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Positive definite class functions on a topological group and characters of factor representations

By

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Introduction

In this paper, we prove four important theorems in the general theory of representations of topological groups. The first one is the combination of Theroems 1.5.4 and 1.6.1, and the second one is Theorem 1.6.2 in Section 1. The third and the fourth ones are Theorems 2.6.1 and 2.6.2 in Section 2. Let us explain a little more in detail.

1. Let G be a Hausdorff topological group, $\mathcal{P}(G)$ the set of continuous positive definite functions on G, $K(G)$ the set of $f \in \mathcal{P}(G)$ which are invariant under inner automorphisms on G, $K_1(G)$ the set of $f \in K(G)$ normalized as $f(e) = 1$ at the identity element $e \in G$, and $E(G)$ the set of extremal points in the convex set $K_1(G)$. For each $f \in \mathcal{P}(G)$ normalized as $f(e) = 1$, Gelfand-Raikov [GR] constructed a cyclic continuous unitary representation (= UR) π_f with a unit cyclic vector v_0 such that $f(g) = \langle \pi_f(g)v_0, v_0\rangle$ $(g \in G)$ (cf. **1.2** below). Let π be a UR of G, and $\mathfrak{U} = \pi(G)''$ the von Neumann algebra generated by $\pi(G) = \{\pi(g); g \in G\}$. Assume that $\mathfrak{U}$ has a faithful normal finite trace t on the set $\mathfrak{U}^+$ of non-negative elements in $\mathfrak{U}$. The unique extention of t to a linear form on $\mathfrak{U}$ is denoted by $\phi = \phi_t$, and the function

$$f(g) = \phi(\pi(g)) \qquad (g \in G) \tag{1}$$

is continuous in g (Proposition 1.5.1) and positive definite and invariant: $f \in K_1(G)$.

Let π_1 and π_2 be two URs of G, and $\mathfrak{U}_i = \pi_i(G)''$ $(i = 1, 2)$ the von Neumann algebra generated by $\pi_i(G)$. We say that π_1 and π_2 are *quasi-equivalent* if there exists an isomorphism Φ from $\mathfrak{U}_1$ onto $\mathfrak{U}_2$ as $*$-algebras such that $\Phi\big(\pi_1(g)\big) = \pi_2(g)$ for $g \in G$.

Theorem A (from Theorems 1.5.4 and 1.6.1)**.** *Let π be a continuous unitary representation (= UR) of G such that the von Neumann algebra $\mathfrak{U} = \pi(G)''$ has a faithful normal finite trace t on the set $\mathfrak{U}^+$ of positive elements. Normalize t as $t(I) = 1$ and put $f = \phi \circ \pi \in K_1(G)$ as in* (1)*. Then, UR π*

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is quasi-equivalent to the UR π_f associated to f in [GR] (*cf.* **1.2** *below*). *An isomorphism Φ from $\mathfrak{U}$ to $\mathfrak{U}_f = \pi_f(G)''$ can be given explicitly.*

A UR π is called *factorial* if $\mathfrak{U}$ is a factor. If the factor is of finite type, there exists a unique faithful finite normal trace t normalized as $t(I) = 1$. Then the function $f(g) = \phi(\pi(g)) \in K_1(G)$ is called a *character* of π. Our second main theorem (Theorem 1.6.2) says the following.

Theorem B. *For a Hausdorff topological group G, let* URff(G) *be the set of all quasi-equivalence classes of URs of G, factorial of finite type. Then there exists a canonical bijective correspondence between* URff(G) *and $E(G)$ through* (1) *above.*

In [Dix2, 17.3], the above canonical bijection is asserted under the condition that G is locally compact and unimodular.

2. Now let $K_{\leq 1}(G) \supset K_1(G)$ be the set of $f \in K(G)$ such that $f(e) \leq 1$. Then the set of extremal points of $K_{\leq 1}(G)$ is the union of $E(G)$ and 0. In the case where G is locally compact, it is known that the weak topology $\sigma(L^\infty(G), L^1(G))$ in $K_1(G)$ is equivalent to the compact uniform topology (cf. [Dix2, 13.5]), and that the convex set $K_{\leq 1}(G)$ is weakly compact (cf. [Dix2, 17.3]). We extend these results to the case where $G = \lim_{n\to\infty} G_n$ is the inductive limit of a countable inductive system $G_1 \to G_2 \to \cdots \to G_n \to \cdots$ of locally compact groups, where each homomorphism from G_n into G_{n+1} is assumed to be homeomorphic. In [TSH], this kind of inductive system is called a *countable LCG inductive system* and there were proved that G with the inductive limit topology τ_{ind} becomes a topological group and that G has sufficiently many continuous positive definite functions and sufficiently many unitary representations.

For this kind of group G, let $\mathcal{C}$ be the family of compact subsets of G, $\boldsymbol{B}_{\mathcal{C}}$ the σ-ring generated by $\mathcal{C}$, and $\mathfrak{M}_{\mathcal{C}}(G)$ be the space of bounded measures on $(G, \boldsymbol{B}_{\mathcal{C}})$. Further let $C_b(G)$ be the space of bounded continuous functions on G, then $C_b(G) \supset K(G)$ and we have a natural pairing of $C_b(G)$ and $\mathfrak{M}_{\mathcal{C}}(G)$ through integration.

The principal parts of our third and fourth main theorems (Theorems 2.6.1 and 2.6.2) are stated as follows, which are generalizations of the corresponding results in the case of locally compact groups.

Theorem C. *Let G be the limit group of a countable LCG inductive system with the inductive limit topology τ_{ind}. Then the convex sets $K_{\leq 1}(G)$ and $K_1(G)$ are compact in the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$.*

Theorem D. *Let G be as in the above theorem, and $\mathcal{P}(G)$ the set of all continuous positive definite functions on G. Then, on every bounded subset of $\mathcal{P}(G)$, the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ is equivalent to the compact uniform topology.*

By Theorem C, there holds for the compact convex set $K_1(G)$ and the set of its extremal points $E(G)$ the integral expression theorem of Choquet-Bishop-K. de Leeuw (Theorem 5.6 in [BL]), which will be applied in [HH2].

3. We know that general theories for traces of von Neumann algebras and C^*-algebras and for characters of their factor representations have been well studied and also well exposed in several text books such as [Dix1], [Dix2] and [Pede]. However, for the level of topological groups, the situation is not yet satisfactory at the point connecting the theory for representations of groups to those of von Neumann algebras and C^*-algebras. Thus we prepared this paper.

We will apply the results in this paper to our succeeding work for determining explicitly all the characters of factor representations of finite type for the wreath product group $G = \mathfrak{S}_\infty(T)$ of a compact group T with the infinite symmetric group $\mathfrak{S}_\infty$. This group is the inductive limit of $G_n = \mathfrak{S}_n(T) \cong T^n \rtimes \mathfrak{S}_n$ and is not locally compact if T is not discrete.

1. Positive definite functions and characters of factor representations

1.0. Positive definite functions on a topological group

Let G be a Hausdorff topological group, and $\mathcal{P}(G)$ the set of all continuous positive definite functions on G. For an $f \in \mathcal{P}(G)$, we have $f(g^{-1}) = \overline{f(g)}$, and Kreĭn's inequality [Krei]

$$|f(g) - f(h)|^2 \leq 2f(e)\{f(e) - \Re(f(gh^{-1}))\} \quad (g, h \in G),$$

where e denotes the identity element of G. Define the kernel of f as $N_f := \{g \in G; f(g) = f(e)\}$, then, $f(gk) = f(kg) = f(g)$ $(k \in N_f, g \in G)$, and especially N_f is a group. The intersection $N = \bigcap_{f \in \mathcal{P}(G)} N_f$ is a closed normal subgroup of G. Introduce in the quotient group $\widetilde{G} := G/N$ the quotient topology, then any f becomes a continuous positive definite function on $\widetilde{G}$, and any continuous unitary representation (= UR) of G becomes a UR of $\widetilde{G}$.

In this paper we study continuous positive definite functions on a group G and URs of G, so it is essentially sufficient for us to consider $\widetilde{G}$ in place of G. The quotient group $\widetilde{G}$ has sufficiently many continuous positive definite functions and sufficiently many URs. Furthermore for any two different $\widetilde{g}, \widetilde{h} \in \widetilde{G}$, there exists a continuous positive definite function $\widetilde{f}$ such that $\widetilde{f}(\widetilde{g}) \neq \widetilde{f}(\widetilde{h})$, because they can be separated by a UR.

A Hausdorff topological group is completely regular as a topological space, and we may ask *if it has sufficiently many URs, what kind of characterization is possible as a topological space.*

1.1. Weak topologies and representations

Let G be a topological group with a Hausdorff topology τ_G, and assume that it has sufficiently many positive definite functions. Denote by $C_b(G)$ the space of all bounded continuous functions on G, and by $\mathfrak{B}(G)$ the σ-field of Borel measurable subsets of G. Since we do not know much about the topological characterization of such a group, we introduce here a sub-σ-field $\mathfrak{B}^0(G)$, the smallest σ-field making all $\varphi \in C_b(G)$ measurable.

Denote by $M_b(G)$ (resp. $M_b^0(G)$) the space of all bounded Borel measurable (resp. $\mathfrak{B}^0(G)$-measurable) functions on G, by $\mathfrak{M}_b(G)$ (resp. $\mathfrak{M}_b^0(G)$) that of all bounded complex Borel measures (resp. $\mathfrak{B}^0(G)$-measures) on G, and by $\mathfrak{F}(G)$ the space of finite linear combinations of unit point masses δ_g giving a mass 1 on a point $g \in G$. Then $M_b(G) \supset M_b^0(G) \supset C_b(G) \supset \mathcal{P}(G)$, and $\mathcal{P}(G)$ separates any two points of G. Put

$$\begin{aligned} &\|\psi\| = \sup\nolimits_{g\in G} |\psi(g)| \quad (\psi \in M_b(G)), \\ &\|\mu\| = \sup\nolimits_{\|\psi\|\le 1, \psi\in M_b(G)} |\mu(\psi)| \quad (\mu \in \mathfrak{M}_b(G)). \end{aligned} \tag{2}$$

The norm $\|\psi\|$ is natural on the space $C_b(G)$, and $\|\mu\|$ is natural on the space $\mathfrak{M}_b(G)$. Put $\|\mu\|_0 = \sup_{\|\varphi\|\le 1} |\mu(\varphi)|$, where φ varies in $C_b(G)$, then it gives a norm on $\mathfrak{M}_b^0(G)$ but may not be a norm (but seminorm) on $\mathfrak{M}_b(G)$, because functions in $C_b(G)$ may not be able to separate two measures in $\mathfrak{M}_b(G)$ in the case where $\mathfrak{B}^0(G) \subsetneqq \mathfrak{B}(G)$. In that case, we have a question: *Under what condition, can any $\mu \in \mathfrak{M}_b^0(G)$ be uniquely extended to a Borel measure in $\mathfrak{M}_b(G)$?*

For a real-valued $\psi \in M_b(G)$, put $\mu(\psi) = (\Re\mu)(\psi) + \sqrt{-1}(\Im\mu)(\psi)$, and define real-valued measures $\Re\mu, \Im\mu$. Put $|\mu| = |\Re\mu| + |\Im\mu|$, then $\|\mu\| \le \|\,|\mu|\,\| \le 2\|\mu\|$. We consider a pairing between $\mathfrak{M}_b(G)$ and $M_b(G)$ given by

$$\mathfrak{M}_b(G) \times M_b(G) \ni (\mu, \psi) \longmapsto \mu(\psi) = \psi(\mu) := \int_G \psi(g)\, d\mu(g) \in \boldsymbol{C}, \tag{3}$$

and denote by $\sigma(\mathfrak{M}_b(G), M_b(G))$ the weak topology induced on $\mathfrak{M}_b(G)$, and also by $\sigma(\mathfrak{M}_b(G), C_b(G))$ the one obtained by restricting $M_b(G)$ to $C_b(G)$.

Lemma 1.1.1. *The space of point masses $\mathfrak{F}(G)$ is everywhere dense in $\mathfrak{M}_b(G)$ (resp. $\mathfrak{M}_b^0(G)$) in the weak topology $\sigma(\mathfrak{M}_b(G), M_b(G))$ (resp. $\sigma(\mathfrak{M}_b^0(G), M_b^0(G))$). Moreover let B_L be the bounded subset of $\mathfrak{M}_b(G)$ (resp. $\mathfrak{M}_b^0(G)$) defined by $\||\mu|\| \le L$, then $\mathfrak{F}(G) \cap B_L$ is dense in B_L.*

Proof. Take a $\mu \in \mathfrak{M}_b(G)$. For a real valued function $\psi \in M_b(G)$, the integral $\mu(\psi) = \int_G \psi(g)\, d\mu(g)$ is defined as follows. For $a < b$ in $\boldsymbol{R}$, put $[a < \psi \le b] = \{g \in G\,;\, a < \psi(g) \le b\}$. For an integer $n > 0$, we take a finite decomposition Δ_n of G given by $E_{n,i} = [i/n < \psi \le (i+1)/n]$, and then corresponding to a choice of elements $g_{n,i} \in E_{n,i}$, we define a Riemannian sum as $\Sigma_{\Delta_n} = \sum_i \psi(g_{n,i})\, \mu(E_{n,i})$. Then $\int_G \psi(g)\ d\mu(g) = \lim_{n\to\infty} \Sigma_{\Delta_n}$. Put $a_{n,i} = \mu(E_{n,i})$, then we have $\Sigma_{\Delta_n} = \mu_n(\psi)$ with $\mu_n = \sum_i a_{n,i}\delta_{g_{n,i}} \in \mathfrak{F}(G)$ and $\mu(\psi) = \lim_{n\to\infty} \mu_n(\psi)$.

A fundamental neighbourhood of μ is given by a finite number of real $\psi_k \in M_b(G), 1 \le k \le N$, and $\varepsilon > 0$ as $U(\mu; (\psi_k); \varepsilon) = \{\nu \in \mathfrak{M}_b(X); |(\mu - \nu)(\psi_k)| < \varepsilon\ (\forall k)\}$. Considering a measurable decomposition of G finer than any of $\{[i/n < \psi_k \le (i+1)/n]\}, 1 \le k \le N$, and a Riemannian sum corresponding to it, we see that the neighbourhood $U(\mu; (\psi_k); \varepsilon)$ contains an element in $\mathfrak{F}(G)$.

For the denseness of $\mathfrak{F}(G) \cap B_L$ in B_L, it is enough to note that $\||\mu_n|\| \le \sum_i (|\Re\mu| + |\Im\mu|)(E_{n,i}) = \||\mu|\|$. $\square$

Remark 1.1.1. A Hausdorff topological group is necessarily completely regular, and for a point g and a closed set F not containing g, there exists a continuous function $\varphi \in C_b(G)$ such that $\varphi(g) = 0$ and $\varphi = 1$ on F. This means that the weak topology $\sigma(\mathfrak{M}_b(G), C_b(G))$ restricted on $\mathfrak{F}(G)$ is Hausdorff.

Suppose that the weak topology $\sigma(\mathfrak{M}_b(G), C_b(G))$ is not Hausdorff. Put $\mathfrak{K}(G) = \{\mu \in \mathfrak{M}_b(G); \mu(\varphi) = 0 \ (\varphi \in C_b(G))\}$. Then, $\mathfrak{K}(G) \cap \mathfrak{F}(G) = \{0\}$, and any $\sigma(\mathfrak{M}_b(G), C_b(G))$-continuous linear form F on $\mathfrak{F}(G)$ can be extended uniquely to a continuous linear form F' on $\mathfrak{M}_b(G)/\mathfrak{K}(G)$ and so such a one F'' on $\mathfrak{M}_b(G)$ vanishing on $\mathfrak{K}(G)$.

A unitary representation (= UR) π of G on a Hilbert space $\mathfrak{H} = V(\pi)$ is by definition assumed to be weakly continuous, that is, for any $v_1, v_2 \in \mathfrak{H}$, the map $G \ni g \mapsto \langle \pi(g)v_1, v_2\rangle \in \boldsymbol{C}$ is continuous. In other words, let τ_w be the weak topology in the space $\mathfrak{B}(\mathfrak{H})$ of all bounded linear operators on $\mathfrak{H}$, then the map $G \ni g \mapsto \pi(g) \in \mathfrak{B}(\mathfrak{H})$ is continuous in τ_G and τ_w.

We define $\pi(\mu)$ for $\mu \in \mathfrak{M}_b(G)$ by the following integral which converges in τ_w:

$$\langle \pi(\mu)v_1, v_2\rangle = \int_G \langle \pi(g)v_1, v_2\rangle \, d\mu(g) \quad (v_1, v_2 \in V(\pi)). \tag{4}$$

Lemma 1.1.2. *The map* $\mathfrak{M}_b(G) \ni \mu \mapsto \pi(\mu) \in \mathfrak{B}(\mathfrak{H})$ *is continuous in the topologies* $\sigma(\mathfrak{M}_b(G), C_b(G))$ (*or* $\sigma(\mathfrak{M}_b(G), M_b(G))$) *and* τ_w.

We also consider the *ultra-weak topology* τ_{uw} (resp. *ultra-strong topology* τ_{us}) on $\mathfrak{B}(\mathfrak{H})$ which is defined by the family of seminorms given by

$$s(T) = \big|\textstyle\sum_{1\le i\le\infty} \langle Tv_i, v_i\rangle\big| \quad \Big(\text{resp. } s(T) = \big(\textstyle\sum_{1\le i\le\infty} \|Tv_i\|^2\big)^{1/2}\Big)$$

for every series $v_i \in \mathfrak{H}, \sum_{i\ge1} \|v_i\|^2 < \infty$. Then, on every bounded set of $\mathfrak{B}(\mathfrak{H})$, the topology τ_{uw} coincides with the one τ_w. Since $\|\pi(\mu)\| \le \|\mu\|$, we see from Lemma 1.1.2 the following.

Lemma 1.1.3. *The map* $\mu \mapsto \pi(\mu)$ *is continuous in* $\sigma(\mathfrak{M}_b(G), C_b(G))$ (*or* $\sigma(\mathfrak{M}_b(G), M_b(G))$) *and* τ_{uw}, *on every bounded subset* (*with respect to* $\|\mu\|$).

For $\mu \in \mathfrak{M}_b(G)$, put $\mu^*(\psi) = \overline{\mu(\psi^*)}$ with $\psi^*(g) := \overline{\psi(g^{-1})}$. Further we wish to introduce in $\mathfrak{M}_b(G)$ (or $\mathfrak{M}_b^0(G)$) a convolution product $\mu_1 * \mu_2$ for $\mu_1, \mu_2 \in \mathfrak{M}_b(G)$ (resp. $\mathfrak{M}_b^0(G)$) by

$$(\mu_1 * \mu_2)(\psi) = \iint_{G\times G} \psi(g_1g_2) \, d\mu_1(g_1)d\mu_2(g_2) \tag{5}$$

for any $\psi \in M_b(G)$ (resp. $\psi \in C_b(G)$). If this is possible, then $\mathfrak{M}_b(G)$ (resp. $\mathfrak{M}_b^0(G)$) becomes a $*$-algebra.

To check this possibility, we should analyse the measurability of the product map $\Lambda : G\times G \ni (g_1, g_2) \mapsto g_1g_2 \in G$. For a σ-ring $\boldsymbol{B}(G)$ of subsets of G, we say that $(G, \boldsymbol{B}(G))$ is a *measurable group* if the inverse map $G \ni g \to g^{-1} \in G$

and the product map Λ are measurable. Let $\mathfrak{M}_b(\boldsymbol{B}(G))$ and $M_b(\boldsymbol{B}(G))$ be the spaces of bounded measures and that of bounded measurable functions on $(G, \boldsymbol{B}(G))$ respectively.

If $\mathfrak{B}(G) \times \mathfrak{B}(G) = \mathfrak{B}(G \times G)$, then $(G, \mathfrak{B}(G))$ is measurable.

Lemma 1.1.4. *Let $(G, \boldsymbol{B}(G))$ be a measurable group. Assume that $M_b(\boldsymbol{B}(G))$ can separate any two points of $\mathfrak{M}_b(\boldsymbol{B}(G))$, then the formula* (5) *for $\psi \in M_b(\boldsymbol{B}(G))$ defines the convolution product and $\mathfrak{M}_b(\boldsymbol{B}(G))$ becomes a $*$-algebra.*

In this case, for $\psi \in M_b(\boldsymbol{B}(G))$, the integration on $G \times G$ in the right hand side of (5) can be rewritten as an iterated integral. Hence, for a UR π of G, $\mu \mapsto \pi(\mu)$ gives a representation of $*$-algebra $\mathfrak{M}_b(\boldsymbol{B}(G))$. If $\boldsymbol{B}(G) \subset \mathfrak{B}(G)$, then $\mathfrak{M}_b(\boldsymbol{B}(G)) \subset \mathfrak{M}_b(G)$. We have a question: *Under what topological condition on G, is $(G, \mathfrak{B}(G))$ or $(G, \mathfrak{B}^0(G))$ a measurable group ?*

Note 1.1.2. (i) Representations of measurable groups were studied in [Mack].

(ii) A weak topology similar to $\sigma(\mathfrak{M}_b(G), C_b(G))$ is utilized in the definition of hypergroups in [BH, §1]. However the base spaces for hypergroups are always assumed to be locally compact.

1.2. Construction of cyclic representations

Let $\mathcal{P}_1(G)$ be the set of $f \in \mathcal{P}(G)$ normalized as $f(e) = 1$, and $\mathcal{E}(G)$ the set of extremal points of the convex set $\mathcal{P}_1(G)$. Take an $f \in \mathcal{P}_1(G)$. As in [GR], we introduce in $\mathfrak{F}(G)$ an inner product by

$$(6) \qquad \begin{aligned}(\mu_1, \mu_2)_f := f(\mu_2^* * \mu_1) &= \iint_{G\times G} f(g_2 g_1)\, d\mu(g_1)\, d\mu_2^*(g_2) \\ &= \iint_{G\times G} f({g_2}^{-1} g_1)\, d\mu(g_1)\, d\overline{\mu_2}(g_2),\end{aligned}$$

where $\overline{\mu_2}(\psi) := \overline{\mu_2(\overline{\psi})}$. Put $J_f = \{\, \mu\,; (\mu_1, \mu)_f = 0 \ (\forall \mu_1 \in \mathfrak{F}(G))\}$, then J_f is a left ideal, and a positive definite inner product, denoted by $\langle {\mu_1}^f, {\mu_2}^f \rangle_f$, is induced on $\mathfrak{F}(G)/J_f$, where ${\mu_1}^f$ denotes the canonical image of μ_1 in $\mathfrak{F}(G)/J_f$. Here we have

$$(7) \qquad \langle \delta_{g_1}^f, \delta_{g_2}^f \rangle_f = (\delta_{g_1}, \delta_{g_2})_f = f({g_2}^{-1} g_1).$$

This inner product is invariant under left translations $L(g_0)$, where $L(g_0)\mu(g) = \mu({g_0}^{-1} g)$ $(g_0 \in G)$, and the latter induces a unitary representation π_f on the completion $\mathfrak{H}_f$ of $\mathfrak{F}(G)/J_f$. This UR is, above all, continuous thanks to the continuity of f, and have a unit cyclic vector $v_0 = {\delta_e}^f$, and f is recovered from π_f by $f(g) = \langle \pi_f(g) v_0, v_0 \rangle_f$.

For an $f \in \mathcal{P}_1(G)$, the UR π_f is irreducible if and only if f is extremal in $\mathcal{P}_1(G)$ or $f \in \mathcal{E}(G)$ ([GR, Theorems 1 and 2]).

The representation π_f generates a von Neumann algebra $\mathfrak{U}_f = \pi_f(G)''$.

Properties of π_f. (i) Any matrix element $F_{v_1,v_2}(g) = \langle \pi_f(g)v_1, v_2\rangle_f$ ($v_1, v_2 \in \mathfrak{H}_f$) is continuous in g as a uniform convergence limit of linear combinations of $f(g_2{}^{-1}gg_1) = \langle \pi_f(g)\delta^f_{g_1}, \delta^f_{g_2}\rangle_f$ ($g_1, g_2 \in G$), and moreover F_{v_1,v_2} is bounded.

(ii) For $\mu \in \mathfrak{M}_b(G)$, the operator $\pi_f(\mu)$ is defined weakly by $\langle \pi(\mu)v_1, v_2\rangle_f = \int_G F_{v_1,v_2}(g)\, d\mu(g)$. Since $\langle \pi(\mu)v_1, \pi_f(g')v_2\rangle_f$ is continuous in g' and is bounded, we have for $\nu \in \mathfrak{M}_b(G)$,

$$\langle \pi_f(\mu)v_1, \pi_f(\nu)v_2\rangle_f = \int_G d\nu^*(g') \int_G F_{v_1,v_2}(g'g)\, d\mu(g). \tag{8}$$

Proposition 1.2.1. (i) *For $\mu \in \mathfrak{M}_b(G)$, the integral $\int_G f(g'g)\, d\mu(g)$ is bounded and continuous in $g' \in G$, and for $\mu, \nu \in \mathfrak{M}_b(G)$,*

$$\int_G d\mu^*(g') \int_G f(g'g)\, d\mu(g) = \langle \pi_f(\mu)v_0, \pi_f(\mu)v_0\rangle_f = \|\pi_f(\mu)v_0\|_f^2 \geq 0, \tag{9}$$

$$\langle \pi_f(\mu)v_0, \pi_f(\nu)v_0\rangle_f = \int_G d\nu^*(g') \int_G f(g'g)\, d\mu(g). \tag{10}$$

(ii) *The integrals* (9) *and* (10) *can be rewritten as a double integral on $G \times G$ if $f(g'g)$ is $(\mathfrak{B}(G) \times \mathfrak{B}(G))$-measurable in (g', g), and it is the case if $\Lambda^{-1}(\mathfrak{B}^0(G)) \subset \mathfrak{B}(G) \times \mathfrak{B}(G)$ or if $(G, \mathfrak{B}(G))$ is a measurable group.*

Remark 1.2.1 (Equivalence to GNS construction)**.** Let G be locally compact and $C_c(G)$ the space of continuous functions on G with compact supports. Denote by dg a left-invariant Haar measure on G and let $\Delta(h) = d(gh)/dg$ ($h \in G$) be the modular function. Introduce in $C_c(G)$ two operations

$$\varphi^*(g) = \Delta(g)^{-1}\overline{\varphi(g^{-1})}, \qquad \varphi * \psi(g) = \int_G \varphi(h^{-1}g)\, \psi(h)\, dh,$$

for $\varphi, \psi \in C_c(G)$. Then, $\|\varphi^*\|_1 = \|\varphi\|_1$, $\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \, \|\psi\|_1$, for the L^1-norm $\|\cdot\|_1$, and $C_c(G)$ is a $*$-subalgebra of the $*$-Banach algebra $L^1(G)$. The spaces $C_c(G)$ and $L^1(G)$ are embedded naturally into $\mathfrak{M}_b(G)$ through $\psi(g) \mapsto \psi(g)dg$, and the norm $\|\psi(g)dg\|$ is equivalent to $\|\psi\|_1$.

In this case, the so-called GNS construction of a cyclic representation, associated to an $f \in \mathcal{P}_1(G)$, is given by using integration with respect to a Haar measure. We remark here that the UR obtained by GNS construction is equivalent to the previous Gelfand-Raikov representation $(\pi_f, \mathfrak{H}_f)$ constructed by using $\mathfrak{F}(G)$ but not integration.

Fix an $f \in \mathcal{P}_1(G)$. Introduce in $C_c(G)$ a positive semidefinite inner product as

$$(\psi_1, \psi_2)'_f := \iint_{G\times G} f(h^{-1}g)\psi_1(g)\overline{\psi_2(h)}\, dg\, dh \qquad (\psi_1, \psi_2 \in C_c(G)\,).$$

Let J'_f be the kernel of $(\cdot\,,\cdot)'_f$, and take a quotient $C_c(G)/J'_f$. Completing it with respect to the positive definite inner product, we get a Hilbert space $\mathfrak{H}'_f$.

The left multiplication of $C_c(G)$ generates a representation π'_f of $C_c(G)$ and also a UR of G on $\mathfrak{H}'_f$.

We define a linear map Φ' of $C_c(G)$ into $\mathfrak{H}_f$, the completion of $\mathfrak{F}(G)/J_f$, as follows. As an operator-valued function on G, $G \ni g \mapsto \pi_f(g)$ is weakly continuous and, since $\pi_f(g)$'s are unitary, is strongly continuous. So, for every $\psi \in C_c(G)$, the operator-valued integration $\pi_f(\psi) = \int_G \pi_f(g)\,\psi(g)\,dg$ is strongly convergent and $\|\pi_f(\psi)\| \le \|\psi\|_1$. It defines a representation of $*$-algebra $C_c(G)$ and also of $L^1(G)$ on the space $\mathfrak{H}_f$. For $v_0 = \delta_e^f \in \mathfrak{H}_f$, put $\Phi'(\psi) := \pi_f(\psi)v_0$. Then, for $\psi_1, \psi_2 \in C_c(G)$,

$$\begin{aligned}\langle \Phi'(\psi_1), \Phi'(\psi_2)\rangle_f &= \int_G \int_G \psi_1(g)\,\overline{\psi_2(h)}\,\langle \pi_f(g)v_0, \pi_f(h)v_0\rangle_f\,dg\,dh \\ &= \int_G \int_G \psi_1(g)\,\overline{\psi_2(h)}\,f(h^{-1}g)\,dg\,dh \;=\; (\psi_1, \psi_2)'_f\,.\end{aligned}$$

Hence Φ' induces a linear map Φ'' from $C_c(G)/J'_f$ into $\mathfrak{H}_f$ which can be extended to an isomorphism $\Phi : \mathfrak{H}'_f \to \mathfrak{H}_f$ of two Hilbert spaces.

($\bullet$) *The extended linear map $\Phi : \mathfrak{H}'_f \to \mathfrak{H}_f$ intertwines two unitary representations π'_f and π_f of G as* $\Phi \cdot \pi'_f(g) = \pi_f(g) \cdot \Phi$ $(g \in G)$.

The inverse isomorphism Φ^{-1} from $\mathfrak{H}_f$ to $\mathfrak{H}'_f$ is given already in [GR]. Let $\{V\}$ be the net of all relatively compact neighbourhoods of $e \in G$ with the order of inclusion. Take functions $\psi_V \in C_c(G)$ such that $\psi_V \ge 0$, $\mathrm{supp}(\psi_V) \subset V$, $\int_G \psi_V(g)\,dg = 1$. Then,

$$\|\pi_f(\psi_V)v_0 - v_0\|^2 = \iint_{G\times G} \langle \pi_f(g)v_0 - v_0, \pi_f(h)v_0 - v_0\rangle\,\psi_V(g)\psi_V(h)\,dg\,dh \;\to\; 0$$

in $\mathfrak{H}_f$ as $V \to e$. Corresponding to this strong convergence of $\pi_f(\psi_V)v_0$ to v_0, through the isomorphism Φ^{-1}, the image of ψ_V in $C_c(G)/J'_f$ converges strongly to an element $\xi_0 = \Phi^{-1}(v_0) \in \mathfrak{H}'_f$. To prove directly this strong convergence of $\psi_V + J'_f$ in $\mathfrak{H}'_f$ is not so simple and it is given in the proof of [GR, Theorem 4, p.7].

1.3. Normal traces on von Neumann algebras

Let $\mathfrak{U}$ be a von Neumann algebra contained in the algebra $\mathfrak{B}(\mathfrak{H})$ of all bounded linear operators on a Hilbert space $\mathfrak{H}$, and $\mathfrak{U}^+$ be its subset consisting of all non-negative operators. A *trace* t on $\mathfrak{U}^+$ is by definition a map to $\boldsymbol{R}_{\ge 0} \cup \{+\infty\}$ such that $t(S+T) = t(S) + t(T)$ $(S, T \in \mathfrak{U}^+)$, $t(\lambda T) = \lambda t(T)$ $(\lambda \ge 0, T \in \mathfrak{U}^+)$ and $t(UTU^{-1}) = t(T)$ for any unitary element $U \in \mathfrak{U}$. It is called *finite* if $t(S) < +\infty$ for any $S \in \mathfrak{U}^+$, and *semifinite* if, for each $S \in \mathfrak{U}^+$, $t(S)$ is the supremum of $t(T)$ for those $T \in \mathfrak{U}^+$ such that $T \le S$ and $t(T) < \infty$. It is called *faithful* if $t(S) = 0$ for an $S \in \mathfrak{U}^+$ implies $S = 0$.

A semifinite trace t on $\mathfrak{U}^+$ is called *normal* if, for each increasing net T_α of $\mathfrak{U}^+$ with supremum $S \in \mathfrak{U}^+$, $t(S)$ is the supremum of $t(T_\alpha)$. A positive linear form ϕ on $\mathfrak{U}$ is called *normal* if $t = \phi|_{\mathfrak{U}^+}$ is normal. A trace t' is said to be majorized by t (notation: $t' \le t$) if $t'(T) \le t(T)$ $(T \in \mathfrak{U}^+)$ or $t - t'$ is again a trace. A finite trace t on $\mathfrak{A}^+$ can be extended uniquely to a linear form $\phi = \phi_t$ on $\mathfrak{A}$.

On a factor, two semifinite faithful normal traces are proportional ([Dix1, I.6.4]).

Lemma 1.3.1 ([Dix1, I.4.2]). *For a positive linear form ϕ on a von Neumann algebra $\mathfrak{A}$, the following conditions are mutually equivalent*:

(i) *ϕ is normal*;

(ii) *ϕ is ultra-weakly continuous*;

(iii) $\phi(S) = \sum_{1\le i<\infty} \langle Sv_i, v_i\rangle$ $(S \in \mathfrak{A})$ *with* $\sum_{1\le i<\infty} \|v_i\|^2 < \infty$.

Lemma 1.3.2 ([Dix1, I.6.4]). *Let $\mathfrak{Z}$ be the center of $\mathfrak{A}$ and t a semifinite normal trace on $\mathfrak{A}^+$. For a fixed $A \in \mathfrak{Z}, 0 \le A \le I$, put $t_A(T) = t(AT)$ $(T \in \mathfrak{A}^+)$. Then it is a normal trace majorized by $t : t_A \le t$. Conversely any normal trace majorized by t is given in this form.*

Moreover we need the following fact which we quote from [Dix1] for exactness:

Lemma 1.3.3 (from [Dix1, I.3.3, Theorem 1]). *Let $\mathcal{M}$ be a ultra-weakly closed linear subspace of $\mathfrak{B}(\mathfrak{H})$, $\mathcal{M}^*$ the dual of the Banach space $\mathcal{M}$, $\mathcal{M}_r$ the ball $\|T\| \le r$ of $\mathcal{M}$, and ϕ a linear form on $\mathcal{M}$. Put $\omega_{x,y}(T) = \langle Tx, y\rangle$ $(x, y \in \mathfrak{H})$.*

(i) *The following conditions are equivalent*:

(i1) *ϕ is weakly continuous*;

(i2) *ϕ is strongly continuous*;

(i3) $\phi = \sum_{\text{finite}} \omega_{x_i,y_i}$.

(ii) *The following conditions are equivalent*:

(ii1) *ϕ is ultra-weakly continuous*;

(ii2) *ϕ is ultra-strongly continuous*;

(ii3) $\phi = \sum_{1\le i<\infty} \omega_{x_i,y_i}$, *with* $\sum_{1\le i<\infty} \|x_i\|^2 < +\infty$, $\sum_{1\le i<\infty} \|y_i\|^2 < +\infty$;

(ii4) [resp. (ii5)] *The restriction of ϕ on $\mathcal{M}_1$ is ultra-weakly* (*resp. weakly*) *continuous*;

(ii6) [resp. (ii7)] *The restriction of ϕ on $\mathcal{M}_1$ is ultra-strongly* (*resp. strongly*) *continuous.*

1.4. Factoriality of the representation π_f

For an $f \in \mathcal{P}(G)$, we give in **1.2** a cyclic UR π_f on the Hilbert space $\mathfrak{H}_f$, which is the completion of $\mathfrak{F}(G)/J_f$, with J_f the kernel of the Hermitian form $(\mu, \nu)_f = f(\nu^* * \mu), \mu, \nu \in \mathfrak{F}(G)$ (cf. [GR]).

Let $K(G)$ be the set of all continuous positive definite invariant functions on G, $K_1(G)$ the subset of $K(G)$ consisting of all normalized ones as $f(e) = 1$, and $E(G)$ the set of all extremal points in the convex set $K_1(G)$. For two positive definite functions f and f', we say that f' is *majorized* by f (notation: $f' \le f$) if $f - f'$ is again positive definite. If f is continuous, then any f'

majorized by f is automatically continuous [GR, p.3]. For $f, f' \in K(G)$, we say that f' is *majorized* by f (notation: $f' \preceq f$) if $f - f'$ is again in $K(G)$.

Suppose $f \in K(G) \subset \mathcal{P}(G)$. Then the kernel J_f is a two-sided ideal and the inner product is invariant not only under left translations but also under right translations $R(g_0)$, where $R(g_0)\mu(g) = \mu(gg_0)$. They induce a UR on $\mathfrak{H}_f$ denoted by ρ_f, which generates von Neumann algebra $\mathfrak{V}_f = \rho_f(G)''$ (bicommutant). It is proved that $\mathfrak{V}_f$ is equal to the commutant algebra $(\mathfrak{U}_f)'$ of $\mathfrak{U}_f = \pi_f(G)''$ (cf. Remark 1.4.1). The common center $\mathfrak{Z}_f = \mathfrak{U}_f \cap \mathfrak{V}_f$ of $\mathfrak{U}_f$ and $\mathfrak{V}_f$ is descrived as follows (cf. [Tho, Lemma 2]).

Lemma 1.4.1. *Let $\mathfrak{Z}_f^+$ be the set of positive hermitian operators in $\mathfrak{Z}_f = \mathfrak{U}_f \cap \mathfrak{V}_f$. Then there exists a bijective correspondence between $\mathfrak{Z}_f^+$ and the subset $M(f) \subset K(G)$ given as*

$$M(f) = \{\, f' \in K(G)\,;\, f' \preceq \lambda f \ \text{ for some } \ \lambda \geq 0\}. \tag{11}$$

The correspondence $C \mapsto f'$ is given by $f'(\mu_2^* * \mu_1) = (\mu_1, \mu_2)_{f'} = \langle C\mu_1^f, \mu_2^f\rangle_f$.

Proof. Take an $f' \in M(f)$, then $\lambda f - f' \in K(G)$ for some $\lambda > 0$, and so $0 \leq (\mu, \mu)_{f'} \leq \lambda(\mu, \mu)_f$. Therefore there exists a unique positive hermitian operator $0 \leq C \leq \lambda I$ on $\mathfrak{H}_f$ such that $(\mu_1, \mu_2)_{f'} = \langle C\mu_1^f, \mu_2^f\rangle_f$ $(\mu_1, \mu_2 \in \mathfrak{F}(G))$. Then, for $\mu_1, \mu_2 \in \mathfrak{F}(G)$ and $g \in G$,

$$(L(g)\mu_1, L(g)\mu_2)_{f'} = (\mu_1, \mu_2)_{f'}, \ \ (R(g)\mu_1, R(g)\mu_2)_{f'} = (\mu_1, \mu_2)_{f'}.$$

Therefore $C \in \mathfrak{U}_f' \cap \mathfrak{V}_f' = \mathfrak{V}_f \cap \mathfrak{U}_f = \mathfrak{Z}_f$, whence $C \in \mathfrak{Z}_f^+$.

Conversely take a $C \in \mathfrak{Z}_f^+$. Put

$$f'(g) = \langle C\,\pi_f(g)v_0, v_0\rangle_f = \langle \pi_f(g)\sqrt{C}v_0, \ \sqrt{C}v_0\rangle_f \ \ (g \in G)$$

with $v_0 = \delta_e^f$. Then f' is continuous and positive definite. Since C and so $\sqrt{C}$ commute with left translations $\pi_f(g_0), g_0 \in G$, we have

$$f'(g) = \big\langle \sqrt{C}\pi_f(g_0 g)\delta_e^f, \sqrt{C}\pi_f(g_0)\delta_e^f\big\rangle_f = \big\langle \sqrt{C}\big(\delta_{g_0 g}\big)^f, \sqrt{C}\big(\delta_{g_0}\big)^f\big\rangle_f. \tag{12}$$

Since $\sqrt{C}$ commutes with right translations $\rho_f(g_1), g_1 \in G$, and since $R(g_1)\delta_g = \delta_{g{g_1}^{-1}}$, we get

$$f'(g) = \big\langle \sqrt{C}\big(\delta_{g{g_1}^{-1}}\big)^f, \sqrt{C}\big(\delta_{{g_1}^{-1}}\big)^f\big\rangle_f = f'(g_1 g {g_1}^{-1}),$$

by (12), whence f' is invariant. Moreover, $\|C\|\, f - f'$ is positive definite because, $(\|C\|\, f - f')(g) = \langle \pi_f(g)Dv_0, Dv_0\rangle_f$ with $D = \sqrt{\|C\|\, I - C}$. Hence $f' \in M(f)$. □

Thus the center $\mathfrak{Z}_f = \mathfrak{U}_f \cap \mathfrak{V}_f$ is reduced to $\boldsymbol{C}I$ if and only if any $f' \in K(G)$ majorized by f is a scalar multiple of f. This gives us a criterion for that the representation π_f is factorial or the von Neumann algebra $\mathfrak{U}_f = \pi_f(G)''$ is a factor. Recall that a von Neumamm algebra $\mathfrak{U}$ is called a *factor* if its center $\mathfrak{U} \cap \mathfrak{U}'$ is trivial.

Theorem 1.4.2. *Let $f \in K_1(G)$. Then the representation π_f is factorial* (*of finite type*) *if and only if f is extremal or $f \in E(G)$. If π_f is factorial, then it is of finite type, and the unique normalized faithful finite normal trace of $\mathfrak{U}_f = \pi_f(G)''$ is given as $\phi(T) = \langle Tv_0, v_0\rangle_f$, and there holds $f(\mu) = \langle \pi_f(\mu)v_0, v_0\rangle_f$ $(\mu \in \mathfrak{M}_b(G))$ with $v_0 = \delta_e^f$.*

For a von Neumann algebra $\mathfrak{V}$ on a Hilbert space $\mathfrak{H}$, a vector $a \in \mathfrak{H}$ is called in [Dix1, I.6.3] a *trace-element* for $\mathfrak{V}$ if $\omega_a(T) := \langle Ta, a\rangle$ $(T \in \mathfrak{V})$ is a trace on $\mathfrak{V}$. Thus the unit vector $v_0 = \delta_e^f$ is a trace-element for $\mathfrak{U}_f = \pi_f(G)''$.

Remark 1.4.1. For the completeness, we give a proof for $(\mathfrak{U}_f)' = \mathfrak{V}_f$. Let $\mathcal{A}$ be an associative algebra over $\boldsymbol{C}$ with an involutive anti-automorphism $x \to x^*$, and a positive definite inner product $(x|y)$ which makes it a pre-Hilbert space. We call $\mathcal{A}$ a *Hilbert algebra* if it satisfies the following axioms ([Dix1, I.5]):

(i) $(x|y) = (y^*|x^*)$;
(ii) $(xy|z) = (y|x^*z)$;
(iii) For any $x \in \mathcal{A}$, the map $\mathcal{A} \ni y \mapsto xy \in \mathcal{A}$ is continuous;
(iv) The set of elements $xy\,(x, y \in \mathcal{A})$ is total in $\mathcal{A}$.

Denote by $\mathfrak{H}$ the Hilbert space obtained by completing $\mathcal{A}$. The mappings $y \to xy$, $y \to yx$ extend uniquely to elements U_x, V_x in $\mathfrak{B}(\mathfrak{H})$ respectively. By (iv), the weak closure of the set U_x (resp. V_x), $x \in \mathcal{A}$, gives a von Neumann algebra $\mathfrak{U}(\mathcal{A})$ (resp. $\mathfrak{V}(\mathcal{A})$). Theorem 1 in [Dix1, I.5.2] asserts that

$$\mathfrak{U}(\mathcal{A})' = \mathfrak{V}(\mathcal{A}), \quad \mathfrak{V}(\mathcal{A})' = \mathfrak{U}(\mathcal{A}). \tag{13}$$

In our present situation, for a fixed $f \in K(G)$, put $\mathcal{A} = \mathfrak{F}(G)/J_f$ with the two-sided ideal J_f and, for $x = \mu_1^f$ and $y = \mu_2^f$ in it, $(x|y) = \langle \mu_1^f, \mu_2^f\rangle_f$. For the axiom (iii), put $z = \nu^f$, then $U_z x = zx = \nu^f \mu_1^f = (\nu * \mu_1)^f = \pi_f(\nu)\mu_1^f$ and

$$\begin{aligned}
(zx|y) &= \int_G \langle \pi_f(g)\mu_1^f, \mu_2^f\rangle_f \; d\nu(g),\\
\therefore \quad |(zx|y)| &\le \int_G |\langle \pi_f(g)\mu_1^f, \mu_2^f\rangle_f| \; d|\nu|(g)\\
&\le \int_G \|\mu_1^f\|_f \|\mu_2^f\|_f \; d|\nu|(g) = C_\nu \,\|x\|\,\|y\|,
\end{aligned}$$

with $C_\nu = |\nu|(G)$ and $\|x\| = \|\mu_1^f\|_f$. We see from $|(zx|y)| \le C_\nu \,\|x\|\,\|y\|$ that $\|zx\| \le C_\nu \,\|x\|$ and so $\|U_z\| \le C_\nu$, whence the axiom (iii).

In this case, we have $\mathfrak{U}(\mathcal{A}) = \mathfrak{U}_f, \mathfrak{V}(\mathcal{A}) = \mathfrak{V}_f$, and (13) above gives $(\mathfrak{U}_f)' = \mathfrak{V}_f$ as is desired.

1.5. Standard realization of URs with finite normal traces

Let π be a UR of G and $\mathfrak{U} = \pi(G)''$ the von Neumann algebra generated by $\pi(G)$, and $\mathfrak{Z}$ the center of $\mathfrak{U}$. Take a finite trace t on $\mathfrak{U}^+$, if exists, and extend it uniquely to a linear form $\phi = \phi_t$ on $\mathfrak{U}$ and put

$$f(g) = \phi\big(\pi(g)\big) \qquad (g \in G). \tag{14}$$

Proposition 1.5.1. *The function $f = \phi \circ \pi$ on G is positive definite and invariant. Suppose t or eqivalently ϕ is normal. Then f is continuous and so $f \in K(G)$, and for $\mu \in \mathfrak{M}_b(G)$, we have $\phi\big(\pi(\mu)\big) = f(\mu) = \int_G f(g)\, d\mu(g)$.*

Proof. The positive definiteness of f comes from that of ϕ. Further, since $\phi(S_1S_2) = \phi(S_2S_1)$, we have $f(g_1g_2) = f(g_2g_1)$ $(g_1, g_2 \in G)$, or $f(g_1g_2{g_1}^{-1}) = f(g_2)$.

For the second assertion, note that the map $G \ni g \mapsto \pi(g) \in \mathfrak{B}(\mathfrak{H})$ is weakly continuous, and that ϕ is ultra-weakly continuous by Lemma 1.3.1 and so weakly continuous on every bounded set by Lemma 1.3.3 (ii). Then f is continuous, as a composition of two continuous maps.

The map $\mathfrak{M}_b(G) \ni \mu \mapsto \pi(\mu) \in \mathfrak{B}(\mathfrak{H})$ is continuous for $\sigma(\mathfrak{M}_b(G), M_b(G))$ and τ_{uw} on the bounded set B_L by Lemma 1.1.3. Then, by Lemma 1.3.3 (ii), the linear map $\mu \mapsto \phi(\pi(\mu))$ is continuous on B_L. On the other hand, the map $\mu \mapsto f(\mu)$ is continuous by itself. Two maps $\phi(\pi(\mu))$ and $f(\mu)$, both continuous on B_L, coincide with each other on the subspace $\mathfrak{F}(G)$, and since $B_L \cap \mathfrak{F}(G)$ is dense in B_L by Lemma 1.1.1, they are identical. □

Let π be a UR of G, t its faithful normal finite trace on $\mathfrak{U} = \pi(G)''$ normalized as $t(I) = 1$, $\phi = \phi_t$, and $f = \phi \circ \pi \in K_1(G)$. Let us compair these things with the corresponding ones for the cyclic representation $(\pi_f, \mathfrak{H}_f)$ in **1.2** associated to f.

First introduce in $\mathfrak{U}$ a Hermitian inner product by

$$\langle T_1, T_2\rangle_\phi := \phi({T_2}^*T_1) = \phi(T_1{T_2}^*) \qquad (T_1, T_2 \in \mathfrak{U}). \tag{15}$$

Then, since t is faithful, we have $\|T\|_\phi^2 := \langle T, T\rangle_\phi = t(T^*T) > 0$ for any $T \neq 0$. Therefore $\mathfrak{U}$ becomes a pre-Hilbert space which we denote by $\mathfrak{U}^\phi$ and a $T \in \mathfrak{U}$ considered as an element of $\mathfrak{U}^\phi$ is denoted by T^ϕ (but we omit the superfix ϕ if it is too cumbersome). The Hilbert space obtained by completing $\mathfrak{U}^\phi$ is denoted by $\mathfrak{H}^\phi$. Note that if $\|T\| \leq M$, then $0 \leq T^*T \leq M^2 I$ with I the identity operator, and $0 \leq \phi(T^*T) \leq \phi(M^2 I) = M^2 \phi(I) = M^2$, whence $\|T\|_\phi \leq \|T\|$. The identical injective map $T \to T^\phi = T$ from $(\mathfrak{U}, \|\cdot\|)$ into $(\mathfrak{H}^\phi, \|\cdot\|_\phi)$ is continuous with dense image.

On $\mathfrak{U}$, we have the right regular representation U_S and the left regular anti-representation V_S $(S \in \mathfrak{U})$ as

$$U_S(T) := ST, \quad V_S(T) := TS \quad (T \in \mathfrak{U}), \tag{16}$$

and, on the level of group representations, $L_\pi(g)$ and $R_\pi(g)$ given by

$$L_\pi(g)T := \pi(g)T, \quad R_\pi(g)T := T\pi(g^{-1}) = T\pi(g)^* \quad (g \in G). \tag{17}$$

Then we have, for $S \in \mathfrak{U}$ such that $\|S\| \leq M$,

$$\|U_S(T)\|_\phi^2 = \phi((ST)^*(ST)) = \phi(TT^*S^*S) \leq \phi(M^2 \cdot TT^*) \leq M^2\phi(TT^*),$$

whence $\|U_S(T)\|_\phi \leq M\,\|T\|_\phi$, because $0 \leq TT^*S^*S \leq M^2 \cdot TT^*$. This means that $\|U_S(T^\phi)\|_\phi \leq \|S\|\,\|T^\phi\|_\phi$, and so $\|U_S\| \leq \|S\|$, where $\|U_S\|$ denotes the operator norm on the Hilbert space $\mathfrak{H}^{\phi}$. Hence U_S is continuous in the norm $\|\cdot\|_{\phi}$. Its natural extension onto $\mathfrak{H}^{\phi}$ is denoted by $\widetilde{U}_S$. Similarly we have a natural extension $\widetilde{V}_S$ of V_S onto $\mathfrak{H}^{\phi}$. On the other hand, $L_{\pi}(g)$ and $R_{\pi}(g)$ are both unitary on $\mathfrak{U}^{\phi}$, and can be extended respectively to unitary representations $\widetilde{L}_{\pi}(g) = \widetilde{U}_{\pi(g)}$ and $\widetilde{R}_{\pi}(g) = \widetilde{V}_{\pi(g)^*}$ on $\mathfrak{H}^{\phi}$.

Here we quote for exactness two fundamental facts from [Dix1] as follows. For von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, a map Φ from $\mathcal{A}$ into $\mathcal{B}$ is called a *homomorphism* (resp. *anti-homomorphism*) if it is a homomorphism (resp. anti-homomorphism) for the $*$-algebra structures of $\mathcal{A}$ and $\mathcal{B}$.

Lemma 1.5.2 ([Dix1, I.1.5, Proposition 8]). *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras and Φ a homomorphism or anti-homomorphism of $\mathcal{A}$ into $\mathcal{B}$. Then,*

(i) $\Phi(\mathcal{A}^+) \subset \mathcal{B}^+$;

(ii) *If E is a projection of $\mathcal{A}$, $\Phi(E)$ is a projection of $\mathcal{B}$*;

(iii) *For each $S \in \mathcal{A}$, we have $\|\Phi(S)\| \le \|S\|$; if Φ is injective, we have $\|\Phi(S)\| = \|S\|$*;

(iv) *If S is a hermitian operator of $\mathcal{A}$, then $\Phi(S)$ is an hermitian operator of $\mathcal{B}$. If h is a (complex-valued) continuous function of a real variable such that $h(0) = 0$, then $\Phi(h(S)) = h(\Phi(S))$.*

Lemma 1.5.3 ([Dix1, I.3.4, Theorem 2(i)]). *Let $\mathcal{A}$ be a $*$-algebra of operators in a Hilbert space $\mathfrak{H}$, and $\mathcal{A}_1$ the unit ball of $\mathcal{A}$. Then the following eight conditions are equivalent*:

(1) (*resp.* (2)) $\mathcal{A}$ (*resp.* $\mathcal{A}_1$) *is weakly closed*;

(3) (*resp.* (4)) $\mathcal{A}$ (*resp.* $\mathcal{A}_1$) *is strongly closed*;

(5) (*resp.* (6)) $\mathcal{A}$ (*resp.* $\mathcal{A}_1$) *is ultra-weakly closed*;

(7) (*resp.* (8)) $\mathcal{A}$ (*resp.* $\mathcal{A}_1$) *is ultra-strongly closed.*

By Lemma 1.5.3, we know that $\mathfrak{U}$ is a strong closure of $\pi(\mathfrak{F}(G))$. For any fixed $T \in \mathfrak{U}$, take a net $A_{\alpha} = \pi(\nu_{\alpha}) \in \pi(\mathfrak{F}(G))$ strongly convergent to T. Then there exists an $M > 0$ such that $\|A_{\alpha}\|, \|T\| \le M$, and we have for $v_1, v_2 \in \mathfrak{H}$,

$$|\langle (A_{\alpha} - T)^*(A_{\alpha} - T)v_1, v_2\rangle| \le \|(A_{\alpha} - T)v_1\| \cdot \|(A_{\alpha} - T)v_2\| \longrightarrow 0,$$

whence $(A_{\alpha} - T)^*(A_{\alpha} - T)$ converges weakly to 0. Therefore, by Lemma 1.3.3 (ii), $\phi\big((A_{\alpha} - T)^*(A_{\alpha} - T)\big) = \|A_{\alpha} - T\|_{\phi}^2 \to 0$. This means that $A_{\alpha}^{\phi} \to T^{\phi}$ in $\mathfrak{U}^{\phi}$, and that $\pi(\mathfrak{F}(G))^{\phi} := \{A^{\phi}; A \in \pi(\mathfrak{F}(G))\}$ is dense in $\mathfrak{U}^{\phi}$. in the norm $\|\cdot\|_{\phi}$.

Now consider the dense subspace $\pi(\mathfrak{F}(G))^{\phi} \supset \pi(G)^{\phi}$ of $\mathfrak{H}^{\phi}$. Then, for $g_1, g_2 \in G$,

$$\langle \pi(g_1)^{\phi}, \pi(g_2)^{\phi}\rangle_{\phi} = \langle \pi(g_1), \pi(g_2)\rangle_{\phi} = \phi(\pi(g_2)^*\pi(g_1)) = f({g_2}^{-1}g_1). \tag{18}$$

On the other hand, $f({g_2}^{-1}g_1) = (\delta_{g_1}, \delta_{g_2})_f = \langle \delta_{g_1}^f, \delta_{g_2}^f\rangle_f$. So $\langle \pi(g_1)^{\phi}, \pi(g_2)^{\phi}\rangle_{\phi} = \langle \delta_{g_1}^f, \delta_{g_2}^f\rangle_f$, and accordingly $\langle \pi(\mu_1)^{\phi}, \pi(\mu_2)^{\phi}\rangle_{\phi} = \langle {\mu_1}^f, {\mu_2}^f\rangle_f (\mu_1, \mu_2 \in \pi(\mathfrak{F}(G)))$. This means that the map

$$\Gamma \,:\, \pi(\mathfrak{F}(G))^{\phi} \ni \pi(\mu)^{\phi} \longrightarrow \mu^f \in \mathfrak{H}_f \tag{19}$$

is an isomorphism from the subspace $\pi(\mathfrak{F}(G))^\phi$ of $\mathfrak{U}^\phi$ into $\mathfrak{H}_f$. Hence by natural extension we get an isomorphism of Hilbert spaces from $\mathfrak{H}^\phi$ onto $\mathfrak{H}_f$. Denote it again by Γ.

Let us transform through Γ the representations $\widetilde{U}_S, \widetilde{V}_S$ and $\widetilde{L}_\pi(g), \widetilde{R}_\pi(g)$ from the space $\mathfrak{H}^\phi$ onto the space $\mathfrak{H}_f$. The following is one of our main results in Section 1.

Theorem 1.5.4. *Let π be a UR of G with a faithful normal finite trace t on $\mathfrak{U} = \pi(G)''$. Normalize t as $t(I) = 1$ and put $\phi = \phi_t$, $f = \phi \circ \pi \in K_1(G)$.*

(i) *The UR $\widetilde{L}_\pi(g)$ (resp. $\widetilde{R}_\pi(g)$) of G on the space $\mathfrak{H}^\phi$ is equivalent to the UR π_f (resp. ρ_f) on the space $\mathfrak{H}_f$ through Γ: for $g \in G$,*

$$
\begin{aligned}
\Gamma \cdot \widetilde{L}_\pi(g) \cdot \Gamma^{-1} &= \Gamma \cdot \widetilde{U}_{\pi(g)} \cdot \Gamma^{-1} = \pi_f(g), \\
\Gamma \cdot \widetilde{R}_\pi(g) \cdot \Gamma^{-1} &= \Gamma \cdot \widetilde{V}_{\pi(g)^*} \cdot \Gamma^{-1} = \rho_f(g).
\end{aligned}
\tag{20}
$$

The UR $\widetilde{L}_\pi$ of G generates the von Neumann algebra $\widetilde{\mathfrak{U}}$ isomorphic to $\mathfrak{U}_f = \pi_f(G)''$. Similarly for the UR $\widetilde{R}_\pi$ and $\mathfrak{V}_f = \rho_f(G)''$.

(ii) *The map $\Phi : S \to \Gamma \cdot \widetilde{U}_S \cdot \Gamma^{-1}$ (resp. $\Phi' : S \to \Gamma \cdot \widetilde{V}_S \cdot \Gamma^{-1}$) is a quasi-isomorphism (resp. quasi-anti-isomorphism) from the von Neumann algebra $\mathfrak{U}$ onto the von Neumann algebra $\mathfrak{U}_f$ (resp. $\mathfrak{V}_f$) which has a trace-element and also is cyclic. Moreover*

$$
\Phi\bigl(\pi(g)\bigr) = \pi_f(g), \qquad \Phi'\bigl(\pi(g)^*\bigr) = \rho_f(g) \quad (g \in G).
$$

Proof. (i) Note that the subset $\pi(G)^\phi = \{\pi(h)^\phi; h \in G\}$ of $\mathfrak{U}^\phi$ is total in $\mathfrak{H}^\phi$. The UR $\widetilde{L}_\pi(g)$ is expressed for the element $\pi(h)^\phi$ as $\pi(h)^\phi \to \bigl(\pi(g)\pi(h)\bigr)^\phi = \pi(gh)^\phi$. Through the isomorphism Γ, this is written as $\delta_h^f \to (\delta_{gh})^f = \pi_f(g)(\delta_h^f)$. This means that $\Gamma \cdot \widetilde{L}_\pi(g) \cdot \Gamma^{-1} = \pi_f(g)$. Similarly for $\widetilde{R}_\pi(g)$ and $\pi_f(g)$.

From this, the statement for the isomorphism of $\widetilde{\mathfrak{U}}$ and $\mathfrak{U}_f$ is now clear.

(ii) Since $\mathfrak{U}_f$ is isomorphic to $\widetilde{\mathfrak{U}}$, it is sufficient for us to prove that the map $\Omega : \mathfrak{U} \ni S \to \widetilde{U}_S \in \widetilde{\mathfrak{U}}$ gives a quasi-isomorphism. To this, it is enough to get $\Omega(\mathfrak{U}) = \widetilde{\mathfrak{U}}$. For calculations here, we introduce a compact notation as ${}^\Omega S := \widetilde{U}_S$. We know that any projection $E \in \mathfrak{U}$ is mapped to a projection ${}^\Omega E \in \widetilde{\mathfrak{U}}$, and that $\|{}^\Omega S\| = \|\widetilde{U}_S\| = \|S\|$, by Lemma 1.5.2 (ii) and (iii). Note further that if $S_\alpha \to S$ strongly in $\mathfrak{U}$, then ${}^\Omega S_\alpha \to {}^\Omega S$ strongly in $\widetilde{\mathfrak{U}}$, by Lemma 1.5.5 below.

On the other hand, the set of unitary operators $\{\pi(g); g \in G\}$ (resp. $\{{}^\Omega\pi(g) = \widetilde{U}_{\pi(g)}; g \in G\}$) generates strongly the von Neumann algebra $\mathfrak{U}$ (resp. $\widetilde{\mathfrak{U}}$), by Lemma 1.5.3. By Lemma 1.5.7 below, we know that the image $\Omega(\mathfrak{U}) \subset \widetilde{\mathfrak{U}}$ is a von Neumann algebra. It contains the generating subset ${}^\Omega\pi(G)$ of $\widetilde{\mathfrak{U}}$, whence $\Omega(\mathfrak{U}) = \widetilde{\mathfrak{U}}$. □

Note that the theorem above shows that the Gelfand-Raikov representation π_f, which has a trace-element and is cyclic, is *standard* among URs corresponding to the same $f \in K_1(G)$ (cf. Remark 1.5.1 below).

Lemma 1.5.5. *Suppose that $S_\alpha \to S$ strongly in $\mathfrak{U}$. Then, ${}^{\Omega}S_\alpha \to {}^{\Omega}S$ strongly in $\widetilde{\mathfrak{U}}$.*

Proof. First note that $\{S_\alpha\}$ is bounded. For $v_1, v_2 \in \mathfrak{H}$,

$$|\langle (S_\alpha - S)^*(S_\alpha - S)v_1, v_2\rangle| \ \le \ \|(S_\alpha - S)v_1\| \, \|(S_\alpha - S)v_2\| \ \to \ 0,$$

whence $(S_\alpha - S)^*(S_\alpha - S) \to 0$ weakly and is bounded in $\mathfrak{U}$. Therefore we have $T^*(S_\alpha - S)^*(S_\alpha - S)T \to 0$ weakly. Thus we get $\phi\big(T^*(S_\alpha - S)^*(S_\alpha - S)T\big) \to 0$. On the other hand, $\|({}^{\Omega}S_\alpha - {}^{\Omega}S)T^\phi\|_\phi^2 = \phi(T^*(S_\alpha - S)^*(S_\alpha - S)T)$. Hence $\|({}^{\Omega}S_\alpha - {}^{\Omega}S)T^\phi\|_\phi \to 0$. Note that $\{{}^{\Omega}S_\alpha\}$ is bounded and that $\{T^\phi\}$ is dense in $\mathfrak{H}^\phi$, we see that ${}^{\Omega}S_\alpha \to {}^{\Omega}S$ strongly. □

Lemma 1.5.6 ([Dix1, I.4.1, Lemma 4]). *Let $\mathcal{A}$ be a $*$-algebra of operators in $\mathfrak{H}$ containing the identity operator $I_{\mathfrak{H}}$ on $\mathfrak{H}$. Every positive linear form ϕ on $\mathcal{A}$ defines a Hilbert space $\mathfrak{K}$, a linear mapping Γ of $\mathcal{A}$ onto a dense linear subspace of $\mathfrak{K}$, and a norm-decreasing homomorphism Φ of $\mathcal{A}$ into $\mathfrak{B}(\mathfrak{K})$, such that, if we put $x = \Gamma(I_{\mathfrak{H}}) \in \mathfrak{K}$, we have $\Gamma(T) = \Phi(T)x$ and $\phi(T) = \langle \Phi(T)x, x\rangle$ for each $T \in \mathcal{A}$. Furthermore, $\Phi(I_{\mathfrak{H}}) = I_{\mathfrak{K}}$. If ϕ is faithful, Φ is an isomorphism of $\mathcal{A}$ onto $\Phi(\mathcal{A})$, and x is separating for $\Phi(\mathcal{A})$.*

Definition 1.5.1. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras. A linear mapping Φ of $\mathcal{A}$ into $\mathcal{B}$ is said to be *positive* if $\Phi(\mathcal{A}^+) \subset \mathcal{B}^+$. We say that Φ is *normal positive* if, further, for every increasing filtering set $\mathcal{F} \subset \mathcal{A}$ with supremum $T \in \mathcal{A}^+$, $\Phi(\mathcal{F}) \subset \mathcal{B}$ has a supremum $\Phi(T)$.

Lemma 1.5.7 ([Dix1, I.4.3, Proposition 1]). *Let $\mathcal{A}$ be a von Neumann algebra, ϕ a normal positive linear form on $\mathcal{A}$, and Φ the canonical homomorphism defined above by ϕ. Then, Φ is normal and $\Phi(\mathcal{A})$ is a von Neumann algebra.*

Remark 1.5.1. Let G be a compact group and π a multiple of an irreducible UR δ with multiplicity m, $1 \le m \le \infty$. Then, t corresponds to the normalized character $\chi_\delta(g)/\dim\delta$. The representation π has a trace-element if and only if $m \ge \dim\delta$, and it is cyclic if and only if $m \le \dim\delta$. The representation $\widetilde{L}_\pi$ ($\cong \pi_f$) on $\mathfrak{U}^\phi = \mathfrak{H}^\phi$ is equivalent to $(\dim\delta)$-multiple of δ acting on the space of matrix elements of δ. Therefore, in this case, the transition from π to $\widetilde{L}_\pi$ is nothing but an adjustment of multiplicity (from m to $\dim\delta$), to have a trace-element and at the same time to be cyclic.

Remark 1.5.2. The von Neumann algebra $\mathfrak{U} = \pi(G)''$ with a faithful trace t is a typical example of Hilbert algebras in [Dix1, I.5], when it is equipped with the innner product $\langle\cdot,\cdot\rangle_\phi$ with $\phi = \phi_t$. However, in our discussions, the detailed result in [Dix1, I.6.2] on traces of Hilbert algebras is not necessary.

1.6. Extremal positive definite class functions as characters

By the discussions until now, we see that the $*$-algebra $\mathfrak{F}(G)$ plays a decisive role. Therefore we apply the definition of quasi-equivalence in [Dix2, V.5.3.2] for the $*$-algebra $A = \mathfrak{F}(G)$.

Definition 1.6.1. Let π_1 and π_2 be two URs of G. Put $\mathfrak{U}_i = \pi_i(G)'' = \pi_i(A)''\ (i = 1, 2)$. Then, π_1 and π_2 are said to be *quasi-equivalent* if there exists an isomorphism Φ (for the $*$-algebra structures) from $\mathfrak{U}_1$ onto $\mathfrak{U}_2$ such that $\Phi(\pi_1(a)) = \pi_2(a)$ for $a \in A = \mathfrak{F}(G)$, or equivalently $\Phi(\pi_1(g)) = \pi_2(g)$ for $g \in G$.

With this notion of quasi-equivalence, we see easily from Theorem 1.5.4 the following.

Theorem 1.6.1. *Let the assumptions and the notations be as in Theorem* 1.5.4. *Then, the unitary representation $\widetilde{L}_\pi$ of G is equivalent to π_f, and is quasi-equivalent to the original π. So π is quasi-equivalent to π_f.*

According to [Dix2, 5.3], we know that a UR quasi-equivalent to a factor representation is also factorial, and that a factor representation is quasi-equivalent to a subrepresentation on any non-zero invariant closed subspace, and so quasi-equivalent to a cyclic one. Our second main theorem in Section 1 is given as follows.

Theorem 1.6.2. *For a Hausdorff topological group G, let* $\mathrm{URff}(G)$ *be the set of all quasi-equivalence classes of continuous URs of G, factorial of finite type. Then there exists a canonical bijection between* $\mathrm{URff}(G)$ *and $E(G)$ through* (21) *below.*

Let π be a UR of G, factorial of finite type, and t the unique faithful finite normal trace on $\mathfrak{U}^+$ normalized as $t(I) = 1$, where $\mathfrak{U} = \pi(G)''$. We put

$$f(g) = \phi(\pi(g)) \quad (g \in G), \tag{21}$$

with $\phi = \phi_t$ the linear extension of t to $\mathfrak{U}$. The quasi-equivalence calss $[\pi] \in \mathrm{URff}(G)$ of π corresponds bijectively to $f \in E(G)$. In this connection, every element f in $E(G)$ is called a *character* of G of finite type. For an $f \in E(G)$, the Gelfand-Raikov representation π_f is a standard representative of the quasi-equivalent class in $\mathrm{URff}(G)$ having the character f, and π_f has a trace-element which is also a cyclic vector.

Note 1.6.1. In [Dix2, 17.3], the above canonical bijective correspondence is asserted under the condition that G is locally compact and unimodular (cf. also [Gode]).

2. Topologies on the spaces of continuous positive definite functions $\mathcal{P}(G)$ and of such class functions $K_{\leq 1}(G)$, $K_1(G)$

2.1. Weak topologies and compact uniform topology

In this section, we study topologies in $\mathcal{P}(G) \supset K(G) \supset K_{\leq 1}(G) \supset K_1(G)$, several weak topologies and also compact uniform topology. Our final aim is to establish Theorems 2.6.1 and 2.6.2.

First take the weak topology $\sigma(C_b(G), \mathfrak{M}_b(G))$ and its restrictions onto $K(G) \subset \mathcal{P}(G) \subset C_b(G)$. For $\mu \in \mathfrak{M}_b(G)$, put $D_\mu = \{z \in \boldsymbol{C}; |z| \leq \|\mu\|\}$ and then put $\mathcal{D} = \prod_{\mu \in \mathfrak{M}_b(G)} D_\mu$, which is compact with the product topology. We can define a map $\Psi : K_{\leq 1}(G) \to \mathcal{D}$ as $\Psi(f) = (\mu(f))_{\mu \in \mathfrak{M}_b(G)}$, because $|\mu(f)| \leq f(e)\|\mu\| \leq \|\mu\|$. Through the map Ψ, the set $K_{\leq 1}(G)$ with the weak topology is homeomorphically imbedded into the compact set $\mathcal{D}$.

We ask if the image $\Psi(K_{\leq 1}(G))$ is closed or equivalently if it is compact.

For a boundary point $b = (b(\mu))$ of it, we see that the map $\mu \to b(\mu)$ is a linear map on $\mathfrak{M}_b(G)$, continuous in the norm, since $|b(\mu)| \leq \|\mu\|$. Furthermore it is positive and invariant in the sense that $b(\mu^* {*} \mu) \geq 0$, $b(\mu_1 {*} \mu_2) = b(\mu_2 {*} \mu_1)$, if the convolution product here is well-defined. For $\mu = \delta_g \in \mathfrak{F}(G)$, we get a positive definite, invariant function $g \mapsto b(\delta_g)$ on G. The above question is devided into two questions as follows:

(a) *Is $g \mapsto b(\delta_g)$ continuous on G ?*

(b) *For $\mu \in \mathfrak{M}_b(G)$, is $b(\mu)$ given by an integral as $b(\mu) = \int_G b(\delta_g)\, d\mu(g)$?*

We also introduce other weak topologies in certain restricted cases, and compair them with the compact uniform topology in $\mathcal{P}(G)$ and $K(G) \supset K_{\leq 1}(G)$.

2.2. Case of locally compact groups and its generalization

From now on we restrict ourselves to the case of locally compact groups G and also to the case of limit groups $G = \lim_{n\to\infty} G_n$ of countable LCG inductive systems $(G_n)_{n\geq 1}$. In these cases we introduce another weak topology more suitable to the situation, and compair it later with the compact uniform topology in the bounded subsets $\mathcal{P}_{\leq M}(G)$ and $K_{\leq M}(G)$ defined by $f(e) \leq M$.

Let G be as above. Denote by $\mathcal{C}$ the family of all compact subsets of G, and by $\boldsymbol{B}_{\mathcal{C}}$ the σ-ring of subsets of G generated by $\mathcal{C}$. Note that every $B \in \boldsymbol{B}_{\mathcal{C}}$ is covered by a σ-compact set. Let $\mathfrak{M}_{\mathcal{C}}(G)$ be the set of all bounded complex measures on $(G, \boldsymbol{B}_{\mathcal{C}})$. When G is locally compact, its Haar measures are regular measures defined on $\boldsymbol{B}_{\mathcal{C}}$ but not on the whole of $\mathfrak{B}(G)$ when G is not σ-compact (cf. Remark 2.2.2).

For a $\mu \in \mathfrak{M}_{\mathcal{C}}(G)$, there exists a (not necessarily unique) set $A \in \boldsymbol{B}_{\mathcal{C}}$ such that $\mu(B \setminus A) = 0$ for any $B \in \boldsymbol{B}_{\mathcal{C}}$. In this case, we say that μ *is supported by* A. For every $\varphi \in C_b(G)$, its restriction on a measurable subset $A \in \boldsymbol{B}_{\mathcal{C}}$ is $\boldsymbol{B}_{\mathcal{C}}$-measurable on A. For a measure $\mu \in \mathfrak{M}_{\mathcal{C}}(G)$, take a measurable set $A \in \boldsymbol{B}_{\mathcal{C}}$ supporting μ. Then the integral $\int_A \varphi(g)\, d\mu(g)$ is independent of the choice of A, and is denoted simply by $\int_G \varphi(g)\, d\mu(g)$.

We have a natural pairing

$$\mathfrak{M}_{\mathcal{C}}(G) \times C_b(G) \ni (\mu, \varphi) \longmapsto \mu(\varphi) = \varphi(\mu) := \int_G \varphi(g)\, d\mu(g),$$

and so get a weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ which we restrict on $\mathcal{P}(G)$ and $K(G)$. Replacing $\mathfrak{M}_b(G)$ by $\mathfrak{M}_{\mathcal{C}}(G)$, we have similar assertions as Lemmas 1.1.1, 1.1.2, 1.1.3 in Section 1, and Lemma 2.2.1 below.

Lemma 2.2.1. *Let G be locally compact. The space of measures on G*

corresponding to $L^1(G)$ *is everywhere dense in* $\mathfrak{M}_{\mathcal{C}}(G)$ *in the topology* $\sigma(\mathfrak{M}_{\mathcal{C}}(G), C_b(G))$.

Proof. Take an approximate identity given as follows. For a compact neighbourhood V of e, let $\psi_V(g) \geq 0$ be a continuous function with support contained in V such that $\int_V \psi_V(g)\, dg = 1$. Then $\psi_V(g)dg$ converges in $\sigma(\mathfrak{M}_{\mathcal{C}}(G), C_b(G))$ to the delta measure δ_e supported by $\{e\}$. For a fixed $g_0 \in G$, put $L(g_0)\psi(g) := \psi({g_0}^{-1}g)$ $(g \in G)$. Then the net $\{L(g_0)\psi_V dg\}$ converges to δ_{g_0}. Hence the weak closure of $L^1(G)$ contains $\mathfrak{F}(G)$.

On the other hand, we can prove as for Lemma 1.1.1 that $\mathfrak{F}(G)$ is everywhere dense in $\mathfrak{M}_{\mathcal{C}}(G)$. □

According to the restriction of μ from $\mathfrak{M}_b(G)$ to $\mathfrak{M}_{\mathcal{C}}(G)$, we modify the homeomorphic imbedding Ψ into $\mathcal{D}$ as follows. Put $\Psi' : K_{\leq 1}(G) \to \mathcal{D}'$ as

$$\mathcal{D}' = \prod_{\mu \in \mathfrak{M}_{\mathcal{C}}(G)} D_\mu, \qquad \Psi'(f) = (\mu(f))_{\mu \in \mathfrak{M}_{\mathcal{C}}(G)}.$$

We take a boundary point $b = (b(\mu))$ of $\Psi'(K_{\leq 1}(G))$ in $\mathcal{D}'$ similarly as for $\mathcal{D}$ and Ψ.

Remark 2.2.1. Let $\mathfrak{M}^0_{\mathcal{C}}(G)$ be the set of bounded complex measures μ on $(G, \boldsymbol{B}_{\mathcal{C}})$ such that μ is regular [Halm, Chapter X] in case G is locally compact, and such that $\mu|_{G_n}$ is regular for any n in case $G = \lim_{n\to\infty} G_n$.

If G is locally compact, it is known that the space of compactly-supported continuous functions $C_c(G)$ can separate two elements of $\mathfrak{M}^0_{\mathcal{C}}(G)$ and so the weak topology $\sigma(\mathfrak{M}^0_{\mathcal{C}}(G), C_b(G))$ is Hausdorff, and that the convolution product in $\mathfrak{M}^0_{\mathcal{C}}(G)$ can be naturally defined (cf. [Halm, §51, Theorem E]).

Consider the case of $G = \lim_{n\to\infty} G_n$. Note that any compact subset is contained in some G_n. Moreover any $\varphi_n \in C_c(G_n)$ can be extended to a $\varphi_{n+1} \in C_c(G_{n+1})$ in such a way that $\|\varphi_n\|_\infty = \|\varphi_{n+1}\|_\infty$, and accordingly we get $\varphi = \lim_{k\to\infty} \varphi_k \in C_b(G)$ extending φ_n. This means that $C_b(G)$ can separate two elements of $\mathfrak{M}^0_{\mathcal{C}}(G)$.

For $\mu, \nu \in \mathfrak{M}^0_{\mathcal{C}}(G)$, we can choose an $A \in \boldsymbol{B}_{\mathcal{C}}$ supporting both of μ and ν which is a union of countable number of compact sets $C_i, i \geq 1$. Then the convolution $\mu * \nu$, supported by $AA = \cup_{i,j\geq 1} C_iC_j$, can be defined as follows: Put $D_i = C_i \setminus \cup_{1\leq k<i} C_k \subset C_i$ and for $\varphi \in C_b(G)$

$$\int_G \varphi(g)\, d(\mu * \nu)(g) := \sum_{i,j\geq 1} \iint_{D_i \times D_j} \varphi(g'g)\, d\mu(g')\, d\nu(g).$$

Choose a $G_m \supset D_i, D_j$, then the integral $\iint_{D_i\times D_j}$ can be considered on $G_m \times G_m$ because $\varphi(g'g)$ is mesurable in $(g', g) \in G_m \times G_m$ with respect to $(\mu|_{G_m}) \times (\nu|_{G_m})$, thanks to the regularity of $\mu|_{G_m}, \nu|_{G_m}$. Moreover this integral is equal to the repeated integral $\int_{D_i} \int_{D_j}$.

Remark 2.2.2. In the case where G is not σ-compact, the whole space G does not belong to $\boldsymbol{B}_{\mathcal{C}}$, and a continuous function on G is not necessarily $\boldsymbol{B}_{\mathcal{C}}$-measurable.

Let G be locally compact. A left invariant Haar measure dg on $(G, \boldsymbol{B}_{\mathcal{C}})$ is σ-finite if and only if G is σ-compact. When G is not σ-compact, to give the dual space $L^1(G)^*$ of $L^1(G)$, which contains $C_b(G)$ naturally, we should be careful. We call a function f $\boldsymbol{B}_{\mathcal{C}}$*-locally-measurable* if $f|_B$ is measurable for any $B \in \boldsymbol{B}_{\mathcal{C}}$.

For $1 \leq p < \infty$, $\mathcal{L}^p(G)$ is defined as the space of functions such that f is measurable (and so $S_f := \{g \in G; f(x) \neq 0\} \in \boldsymbol{B}_{\mathcal{C}}$), and $\|f\|_p := \left(\int_{S_f} |f(g))|^p \, dg\right)^{1/p} < \infty$. The space $L^p(G)$ is its quotient space under the equivalence relation $\sim$, where $f_1 \sim f_2$ if $f_1 = f_2$ almost everywhere (a.e.).

For $p = \infty$, let $\widetilde{\mathcal{L}}^\infty(G)$ be the space of all $\boldsymbol{B}_{\mathcal{C}}$-locally-measurable functions for which $\|f\|_\infty := \sup\{\, \|f|_B\|_\infty \,;\, B \in \boldsymbol{B}_{\mathcal{C}}\} < \infty$, where $\|f|_B\|_\infty$ denotes the L^∞-norm of $f|_B$ on a set B. The equivalence relation needed here is $f_1 \approx f_2$, rougher than $f_1 \sim f_2$ (i.e., $f_1 \sim f_2$ implies $f_1 \approx f_2$), given as follows. For two $\boldsymbol{B}_{\mathcal{C}}$-locally-measurable functions f_1 and f_2, if $f_1|_B = f_2|_B$ (a.e.) for any $B \in \boldsymbol{B}_{\mathcal{C}}$, we say that f_1 is equal to f_2 $\boldsymbol{B}_{\mathcal{C}}$*-locally-almost-everywhere*, and denote this equivalence relation by $f_1 \approx f_2$. The quotient space $\widetilde{\mathcal{L}}^\infty(G)/\approx$ is denoted by $\widetilde{L}^\infty(G)$, and it is naturally isomorphic to the dual space $L^1(G)^*$.

In Dixmier's book [Dix2, 17], the dual space $L^1(G)^*$ of $L^1(G)$ is denoted simply by $L^\infty(G)$ eventhough G is not assumed to be σ-compact. Therefore $L^\infty(G)$ there is equal to $\widetilde{L}^\infty(G)$ above.

2.3. Problem in the case of a locally compact G

In this case, a kind of affirmative answers to the problems (a) and (b) were obtained already in [GR] and is exposed in [Dix2] as follows.

Proposition 2.3.1 ([GR, Theorem 4], [Dix2, 13.4.5 (i)])**.** *Let $f \in L^\infty(G)$ and put $\omega(\psi) = \int_G f(g)\, \psi(g) dg$ ($\psi \in L^1(G)$) with a left-invariant Haar measure dg, the continuous linear form defined by f. Then ω is positive if and only if f coincides with a continuous positive definite function $\boldsymbol{B}_{\mathcal{C}}$-locally-almost-everywhere* (*cf. Remark* 2.2.2)*.*

Proposition 2.3.2 ([GR, Theorem 5], [Dix2, 13.5.2])**.** *For a locally compact group G, the weak topology $\sigma(L^\infty(G), L^1(G))$ on the set $\mathcal{P}_1(G) = \{f \in \mathcal{P}(G); f(e) = 1\}$ coincides with the compact uniform topology.*

However we consider here another weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ in $K_{\leq 1}(G)$ stronger than $\sigma(L^\infty(G), L^1(G))$. A boundary point b of $\Psi'(K_{\leq 1}(G)) \subset \mathcal{D}'$ gives a positive linear form on $L^1(G)$, continuous in the norm. Then, by Proposition 2.3.1, the latter coincides with a linear form given by a continuous positive definite function f^b on G in such a way that $b(\mu) = \int_G f^b(g)\, \psi(g) dg$ for $\mu = \psi(g)dg$ with $\psi \in L^1(G)$. Actually f^b is an element of $K_{\leq 1}(G)$.

In turn, f^b gives a positive invariant linear form $\widetilde{f^b} : \mu \mapsto f^b(\mu) := \int_G f^b(g)\, d\mu(g)$ on $\mathfrak{M}_{\mathcal{C}}(G)$, which coincides with $\mu \mapsto b(\mu)$ on $L^1(G)(\hookrightarrow \mathfrak{M}_{\mathcal{C}}(G))$, and is continuous in the weak topology $\sigma(\mathfrak{M}_{\mathcal{C}}(G), C_b(G))$. By Lemma 2.2.1 we see that the linear form $\mu \mapsto b(\mu)$ on $L^1(G)$ has a *unique* extension to a weakly continuous one $\widetilde{f^b}$ on $\mathfrak{M}_{\mathcal{C}}(G)$.

We ask *if the linear form $\mu \mapsto b(\mu)$ coincides with $\widetilde{f^b}$ on the whole of $\mathfrak{M}_{\mathcal{C}}(G)$, and especially if there holds that $f^b(g) = b(\delta_g)$ $(g \in G)$.*

2.4. Problem for $G = \lim_{n\to\infty} G_n$ of a countable LCG inductive system

Let $(G_n)_{n\geq 1}$ be a countable LCG inductive system and $G = \lim_{n\to\infty} G_n$ its limit group equipped with the inductive limit topology τ_{ind}. Here it is assumed by definition that G_n's are locally compact and each homomorphism $G_n \to G_{n+1}$ is homeomorphic into. Note that, by [TSH, Theorem 5.7], G has sufficiently many continuous positive definite functions. Note further that the space of measures $\mathfrak{M}_{\mathcal{C}}(G_n)$ on G_n is canonically imbedded into $\mathfrak{M}_{\mathcal{C}}(G)$.

Take a boundary element b of $\Psi'(K_{\leq 1}(G)) \subset \mathcal{D}'$. Then, discussing only on G_n, we get for each n a continuous positive definite class function $f_n \in K_{\leq 1}(G_n)$ such that $b(\mu) = \int_{G_n} f_n(g_n)\,\psi_n(g_n)dg_n$ for $d\mu(g_n) = \psi_n(g_n)dg_n$ ($\psi_n \in L^1(G_n; dg_n)$), where dg_n $(g_n \in G_n)$ denotes a left invariant Haar measure on G_n.

We ask *if the consistency condition $f_{n+1}|_{G_n} = f_n$ holds for the system of functions $\{f_n(g_n)\}_{n\geq 1}$.*

2.5. Topologies in $\mathcal{P}(G)$ and $K(G)$ for a locally compact group G

Proposition 2.5.1. *Let G be locally compact, and $M > 0$.*

(i) *On $\mathcal{P}_{\leq M}(G) := \{f \in \mathcal{P}(G); \|f\| = f(e) \leq M\}$, the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ is weaker than or equivalent to the compact uniform topology τ_{cu}.*

(ii) *On $\mathcal{P}_{\leq M}(G)$, the weak topology $\sigma(C_b(G), \boldsymbol{C}\delta_e + L^1(G))$ is stronger than or equivalent to τ_{cu}.*

Proof. (i) Take a neighbourhood of 0 in the topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ as

$$U((\mu_i)_{1\leq i\leq N}; \varepsilon) = \{f \in C_b(G); |\mu_i(f)| < \varepsilon\,(1 \leq i \leq N)\},$$

where $\mu_1, \mu_2, \ldots, \mu_N \in \mathfrak{M}_{\mathcal{C}}(G)$ and $\varepsilon > 0$. Then, for any $\varepsilon' > 0$, there exists a compact set C such that $|\mu_i|(B \setminus C) < \varepsilon'$ for any i and $B \in \boldsymbol{B}_{\mathcal{C}}$, where $|\mu_i| = |\Re(\mu_i)| + |\Im(\mu_i)|$. Take a $B \in \boldsymbol{B}_{\mathcal{C}}$ which supports any of μ_i. Suppose that $\max_{g\in C} |f(g)| < \varepsilon'$, then

$$|\mu_i(f)| \leq \left|\int_{B\cap C} f(g)\,d\mu_i(g)\right| + M \cdot |\mu_i|(B \setminus C) \leq \varepsilon'(|\mu_i|(B) + M).$$

If $\varepsilon'(|\mu_i|(B) + M) < \varepsilon$, then f belongs to the neighbourhood $U((\mu_i)_{1\leq i\leq N}; \varepsilon)$.

(ii) (After the proof of [Dix2, 13.5.2]) Let us prove that, for a fixed $f_0 \in \mathcal{P}_{\leq M}(G)$ and a compact set $C \subset G$, there exists a neighbourhood $U((\mu_i)_{i=1,2}; \varepsilon')$ of 0 in $\sigma(C_b(G), \boldsymbol{C}\delta_e + L^1(G))$ with $\mu_1 = \delta_e$ and $\mu_2 = \psi(g)dg$ with a $\psi \in L^1(G)$, such that $|f(g) - f_0(g)| < \varepsilon\,(g \in C)$ if $f - f_0 \in U((\mu_i)_{i=1,2}; \varepsilon')$.

First there exists a compact neighbourhood V of $e \in G$ such that

$$|f_0(e) - f_0(g)| < \varepsilon' \quad (g \in V).$$

Let $\mu_1 = \delta_e$ and $\mu_2 = a^{-1}\chi_V(g)dg$, where χ_V is the characteristic function of V and $a = \int_V dg$. Let $\mathcal{U}$ be a neighbourhood of f_0 in $\sigma(C_b(G), \boldsymbol{C}\delta_e + L^1(G))$ defined by the following conditions on $f \in \mathcal{P}_{\leq M}(G)$:

$$|\mu_1(f - f_0)| = |f(e) - f_0(e)| < \varepsilon',$$
$$|\mu_2(f - f_0)| = a^{-1}\left|\int_V (f(g) - f_0(g))\, dg\right| < \varepsilon'.$$

Then, for $f \in \mathcal{U}$, we have $a^{-1}\left|\int_V (f(e) - f(g))\, dg\right| < 3\varepsilon'$.

On the other hand, take an $f \in \mathcal{U}$. Then, for any $g \in G$,

$$\begin{aligned} &|(\mu_2 * f)(g) - f(g)| \\ &\quad = a^{-1}\left|\int_V (f(h^{-1}g) - f(g))\, dh\right| \leq a^{-1}\int_V |f(h^{-1}g) - f(g)|\, dh. \end{aligned}$$

By Kreĭn's inequality, we have $|f(h^{-1}g) - f(g)|^2 \leq 2f(e)\{f(e) - \Re f(h)\}$, whence the right hand side is majorized by

$$\begin{aligned} &a^{-1}\sqrt{2M}\int_V |f(e) - \Re f(h)|^{1/2}\, dh \\ &\leq a^{-1}\sqrt{2M}\left(\int_V |f(e) - \Re f(h)|\, dh\right)^{1/2}\left(\int_V dh\right)^{1/2} \\ &\leq a^{-1}\sqrt{2M}\,\sqrt{3a\varepsilon'}\,\sqrt{a} = \sqrt{6M\varepsilon'}, \end{aligned}$$

because $0 \leq f(e) - \Re f(h) = \Re(f(e) - f(h))$. Hence, $|(\mu_2 * f)(g) - f(g)| \leq \sqrt{6M\varepsilon'}$.

Here we apply [Dix2, 13.5.1] which we quote below as Lemma 2.5.2 for the convenience of the reader. Then, there exists a neighbourhood $\mathcal{U}'$ of f_0 in $\mathcal{P}_{\leq M}(G)$ in the weak topology $\sigma(L^\infty(G), L^1(G))$ such that $f \in \mathcal{U}'$ gives for $\mu_2 = a^{-1}\chi_V(g)\, dg$,

$$|\mu_2 * f(g) - \mu_2 * f_0(g)| \leq \varepsilon' \qquad (g \in C).$$

Thus we get for $f \in \mathcal{U} \cap \mathcal{U}'$,

$$|f(g) - f_0(g)| \leq \varepsilon' + 2\sqrt{6M\varepsilon'} \qquad (g \in C).$$

□

Lemma 2.5.2 (from [Dix2, 13.5.1])**.** *Let B be a bounded subset of $L^\infty(G)$, and $\psi \in L^1(G)$. If $f \in B$ converges to $f_0 \in B$ in the weak topology $\sigma(L^\infty(G), L^1(G))$, then $\psi * f$ converges to $\psi * f_0$ in the compact uniform topology τ_{cu}.*

It follows immediately from Proposition 2.5.1 the following variant of Proposition 2.3.2.

Theorem 2.5.3. *Let G be locally compact. On every bounded set of $\mathcal{P}(G)$, the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ is equivalent to the compact uniform topology τ_{cu}.*

We slitely generalize the following result (Theorem 2.5.4) in [Dix2] and get Theorem 2.5.5 below.

Theorem 2.5.4 ([Dix2, 17.3.5]). *Let G be locally compact and unimodular.*

(i) *The convex set $K_{\leq 1}(G)$ is compact in the weak topology $\sigma(L^\infty(G), L^1(G))$.*
(ii) *The extremal points of $K_{\leq 1}(G)$ are 0 and characters of finite type equal to 1 at e.*
(iii) *$K_{\leq 1}(G)$ is the weakly closed convex hull of 0 and the set of characters $E(G)$ of finite type normalized as $f(e) = 1$.*

Theorem 2.5.5. *Let G be locally compact.*

(i) *The convex sets $K_{\leq 1}(G)$ and $K_1(G)$ are compact in the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$.*
(ii) *The set of extremal points of $K_{\leq 1}(G)$ is $\{0\} \cup E(G)$.*
(iii) *$K_{\leq 1}(G)$ (resp. $K_1(G)$) is the closed convex hull of $\{0\} \cup E(G)$ (resp. $E(G)$) in the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$.*

Proof. (i) Imbed $K_{\leq 1}(G)$ homeomorphically into $\mathcal{D}'$ by $\Psi' : f \mapsto (\mu(f))$ $(\mu \in \mathfrak{M}_{\mathcal{C}}(G))$. We denote $\mu(f)$ also by $f(\mu)$. Take a boundary point $b = (b(\mu))$ of $\mathrm{Im}(\Psi') \subset \mathcal{D}'$. Then there exists a net $f_\alpha \in K_{\leq 1}(G)$ for which $\Psi'(f_\alpha) = (f_\alpha(\mu))$ converges to b, or equivalently, for any μ, $\lim_\alpha f_\alpha(\mu) = b(\mu)$.

Since $|b(\mu)| \leq \|\mu\|$, the map $L^1(G) \ni \psi \to F(\psi) := b(\psi(g)dg)$ gives an invariant positive linear form on $L^1(G)$ such that $|F(\psi)| \leq \|\psi\|$. Then by Proposition 2.3.1 it is given by an $f^b \in K_{\leq 1}(G)$ as $F(\psi) = \int_G f^b(g)\,\psi(g)dg = f^b(\mu)$ with $\mu = \psi(g)dg$. In $K_{\leq 1}(G) \subset C_b(G)$, f_α converges to f^b in the topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$, and therefore f_α converges to f^b in $K_{\leq 1}(G) \subset \mathcal{P}_{\leq 1}(G)$ in the topology $\sigma(C_b(G), \boldsymbol{C}\delta_e + L^1(G))$. Hence, by (ii) of Proposition 2.5.1, f_α converges to f^b uniformly on every compact.

Take a $\mu \in \mathfrak{M}_{\mathcal{C}}(G)$. Then there exists an $A \in \boldsymbol{B}_{\mathcal{C}}$ such that $|\mu|(B \setminus A) = 0$ for any $B \in \boldsymbol{B}_{\mathcal{C}}$. Moreover, for an $\varepsilon > 0$, there exists a compact set C_ε such that $|\mu|(A \setminus C_\varepsilon) < \varepsilon$. Since $f_\alpha, f^b \in K_{\leq 1}(G)$, we have $|f_\alpha(g)| \leq f_\alpha(e) \leq 1$ and

$$\left| \int_{A \setminus C_\varepsilon} f_\alpha(g)\, d\mu(g) \right| \leq |\mu|(A \setminus C_\varepsilon) < \varepsilon,$$

and similarly for f^b. Hence $|f^b(\mu) - f_\alpha(\mu)|$ is majorized by

$$\left| \int_{A \cap C_\varepsilon} (f^b - f_\alpha)(g)\, d\mu(g) \right| + \left| \int_{A \setminus C_\varepsilon} f_\alpha(g)\, d\mu(g) \right| + \left| \int_{A \setminus C_\varepsilon} f^b(g)\, d\mu(g) \right|$$
$$\leq\ |\mu|(A) \times \sup_{g \in C_\varepsilon} |f^b(g) - f_\alpha(g)| + 2\varepsilon.$$

Therefore we see that $\lim_\alpha f_\alpha(\mu) = f^b(\mu)$, whence $b(\mu) = f^b(\mu)$. This means that $b = (b(\mu)) = (f^b(\mu)) = \Psi'(f^b) \in \Psi'(K_{\leq 1}(G))$, and so $\Psi'(K_{\leq 1}(G))$ is compact in $\mathcal{D}'$. This proves the compactness of $K_{\leq 1}(G)$ and so the assertion (i).

We omit the proofs of the assertions (ii) and (iii). □

2.6. The case of the limit of a countable LCG inductive system

Let $G = \lim_{n\to\infty} G_n$ be the limit group of a countable LCG inductive system $(G_n)_{n\geq 1}$. Then $\mathfrak{M}_{\mathcal{C}}(G_n) \subset \mathfrak{M}_{\mathcal{C}}(G_{n+1}) \subset \cdots \subset \mathfrak{M}_{\mathcal{C}}(G)$. Take a boundary point $b = (b(\mu))_{\mu\in\mathfrak{M}_{\mathcal{C}}(G)}$ of $\Psi'(K_{\leq 1}(G))$ in $\mathcal{D}'$. Restricting the range of indices μ of $b(\mu)$ from the whole $\mathfrak{M}_{\mathcal{C}}(G)$ to the subspace $\mathfrak{M}_{\mathcal{C}}(G_n)$, we get a point $b_n = (b(\mu))_{\mu\in\mathfrak{M}_{\mathcal{C}}(G_n)}$ of $\prod_{\mu\in\mathfrak{M}_{\mathcal{C}}(G_n)} D_\mu$. Working for G_n and b_n as a case of a locally compact group, we can apply Theorem 2.5.5 or its proof for (i), and get a continuous positive definite class function $f_n \in K_{\leq 1}(G_n)$ on G_n for which $b_n = (f_n(\mu))_{\mu\in\mathfrak{M}_{\mathcal{C}}(G_n)}$ or

$$b_n(\mu) = b(\mu) = \int_{G_n} f_n(g_n)\, d\mu(g_n) \quad (\forall \mu \in \mathfrak{M}_{\mathcal{C}}(G_n)).$$

Comparing this expression for $\mathfrak{M}_{\mathcal{C}}(G_n)$ and that for $\mathfrak{M}_{\mathcal{C}}(G_{n+1})$, we see from the inclusion $\mathfrak{M}_{\mathcal{C}}(G_n) \subset \mathfrak{M}_{\mathcal{C}}(G_{n+1})$ that the consistency condition $f_{n+1}|_{G_n} = f_n$ holds, and we get a function $f = \lim_{n\to\infty} f_n$ on the whole G. With respect to the inductive limit topology τ_{ind} on G, f is continuous because $f_n = f|_{G_n}$ is continuous on G_n for each n, and then $f \in K_{\leq 1}(G)$. Thus $b = \Psi'(f)$. Hence we have proved that $\Psi'(K_{\leq 1}(G))$ is closed in $\mathcal{D}'$ and consequently that $K_{\leq 1}(G)$ is compact in the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$.

In this way we get one of our main results in this section as follows.

Theorem 2.6.1. *Let $G = \lim_{n\to\infty} G_n$ be the inductive limit of a countable LCG inductive system $(G_n)_{n\geq 1}$. Then the assertions* (i), (ii) *and* (iii) *of Theorem* 2.5.5 *hold for G too.*

Theorem 2.6.1 above answers affirmatively to the questions (a) and (b) in **2.1** in the case of the limit group G of a countable LCG inductive system.

Remark 2.6.1. The wreath product $G = \mathfrak{S}_\infty(T)$ of a compact group T with the infinite symmetric group $\mathfrak{S}_\infty$ is considered as the limit of a countable LCG inductive system of compact groups $G_n = \mathfrak{S}_n(T) \cong T^n \rtimes \mathfrak{S}_n$, the wreath product of T with the n-th symmetric group $\mathfrak{S}_n$. The topological group (G, τ_{ind}) is σ-compact but not locally compact except when T is finite.

By Theorem 2.6.1, there holds for the compact convex set $K_1(G)$ and the set of its extremal points $E(G)$ the integral expression theorem of Choquet-Bishop-K. de Leeuw (Theorem 5.6 in [BL]), which will be applied in [HH2] succeeding [HH1].

For the topologies in the set of continuous positive definite functions $\mathcal{P}(G)$, we have another main theorem, a similar result as Theorem 2.5.3.

Theorem 2.6.2. *Let G be as in Theorem* 2.6.1. *On every bounded set of $\mathcal{P}(G)$, the weak topology $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$ is equivalent to the compact uniform topology τ_{cu}.*

Proof. Let us consider topologies on a bounded set $\mathcal{P}_{\leq M}(G)$. First note that any compact subset C of $G = \lim_{n\to\infty} G_n$ is contained in some G_n (cf. e.g., Proposition 6.5 in [HSTH]). Then we see from Theorem 2.5.3 applied to G_n that the uniform convergence of a net f_α to f on C comes from the convergence of $f_\alpha|_{G_n}$ to $f|_{G_n}$ in the weak topology $\sigma(C_b(G_n), \mathfrak{M}_{\mathcal{C}}(G_n))$. Thus τ_{cu} is weaker than or equivalent to $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$.

Conversely fix a $\mu \in \mathfrak{M}_{\mathcal{C}}(G)$. Take an $A \in \boldsymbol{B}_{\mathcal{C}}$ supporting it. Then, for any $\varepsilon > 0$, there exists a compact subset C_ε such that $|\mu|(A \setminus C_\varepsilon) < \varepsilon$, and we have

$$|\mu(f_\alpha - f)| \leq 2M\varepsilon + \int_{A\cap C_\varepsilon} |f_\alpha(g) - f(g)|\, d|\mu|(g).$$

Hence, if $|f_\alpha(g) - f(g)| \leq \varepsilon'$ on C_ε, then $|\mu(f_\alpha - f)| \leq 2M\varepsilon + |\mu|(A)\varepsilon'$. This evaluation proves that τ_{cu} is stronger than or equivalent to $\sigma(C_b(G), \mathfrak{M}_{\mathcal{C}}(G))$. □

Added in Proof. Very recently we found a counter example to Theorem 2.5.5. We thank Prof. J. Faraut for suggesting it to the first author. At the present moment, we should withdraw Theorem 2.5.5 and 2.6.1 and accordingly Theorem C in Introduction, and we hope that we can present correct versions of them in near future.

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