb{N}$, coincide.

In order to construct the example proving Theorem 1.1 we shall need the following:

Lemma 2.1. *Let* X *be a complex manifold,* $\Omega_n \subset\subset \Omega_{n+1}$ *a sequence of connected open subsets,* $X = \bigcup_{n\in\mathbb{N}} \Omega_n$ *and let* E *be a holomorphic vector bundle over* X*. Assume that*:

1. $H^0(\Omega_j, E) \neq 0$ $\forall j \in \mathbb{N}$
2. $H^0(X, E) = 0$
3. $H^1(\Omega_j, E)$ *is separated* $\forall j \in \mathbb{N}$

Then $H^1(X, E)$ *is not separated.*

Proof. By well-known duality arguments (see [La], [LT-Le]) the condition $H^1(X,E)$ is separated is equivalent to $H^k_c(X, E^* \otimes K_X)$ is separated, where $k = dimX$ and K_X denotes the canonical line bundle of X. Consider the inductive system $\{H^k_c(\Omega_j, E^* \otimes K_X)\}_{j\in\mathbb{N}}$. The separation of $H^k_c(X, E^* \otimes K_X)$ is equivalent to a condition of "essential injectivity": $\forall\, i \in \mathbb{N}$, $\exists\, j \in \mathbb{N}$, $j > i$, such that if an element in $H^k_c(\Omega_i, E^* \otimes K_X)$ has null image in $H^k_c(X, E^* \otimes K_X)$ then necessarily it has null image in $H^k_c(\Omega_j, E^* \otimes K_X)$. By the assumption (c) the cohomology groups with compact support $H^k_c(\Omega_i, E^* \otimes K_X)$, $H^k_c(\Omega_j, E^* \otimes K_X)$ can be identified with the topological duals of $H^0(\Omega_i, E)$ and $H^0(\Omega_j, E)$ respectively. If $H^1(X,E)$ were separated then also $H^k_c(X, E^* \otimes K_X)$ could be identified with the topological dual of $H^0(X,E)$. By (a) and (b) we get easily a contradiction, therefore $H^1(X,E)$ is not separated, as required. □

We can now describe our example. In $\mathbb{C}^k$, $k \geq 3$, we consider a sequence of points $x_n \to 0$, $x_n \neq 0$. We assume that the points of the sequence $\{x_n\}_{n\in\mathbb{N}}$ are chosen in general position, i.e. they cannot be all contained in some complex analytic hypersurface of $\mathbb{C}^k$. We blow-up $\mathbb{C}^k \setminus \{0\}$ at the closed analytic set $A = \{x_n\}_{n\in\mathbb{N}}$ and we get a complex manifold X of dimension k. We shall show that this X has the required properties. We have to define also a suitable holomorphic line bundle L over X. From the definition of X it follows that it is a proper modification of $\mathbb{C}^k \setminus \{0\}$ via the projection map $p : X \to \mathbb{C}^k \setminus \{0\}$, at the discrete subset A of $\mathbb{C}^k \setminus \{0\}$. The exceptional divisor of this modification is $D = \sum_{n\in\mathbb{N}} A_n$ where $A_n = \mathbb{P}^{k-1}$ and we define $L = \mathcal{O}(D)$ the corresponding line bundle. Let now $\{B_i\}_{i\in\mathbb{N}}$ be a decreasing sequence of closed balls of positive radius centered at the origin 0 of $\mathbb{C}^k$ such that ∂B_i does not contain any point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\bigcap_{i\in\mathbb{N}} B_i = \{0\}$. Put $Q_i = \mathbb{C}^k \setminus B_i$ and $X_i = p^{-1}(Q_i)$. Then X_i is an increasing sequence of $(1,1)$ convex-concave domains in X, $X = \bigcup_{i\in\mathbb{N}} X_i$ and clearly they all have the same convex boundary. One has $H^0(X,L) = 0$ since the points of the given sequence $\{x_n\}_{n\in\mathbb{N}}$ cannot be contained in any hypersurface of $\mathbb{C}^k$. On the other hand $H^0(X_i,L) \neq 0$, $\forall i$, since any finite part of this sequence is contained in a suitable hypersurface of $\mathbb{C}^k$. It also follows that $H^1(X_i,L)$ is separated [Ra] because every X_i is $(1,1)$ convex-concave. We still cannot apply Lemma 2.1 since the assumption $X_i \subset\subset X_{i+1}$ is not satisfied. But approximating X_i from inside with a suitable Ω_i (near the convex part of the boundary of X_i) we may achieve that also this assumption is verified, consequently by Lemma 2.1 it follows that the cohomology group $H^1(X,L)$ is not separated, as required. The proof of Theorem 1.1 is complete.

Remark 1. In the previous example one has also non-separation for $H^1(X,F)$ for $n = 2$ but in this case one has to use the results in [LT-Le2] instead of [Ra] in order to apply Lemma 2.1.

Remark 2. Instead of choosing a generic sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to 0 we may fix from the beginning an arbitrary sequence $x_n \to 0$, $x_n \neq 0$, but in this case we have to replace the divisor $D = \sum_{n\in\mathbb{N}} A_n$ (and its corresponding line bundle $\mathcal{O}(D)$) by the divisor $\sum_{n\in\mathbb{N}} \alpha_n A_n$ where α_n is a suitable chosen sequence of positive integers.

If all points x_n are on the same complex linear hyperplane passing through the origin $\triangle \subset \mathbb{C}^k$ then $Y = X \setminus \tilde{\triangle}$, where $\tilde{\triangle}$ is the proper transform of $\triangle$ via the projection map p, is the example obtained by Fornæss [Fo] of a non-Stein manifold Y which is an increasing sequences of Stein open sets.

Remark 3. If $\{L_i\}_{i\in\mathbb{N}}$ is an inductive system of finite dimensional vector spaces and L is its limit then always the condition of "essential injectivity", as described above, is satisfied, since $\forall\, i$ the map $\varphi : L_i \to L$ has kernel of finite dimension. Together with the duality results already discussed this simple remark shows that for any increasing union X of $(1,1)$ convex-concave manifolds, with fixed convex boundary, and any vector bundle E on X, the cohomology groups $H^i(X, L)$, $i \geq 2$, are always separated. Therefore the given example is optimal in this sense. If X is an increasing sequence of 1-concave manifolds then the same simple remark shows that all cohomology groups $H^i(X, L)$ are separated, so it is no possible to construct an example as in Theorem 1.1 with an increasing sequence of 1-concave manifolds.

Remark 4. If X is a 1-concave manifold of dimension $n \geq 2$ then for every holomorphic vector bundle L on X one has: $H^i(X, L)$ is of finite dimension for $i \leq n-2$ and $H^{n-1}(X, L)$ is separated ([A-G], [A-V]). However the converse of this statement does not hold. One may take as X the blowing-up of $\mathbb{P}^2 \setminus \{0\}$ at a discrete sequence converging to 0. Clearly X is not 1-concave, $H^0(X, L)$ is of finite dimension and $H^1(X, L)$ is separated by the previous remark.

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Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
RO-014700 Bucureşti, Romania
e-mail: Mihnea.Coltoiu@imar.ro

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