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On the squarefree and squarefull numbers

By

Huaning Liu and Wenpeng Zhang

Abstract

The main purpose of this paper is using the important works of Heath-Brown, Yoichi Motohashi, Masanori Katsurada and Kohji Matsumoto, and the properties of Dirichlet L-functions to study the number of squarefree primitive roots, squarefull primitive roots and squarefree quadratic residues modulo a prime p, and give three much sharper asymptotic formulae.

1. Introduction

Let p be an odd prime. For any integer n with $(p, n) = 1$, the smallest positive integer f such that $a^f \equiv 1 \bmod p$ is called the exponent of n modulo p. If $f = p - 1$, then n is called a primitive root mod p. Let $prim(x)$ denote the number of positive primitive roots modulo p not exceeding x. From [1] we have

$$prim(x) = \frac{\phi(p-1)}{p-1}\left(x + O\left(2^{\omega(p-1)} \cdot \sqrt{p}\log p\right)\right),$$

where $\phi(q)$ is the Euler function, $\omega(q)$ denotes the number of all distinct prime divisors of q and the O term is uniform in x and p.

An integer is called k-free ($k \geq 2$, integer) if it is not divisible by the k-powers of any prime. Also an integer q is called a k-full integer if it satisfies that $p|q$ if and only if $p^k|q$. The properties of k-free integer and k-full integer were studied by many authors. For example, let $Q_k(x)$ denote the number of k-free integers $\leq x$, L. Gegenbauer [2] gave the following estimate:

$$Q_k(x) = \frac{x}{\zeta(k)} + O\left(x^{1/k}\right),$$

where $\zeta(k)$ is the Riemann zeta- function.

Now we consider the number of squarefree (squarefull) primitive roots modulo p not exceeding x. From [1] we have the following two Propositions:

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Proposition 1.1. *The number of positive squarefree primitive roots modulo p not exceeding x equals*

$$\frac{\phi(p-1)}{p-1}\left(C_1x + O\left(2^{\omega(p-1)} \cdot p^{1/4} \cdot (\log p)^{1/2} \cdot x^{1/2}\right)\right),$$

where $C_1 = \prod_p \left(1 - 1/p^2\right)$.

Proposition 1.2. *The number of positive primitive roots modulo p not exceeding x which are squarefull numbers equals*

$$\frac{\phi(p-1)}{p-1}\left(C_2x^{1/2} + O\left(2^{\omega(p-1)} \cdot p^{1/6} \cdot (\log p)^{1/3} \cdot x^{1/3}\right)\right),$$

where $C_2 = 2\left(\sum_{\substack{q \text{ squarefree} \\ \left(\frac{q}{p}\right)=-1}} 1/q^{3/2}\right)\left(1 - \frac{1}{p}\right)$ *and* $\left(\frac{q}{p}\right)$ *is the Legendre's symbol.*

If congruence $x^2 \equiv n \bmod p$ has a solution we say that n is a quadratic residue modulo p. From [3] we have the following:

Proposition 1.3. *Let p be a prime,* $0 < a \le 1/128$ *and* $x > p^{1/4+b}$ *with* $b = b(a) > 0$. *Then the number of squarefree numbers not exceeding x which are quadratic residues modulo p equals*

$$\frac{3}{\pi^2}x + O\left(x/p^a\right).$$

The error terms in Proposition 1.1, Proposition 1.2 and Proposition 1.3 are not best possible. The main purpose of this paper is to show the point. In this paper, we use the important works of Heath-Brown [4], Yoichi Motohashi [5], Masanori Katsurada and Kohji Matsumoto [6], and the properties of Dirichlet L-functions to give three much sharper asymptotic formulae. That is, we shall prove the following theorems.

Theorem 1.1. *The number of positive squarefree primitive roots modulo p that are* $\le x$ *equals*

$$\frac{p\phi(p-1)}{(p^2-1)\zeta(2)}x + O\left(p^{9/44+\epsilon}x^{1/2+\epsilon}\right),$$

where ϵ *is any fixed positive number.*

Theorem 1.2. *The number of positive primitive roots* $\le x$ *which are squallfull numbers equals*

$$\frac{2C_3p\phi(p-1)}{(p^2-1)\,\zeta(2)}x^{1/2} + O\left(p^{9/44+\epsilon}x^{1/4+\epsilon}\right),$$

where $C_3 = \left[\prod_{p_1 \ne p}\left(1 + \frac{1}{\left(p_1^{1/2}-1\right)(p_1+1)}\right) - \prod_{p_1 \ne p}\left(1 + \frac{\left(\frac{p_1}{p}\right)}{\left(p_1^{1/2}-\left(\frac{p_1}{p}\right)\right)(p_1+1)}\right)\right]$.

Theorem 1.3. *The number of squarefree numbers $\leq x$ which are quadratic residues modulo p equals*

$$\frac{3}{\pi^2}x + O\left(p^{9/44+\epsilon}x^{1/2+\epsilon}\right).$$

2. Some Lemmas

To complete the proof of the theorems, we need following several lemmas.

Lemma 2.1. *Let prime $p > 2$. Then*

$$\sum_{k|p-1} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{a=1}^{k} \mathrm{e}\left(\frac{a\mathrm{ind}n}{k}\right) = \begin{cases} \dfrac{p-1}{\phi(p-1)}, & \text{if } n \text{ is a primitive root of } p; \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi iy}$, *$\mu(q)$ is the Möbius function,* $\mathrm{ind}n$ *denotes the index of n relative to some fixed primitive root of p, and* $\sideset{}{'}\sum_{a=1}^{k}$ *demotes the summation over a reduced residue system modulo k.*

Proof. See reference [7]. $\square$

The following two lemmas are the important works of Heath-Brown, Yoichi Motohashi, Masanori Katsurada and Kohji Matsumoto on Dirichlet L-functions.

Lemma 2.2. *For any χ modulo q, we have*

$$L(1/2 + it, \chi) \ll (q(|t| + 1))^{3/16+\epsilon},$$

where $L(s, \chi)$ is the Dirichlet L-function corresponding to χ.

Proof. See reference [4]. $\square$

Lemma 2.3. *Let χ be any primitive character modulo a prime p, then*

$$\int_0^T |L(1/2 + it, \chi)|^2 \, dt - \frac{(p-1)}{p}T\left[\log(pT/2\pi) + 2\gamma - 1 + 2(\log p)/(p-1)\right]$$
$$\ll (pT)^{1/3}(\log pT)^2 + p^{1/2}(\log pT)^3 \log T,$$

where $T \geq 1$, and γ is the Euler constant.

Proof. Yoichi Motohashi [5] treated this formula, which was developed by Masanori Katsurada and Kohji Matsumoto [6]. $\square$

Now we can get the following estimates:

Lemma 2.4. *Let χ be a primitive character modulo p and $T \geq 1$, then we have*

$$\int_0^T \left| \frac{L(1/2+it,\chi)}{t+1} \right| \mathrm{d}t \ll p^{9/44+\epsilon}$$

and

$$\int_0^T \left| \frac{\zeta(1/2+it)}{t+1} \right| \mathrm{d}t \ll T^{\epsilon}.$$

Proof. Let $0 < u < p$, from Cauchy inequality, Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned}
&\int_0^T \left| \frac{L(1/2+it,\chi)}{t+1} \right| \mathrm{d}t = \int_0^u \left| \frac{L(1/2+it,\chi)}{t+1} \right| \mathrm{d}t + \int_u^T \left| \frac{L(1/2+it,\chi)}{t+1} \right| \mathrm{d}t \\
&\ll u^{3/16+\epsilon} p^{3/16+\epsilon} + \left(\int_u^T \frac{1}{t+1} \mathrm{d}t \right)^{1/2} \left[\int_u^T \frac{|L(1/2+it,\chi)|^2}{t+1} \mathrm{d}t \right]^{1/2} \\
&\ll u^{3/16+\epsilon} p^{3/16+\epsilon} + T^{\epsilon} \left[\int_u^T \frac{\mathrm{d}\left(\int_u^t |L(1/2+is,\chi)|^2 \,\mathrm{d}s \right)}{t+1} \right]^{1/2} \\
&\ll u^{3/16+\epsilon} p^{3/16+\epsilon} + \left[p^{1/3+\epsilon} T^{\epsilon-2/3} + p^{1/2+\epsilon} T^{\epsilon-1} + p^{1/3+\epsilon} u^{\epsilon-2/3} + p^{1/2+\epsilon} u^{\epsilon-1} \right]^{1/2} \\
&\ll u^{3/16+\epsilon} p^{3/16+\epsilon} + p^{1/4+\epsilon} u^{\epsilon-1/2}.
\end{aligned}$$

Now taking $u = p^{1/11}$ in the above, then we have

$$\int_0^T \left| \frac{L(1/2+it,\chi)}{t+1} \right| \mathrm{d}t \ll p^{9/44+\epsilon}.$$

Using the same methods we can get

$$\int_0^T \left| \frac{\zeta(1/2+it)}{t+1} \right| \mathrm{d}t \ll T^{\epsilon}.$$

This proves Lemma 2.4. □

Lemma 2.5. *Let A denote the set of squarefree integers, and χ be any character modulo p, then*

$$\sum_{\substack{n \leq x \\ n \in A}} \chi(n) = \begin{cases} \dfrac{p}{p+1} \cdot \dfrac{x}{\zeta(2)} + O(x^{1/2+\epsilon}), \\ \qquad \text{if } \chi \text{ is principal character modulo } p; \\ O(p^{9/44+\epsilon} x^{1/2+\epsilon}), \\ \qquad \text{otherwise.} \end{cases}$$

Proof. It is obvious that

$$\sum_{\substack{n\le x\\ n\in A}} \chi(n) = \sum_{n\le x} \mu^2(n)\chi(n).$$

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{\mu^2(n)\chi(n)}{n^s},$$

by the Euler product formula [8] we have

$$f(s) = \prod_{p_1}\left(1 + \frac{\chi(p_1)}{{p_1}^s}\right) = \frac{L(s,\chi)}{L(2s,\chi^2)}.$$

Note that $\left|\mu^2(n)\chi(n)\right| \le 1$ and $\sum\limits_{n=1}^{\infty}\left|\mu^2(n)\chi(n)\right| n^{-\sigma} \le \zeta(\sigma)$. For any complex $s_0 = \sigma_0 + it_0$, and real number $b > 0$, by Perron formula [9] we have

$$\sum_{n\le x} \frac{\mu^2(n)\chi(n)}{n^{s_0}} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^b\zeta(b+\sigma_0)}{T}\right) + O\left(x^{1-\sigma_0}\min\left(1, \frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0}\min\left(1, \frac{x}{||x||}\right)\right).$$

That is

$$\sum_{\substack{n\le x\\ n\in A}} \chi(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{L(s,\chi)}{L(2s,\chi^2)}\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^b\zeta(b)}{T}\right) + O\left(x\min\left(1, \frac{\log x}{T}\right)\right). \tag{2.1}$$

If χ is nonprincipal character modulo p, then taking $b = 1/2$, $T = x^{1/2}$ in the above formula, by Lemma 2.4 we easily have

$$\sum_{\substack{n\le x\\ n\in A}} \chi(n) \ll \int_0^T \left|\frac{L(1/2+it,\chi)}{L(1+2it,\chi^2)}\frac{x^{1/2+\epsilon}}{(t+1)}\right|\mathrm{d}t + O\left(x^{1/2+\epsilon}\right) \ll p^{9/44+\epsilon}x^{1/2+\epsilon}.$$

On the other hand, if χ is principal character modulo p, note that $L(s,\chi) = \zeta(s)(1-p^{-s})$, then taking $b = 2$, $T = x^{3/2}$ in formula (2.1), we have

$$\sum_{\substack{n\le x\\ n\in A}} \chi(n) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}\mathrm{d}s + O\left(x^{1/2+\epsilon}\right).$$

We move the integral line from $s = 2 \pm iT$ to $s = 1/2 \pm iT$. This time, the function $\frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}$ has a simple pole point at $s=1$ with residue $\frac{p}{p+1}\frac{x}{\zeta(2)}$. So we have

$$\begin{aligned}\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{1/2+iT}+\int_{1/2+iT}^{1/2-iT}+\int_{1/2-iT}^{2-iT}\right)\frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}\mathrm{d}s \\ = \frac{p}{p+1}\frac{x}{\zeta(2)}.\end{aligned}$$

Note that

$$\zeta(\sigma+it) \ll |t|^{(1-\sigma)/2}\log|t|, \quad 0 \le \sigma \le 1, \quad |t| \ge 2,$$

then we get the estimates

$$\begin{aligned}\frac{1}{2\pi i}\int_{2+iT}^{1/2+iT}\frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}\mathrm{d}s &\ll \int_{1/2}^{2}\frac{|\zeta(\sigma+iT)|\,x^\sigma}{T}\mathrm{d}\sigma \\ &\ll \frac{x^{1/2}}{T^{3/4-\epsilon}}+\frac{x^2}{T^{1-\epsilon}} \ll x^{1/2+\epsilon}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_{1/2-iT}^{2-iT}\frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}\mathrm{d}s &\ll \int_{1/2}^{2}\frac{|\zeta(\sigma-iT)|\,x^\sigma}{T}\mathrm{d}\sigma \\ &\ll \frac{x^{1/2}}{T^{3/4-\epsilon}}+\frac{x^2}{T^{1-\epsilon}} \ll x^{1/2+\epsilon}.\end{aligned}$$

By Lemma 2.4 we also get

$$\frac{1}{2\pi i}\int_{1/2+iT}^{1/2-iT}\frac{\zeta(s)(1-p^{-s})}{\zeta(2s)(1-p^{-2s})}\frac{x^s}{s}\mathrm{d}s \ll \int_0^T\left|\frac{\zeta(1/2+it)}{\zeta(1+2it)}\frac{x^{1/2+\epsilon}}{(t+1)}\right|\mathrm{d}t \ll x^{1/2+\epsilon},$$

then from the above we have

$$\sum_{\substack{n\le x\\ n\in A}}\chi(n) = \frac{p}{p+1}\cdot\frac{x}{\zeta(2)}+O\left(x^{1/2+\epsilon}\right), \quad \text{if } \chi \text{ is principal character modulo } p.$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let B denote the set of squarefull integers, and χ be any character modulo p, then*

$$\sum_{\substack{n\le x\\ n\in B}}\chi(n) = \begin{cases}\dfrac{2f(\chi)px^{1/2}}{(p+1)\,\zeta(2)}+O(x^{1/4+\epsilon}), \\ \quad \textit{if } \chi^2 \textit{ is principal character modulo } p; \\ O(p^{9/44+\epsilon}x^{1/4+\epsilon}), \\ \quad \textit{otherwise,}\end{cases}$$

where $f(\chi) = \prod\limits_{p_1 \neq p} \left(1 + \frac{\chi(p_1)}{\left(p_1^{1/2} - \chi(p_1)\right)(p_1+1)}\right)$.

Proof. We define a new arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } n = 1; \\ \chi(n), & \text{if } n \text{ is a squarefull number;} \\ 0, & \text{if } n \text{ is not a squarefull number.} \end{cases}$$

It is clear that

$$\sum_{\substack{n \leq x \\ n \in B}} \chi(n) = \sum_{n \leq x} a(n).$$

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

by the Euler product formula [8] we have

$$\begin{aligned} f(s) &= \prod_{p_1} \left(1 + \frac{\chi^2(p_1)}{{p_1}^{2s}} + \frac{\chi^3(p_1)}{{p_1}^{3s}} + \cdots\right) = \prod_{p_1} \left(1 + \frac{\chi^2(p_1)}{p_1^{2s}} \cdot \frac{p^s}{p^s - \chi(p)}\right) \\ &= \prod_{p_1} \left(1 + \frac{\chi^2(p_1)}{p_1^{2s}}\right) \prod_{p_1} \left(1 + \frac{\chi^3(p_1)}{(p_1^s - \chi(p))\,(p_1^{2s} + \chi^2(p))}\right) \\ &= \frac{L(2s, \chi^2)}{L(4s, \chi^4)} \prod_{p_1} \left(1 + \frac{\chi^3(p_1)}{(p_1^s - \chi(p))\,(p_1^{2s} + \chi^2(p))}\right). \end{aligned}$$

Note that $|a(n)| \leq 1$ and $\sum\limits_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \leq \zeta(\sigma)$. For any complex $s_0 = \sigma_0 + it_0$, and real number $b > 0$, by Perron formula [9] we have

$$\begin{aligned} \sum_{n \leq x} \frac{a(n)}{n^{s_0}} &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s + s_0) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^b \zeta(b + \sigma_0)}{T}\right) \\ &\quad + O\left(x^{1-\sigma_0} \min\left(1, \frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0} \min\left(1, \frac{x}{||x||}\right)\right). \end{aligned}$$

That is

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in B}} \chi(n) &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{L(2s, \chi^2)}{L(4s, \chi^4)} \prod_{p_1} \left(1 + \frac{\chi^3(p_1)}{(p_1^s - \chi(p))\,(p_1^{2s} + \chi^2(p))}\right) \frac{x^s}{s} \mathrm{d}s \\ &\quad + O\left(\frac{x^b \zeta(b)}{T}\right) + O\left(x \min\left(1, \frac{\log x}{T}\right)\right). \end{aligned}$$

Using the same methods in Lemma 2.5 we can have

$$\sum_{\substack{n\le x\\ n\in B}} \chi(n) = \begin{cases} \dfrac{2f(\chi)px^{1/2}}{(p+1)\,\zeta(2)} + O(x^{1/4+\epsilon}), \\ \quad \text{if } \chi^2 \text{ is principal character modulo } p; \\ O(p^{9/44+\epsilon}x^{1/4+\epsilon}), \\ \quad \text{otherwise.} \end{cases}$$

This completes the proof of Lemma 2.6. □

3. Proof of the theorems

In this section, we complete the proof of the theorems. First we prove Theorem 1.1. Let C denote the set of primitive roots modulo p. Note that $\chi_{a,k}(n) = \mathrm{e}\left(\frac{a\mathrm{ind}n}{k}\right)$ is a character modulo p, and $\chi_{a,k}$ is principal character if and only if $k=1$. So from Lemma 2.1 and Lemma 2.5 we have

$$\begin{aligned} \sum_{\substack{n\le x\\ n\in A\\ n\in C}} 1 &= \frac{\phi(p-1)}{p-1} \sum_{k|p-1} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{a=1}^{k} \sum_{\substack{n\le x\\ n\in A\\ (n,p)=1}} \mathrm{e}\left(\frac{a\mathrm{ind}n}{k}\right) \\ &= \frac{\phi(p-1)}{p-1} \sum_{k|p-1} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{a=1}^{k} \sum_{\substack{n\le x\\ n\in A}} \chi_{a,k}(n) \\ &= \frac{p\phi(p-1)}{(p^2-1)\zeta(2)}x + O\left(p^{9/44+\epsilon}x^{1/2+\epsilon}\right). \end{aligned}$$

This proves Theorem 1.1.

Now we prove Theorem 1.2. Note that $\chi_{a,k}^2$ is principal character if and only if $k=1$ or $k=2$. So from Lemma 2.1 and Lemma 2.6 we have

$$\begin{aligned} \sum_{\substack{n\le x\\ n\in B\\ n\in C}} 1 &= \frac{\phi(p-1)}{p-1} \sum_{k|p-1} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{a=1}^{k} \sum_{\substack{n\le x\\ n\in B\\ (n,p)=1}} \mathrm{e}\left(\frac{a\mathrm{ind}n}{k}\right) \\ &= \frac{\phi(p-1)}{p-1} \sum_{k|p-1} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{a=1}^{k} \sum_{\substack{n\le x\\ n\in B}} \chi_{a,k}(n) \\ &= \frac{2C_3p\phi(p-1)}{(p^2-1)\,\zeta(2)}x^{1/2} + O\left(p^{9/44+\epsilon}x^{1/4+\epsilon}\right). \end{aligned}$$

This completes the proof of Theorem 1.2.

Let D denote the set of quadratic residues modulo p, then from [2] and Lemma 2.5 we have

$$\sum_{\substack{n\le x\\ n\in A\\ n\in D}} 1 = \sum_{\substack{n\le x\\ n\in A}} \frac{1}{2}\left(1+\left(\frac{n}{p}\right)\right) = \frac{x}{2\zeta(2)}+\frac{1}{2}\sum_{\substack{n\le x\\ n\in A}}\left(\frac{n}{p}\right) = \frac{3}{\pi^2}x+O\left(p^{9/44+\epsilon}x^{1/2+\epsilon}\right).$$

This completes the proof of Theorem 1.3.

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Department of Mathematics
Northwest University
Xi'an, Shaanxi, P. R. China
e-mail: hnliu@nwu.edu.cn
wpzhang@nwu.edu.cn

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